Additional Problems
Math 346 – Prof. Santoro

Instructions: This document contains sample problems, to be used in preparation for your final exam. Allow yourself at most 15 minutes per question when attempting to solve the problems.

[**1**] Solve the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 6 \\ 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 26 \\ 34 \end{bmatrix}.$$

[**2**]

(a) Write down the definition of a subspace of a vector space.

(b) Prove that the set S of all vectors (x_1, x_2, x_3) in $\mathbb{R}^3$ such that $x_1 + x_2 = 0$ is a subspace of $\mathbb{R}^3$.

[**3**] Find bases for the four fundamental subspaces of A:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}$$

[**4**]

(a) Find the orthogonal projection of the vector $b = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ on the subspace spanned by the vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(b) If P is the plane spanned by the vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ in $\mathbb{R}^3$, find a basis for the orthogonal complement $P^\perp$.

[**5**] Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 3 \\ 1 & 1 & 4 & 8 \\ 1 & 1 & 1 & 5 \end{bmatrix},$$

Compute the determinant of the matrix $B = A^4 (A^T)^3 A^{-5}$, justifying all of your steps.

[**6**]

(a) Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$$

(b) Find $\lim_{k\to\infty} A^k$, justifying your answer.

[7]

Let $\mathcal{P}^3$ (resp. $\mathcal{P}^4$) be the vector space of polynomials of degree less than or equal to 3 (resp. 4).

Let $T : \mathcal{P}^3 \to \mathcal{P}^4$ be the linear transformation that assigns to each polynomial $p(x)$ in $\mathcal{P}^3$ its only antiderivative which vanishes at $x = 0$:

$$T(p) = \int_0^x p(t)dt.$$

For example, if $p(x) = 1 + 2x + 3x^2$, then $T(p(x)) = x + x^2 + x^3$.

(a) Write down a basis for $\mathcal{P}^3$.

(b) Write down a basis for $\mathcal{P}^4$.

(c) For your choice of basis, find the matrix M_T that represents the linear transformation T.

[8] Suppose A is a 3×3 matrix with eigenvalues $1, 2$ and 3, and associated eigenvectors v_1, v_2, v_3.

Define the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(v) = Av.$$

(a) Prove that T is a linear transformation.

(b) Choose v_1, v_2, v_3 to be a basis for $\mathbb{R}^3$ (the input and output space for T). For this choice of basis, what is the matrix that represents the linear transformation T?

[9] Suppose a 4×4 matrix A has eigenvectors $0, 0, 1$ and 2, associated to the eigenvectors z, u, v and w, respectively. Assume that z and u are linearly independent.

(a) (5 pts) Find a basis for the nullspace of A, and a basis for the column space of A.

(b) (5 pts) Find the **complete** solution to the system $Ax = v + w$.

[10] Solve the linear system of differential equations

$$\begin{cases} x' = 3x - y \\ y' = -x + 3y \\ x(0) = 1 \\ y(0) = 0 \end{cases}$$

[11] Suppose a 4×4 matrix A has eigenvalues $0, 0, 1$ and 2. What is the determinant of the matrix $B = (A^2 + I)^{-1}$? Justify your answer.

[12] Let a_1, a_2, a_3 be linearly independent vectors in $\mathbb{R}^3$, and let q_1, q_2, q_3 be the vectors obtained from a_1, a_2, a_3 by the Gram-Schmidt algorithm.

Define the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(q_1) = a_1$, $T(q_2) = a_2$ and $T(q_3) = a_3$. For the choice of basis $\{q_1, q_2, q_3\}$ both for the input and output spaces, find the matrix M_T which represents the linear transformation T for this choice of basis.

[13] Let A be a matrix with orthogonal columns $w_1, \cdots, w_n$, of lengths $\sigma_1, \cdots, \sigma_n$. What are U, Σ and V in the singular value decomposition of A?

[14] If A is the matrix

$$A = \begin{bmatrix} a & b & c & d+1 \\ a & b & c+1 & d \\ a & b+1 & c & d \\ a+1 & b & c & d \end{bmatrix}$$

find the determinant of $B = 3A^5 A^t A^{-1}$. Please justify your answer.

[15]

(a) Write down the definition of an eigenvector v associated to an eigenvalue λ of a matrix A.

(b) Find the eigenvalues and eigenvectors for the matrix $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

[16]

(a) Define **linear independence** for a set of vectors $v_1, \ldots, v_k \in \mathbb{R}^n$.

(b) Give an example of three linearly **independent** vectors in $\mathbb{R}^4$, and prove that they are indeed linearly independent.

(c) Give an example of three linearly **dependent** vectors in $\mathbb{R}^5$, and prove that they are indeed linearly dependent.

[17]

(a) Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}.$$

(b) Use your answer above to solve the system $Ax = b$, where $b = [1\,2\,3\,4]^t$.

[18]

(a) Let A be the matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Give sufficient conditions on those constants in order to guarantee that the matrix A is invertible.

(b) Assuming the conditions on a), solve the system

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2a \\ a+b \\ a+b \\ a+b \end{bmatrix}.$$

[19]

(a) Write down the definition of a *subspace* S of a vector space V.

(b) Recall that a *diagonal matrix* $A = (a_{ij})$ is a square matrix such that the entry a_{ij} is zero whenever $i \neq j$.

Show that the set D of all 3×3 diagonal matrices is a subspace of $\mathcal{M}_{3\times3}$.

(c) Let S be the set of 3×3 matrices A such that the sum of the entries of A is exactly 1. Is S a subspace of $\mathcal{M}_{3\times3}$? Justify your answer.

[20]

(a) Give an example of a 4×4 matrix A such that the system $Ax = b$ is solvable if $b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ or $b = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, but it is **not** solvable if $b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$.

Please justify your answer.

(b) Is it possible to construct a 3×3 matrix B such that the system $Ax = b$ is solvable if $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ or $b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, but it is **not** solvable if $b = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$?

Please justify your answer.

[21]

Let A be a matrix. Show that $A^t A$ is invertible if and only if A has linearly independent columns.

Hint: Show that the nullspaces of A^tA and A must be the same.

[22]

Suppose you are given four nonzero vectors r, n, c, ℓ.

(a) What are the conditions for those vectors to be bases for the four fundamental subspaces $R(A)$, $N(A)$, $C(A)$ and $N(A^t)$ of a 2×2 matrix A, respectively? You must justify your work.

(b) What is one possible matrix A? The matrix A may depend on r, n, c, ℓ. You must justify your work.

[23]

Let A be a 4×5 matrix such that the vector $s_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ is the only special solution.

(a) What is the rank of the matrix A? Justify your answer.

(b) Is it possible to solve $Ax = b$ for any $b \in \mathbb{R}^4$? Justify your answer.

(c) Find the complete solution to the system $Ax = b$, where $b =$ (column 1 of A) + (column 3 of A) + (column 5 of A). You must justify your work.

[24]

Let S be the subspace of $\mathcal{M}_{3\times 3}$ defined by

$$S = \{A \in \mathcal{M}_{3\times 3}; A^t = -A\}.$$

(a) Find a basis for S.

(b) Show that the basis you found is indeed a basis for S.

[25]

Let A be a 3×3 matrix such that there exist three nonzero vectors v_1, v_2 and v_3 which satisfy

$$Av_1 = 3v_1, \quad Av_2 = 4v_2 \text{ and } Av_3 = 5v_3.$$

(a) Find the determinant of the matrix $B = (A - 2I)^{-1}A^t(3A)^2$.

Note: No credit will be given if you use a particular matrix A to find your answer.

(b) Find bases for the nullspace and the column space of the matrix $C = A - 4I$. Justify your answer.

[26]

Let A be the following 3×3 matrix:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

It is known that $\det(A) = 5$.

Find the determinant of the matrix B, given by

$$B = \begin{bmatrix} g+a & h+b & i+c \\ 3a & 3b & 3c \\ a+d+g & b+e+h & c+f+i \end{bmatrix}.$$

You need to justify your answer.

[**27**]

Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

(a) Find the four eigenvalues of A. Hint: what is the rank of A?

(b) Let $B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$. Find the eigenvalues of B, and justify your answer.

[**28**]

Let $r(t)$ denote the rabbit population at time t, and let $w(t)$ denote the wolf population at time t. Those functions satisfy the following differential equation:

$$\begin{cases} r'(t) = 6r - 2w \\ w'(t) = 2r + w \end{cases}$$

Given that $r(0) = w(0) = 30$, find the populations $r(t)$ and $w(t)$ after time t.

[**29**] Find four vectors u_1, u_2, v_1 and v_2, and two real numbers σ_1 and σ_2, such that the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 2 & 2 \end{bmatrix}$$

can be written as

$$A = \sigma_1 u_1 v_1^t + \sigma_2 u_2 v_2^t.$$

The following problems are a courtesy of Prof. Merenkov.

[M1] Find the set of all solutions to the following system. If there are no solutions, state so and justify.

$$\begin{cases} 2x_2 - x_3 - x_4 = 1, \\ x_1 + x_2 - x_3 + x_4 = 0, \\ 3x_1 + 3x_2 - 2x_3 - x_4 = -2. \end{cases}$$

[M2] Let

$$A = \begin{bmatrix} -2 & 2 & -1 \\ 1 & 1 & 2 \\ 2 & -2 & 3 \end{bmatrix}.$$

(a) Find A^{-1}.

(b) Use the inverse matrix above to solve the system

$$\begin{cases} -2x_1 + 2x_2 - x_3 = 2, \\ x_1 + x_2 + 2x_3 = -1, \\ 2x_1 - 2x_2 + 3x_3 = 5. \end{cases}$$

(c) Write the following matrix A as a product of elementary matrices.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 3 & 5 \end{bmatrix}.$$

[M3] Let $\vec{v}_1 = (3, -1, 2), \vec{v}_2 = (-1, 0, -3), \vec{v}_3 = (3, -2, -5)$ be vectors in $\mathbb{R}^3$.

(a) Find the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

(b) Does the system

$$\begin{cases} 3x_1 - 1x_2 + 3x_3 = 3, \\ -1x_1 - 2x_3 = -1, \\ 2x_1 - 3x_2 - 5x_3 = 5 \end{cases}$$

have a solution? Justify your claim.

[M4] Let $V = M_{2\times 2}$ be the vector space of all 2×2 matrices and let W consist of all 2×2 matrices

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

such that $a_{21} = -2 \cdot a_{12} + 1$. Is W a vector subspace of V? Answer 'yes' or 'no' and justify your claim.

[M5] Let p_1, p_2, p_3 be polynomials defined by

$$p_1(x) = x^2 + x + 1, \quad p_2(x) = 2x^2 + 1, \quad p_3(x) = 2x.$$

(a) Verify that p_1, p_2, p_3 are linearly independent in the space P_2 of all polynomials of degree at most 2.

(b) Express the polynomial $p(x) = x^2 - x + 1$ as a linear combination of p_1, p_2, p_3.

[M6] Let

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 3 & 1 & -3 \\ -3 & -3 & 0 & 6 \end{bmatrix}.$$

(a) Which columns of A form a basis for the column space $C(A)$ of the matrix A?

(b) Express the non-basis columns of A as linear combinations of the columns of A from the basis for $C(A)$.

[M7] Find a basis for the orthogonal complement $W^{\perp}$ of $W = \text{span}\{\vec{v}_1, \vec{v}_2\}$, where

$$\vec{v}_1 = (-1, 2, 3, 0), \quad \vec{v}_2 = (2, 0, -1, 1).$$

What is the dimension of $W^{\perp}$?

[M8] Use the Least Squares Approximation to find a line $y = \alpha + \beta t$ that best fits the data $(-1, 1), (0, 2), (1, 1), (2, 0)$. Here, the first component is the t-value and the second is the y-value.

[M9] Use the Gram–Schmidt process to find an orthonormal basis for

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

The following problems are a courtesy of Prof. Steinberg.

[S1] Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and use it to solve the system

$$\begin{aligned} x + 2y &= 1 \\ 3x + 5y &= 2 \end{aligned}$$

[S2] Find the complete solution to the linear system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 1 & 2 & 5 \\ 2 & 2 & 3 & 9 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

[S3] Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 1 \\ 4 & 7 & 5 \end{bmatrix}.$$

Find a basis for the column space of A consisting of columns from A and determine the rank of A.

[S4] Compute the inverse of $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ or show that A is not invertible.

[S5] Find numbers a, b, c, d not all zero such that the plane

$$ax + by + cz = d$$

contains the points $(1, 2, 3)$, $(0, 1, 0)$ and $(1, 0, 1)$. (Hint: Rewrite the equation as $ax + by + cz - d = 0$ and obtain a linear system of three equations in four unknowns using the three points. Choose a special solution to obtain a non-zero solution.)

[S6] Compute the determinant of the matrix $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 4 & 1 & 3 & 2 \end{bmatrix}$ and determine whether A is invertible.

[S7] Use the Gram-Schmidt process to find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $(1, 0, 0, 1), (1, 1, 0, 0), (1, 1, 1, 1)$.

[S8] Let

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 & 1 \\ 2 & 1 & 1 & 1 & -1 \\ 1 & 1 & 0 & 2 & -2 \end{bmatrix}$$

and you are given that the row reduced echelon form of A is

$$R = \begin{bmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

1. Compute the rank of A and the dimension of $N(A)$. [2 points]
2. Find a basis for null(A). [3 points]
3. Find a basis for row(A). [2 points]
4. Find a basis for col(A) consisting of columns of A. [3 points]

[S9] Find the line of best fit $y = a + bt$ for the data points $(1, 3)$, $(2, 4)$ and $(-1, -1)$.

[S10] Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ and an eigenvector for each eigenvalue.