Instructions: Show your work. If you use a theorem or a test to help solve a problem, state the name of the theorem or test.

Question 1 Give an example of a non-constant vector field $\vec{F}$ which is defined in the whole plane and which has zero circulation along the following simple closed curve:

$$\begin{cases} x = 7\sqrt[3]{\sin(t)}, \\ y = 11\sqrt[3]{\cos(t)}, \end{cases}$$

$t \in [0, 2\pi]$.

Question 2 Let E be the solid bounded by $y = x^2$, $y = x$, $x = z$, and $z = 0$ whose mass density is given by $\rho(x, y, z) = x$. Sketch E and find its mass.

Question 3 Use the method of Lagrange multipliers to set up finding the point on the ellipsoid $x^2 + y^2 + 2z^2 = 4$ closest to the point $(2, 2, 2)$. More specifically, you need to write down the system of equations in x, y, z and the parameter λ, solve for x, y and z in terms of λ, and obtain a single polynomial equation for λ. Do not solve this equation.

Question 4 Compute $\iiint_H z\, dV$, where H is the solid region bounded above by the xy-plane and below by the sphere of radius 4 centered at the origin.

Question 5 The parametric surface S is given by the vector function

$$\vec{r}(u, v) = uv\hat{\imath} + (u + v)\hat{\jmath} + (u - v)\hat{k}$$

with the domain $1 \le u^2 + v^2 \le 4$. Give an equation of the tangent plane to the surface at the point where $u = 1, v = 1$, and find the area of the surface.

Question 6 Use Green's Theorem to find the work done by the vector field $\vec{F} = (4x\sin(2x) - 2y)\hat{\imath} + 4x^2\hat{\jmath}$ along the path connecting $(0, 0)$ to $(0, 4)$ to $(3, 4)$ to $(3, 0)$ back to $(0, 0)$ by four straight segments.

Question 7 Evaluate

$$\oint_C \arctan(x)\, dx + (3x - 4 - 5y)\, dy,$$

where C is the circle of radius 4 centered at $(2, 5)$ parameterized counterclockwise.

Question 8 Let K be the surface defined by

$$\left\{x^2 + y^2 + z^2 = 4, \quad x > 0, y < 0, z < 0\right\}.$$

Compute the following surface integral: $\iint_K \vec{F} \cdot \vec{n}\, dS$, for $\vec{F} = 3x\hat{\imath} + 2\hat{k}$, and the normal $\vec{n}$ pointing towards the origin.

Question 9 Let S be the sphere centered at the origin of radius 3. Compute $\int\int_S y^2 \, dS$ via whatever method seems most convenient.

Question 10 Suppose $\vec{F}$ and $\vec{G}$ are vector fields such that for some non-zero real number r, $\nabla \times (\vec{F} - r\vec{G}) = 0$ for all of space. Suppose that for some simple closed curve C in space,

$$\oint_C \vec{F} \cdot \mathrm{T} \, ds = a.$$

Prove that

$$\oint_C \vec{G} \cdot \mathrm{T} \, ds = \frac{a}{r}.$$