

Math 39100 K (26005)

- Homework Solutions 02

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Example 3.4/ BD25, BDM20 : $t^2 y'' + 3ty' + y = 0, t > 1$
with $y_1(t) = t^{-1}$.

$$\begin{array}{rclclcl} 1 \times & y_2 & = & ut^{-1} & & \\ 3t \times & y_2' & = & -ut^{-2} & + & u't^{-1} \\ t^2 \times & y_2'' & = & 2ut^{-3} & - & 2u't^{-2} & + & u''t^{-1} \\ & & & 0 & = & 0 & + & u' & + & u''t \end{array}$$

Let $v = u'$ and $v' = u''$. $t \frac{dv}{dt} = -v$.

$$\ln v = -\ln t, \quad \text{and so} \quad \frac{du}{dt} = v = t^{-1}.$$

Hence, $u = \int t^{-1} dt = \ln t$.

$$y_2 = ut^{-1} = t^{-1} \ln t.$$

Example 3.4/ BD27; BDM21 : $xy'' - y' + 4x^3y = 0$, with $y_1(t) = \sin(x^2)$.

$$4x^3 \times y_2 = 4x^3 u \sin(x^2)$$

$$- \times y_2' = -u2x \cos(x^2) + u' \sin(x^2)$$

$$x \times y_2'' = xu(2 \cos(x^2) - 4x^2 \sin(x^2)) + xu'4x \cos(x^2) + xu'' \sin(x^2)$$

$$0 = 0 + u'[-\sin(x^2) + 4x^2 \cos(x^2)] + u''[x \sin(x^2)].$$

Let $v = u'$ and $v' = u''$,

$$0 = v[-\sin(x^2) + 4x^2 \cos(x^2)] + \frac{dv}{dx}[x \sin(x^2)].$$

$$\frac{dv}{v} = \left[\frac{1}{x} - 4x^2 \cot(x^2) \right] dx.$$

Recall (or check) $\int \cot(x) dx = -\ln(\csc(x))$.

$$\ln(v) = \ln(x) + 2 \ln(\csc(x^2)) \quad \text{so} \quad \frac{du}{dx} = v = x \csc^2(x^2).$$

Integrating, we get $u = -\frac{1}{2} \cot(x^2)$, and so

$$y_2 = uy_1 = -\frac{1}{2} \cos(x^2).$$

Example 3.5/ BD9 : $2y'' + 3y' + y = t^2 + 3 \sin t$.

1. Homogeneous Equation $2y'' + 3y' + y = 0$ has characteristic equation $2r^2 + 3r + 1 = (2r + 1)(r + 1) = 0$. So $y_h = C_1 e^{-t/2} + C_2 e^{-t}$.

2. t^2 has associated root 0 and $3 \sin t$ has associated root the conjugate pair $\pm i$. So our first guess for the test solution is $Y = At^2 + Bt + C + D \cos t + E \sin t$. Since none of the associated roots is a root of the characteristic equation, this is what we use for Y_p .

3. Substitute in the equation.

$$\begin{aligned} 1 \times Y_p &= At^2 + Bt + C + D \cos t + E \sin t, \\ 3 \times Y'_p &= 6At + 3B + 3E \cos t - 3D \sin t, \\ 2 \times Y''_p &= +4A - 2D \cos t - 2E \sin t. \end{aligned}$$

$$\text{So } t^2 + 3 \sin t = At^2 + (B + 6A)t + (C + 3B + 4A) \\ + (-D + 3E) \cos t + (-E - 3D) \sin t.$$

$$A = 1.$$

$$B + 6A = 0, \text{ and so } B = -6.$$

$$C + 3B + 4A = 0, \text{ and so } C = 14.$$

$$-D + 3E = 0 \text{ and so } D = 3E.$$

$$-E - 3D = 3 \text{ and so } -10E = 3.$$

$$\text{Thus, } E = -3/10, \quad D = -9/10.$$

$$y_g = C_1 e^{-t/2} + C_2 e^{-t} + t^2 - 6t + 14 - (9/10) \cos t - (3/10) \sin t.$$

Example 3.5/ BD17; BDM13 :

$$y'' - 2y' + y = te^t + 4, \quad y(0) = 1, y'(0) = 1.$$

1. Homogeneous Equation $y'' - 2y' + y = 0$ has characteristic equation $r^2 - 2r + 1 = (r - 1)^2 = 0$. So $y_h = C_1 e^t + C_2 t e^t$.

2. te^t has associated root 1 and 4 has associated root 0. So our first guess for the test solution is $Y = Ate^t + Be^t + C$. Since 1 is a root for the homogeneous, we must multiply the block $Ate^t + Be^t$ first by t and then by another t since 1 is a repeated root. So $Y_p = At^3 e^t + Bt^2 e^t + C$.

3. Substitute in the equation. Leave columns for the te^t and e^t terms.

$$\begin{aligned} 1 \times Y_p &= At^3 e^t + Bt^2 e^t + C, \\ -2 \times Y_p' &= -2At^3 e^t + (-6A - 2B)t^2 e^t - 4Bte^t, \\ 1 \times Y_p'' &= At^3 e^t + (6A + B)t^2 e^t + (6A + 4B)te^t + 2Be^t. \end{aligned}$$

So $te^t + 4 = 0 + 0 + 6Ate^t + 2Be^t + C$, Notice that the first two columns on the right add up to 0.

$A = 1/6, B = 0, C = 4$, So that

$$y_g = C_1e^t + C_2te^t + (1/6)t^3e^t + 4,$$
$$y'_g = (C_1 + C_2)e^t + C_2te^t + (1/2)t^2e^t + (1/6)t^3e^t.$$

$$1 = y(0) = C_1 + 4, \quad 1 = y'(0) = C_1 + C_2.$$

$$C_1 = -3, \quad C_2 = 4.$$

$$y = -3e^t + 4te^t + (1/6)t^3e^t + 4.$$

For BD 19-23 (BDM 16-21), just solve the homogeneous equation and get the test function $Y(t)$. Don't try to solve for the coefficients.

Example 3.5/ BD21; BDM16 :

$y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin 3t$. The characteristic equation of the homogeneous equation is $r^2 + 3r = 0$ with roots $r = 0, -3$. So $y_h = C_1 + C_2e^{-3t}$.

For $2t^4$ the associated root is 0

For t^2e^{-3t} the associated root is -3

For $\sin 3t$ the associated root(s) are $\pm 3i$.

First version of test function is $\bar{Y}(t) =$

$$(At^4 + Bt^3 + Ct^2 + Dt + E) + (Ft^2 + Gt + H)e^{-3t} + (I \cos 3t + J \sin 3t).$$

Because 0 and -3 are roots from the homogeneous equation, each of those blocks must be multiplied by t to get $Y(t) =$

$$t(At^4 + Bt^3 + Ct^2 + Dt + E) + t(Ft^2 + Gt + H)e^{-3t} + (I \cos 3t + J \sin 3t).$$

Example 3.5/ BD22 : $y'' + y = t(1 + \sin t) = t + t \sin t$.

Characteristic equation is $r^2 + 1 = 0$ with roots $r = \pm i$.

$$y_h = C_1 \cos t + C_2 \sin t.$$

For t the associated root is 0.

For $t \sin t$ the associated root is $\pm i$.

First version of test function is

$$\bar{Y}(t) = (At + B) + ((Ct + D) \cos t + (Et + F) \sin t).$$

Because $\pm i$ are roots from the homogeneous equation, the corresponding block must be multiplied by t to get

$$Y(t) = (At + B) + t((Ct + D) \cos t + (Et + F) \sin t).$$

Example 3.5/ BD25 : $y'' - 4y' + 4y = 2t^2 + 4te^{2t} + t \sin 2t$.
Characteristic equation is $r^2 - 4r + 4 = (r - 2)^2 = 0$ with roots $r = 2, 2$. $y_h = C_1 e^{2t} + C_2 t e^{2t}$.

For $2t^2$ the associated root is 0.

For $4te^{2t}$ the associated root is 2.

For $t \sin 2t$ the associated root is $\pm 2i$.

First version of test function is $\bar{Y}(t) =$

$$(At^2 + Bt + C) + (Dt + E)e^{2t} + ((Ft + G) \cos 2t + (Ht + I) \sin 2t).$$

Because 2 is a twice repeated root of the homogeneous equation, the corresponding block must be multiplied by t^2 . The test function $Y(t) =$

$$(At^2 + Bt + C) + t^2(Dt + E)e^{2t} + ((Ft + G) \cos 2t + (Ht + I) \sin 2t).$$

Example 4.2/ BD11; BDM8 : $y''' - y'' - y + 1 = 0$.

Characteristic equation is $r^3 - r^2 - r + 1 = 0$. Factor by grouping: $r^3 - r^2 - r + 1 = r^2(r - 1) - (r - 1) = (r^2 - 1)(r - 1) = (r + 1)(r - 1)(r - 1)$ with roots $-1, 1, 1$.

$$y_g = C_1 e^{-t} + C_2 e^t + C_3 t e^t.$$

Example 4.2/ BD13 : $2y''' - 4y'' - 2y' + 4y = 0$. Divide the characteristic equation by 2 and factor by grouping to get $r^3 - 2r^2 - r + 2 = (r^2 - 1)(r - 2)$ with roots $1, -1, 2$.

Example 4.2/ BD15; BDM11 : $y^{(6)} + y = 0$. The characteristic equation $r^6 = -1$ requires DeMoivre's Theorem. The modulus is $1^{1/6} = 1$. Each step is 60° , beginning with half-step 30° . The six roots are

$$\begin{aligned}(\cos 30) \pm \mathbf{i}(\sin 30) &= (\sqrt{3}/2) \pm \mathbf{i}(1/2), (\cos 90) \pm \mathbf{i}(\sin 90) = \pm \mathbf{i}, \\(\cos 150) \pm \mathbf{i}(\sin 150) &= (-\cos 30) \pm \mathbf{i}(\sin 30) = (-\sqrt{3}/2) \pm \mathbf{i}(1/2).\end{aligned}$$

$$\begin{aligned}y_g &= C_1 e^{t\sqrt{3}/2} \cos(t/2) + C_2 e^{t\sqrt{3}/2} \sin(t/2) + C_3 \cos(t) + C_4 \sin(t) \\&\quad + C_5 e^{-t\sqrt{3}/2} \cos(t/2) + C_6 e^{-t\sqrt{3}/2} \sin(t/2).\end{aligned}$$

Example 4.2/21 : $y^{(8)} + 8y^{(4)} + 16y = 0$. The characteristic equation is $r^8 + 8r^4 + 16 = (r^4 + 4)^2 = 0$. $r^4 = -4$ requires DeMoivre's Theorem. The modulus is $4^{1/4} = \sqrt{2}$. Each step is 90° , beginning with half-step 45° . The four roots are

$$\sqrt{2}((\cos 45) \pm \mathbf{i}(\sin 45)) = 1 \pm \mathbf{i},$$

$$\sqrt{2}((\cos 135) \pm \mathbf{i}(\sin 135)) = \sqrt{2}(-(\cos 45) \pm \mathbf{i}(\sin 45)) = -1 \pm \mathbf{i}.$$

Each pair of complex roots is repeated.

$$y_g = C_1 e^t \cos t + C_2 e^t \sin t + C_3 t e^t \cos t + C_4 t e^t \sin t + C_5 e^{-t} \cos t + C_6 e^{-t} \sin t + C_7 t e^{-t} \cos t + C_8 t e^{-t} \sin t$$

Example 4.3/ BD13; BDM10 : $y''' - 2y'' + y' = t^3 + 2e^t$. The characteristic equation for the homogeneous equation is $r^3 - 2r^2 + r = r(r - 1)^2 = 0$ with roots 0, 1, 1.

$$y_h = C_1 + C_2e^t + C_3te^t.$$

For t^3 the associated root is 0.

For $2e^t$ the associated root is 1.

The first guess for the test function Y_p^1 is thus

$$Y_p^1 = (At^3 + Bt^2 + Ct + D) + (Ee^t).$$

Because 0 is a root for the homogeneous equation, we must multiply the first block by t .

Because 1 is a repeated root for the homogeneous equation, we must multiply the second block by t^2 .

$$Y_p = t(At^3 + Bt^2 + Ct + D) + t^2(Ee^t).$$

Example 4.3/15 : $y^{(4)} - 2y'' + y = e^t + \sin t$. The characteristic equation for the homogeneous equation is $r^4 - 2r^2 + 1 = (r^2 - 1)^2 = 0$ with roots $1, 1, -1, -1$.

$$y_h = C_1 e^t + C_2 t e^t + C_3 e^{-t} + C_4 t e^{-t}.$$

For e^t the associated root is 1.

For $\sin t$ the associated root is the complex pair $\pm i$.

The first guess for the test function Y_p^1 is thus

$$Y_p^1 = (Ae^t) + (B \cos(t) + C \sin(t)).$$

Because 1 is a repeated root for the homogeneous equation, we must multiply the first block by t^2 .

$$Y_p = t^2(Ae^t) + (B \cos(t) + C \sin(t)).$$

Example 4.3/ BD16 : $y^{(4)} + 4y'' = \sin 2t + te^t + 4$. The characteristic equation for the homogeneous equation is $r^4 + 4r^2 = r^2(r^2 + 4)$ with roots $0, 0, \pm 2i$.

$$y_h = C_1 + C_2 t + C_3 \cos(2t) + C_4 \sin(2t).$$

For $\sin 2t$ the associated root is the complex pair $\pm 2i$.

For te^t the associated root is 1.

For 4 the associated root is 0.

The first guess for the test function Y_p^1 is thus

$$Y_p^1 = (A \cos(2t) + B \sin(2t)) + (Ct + D)e^t + (F).$$

Because $\pm 2i$ are roots for the homogeneous equation, we must multiply the first block by t .

Because 0 is a repeated root for the homogeneous equation, we must multiply the third block by t^2 .

$$Y_p = t(A \cos(2t) + B \sin(2t)) + (Ct + D)e^t + t^2(F).$$

Example 4.3/ BD18; BDM13 :

$y^{(4)} + 2y''' + 2y'' = 3e^t + 2te^{-t} + e^{-t} \sin t$. The characteristic equation for the homogeneous equation is

$r^4 + 2r^3 + 2r^2 = r^2(r^2 + 2r + 2) = 0$. For the quadratic we use the Quadratic Formula, to get roots $0, 0, -1 \pm \mathbf{i}$.

$$y_h = C_1 + C_2 t + C_3 e^{-t} \cos(t) + C_4 e^{-t} \sin(t).$$

For $3e^t$ the associated root is 1.

For $2te^{-t}$ the associated root is -1 .

For $e^{-t} \sin t$ the associated root is $-1 \pm \mathbf{i}$.

The first guess for the test function Y_p^1 is thus

$$Y_p^1 = (Ae^t) + (Bt + C)e^{-t} + (De^{-t} \cos(t) + Ee^{-t} \sin(t)).$$

Because $-1 \pm \mathbf{i}$ are roots for the homogeneous equation, we must multiply the third block by t .

$$Y_p = (Ae^t) + (Bt + C)e^{-t} + t(De^{-t} \cos(t) + Ee^{-t} \sin(t)).$$

Example 3.6/ BD3 : $y'' + 2y' + y = 3e^{-t}$. The roots of the characteristic equation for the homogeneous equation $r^2 + 2r + 1 = (r + 1)^2 = 0$ are $-1, -1$ and so $y_h = C_1e^{-t} + C_2te^{-t}$.

Undetermined Coefficients:

The associated root for $3e^{-t}$ is -1 . Our first guess for test function $Y_p^1 = Ae^{-t}$.

Because -1 is a root repeated twice, we must multiply by t^2 . So $Y_p = At^2e^{-t}$.

$$1 \times Y_p = At^2e^{-t},$$

$$2 \times Y_p' = -2At^2e^{-t} + 4Ate^{-t},$$

$$1 \times Y_p'' = At^2e^{-t} - 4Ate^{-t} + 2Ae^{-t}.$$

So $3e^{-t} = 2Ae^{-t}$. So $A = 3/2$ and $y_g = C_1e^{-t} + C_2te^{-t} + (3/2)t^2e^{-t}$.

Variation of Parameters:

We look for $y_p = u_1 e^{-t} + u_2 t e^{-t}$. We have the linear equations:

$$\begin{aligned}u_1'(e^{-t}) + u_2'(te^{-t}) &= 0 \\u_1'(-e^{-t}) + u_2'(e^{-t} - te^{-t}) &= 3e^{-t}.\end{aligned}$$

Wronskian is e^{-2t} so

$$u_1' = \begin{vmatrix} 0 & te^{-t} \\ 3e^{-t} & (e^{-t} - te^{-t}) \end{vmatrix} / e^{-2t},$$

$$u_2' = \begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & (3e^{-t}) \end{vmatrix} / e^{-2t}.$$

$$u_1' = -3t, u_2' = 3. \quad u_1 = -(3/2)t^2, u_2 = 3t.$$

$$Y_p = -(3/2)t^2 e^{-t} + (3t)te^{-t} = (3/2)t^2 e^{-t}.$$

Example 3.6/ BD5; BDM4 : $y'' + y = \tan(t)$. The roots of the characteristic equation for the homogeneous equation $r^2 + 1 = 0$ are $\pm i$ and so $y_h = C_1 \cos(t) + C_2 \sin(t)$.

We look for $y_p = u_1 \cos(t) + u_2 \sin(t)$. We have the linear equations:

$$\begin{aligned}u_1'(\cos(t)) + u_2'(\sin(t)) &= 0 \\u_1'(-\sin(t)) + u_2'(\cos(t)) &= \tan(t).\end{aligned}$$

Wronskian is 1.

$$u_1' = -\sin(t) \tan(t) = -\sin^2(t)/\cos(t) = (\cos^2(t) - 1)/\cos(t) = \cos(t) - \sec(t),$$

$$u_2' = \cos(t) \tan(t) = \sin(t).$$

$$u_1 = \sin(t) - \ln |\sec(t) + \tan(t)|, u_2 = -\cos(t),$$

$$y_p = -\cos(t) \ln |\sec(t) + \tan(t)|.$$

Example 3.6/ BD10; BDM8 : $y'' - 2y' + y = e^t/(1 + t^2)$.

The roots of the characteristic equation for the homogeneous equation $r^2 - 2r + 1 = 0$ are 1, 1 and so $y_h = C_1 e^t + C_2 t e^t$.

$$y_p = u_1 e^t + u_2 t e^t$$

$$u_1'(e^t) + u_2'(te^t) = 0$$

$$u_1'(e^t) + u_2'(e^t + te^t) = e^t/(1 + t^2).$$

Wronskian is e^{2t} .

$$u_1' = -t/(1 + t^2), u_2' = 1/(1 + t^2).$$

$$u_1 = -\frac{1}{2} \ln(1 + t^2), u_2 = \arctan(t).$$

$$y_p = -\frac{1}{2} e^t \ln(1 + t^2) + t e^t \arctan(t).$$

Example 3.6/ BD14 : $t^2y'' - t(t+2)y + (t+2)y = 2t^3$.

Divide by t^2 to get

$$y'' - t^{-1}(t+2)y + t^{-2}(t+2)y = 2t.$$

Given $y_1 = t$, $Y_2 = te^t$. (Check that these are solutions).

$$y_p = u_1t + u_2te^t.$$

$$u_1'(t) + u_2'(te^t) = 0$$

$$u_1'(1) + u_2'(e^t + te^t) = 2t.$$

Wronskian is t^2e^t .

$$u_1' = -2t^2e^t/t^2e^t = -2, u_2' = 2t^2/t^2e^t = 2e^{-t}.$$

$$u_1 = -2t, u_2 = -2e^{-t}.$$

$$y_p = -2t^2 - 2t.$$

Example 3.7/ BD6; BDM3 : Units are centimeters, grams and seconds.

So $g = 980 \text{ dynes}/s^2$. $m = 100 \text{ gr}$, $\Delta L = 5 \text{ cm}$,

From equilibrium, so $y_0 = 0$.

Downward velocity of $10 \text{ cm}/s$, so $y'_0 = -10$.

$w = mg = 98000$ (dynes). $w = k\Delta L$, so

$k = 98000/5 = 19600$.

No damping, so $c = 0$. No external force.

$$100y'' + 19600y = 0, \quad \text{or} \quad y'' + 196y = 0,$$

Characteristic equation: $r^2 + 196 = 0$ with roots $r = \pm 14i$. So

$$\begin{aligned} y &= C_1 \cos 14t + C_2 \sin 14t \\ y' &= 14C_2 \cos 14t - 14C_1 \sin 14t. \end{aligned}$$

$$0 = y(0) = C_1, \quad -10 = y'(0) = 14C_2.$$

So the solution is $y = -(5/7) \sin 14t$.

Example 3.7/ BD7; BDM4 : Units are feet, pounds and seconds.

So $g = 32\text{ft}/\text{s}^2$. $w = 3\text{lb}$, $\Delta L = 3/12 = 1/4\text{ft}$,

Lifted up, so $y_0 = 1/12\text{ft}$.

Downward velocity of $2\text{ft}/\text{s}$, so $y'_0 = -2$. $3w = mg = 32m$, so $m = 3/32$ (slugs). $w = k\Delta L$, so $k = 3 \div (1/4) = 12$.

No damping, so $c = 0$. No external force.

$$3/32y'' + 12y = 0, \quad \text{or} \quad y'' + 128y = 0,$$

Characteristic equation: $r^2 + 128 = 0$ with roots $r = \pm 8\sqrt{2}i$.

So

$$y = C_1 \cos(8\sqrt{2}t) + C_2 \sin(8\sqrt{2}t)$$

$$y' = 8\sqrt{2}C_2 \cos(8\sqrt{2}t) - 8\sqrt{2}C_1 \sin(8\sqrt{2}t).$$

$$\frac{1}{12} = y(0) = C_1, \quad -2 = y'(0) = 8\sqrt{2}C_2.$$

So the solution is $y = \frac{1}{12} \cos(8\sqrt{2}t) - \frac{1}{4\sqrt{2}} \sin(8\sqrt{2}t)$.

$$A = \sqrt{\frac{1}{144} + \frac{1}{32}}, \quad \phi = \arctan\left(-\frac{3}{\sqrt{2}}\right), \quad y = A \cos(8\sqrt{2}t - \phi)$$

[This part I won't ask.]

Example 3.7/ BD9; BDM5 : Units are centimeters, grams and seconds with $g = 980 \text{ dynes}/s^2$.

$$m = 20g, \Delta L = 5cm, c = 400$$

Down $2cm$, so $y_0 = -2$. Released, so $y'_0 = 0$.

$$w = mg = 19600 \text{ (dynes)}. w = k\Delta L, \text{ so } k = 19600/5 = 3920.$$

No external force.

$$20y'' + 400y' + 3920y = 0, \quad \text{or} \quad y'' + 20y' + 196y = 0,$$

Characteristic equation: $r^2 + 20r + 196 = 0$ with roots

$$r = \left(-\frac{20}{2}\right) \pm \frac{\sqrt{384}}{2}i = -10 \pm 4\sqrt{6}i.$$

So

$$y = C_1 e^{-10t} \cos(4\sqrt{6}t) + C_2 e^{-10t} \sin(4\sqrt{6}t)$$
$$y' = (-10C_1 + 4\sqrt{6}C_2)e^{-10t} \cos(4\sqrt{6}t) + (-10C_2 - 4\sqrt{6}C_1)e^{-10t} \sin(4\sqrt{6}t).$$

$$-2 = y(0) = C_1, \quad 0 = y'(0) = -10C_1 + 4\sqrt{6}C_2.$$

So the solution is

$$y = -2e^{-10t} \cos(4\sqrt{6}t) - \frac{5}{\sqrt{6}}e^{-10t} \sin(4\sqrt{6}t).$$

Example 3.8/ BD10; BDM6 : Units are feet, pounds and seconds.

$w = 8lb$. So $m = w/g = 8/32 = 1/4$. $\Delta L = 1/2$. So

$k = w/\Delta L = 16$. $c = 0$.

$y(0) = -1/4, y'(0) = 0$.

$(1/4)y'' + 16y = 8 \sin(8t)$, or $y'' + 64y = 32 \sin(8t)$.

The roots of the characteristic equation for the homogeneous equation $r^2 + 64 = 0$ are $\pm 8i$. So the forcing occurs at resonance.

$Y_p = At \cos(8t) + Bt \sin(8t)$

$Y_p' = -8At \sin(8t) + 8Bt \cos(8t) + A \cos(8t) + B \sin(8t)$.

$Y_p'' = -64At \cos(8t) - 64At \sin(8t) - 16A \sin(8t) + 16B \cos(8t)$.

$32 \sin(8t) = Y_p'' + 64Y_p = -16A \sin(8t) + 16B \cos(8t)$. So

$A = -2, B = 0$.

$y = C_1 \cos(8t) + C_2 \sin(8t) - 2t \cos(8t)$,

$y' = -8C_1 \sin(8t) + 8C_2 \cos(8t) - 2 \cos(8t) + 16t \sin(8t)$.

At $t = 0$: $-1/4 = C_1, 0 = 8C_2 - 2$. So

$$y = -(1/4) \cos(8t) + (1/4) \sin(8t) - 2t \cos(8t).$$

Example 5.2/ BD9 : $(1 + x^2)y'' - 4xy' + 6y = 0$

$$1y'' = \sum 1n(n-1)a_n x^{n-2} [k = n-2] = \sum (k+2)(k+1)a_{k+2} x^k.$$

$$+x^2y'' = \sum n(n-1)a_n x^n [k = n] = \sum k(k-1)a_k x^k.$$

$$-4xy' = \sum 4na_n x^n [k = n] = \sum -4ka_k x^k.$$

$$6y = \sum 6a_n x^n [k = n] = \sum 6a_k x^k.$$

$$a_{k+2} = \frac{1}{(k+2)(k+1)} [(-k(k-1)+4k-6)a_k] = \frac{-k^2 + 5k - 6}{(k+2)(k+1)} a_k$$

$$= -\frac{(k-2)(k-3)}{(k+2)(k+1)} a_k.$$

$$k = 0 : a_2 = -3a_0,$$

$$k = 1 : a_3 = -\frac{1}{3}a_1.$$

$$k = 2 : a_4 = 0, \quad k = 3 : a_5 = 0. \text{ So } a_k = 0 \text{ for } k \geq 2.$$

Example 5.2/ BD12,18; BDM14 :

$$(1 - x)y'' + xy' + -y = 0, y(0) = -3, y'(0) = 2.$$

$$\begin{aligned} 1y'' &= \sum 1n(n-1)a_n x^{n-2} [k = n-2] = \sum (k+2)(k+1)a_{k+2} x^k \\ -xy'' &= \sum -n(n-1)a_n x^{n-1} [k = n-1] = \sum -(k+1)(k)a_{k+1} x^k \\ +xy' &= \sum na_n x^n [k = n] = \sum ka_k x^k \\ -y &= \sum -a_n x^n [k = n] = \sum -a_k x^k \end{aligned}$$

$$a_{k+2} = \frac{1}{(k+2)(k+1)} [(k+1)ka_{k+1} - (k-1)a_k], \quad a_0 = -3, a_1 = 2.$$

$$k = 0 : a_2 = a_0/2 = -3/2,$$

$$k = 1 : a_3 = a_2/3 = a_0/6 = -1/2.$$

$$k = 2 : a_4 = [6a_3 - a_2]/12 = -a_0/24 = 1/8,$$

$$k = 3 : a_5 = [12a_4 - 2a_3]/20 = -a_0/24 = 1/8$$

Where is a_1 ?