MATHEMATICS 391 **FINAL EXAM** **Spring, 2017**

Part I Answer all questions in this part.

(1) Compute the general solution of each of the following (7 points each):

(a) $y^{(4)} - y' = 0.$

(b) $t \dfrac{dy}{dt} + 2ty = 1 - y.$

(c) $x^2\, dy - (x^2 + xy + y^2)\, dx = 0.$

(d) $y'' + 2y' + y = t^{-2}e^{-t}.$

(2) Solve the following initial value problems (7 points each):

(a) $(\cos(xy) - xy\sin(xy) + 1)dx + (2y - x^2\sin(xy))dy = 0 \qquad y(0) = 3$

(b) $y'' - y' = 6t, \quad y(0) = 0,\ y'(0) = 0.$

(3) (9 points)For the differential equation:

$$(2 - x^2)y'' - xy' + x^2y = 0$$

(a) Compute the recursion formula for the coefficients of the power series solution centered at $x_0 = 0$ and use it to compute the first three nonzero terms of the solution with $y(0) = 0, y'(0) = -36$.

(b) Show that the solution given in (a) is an odd function (Hint: what is a_n when n is even ?)

(4) (a) (5 points) Compute the sine series for the function f such that $f(x) = \pi - x$ on the interval $[0, \pi]$.

(b) (4 points) Compute the solution to the partial differential equation with x in the interval $[0, \pi]$ and $t > 0$:

$$\begin{aligned} u_t &= 9u_{xx} \quad \text{with} \\ u(0,t) = u(\pi,t) &= 0 \quad \text{for } t > 0 \qquad \text{(boundary conditions)} \\ u(0,x) &= \pi - x \quad \text{for } 0 < x < \pi \qquad \text{(initial conditions)} \end{aligned}$$

PartII Answer all sections of four (4) questions out of the five (5) questions in this part (10 points each).

(5) For the equation $t^2y'' - 7ty' + 16y = 0 \quad (t > 0)$, $y_1(t) = t^4$ is a solution.

(a) Use the method of Reduction of Order to obtain a second, independent solution.

(b) Solve the equation directly, using that it is an Euler Equation.

(c) Compute the Wronskian of the pair of solutions.

(6) (a) State the definition of the Laplace transform and use it to compute the Laplace transform of the function f with $f(t) = 1$.

(b) Compute the Laplace transform $\mathcal{L}(y)(s)$ where y is the solution of the initial value problem:

$$2y'' - 5y' + y = 1 \qquad \text{with} \quad y(0) = -3 \quad \text{and } y'(0) = 5.$$

You need not compute the inverse transform.

(7) (a) Compute the general solution of the differential equation

$$y^{(4)} + y'' - 6y' + 4y = 0.$$

(Hint: $r^4 + r^2 - 6r + 4 = (r^2 - 2r + 1)(r^2 + 2r + 4)$.)

(b) Determine the test function $Y(t)$ with the fewest terms to be used to obtain a particular solution of the following equation via the *method of undetermined coefficients.* Do not attempt to determine the coefficients.

$$y^{(4)} + y'' - 6y' + 4y = 7e^t + te^t\cos(\sqrt{3}\,t) - e^{-t}\sin(\sqrt{3}\,t) + 5.$$

(8) For the differential equation $3x^2y'' + 2xy' + x^2y = 0$ show that the point $x = 0$ is a regular singular point (either by using the limit definition or by computing the associated Euler equation). Compute the recursion formula for the series solution corresponding to the larger root of the indicial equation. With $a_0 = 1$, compute the first three nonzero terms of the series.

(9) A 200 gallon tank initially contains 50 gallons in which are dissolved 5 pounds of salt. The tank is flushed by pumping pure water into the tank at a rate of 3 gallons per minute and a well-mixed solution is pumped out at a rate of 2 gallons per minute. Compute the time when the tank has filled, then write the initial value problems which describes amount of salt in the tank at any time before the tank is full. You need not solve the equation.