Math 39100 K - Test Solutions, Spring 2019

Ethan Akin

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(1) (12 points each)Compute the general solution of each of the following differential equations:

(a) $$x\frac{dy}{dx} + 3xy = x^2e^{-3x} + y.$$

(Linear Equation) $\frac{dy}{dx} + (3 - \frac{1}{x})y = xe^{-3x}$. $\mu = exp[\int (3 - \frac{1}{x})dx] = exp[3x - \ln x] = x^{-1}e^{3x}$.

$[x^{-1}e^{3x}y]' = 1$. and so $x^{-1}e^{2x}y = x + C$.

$$y = x^2e^{-3x} + Cxe^{-3x}.$$

(b) $$5xy\ dy = (6y^2 + xy - 6x^2)\ dx.$$

(Homogeneous) $\frac{dy}{dx} = \frac{6y^2+xy-6x^2}{5xy}$. So with $z = y/x$,

$x\frac{dz}{dx} + z = \frac{6z^2+z-6}{5z}$

$x\frac{dz}{dx} = \frac{z^2+z-6}{5z} = \frac{(z+3)(z-2)}{5z}$.

(Partial Fractions:) $\frac{5z}{(z+3)(z-2)} = \frac{A}{(z+3)} + \frac{B}{(z-2)}$.

$5z = A(z-2) + B(z+3)$. With $z = 2, -3 : 2 = B, 3 = A$.

$\int \frac{3}{(z+3)} + \frac{2}{(z-2)}dz = \int \frac{dx}{x}$.

(c) $$yy'' + (y')^2 = 0 \qquad (y \text{ is a function of } x).$$

(Reduction of Order) $yv\frac{dv}{dy} = -v^2$. $\frac{dv}{v} = -\frac{dy}{y}$

$\ln(v) = -\ln(y) + C_1$, or $y' = v = \frac{C_1}{y}$

$$\int ydy = \int C_1dx,, \text{ or } y^2 = C_1x + C_2.$$

(d) $$y'' + y' - 6y = 0 \qquad (y \text{ is a function of } x).$$

(Second Order, Linear, Homogeneous, Constant Coefficients) Test solution $y = e^{rx}$ with characteristic equation $r^2 + r - 6 = (r-2)(r+3) = 0$. Roots $r = 2, -3$.

$$y = C_1 e^{2x} + C_2 e^{-3x}.$$

(2) (12 points each) Solve the following initial value problems:

(a) $$(\cos(xy) - xy\sin(xy) + x)\,dx + (2y - x^2\sin(xy))\,dy = 0. \text{ with } \quad y(0) = 5.$$

(Exact) $\partial_y[\cos(xy) - xy\sin(xy) + x] =$
$x\sin(xy) - x\sin(xy) - x^2 y\sin(xy)$
$= \partial_x[2y - x^2\sin(xy)]$.

$F(x,y) = \int(2y - x^2\sin(xy))\,dy = y^2 + x\cos(xy) + H(x)$.

$\partial_x F = \cos(xy) - xy\sin(xy) + H'(x) = \cos(xy) - xy\sin(xy) + x$. and so $H'(x) = x$.

$y^2 + x\cos(xy) + \frac{1}{2}x^2 = C$. With $x = 0, y = 5$, and so $C = 25$.

$$y^2 + x\cos(xy) + \frac{1}{2}x^2 = 25.$$

(b) $$\frac{dy}{dx} = xy^2 + x^2y^2. \text{ with } y(0) = -3.$$

(Variables Separable) $y^{-2}dy = (x + x^2)dx$.

$-y^{-1} = \frac{1}{2}x^2 + \frac{1}{3}x^3 + C$. $\qquad C = \frac{1}{3}$

$y = -\frac{6}{3x^2 + 2x^2 + 2}$.

(3) (10 points) A tank with a capacity of 100 gallons contains 20 gallons of a salt solution with concentration 2 pound per gallon. Pure water is poured in at a rate of 4 gallons per minute and the well stirred solution is pumped out at the same rate.

(a) Set up an initial value problem for $S(t)$ the amount of salt in the tank at time t.

V is constant at 20 gallons. $S_0 = 2$ pounds/gallon $\times 20$ gallons $= 40$ pounds.

$\frac{dS}{dt} = 4$ gallons/minute $\times 0$ pounds/gallon -4 gallons/minute $\times \frac{S}{20}$ pounds/gallon.

$$\frac{dS}{dt} = -\frac{S}{5}, S(0) = 40.$$

(b) Solve the equation to get a formula for $S(t)$.

(Exponential Growth, Variables Separable): $\ln(S) = -\frac{t}{5} + C$.

$$S = 40e^{-t/5}.$$

(4) (8 points) Offred opens a bank account with \$100 and add, continuously, \$500 per year. The bank pays interest at a rate of 3% per year compounded continuously. Set up an initial value problem (equation plus initial conditions) whose solution is $P(t)$ the amount of money in the bank at time t measure in years. You need not solve the equation.

$\frac{dP}{dt} = rP + k$ with $r = .03, k = 500, P_0 = 100$.

$$\frac{dP}{dt} = .03P + 500, \quad P_0 = 100.$$

(5)(10 points) Assume that y_1, y_2 are solutions of the equation $y'' + py' + qy = 0$ where p and q are functions of t defined for all t.

(a) Show that $y_g = C_1y_1 + C_2y_2$ is a solution of the equation for all constant values C_1 and C_2.

$y_g'' + py_g' + qy_g = [C_1y_1 + C_2y_2]'' + p[C_1y_1 + C_2y_2]' + q[C_1y_1 + C_2y_2]$
$= C_1[y_1'' + py_1' + qy_1] \; + \; C_2[y_2'' + py_2' + qy_2] \; = \; C_1[0] + C_2[0] \; = \; 0.$

(b) Define the Wronskian $W(y_1, y_2)$.

$W = y_1y_2' - y_2y_1'$.

(c) Assume the Wronskian is never zero. Explain why $y_g = C_1y_1 + C_2y_2$ is the general solution. That is, these are all the solutions.

For any A, B the initial condition equations $y_g(x_0) = A, y_g'(x_0) = B$, are

$$\begin{aligned} C_1y_1(x_0) + C_2y_2(x_0) &= A, \\ C_1y_1'(x_0) + C_2y_2'(x_0) &= B, \end{aligned}$$

Since the Wronskian is not zero we can solve for C_1, C_2 by using Cramer's Rule. Since every IVP has a solution of the form $C_1y_1 + C_2y_2$, these are all the solutions.

(1)(18 points) Solve the initial value problem: $y'' - 3y' = 36t + 18e^{-3t}$ with $y(0) = 0$ and $y'(0) = 0$.

The characteristic equation is $0 = r^2 - 3r = r(r-3)$ with roots $0, 3$. So the general solution of the homogeneous equation is $y_h = C_1 + C_2 e^{3t}$.

For $36t$ the associated root is 0 which is a root of the homogeneous equation and for $18e^{-3t}$ the associated root is -3. $Y_p = Ae^{-3t} + Bt^2 + Ct$

Substitute

$$\begin{array}{rcl} 0 \times\ Y_p &=& 0 \times [\ Ae^{-3t} \ +\ Bt^2 \ +\ Ct \quad] \\ -3 \times\ Y_p' &=& -3 \times [-3Ae^{-3t} \ + \qquad 2Bt \ +\ C] \\ \underline{1 \times\ Y_p''} &=& \underline{1 \times [\ 9Ae^{-3t} + \qquad\qquad +2B} \\ 36t + 18e^{-3t} &=& 18Ae^{-3t} - 6Bt \ +\ (2B - 3C). \end{array}$$

So $A = 1, B = -6, C = -4$. So

$$\begin{array}{rcl} y &=& C_1 + C_2 e^{3t} + e^{-3t} - 6t^2 - 4t, \\ y' &=& \qquad 3C_2 e^{3t} + -3e^{-3t} \quad - 12t - 4. \end{array}$$

At $t = 0$, $0 = C_1 + C_2 + 1, 0 = 3C_2 - 3 - 4$. So $C_1 = -10/3, C_2 = 7/3$.

$$y = (-10/3) + (7/3)e^{3t} + e^{-3t} - 6t^2 - 4t.$$

(2)(18 points) Consider the seventh order differential equation $y^{(7)} - 2y^{(4)} + y' = 3t^2 + te^{-t/2} - 7e^{-t/2}\sin((\sqrt{3}/2)t) + \cos((\sqrt{3}/2)t) + 1$.

(a) Compute the general solution of the associated homogeneous equation.

The characteristic equation is

$$0 = r^7 - 2r^4 + r = r(r^3 - 1)^2 = r(r-1)^2(r^2 + r + 1)^2$$

The roots are $0, 1, 1, -\frac{1}{2} \pm \mathbf{i}\frac{\sqrt{3}}{2}, -\frac{1}{2} \pm \mathbf{i}\frac{\sqrt{3}}{2}$.

$$\begin{array}{rcl} y_h &=& C_1 \ +\ C_2 e^t \ +\ C_3 te^t + \\ && C_4 e^{-t/2}\cos((\sqrt{3}/2)t) \ +\ C_5 e^{-t/2}\sin((\sqrt{3}/2)t) \ + \\ && C_6 te^{-t/2}\cos((\sqrt{3}/2)t) \ +\ C_7 te^{-t/2}\sin((\sqrt{3}/2)t). \end{array}$$

(b) Write down the test function with the fewest terms which can be used to obtain a particular solution via the Method of Undetermined Coefficients. Do not solve for the constants.

The associated root for $3t^2, +1$ is 0, for $te^{-t/2}$ is $-\frac{1}{2}$, for $-7e^{-t/2}\sin((\sqrt{3}/2)t)$ is $-\frac{1}{2} \pm \mathbf{i}\frac{\sqrt{3}}{2}$, and for $\cos((\sqrt{3}/2)t)$ is $\pm\mathbf{i}\frac{\sqrt{3}}{2}$. So

$$\begin{aligned} Y_p &= t[At^2 + Bt + C] + [(Dt + E)e^{-t/2}] + \\ t^2[Fe^{-t/2}&\cos((\sqrt{3}/2)t) + Ge^{-t/2}\sin((\sqrt{3}/2)t)] \\ &+[H\cos((\sqrt{3}/2)t) + I\sin((\sqrt{3}/2)t)]. \end{aligned}$$

(3)(16 points) Compute the general solution of $y'' + y = \tan(t)$.

The characteristic equation is $0 = r^2 + 1$ with roots $\pm\mathbf{i}$. So $y_h = C_1\cos(t) + C_2\sin(t)$.

We use Variation of Parameters, looking for a solution of the form $y_p = u_1\cos(t) + u_2\sin(t)$.

$$\begin{aligned} u_1'(\cos(t)) \;+\; u_2'(\sin(t)) &= 0 \\ u_1'(-\sin(t)) \;+\; u_2'(\cos(t)) &= \tan(t). \end{aligned}$$

The Wronskian equals $\cos^2(t) + \sin^2(t) = 1$.

$u_1' = -\sin(t)\tan(t) = (\cos^2(t) - 1)/\cos(t) = \cos(t) - \sec(t)$ and so $u_1 = \sin(t) - \ln(\sec(t) + \tan(t))$.

$u_2' = \cos(t)\tan(t) = \sin(t)$ and so $u_2 = -\cos(t)$.

So $y_p = -[\ln(\sec(t) + \tan(t))]\cos(t)$. and

$$y_g = C_1\cos(t) + C_2\sin(t) - [\ln(\sec(t) + \tan(t))]\cos(t).$$

(4)(16 points) For the differential equation, $(x-1)y'' - xy' + y = 0$, $y_1 = e^x$ is a solution. Use the method of Reduction of Order to compute a second solution y_2 which is independent of the first one.

$y_2 = ue^x$.

$$\begin{array}{rrl} 1\times & y_2 = & ue^x \\ -x\times & y_2' = & -x\times\ [ue^x \quad + u'e^x] \\ (x-1)\times & y_2'' = & (x-1)\times [ue^x + 2u'e^x \quad + u''e^x] \\ \hline & 0 = & 0 \;+\; +(x-2)e^x u' + (x-1)e^x u''. \end{array}$$

Let $v = u'$ so that $v' = -\frac{(x-2)}{(x-1)}v$.

$$\frac{dv}{v} = -1 + \frac{1}{(x-1)}dx.$$

$\ln(v) = -x + \ln(x-1)$ and so $\frac{du}{dx} = v = (x-1)e^{-x}$ or $u = -xe^{-x}$.

$$y_2 = uy_1 = -xe^{-x}e^x = -x.$$

(5)(14 points) A hanging spring is stretched 6 inches (= .5 feet) by a weight of 24 pounds.

(a) Set up the initial value problem (differential equation and initial conditions) which describes the motion, neglecting friction, when the weight is pulled down an additional foot and is then released and is subjected to an external force of $6\cos(\omega t)$. You need not solve the equation. (Recall that g, the acceleration due to gravity is 32 feet/second^2.)

$m = w/g = 24/32 = 3/4$. $k = w/\Delta L = 24 \div (1/2) = 48$.

$$(3/4)y'' + 0y' + 48y = 6\cos(\omega t),$$

$$y'' + 64y = 8\cos(\omega t),$$

with initial values $y(0) = -1, y'(0) = 0$.

(b) Write down the test function with the fewest terms which can be used to obtain a particular solution via the Method of Undetermined Coefficients when $\omega = 5$.

$$Y_p = A\cos(5t) + B\sin(5t).$$

(c) Write down the test function with the fewest terms which can be used to obtain a particular solution via the Method of Undetermined Coefficients when ω is the *resonance frequency.*

The characteristic equation for the homogeneous is $r^2 + 64 = 0$ with roots $\pm 8\mathbf{i}$. So the resonance frequency is 8.

$$Y_p = t \times [A\cos(8t) + B\sin(8t)].$$

(6) (18 points) For the differential equation:

$$(2 - x^2)y'' + x^2y' + (x - 2x^3)y = 0$$

Compute the recursion formula for the coefficients of the power series solution centered at $x_0 = 0$. Use it to compute the first three nonzero terms of the series for the solution with $y(0) = -24$ and $y'(0) = 0$.

$$2y'' = \Sigma\ 2n(n-1)a_nx^{n-2}[k = n-2] = \Sigma 2(k+2)(k+1)a_{k+2}x^k.$$
$$-x^2y'' = \Sigma\ -n(n-1)a_nx^n\ [k = n] = \Sigma\ -k(k-1)a_kx^k.$$
$$x^2y' = \Sigma na_nx^{n+1}[k = n+1] = \Sigma(k-1)a_{k-1}x^k.$$
$$xy = \Sigma a_nx^{n+1}[k = n+1] = \Sigma\ a_{k-1}x^k.$$
$$-2x^3y = \Sigma\ -2a_nx^{n+3}\quad [k = n+3] = \Sigma\ -2a_{k-3}x^k.$$

The recursion formula is:

$$a_{k+2} = \frac{1}{2(k+2)(k+1)}[k(k-1)a_k \ + \ (-(k-1)-1)a_{k-1} + 2a_{k-3}]$$

$$= \frac{1}{2(k+2)(k+1)}[k(k-1)a_k \ - ka_{k-1} + 2a_{k-3}].$$

$a_0 = -24, a_1 = 0.$
$k = 0 : a_2 = \frac{1}{4}[0a_0 - 0a_{-1} + 2a_{-3}] = 0.$
$k = 1 : a_3 = \frac{1}{12}[-0a_1 - a_0 + 2a_{-2}] = 2.$
$k = 2 : a_4 = \frac{1}{24}[2a_2 - 2a_1 + 2a_{-1}] = 0.$
$k = 3 : a_5 = \frac{1}{40}[6a_3 - 3a_2 + 2a_0] = \frac{-36}{40} = -\frac{9}{10}.$

$$y = -24 + 2x^3 - \frac{9}{10}x^5 + \dots$$