Hand-in Assignment 1 Solutions (Math 20300 DD, PP, ST)

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Due Date: Friday, February 8, 2008.

SECTION 13.1 (pages 829–834): DO Exercises 1, 3, 4, 5, 12, 18, 19, 36, 40, and 42.

Formulas Used:

Distance Formula in Three Dimensions: The distance $\| \overline{P_1P_2} \|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in 3-Dimensional real space or $\mathbb{R}^3$ is

$$\| \overline{P_1P_2} \| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{1}$$

Equation of a Sphere: An equation of a sphere with center $C(\alpha, \beta, \gamma)$ and radius ρ is

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = \rho^2. \tag{2}$$

In particular, if the center is the origin $O(0, 0, 0)$, then an equation of the sphere is

$$x^2 + y^2 + z^2 = \rho^2 \tag{3}$$

Volume of a spherical cap: The volume V of a cap of a sphere centered at the origin $O(0, 0, 0)$ with radius ρ and height $h < \rho$ of the cap (measured from the endpoint of a diameter of the sphere) is

$$V = \frac{1}{3}\pi h^2(3\rho - h) \text{ (can be derived using calculus).} \tag{4}$$

Distance from a Point to a Plane: Let $\mathscr{P}$ be a plane with equation $ax + by + cz + d = 0$ and $P(x_0, y_0, z_0)$ is a point not in the plane $\mathscr{P}$ then the distance $d(p, \mathscr{P})$ from the point P to the plane $\mathscr{P}$ is given by

$$d(P, \mathscr{P}) = \frac{| ax_0 + by_0 + cz_0 + d |}{\sqrt{a^2 + b^2 + c^2}}. \tag{5}$$

Exercise 1: Suppose you start at the origin, move along the x-axis a distance of 4 units in the positive direction, and then move downward a distance of 3 units. What are the coordinates of your position?

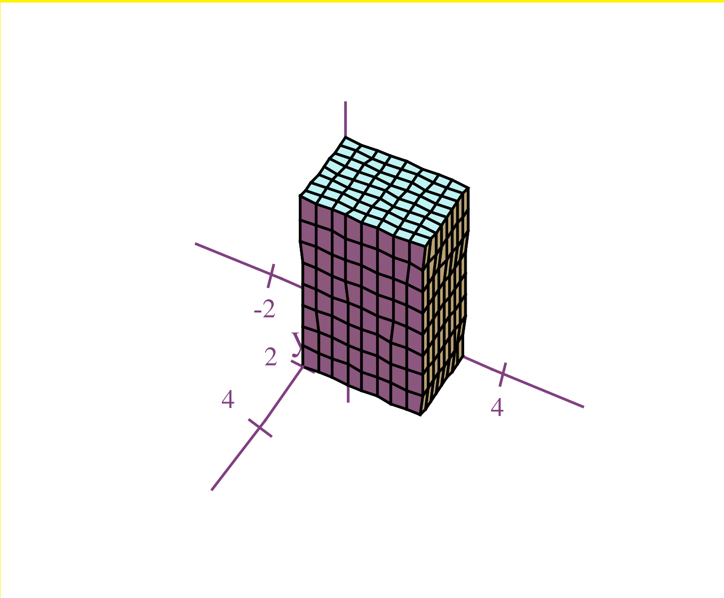
Solution. Without any drawings we start at the origin $O(0,0,0)$ and move along the x-axis a distance of 4 units in the positive direction. This puts us at the point $A(4,0,0)$. From A we move downward a distance of 3 units, this puts our final position at the point $B(4,0,-3)$. Note this is a point in the xz-plane. □

Exercise 3: Which of the points $P(6,2,3)$, $Q(-5,-1,4)$, and $R(0,3,8)$ is closest to the xz-plane? Which point lies in the yz-plane?

Solution. Of the three points P, Q and R we can see that Q is closest to the xz-plane. The distance to the xz-plane from a point $A(x,y,z)$ in space is determined by $\mid y \mid$ (i.e., absolute value of the y-coordinate of the point). Since the absolute value of the y-coordinate in Q is the smallest, then Q is closest to the xz-plane. Only the point $R(0,3,8)$ lies in the yz-plane since only its x-coordinate is 0. □

Exercise 4: What are the projections of the point $(2,3,5)$ on the xy-, yz-, and xz-planes? Draw a rectangular box with the origin and $(2,3,5)$ as opposite vertices and with its faces parallel to the coordinate planes. Label all vertices of the box. Find the length of the diagonal of the box.

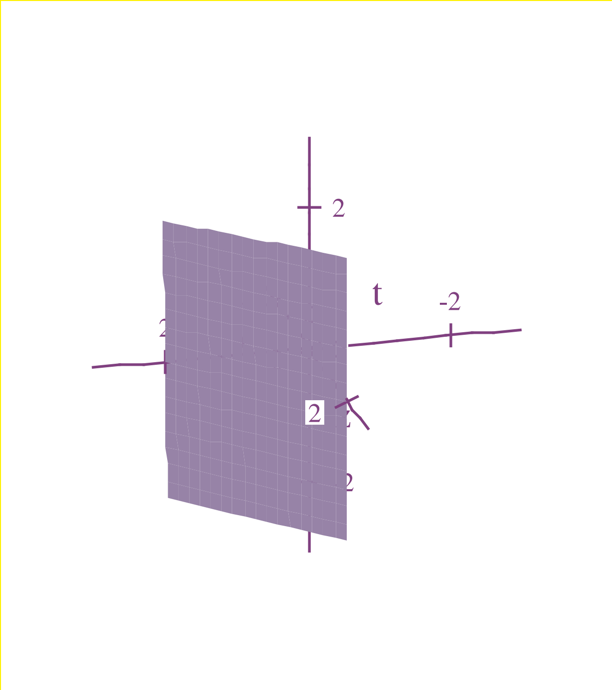
Solution. The projections of the point $(2,3,5)$ on the stated planes are $(2,3,0)$, $(0,3,5)$, and $(2,0,5)$ respectively. There are 8 vertices with coordinates $(0,0,0)$, $(2,0,0)$, $(2,3,0)$, $(0,3,0)$, $(0,0,5)$, $(2,0,5)$, $(2,3,5)$, and $(0,3,5)$. The length of the diagonal of the box is the distance from the origin $(0,0,0)$ to $(2,3,5)$ which is given by $\sqrt{2^2+3^2+5^2} = \sqrt{38}$ units. I did not labeled the vertices of the box.



□

Exercise 5: Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $x + y = 2$.

Solution. Verbal Description: This is a vertical plane which intersects the xy-plane in the line $x + y = 2$.



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Exercise 12: Find an equation of the sphere with center $(6, 5, -2)$ and radius $\sqrt{7}$. Describe its intersection with each of the coordinate planes.

Solution. We are given the center $C(\alpha, \beta, \gamma) = C(6, 5, -2)$ and radius $\rho = \sqrt{7}$. Therefore the equation of the sphere with such a center and radius is given as:

$$(x - 6)^2 + (y - 5)^2 + (z + 2)^2 = 7 \text{ or } x^2 + y^2 + z^2 - 12x - 10y + 4z + 58 = 0.$$

A. The intersection of the sphere with the xy-plane (i.e., the z-coordinates are all 0) is the circle whose equation is $(x - 6)^2 + (y - 5)^2 = 3$ with center $(6, 5, 0)$ and radius $\sqrt{3}$

B. The sphere does not intersect the xz-plane (i.e., the y-coordinates are all 0) at all. Can you see why with a picture?

C. The sphere does not intersect the yz-plane (i.e., the x-coordinates are all 0) either. Can you explain why (without having a picture)?

Exercise 18: Show that equation $4x^2 + 4y^2 + 4z^2 - 8x + 16y = 1$ represents a sphere, and find its center and radius.

Solution.

$$\begin{aligned} 4x^2 + 4y^2 + 4z^2 - 8x + 16y &= 4(x^2 - 2x + 1) + 4(y^2 + 4y + 4) + 4(z - 0)^2 - 4 - 16 \\ &= 4(x - 1)^2 + 4(y + 2)^2 + 4(z - 0)^2 = (\sqrt{21})^2 \\ \text{or} & \\ (x - 1)^2 + (y + 2)^2 + (z - 0)^2 &= \left(\frac{\sqrt{21}}{2}\right)^2 \end{aligned}$$

The last equation represents a sphere with center $(1, -2, 0)$ and radius $\rho = \frac{\sqrt{21}}{2}$. □

EXERCISE 19(a): Prove that the midpoint of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

(b) Find the lengths of the medians of the triangle with vertices $A(1, 2, 3)$, $B(-2, 0, 5)$, and $C(4, 1, 5)$.

SOLUTION.

a) Let $P(\bar{x}, \bar{y}, \bar{z})$ be the midpoint of the line segment $\overline{P_1P_2}$. Then we know that

$$\begin{aligned}
\| \overline{P_1P}\| &= \| \overline{P_2P}\| \\
\| \overline{P_1P}\|^2 &= \| \overline{P_2P}\|^2 \\
(\bar{x} - x_1)^2 + (\bar{y} - y_1)^2 + (\bar{z} - z_1)^2 &= (\bar{x} - x_2)^2 + (\bar{y} - y_2)^2 + (\bar{z} - z_2)^2 \\
-2\bar{x}x_1 + x_1^2 - 2\bar{y}y_1 + y_1^2 - 2\bar{z}z_1 + z_1^2 &= -2\bar{x}x_2 + x_2^2 - 2\bar{y}y_2 + y_2^2 - 2\bar{z}z_2 + z_2^2.
\end{aligned}$$

This means that we have

$$x_1^2 - x_2^2 + 2\bar{x}(x_2 - x_1) + y_1^2 - y_2^2 + 2\bar{y}(y_2 - y_1) + z_1^2 - z_2^2 + 2\bar{z}(z_2 - z_1) = 0$$

or

$$\underbrace{(x_1 - x_2)[(x_2 + x_1) - 2\bar{x}]}_{\text{must be } 0} + \underbrace{(y_1 - y_2)[(y_2 + y_1) - 2\bar{y}]}_{\text{must be } 0} + \underbrace{(z_1 - z_2)[(z_2 + z_1) - 2\bar{z}]}_{\text{must be } 0} = 0$$

however, we are assuming that $x_1 \neq x_2$, $y_1 \neq y_2$, and $z_1 \neq z_2$ and so we must have

$$2\bar{x} = x_1 + x_2, \quad 2\bar{y} = y_1 + y_2, \quad \text{and} \quad 2\bar{z} = z_1 + z_2$$

from which the midpoint formula follows.

b) Recall that a median of a triangle is line segment joining the midpoint of a side of the triangle to the vertex opposite that side. So accordingly, let X, Y and Z denote the midpoints of the sides $\overline{AC}$, $\overline{AB}$, and $\overline{BC}$ of the triangle respectively. Then using our midpoint formula we see that $X = (\frac{5}{2}, \frac{3}{2}, 4)$, $Y = (-\frac{1}{2}, 1, 4)$ and $Z = (1, \frac{1}{2}, 5)$. Now the medians of the triangle are $\overline{BX}$, $\overline{CY}$, and $\overline{AZ}$. The lengths of these medians is then

$$\begin{aligned}
\| \overline{BX} \| &= \sqrt{\frac{81}{4} + \frac{9}{4} + 1} = \frac{\sqrt{94}}{2} \\
\| \overline{CY} \| &= \sqrt{\frac{81}{4} + 1} = \frac{\sqrt{85}}{2} \\
\| \overline{AZ} \| &= \sqrt{\frac{9}{4} + 4} = \frac{\sqrt{25}}{2} = \frac{5}{2}
\end{aligned}$$

□

EXERCISE 36: Write inequalities to describe the solid rectangular box in the first octant bounded by the planes $x = 1$, $y = 2$, and $z = 3$.

SOLUTION. As a set of points in $\mathbb{R}^3$ we could describe the solid rectangular box in the first octant as

$$\text{BOX} := \{(x, y, z) |\ 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}.$$

So the inequalities we seek are $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$, $0 \leqslant z \leqslant 3$. □

EXERCISE 40: Consider the points P such that the distance from P to $A(-1,5,3)$ is twice the distance from P to $B(6,2,-2)$. Show that the set of all such points is a sphere, and find its center and radius.

SOLUTION. Let P have coordinates x, y, and z. Then we would have

$$\begin{aligned}\sqrt{(x+1)^2+(y-5)^2+(z-3)^2} &= \|\overline{PA}\| = 2\|\overline{PB}\| = 2\sqrt{(x-6)^2+(y-2)^2+(z+2)^2}\\ (x+1)^2+(y-5)^2+(z-3)^2 &= \|\overline{PA}\|^2 = 4\|\overline{PB}\|^2 = 4\left((x-6)^2+(y-2)^2+(z+2)^2\right).\end{aligned}$$

So after expanding and collecting and factoring, we arrive at the following

$$\left(x-\frac{25}{3}\right)^2+(y-1)^2+\left(z+\frac{11}{3}\right)^2=\left(\frac{\sqrt{332}}{3}\right)^2$$

which is the equation of a sphere with center at $\left(\frac{25}{3},1,-\frac{11}{3}\right)$ and radius $\rho=\frac{\sqrt{332}}{3}$. □

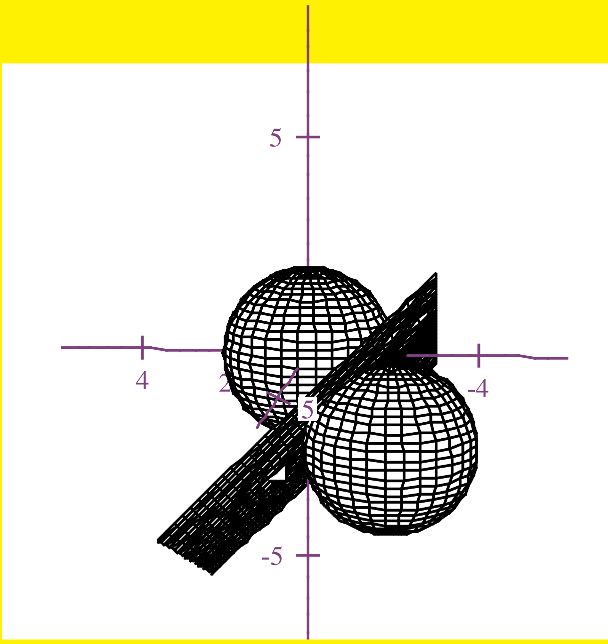
EXERCISE 42: Find the volume of the solid that lies inside both of the spheres

$$\text{SPHERE 1}: \quad x^2+y^2+z^2+4x-2y+4z+5=0$$

and

$$\text{SPHERE 2}: \quad x^2+y^2+z^2=4$$

SOLUTION. Note that sphere 1 has center $(-2,1,-2)$ and sphere 2 has center at the origin $(0,0,0)$. Both spheres has radius 2. The distance, d, between their centers is just the length of the position vector $\langle -2,1,-2\rangle$ which is 3. So we conclude that the two spheres overlap each other (not one sphere inside the other as I originally thought). The volume of the space occupied by the overlap is what we seek.



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Now the two spheres intersect exactly when

$$(x+2)^2+(y-1)^2+(z+2)^2=4=x^2+y^2+z^2$$

i.e., the two spheres intersect in the plane whose equation is:

$$4x-2y+4z+9=0. \qquad (\mathscr{P})$$

The distance, D from the center of each sphere to the plane $(\mathscr{P})$ is given by

$$D = \frac{|\, ax_0 + by_0 + cz_0 + d \,|}{\sqrt{a^2 + b^2 + c^2}} = \frac{9}{\sqrt{36}} = \frac{9}{6} = \frac{3}{2} \text{ (This distance can be found using a fact from plane Geometry!)}$$

where $a = 4$, $b = -2$ and $c = 4$, $d = 9$ and $x_0 = y_0 = z_0 = 0$. Considering just one sphere, say sphere 2 $(x^2 + y^2 + z^2 = 4)$. Let $r = 2$, then $h = \rho - \alpha = \frac{3}{2}$ and $h = \frac{1}{2}$. The contribution of sphere 2 to the volume of the solid is given by

$$V_2 = \frac{1}{3}\pi h^2(3\rho - h) = \frac{1}{3}\pi\left(\frac{1}{2}\right)^2\left(3(2) - \frac{1}{2}\right) = \frac{11\,\pi}{24}.$$

Sphere 1 contributes the same amount as sphere 2, thus the total volume, V_{Total}, is

$$V_{\text{Total}} = V_1 + V_2 = 2V_2 = 2\left(\frac{11\,\pi}{24}\right) = \frac{11\,\pi}{12} \text{ cubic units}$$

Note the formula $\frac{1}{3}\pi h^2(3r - h)$ can be obtained using integration methods (volumes of solids of revolution) studied in your Math 20200 courses.