Instructions: Show your work. If you use a theorem or a test to help solve a problem, state the name of the theorem or test.

Question 1 Fill in the blanks with real numbers to make the following statement as precise as possible:

If at some point (x_0, y_0), $\dfrac{\partial f}{\partial x}(x_0, y_0) = 3$ and $\dfrac{\partial f}{\partial y}(x_0, y_0) = -4$ then

$$\underline{\hspace{4cm}} \leq D_u f(x_0, y_0) \leq \underline{\hspace{4cm}}$$

for any unit vector $\vec{u}$. $D_{\vec{u}} f$ is the directional derivative of f in the direction of $\vec{u}$.

Question 2 The cylindrical solid enclosed by $x^2 + y^2 = 4, z = 0$ and $z = 5$ has mass density given by $\sigma(x, y, z) = 8 - \arctan(z^2)$. Set up integrals to find the mass and z-coordinate of the center of mass. You do not need to actually do any of the integrals.

Question 3 Find and classify the critical points of the function $f(x, y) = x^3 - 12x + 3y^2 - 6y$.

Question 4 A farmer wants to build a box-shaped barn to contain 200 cubic meters of hay. The barn will have a roof, two opposite sides which are walls and two opposite sides which have big barn doors. He does not need to pay for the floor since he will be using dirt, which is essentially free on a farm. The walls on the sides cost \$2 per square meter. The end walls with the large doors cost \$4 per square meter and the roof costs \$8 per square meter. He would like to build the barn for the least total cost. What dimensions will give the least total cost?

Question 5 Evaluate $\displaystyle\int\!\!\int_T (2x + 6y)\ dA$, where T is the triangular region in the xy-plane with corners at the origin, $(0, 4)$ and $(2, 0)$.

Question 6 Find the volume of the solid between the xy-plane and the surface $z = 12x^2y + 4$ enclosed by the planes $x = -2$, $x = 1$, $y = 2$ and $y = 4$.

Question 7 Let $\vec{F} = (2ye^{2xy} + 2)\hat{\imath} + (2xe^{2xy} + 4y)\hat{\jmath}$. Find the work done along the path $\vec{r}(t) = (t^2 + t^3)\hat{\imath} + (t^2 + t^7)\hat{\jmath}$, with t in $[0, 1]$. It may help to notice that $\vec{F}$ is conservative.

Question 8 Let K be the surface defined by

$$\{x + y + z = 4, \quad x > 0, y > 0, z > 0\},$$

with upward normal $\vec{n}$. Compute the following surface integral: $\displaystyle\int\!\!\int_K \vec{F} \cdot \vec{n}\, dS$, for $\vec{F} = 3x\hat{\imath} + 2\hat{k}$.

Question 9 Compute $\int\int_B (\nabla \times \vec{F}) \cdot \hat{n}\, dS$ for the vector field $\vec{F}$ given by

$$\vec{F} = x\hat{\imath} + (y + z + 4)\hat{\jmath} + (2xyz^8 + 6xz)\hat{\mathrm{k}},$$

where B is the part of the sphere $x^2+y^2+z^2 = 25$ which lies below the plane $z = -4$, and $\hat{n}$ is the outward unit normal to the sphere.

Question 10 Let R be the region in space given by $\{y^2 + z^2 \geq 25\}$. If $\vec{F}$ is defined on R and $\nabla \times \vec{F} = 0$ on R, is there possibly any closed curve C in R so that

$$\oint_C \vec{F} \cdot \vec{T}\, ds > 0?$$

Explain your answer carefully.