CCNY 203 Fall 2017 Final Solutions

1a: The direction vectors are scalar multiples, so the lines are parallel.

1b: We use cross product of a direction vector and a displacement vector from a point on one line to the other to get a normal to the plane:

```
Cross[{1, -3, 4}, {2, 0, -1} - {5, 1, 1}]
{10, -10, -10}
```

So an equation of the plane is 10(x-2)-10(y-0)-10(z+1)=0 or more simply, after dividing by 10, (x-2)-(y)-(z+1)=0

2a: We take the gradient

$$f = 10 + \frac{25}{z^2+1} + \text{Sin}\left[2\,x^2 + y^3 + z\right]$$

$$10 + \frac{25}{1+z^2} + \text{Sin}\left[2\,x^2 + y^3 + z\right]$$

```
gradf = {D[f, x], D[f, y], D[f, z]}
```

$$\left\{4\,x\,\text{Cos}\left[2\,x^2+y^3+z\right],\ 3\,y^2\,\text{Cos}\left[2\,x^2+y^3+z\right],\ -\frac{50\,z}{\left(1+z^2\right)^2}+\text{Cos}\left[2\,x^2+y^3+z\right]\right\}$$

evaluated at (1,0,-2) gives:

```
gradf = {D[f, x], D[f, y], D[f, z]} /. {x → 1, y → 0, z → -2}
{4, 0, 5}
```

The directional derivative is obtained by dotting the gradient with a unit vector in the desired direction:

$$u = \frac{\{1, 2, 5\}.\text{gradf}}{\sqrt{1^2+2^2+5^2}}$$

$$\frac{29}{\sqrt{30}}$$

2b: We solve to find that the point of interest is when t= 1 we use the chain rule to get $\frac{dP}{dt} = \frac{\partial P}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + \frac{\partial P}{\partial z}\frac{dz}{dt}$, which gives

$$r = \left\{2\,t-1,\ \text{Exp}[2\,t-2]-1,\ t^3-t-2\right\}$$

$$\left\{-1+2\,t,\ -1+e^{-2+2\,t},\ -2-t+t^3\right\}$$

```
Solve[r == {1, 0, -2}]
{{t → 1}}
```

```
D[r, t]
```

$$\left\{2,\ 2\,e^{-2+2\,t},\ -1+3\,t^2\right\}$$

```
D[r, t] /. t → 1
{2, 2, 2}

gradf = {D[f, x], D[f, y], D[f, z]} /. {x → 1, y → 0, z → -2}
{4, 0, 5}

dPdt = 2 × 4 + 2 × 0 + 2 × 5
18
```

3: Taking the first partials and setting them to zero and solving gives:

```
f = -3 y^3 - 4 x^2 + 8 x + 9 y
8 x - 4 x^2 + 9 y - 3 y^3

Solve[{D[f, x] == 0, D[f, y] == 0}]
{{x → 1, y → -1}, {x → 1, y → 1}}
```

At the first point, we evaluate the discriminant to get:

```
D[f, x, x] D[f, y, y] - D[f, x, y]^2 /. {x → 1, y → -1}
-144
```

So there is a saddle there.

At the second point, we evaluate the discriminant to get:

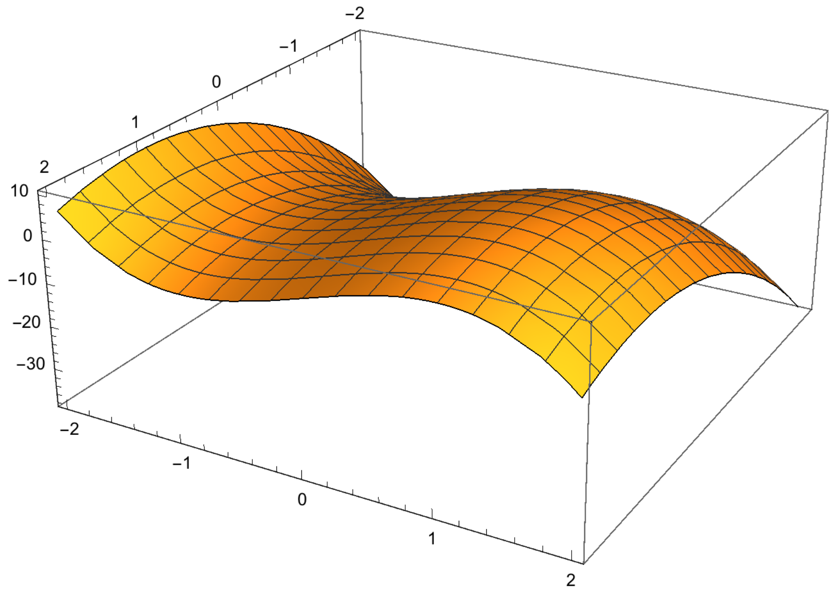
```
D[f, x, x] D[f, y, y] - D[f, x, y]^2 /. {x → 1, y → 1}
144
```

Since it is positive, we look at f_{xx} :

```
D[f, x, x] /. {{x → 1, y → 1}}
{-8}
```

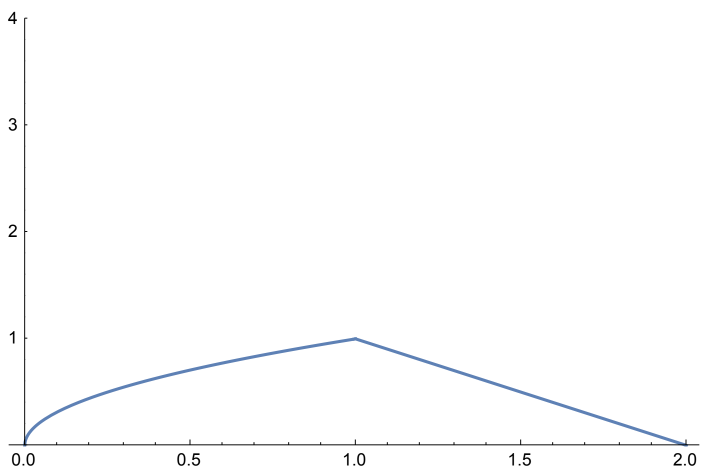
So there is a relative max at (1,1).

```
Plot3D[f, {x, -2, 2}, {y, -2, 2}]
```



4a: This region is described by the region:

```
Show[Plot[√x, {x, 0, 1}, PlotRange → {0, 2}],
 Plot[2 - x, {x, 1, 2}], PlotRange → {0, 4}]
```



This region could be broken up into two parts to get:

$$\int_0^1 \int_0^{\sqrt{x}} x \, dy \, dx + \int_1^2 \int_0^{2-x} x \, dy \, dx$$

$$\frac{16}{15}$$

Or could be done x-integral first to get parts to get:

$$\int_0^1 \int_{y^2}^{2-y} x \, dx \, dy$$

$$\frac{16}{15}$$

4b. Use nearby point (3,4) and adjust:

$$f = \frac{25}{x^2 + y^2}$$

$$\frac{25}{x^2 + y^2}$$

```
dfdx = D[f, x] /. {x → 3, y → 4}
```

$$-\frac{6}{25}$$

```
dfdx = D[f, y] /. {x → 3, y → 4}
```

$$-\frac{8}{25}$$

```
approx = 5 + ( D[f, x] /. {x → 3, y → 4}) .1 + (D[f, y] /. {x → 3, y → 4}) - .2
```

4.456

5a: In polar we get for the volume after finding the intersection to be a circle of radius 2, with the first surface being the lower one and the second one the top one, we integrate the top - bottom to get:

$$\int_0^{2\pi} \int_0^2 \left(8 - r^2 - r^2\right) r \, dr \, d\theta$$

16π

5b: We rewrite to get an implicit description, use the gradient and evaluate to get:

$$f = x y^2 - \text{Log}[2 z - 1]$$

$$x y^2 - \text{Log}[-1 + 2 z]$$

```
D[f, x] /. {x → 2, y → -1, z → 1}
```

1

```
D[f, y] /. {x → 2, y → -1, z → 1}
```

-4

```
D[f, z] /. {x → 2, y → -1, z → 1}
```

-2

which gives an equation of the tangent plane as

$$1 (x - 2) + -4 (y + 1) + -2 (z - 1) == 0$$

6a: Diverges by the test for divergence.
6b: Converges absolutely by the ratio test
6c: Conditionally convergent by: 1) alternating series test 2) integral test

7: We use the ratio test to get convergence on the interval -3<x< -1. For x=-1, divergent by comparison with the harmonic series. For x=-3, convergent by the alternating series test. So the power series converges on the interval (-6,2].

```
a[n_] := (n + 1) (x + 2)^n / (n + 2)^2

Limit[Abs[a[n + 1] / a[n]], n → Infinity]
```

Abs[2 + x]

7b: Approaching along the x-axis gives a limit of 1, and approaching along the y-axis gives a limit of 0, so the limit does not exist.

Alternatively, we can use polar and cancel out r^2 from numerator and denominator to get the numerator is $\cos^2(\theta)$ whose value depends upon which direction is approached, so the limit does not exist.

8a: differentiate to find the tangent vector at the relevant time (t=0)

```
r = {Sqrt[t + Exp[t]], 2 t + 5, t^3 + 2}
```

$\left\{\sqrt{e^t + t},\ 5 + 2t,\ 2 + t^3\right\}$

```
D[r, t] /. t → 0
```

{1, 2, 0}

giving the unit vector after dividing by the length $\sqrt{5}$

8b: rearrange to put into standard form of $\frac{(x-3)^2}{2^2} - \frac{y^2}{1^2} - \frac{z^2}{1^2} = 1$ gives a hyperboloid of two sheets opening up in the +-x directions, with vertices at (5,0,0) and (1,0,0).

9a: We use cylindrical coordinates to find the mass:

$$\text{mass} = \int_0^{2\pi}\int_0^1\int_0^{r^2} 2\,z\,r^2\,r\,dz\,dr\,d\theta$$

$\frac{\pi}{4}$

9b: We use spherical coordinates to find the mass:

$$\text{mass} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \rho\,\text{Cos}[\phi]\ \rho^2\,\text{Sin}[\phi]\,d\rho\,d\phi\,d\theta$$

2π

10a: The series is obtained from known series by substitution and multiplying:

```
Series[Exp[-x^4], {x, 0, 14}]
```

$1 - x^4 + \frac{x^8}{2} - \frac{x^{12}}{6} + O[x]^{15}$

10b: We evaluate the definite integral to get:

$$\int_0^{\frac{1}{2}} \left(1 - x^4\right) \, dx$$

$$\frac{11\,381}{23\,040}$$

Since the series is alternating, the error is less than the next term, which is smaller than $\frac{x^9}{9}\left(\frac{1}{2}\right)^9$ and since 2^9 is 512 which when multiplied by 9 is more than 1000, the error is less than $\frac{1}{1000}$.

10b: integrate over the shadow to get possibly:

$$\text{area} = \int_0^1 \int_0^x \sqrt{1 + 0^2 + (2\,y)^2} \, dy \, dx$$

$$\frac{1}{12}\left(1 + \sqrt{5} + 3\,\text{ArcSinh}[2]\right)$$

Easier with respect to x first to get the integral manageable via substitution:

$$\text{area} = \int_0^1 \int_0^y \sqrt{1 + 0^2 + (2\,y)^2} \, dx \, dy$$

$$\frac{1}{12}\left(-1 + 5\,\sqrt{5}\right)$$