Analytic Geometry: the co-ordinate plane

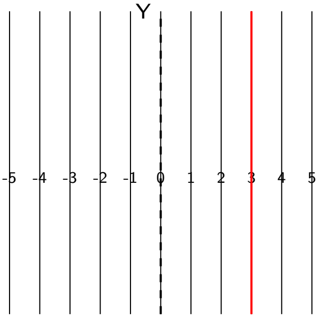
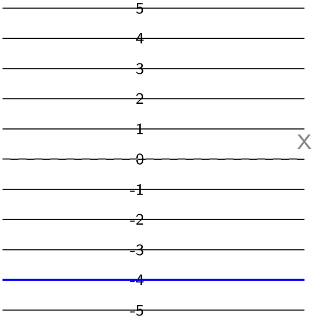
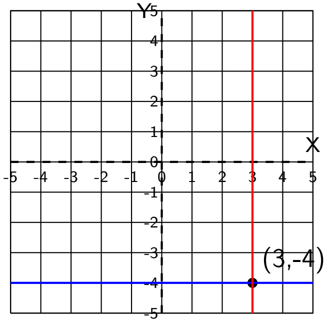
All of mathematics and science relies on analyzing the graphs of equations. To get started, we need to review the co-ordinate plane.

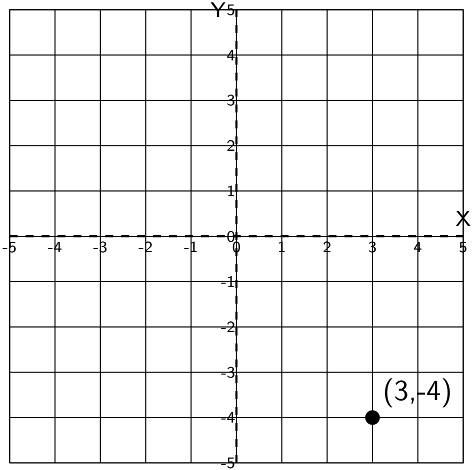
Its easy to locate points on a line. Choose a point 0. The point 3 units (for instance) to the right of 0 is called 3, while the point 3 units to the left of 0 is called -3. Thus every point on the number line is labeled with a single number, positive or negative.

It's harder to locate points in the plane. Fortunately, we are in Manhattan, and you can stand on any corner and identify your location by giving the avenue number and the street number at that corner. Avenues run South-North and streets run East-West. So you can locate yourself by noticing that you are at the corner where 3rd Avenue meets 4th Street, for instance.

The street map of Manhattan gives a model for the co-ordinate plane. Each vertical line has an "avenue name" such as $x = 3$. Each horizontal line as a "street name" such as $y = 4$. The point where Avenue $x = 3$ meets Street $y = 4$ is called the *point* $(3, 4)$.

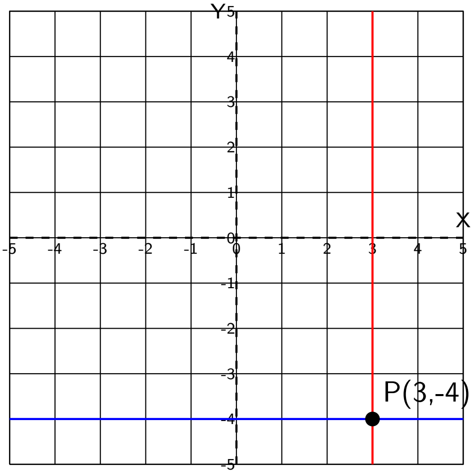
The x-axis is the horizontal line $y = 0$, and the y-axis is the vertical line $x = 0$.
If these are visible, it is customary to write street and avenue numbers near them.

Avenues (South to North)	Streets (East to West)	Combine into Grid
		
Street map: 3rd Avenue	and -4th Street meet at	the corner of 3rd and -4th
Math map: Vertical line x = 3	and Horizontal line y = -4	meet at Point(x,y) =(3,-4)



- The x-axis is a horizontal line, labeled with numbers called x-labels and the letter X at the right. The y-axis is a vertical line, labeled with numbers called y-label and the label Y at the top.
- Each axis is a number line. The numbers on the x-axis are called **x-labels.** They increase from left to right and are usually written below the x-axis.
  The numbers on the y-axis are called **y-labels.** They increase from bottom to top and are usually written to the left of the y-axis.

Use common sense when you label the axes. If an axis runs from -10 to 10, it is reasonable to write labels -10, -5, 0, 5, and 10. Indeed, it would have been fine to label each axis at the left with just -5,0, and 5.



- The vertical red line ($x = 3$) meets the horizontal blue line ($y = -4$) at the point labeled $P(3, -4)$.
- The **x-coordinate of a point** P is the x-label (the number on the x-axis) hit by a vertical line through P. In the diagram, the x-coordinate of P is 3.

You may have encountered a statement that the x-coordinate of point P is the distance from P to the y-axis. This is not correct, because an x-coordinate could be negative, while distance is always positive! It is true that the absolute value of the x-coordinate of point P is the distance from P to the y-axis.

- The **y-coordinate of a point** P in the plane is the y-label(number on the y-axis) hit by a horizontal line through P. In the diagram, the y-coordinate of P is -4.

- All points on the x-axis have y-coordinate zero. All points on the y-axis have x-coordinate zero. The **origin** of the coordinate system is the point (0,0) where the x-axis and y-axis meet.

- The **coordinates of any point P** are written as a pair (x, y), where x is the x-coordinate of P and y is the y-coordinate of P. The co-ordinates of the origin are $(0, 0)$.

When we draw the co-ordinate plane, we label the point as (x, y) or as $P(x, y)$ if we wish to call the point P. In the diagram on the previous page, the point $P(3, -4)$ is the point with x-coordinate 3 and y-coordinate -4.

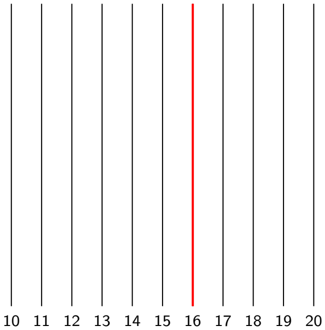
Procedure

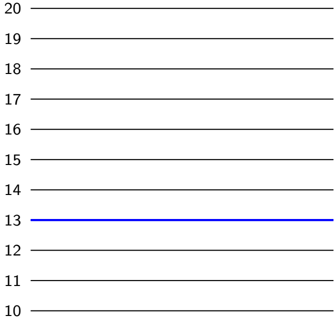
To **plot a point** $(3, -4)$ *draw a dot where the vertical line through the x-label* 3 *meets the horizontal line through the y-label* -4*. If you want to name the dot* P*, label it* $P(3, -4)$*.*

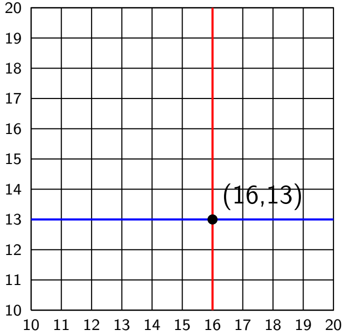
In most graphs you have seen, the x-label -3 is on the x-axis at point $(3, 0)$ and the y-label -4 is on the y-axis at point $(0, -4)$. However, the x-axis and y-axis are not always visible. For example, you might want to draw a graph showing all points (x, y) with $10 \leq x \leq 20$ and $10 \leq y \leq 20$. In that case, put the x- and y-labels below and to the left of the grid, respectively. See how this is done on the next slide.

The grid below shows the region in the plane containing all points (x, y) with$10 \leq x \leq 20$ and $10 \leq y \leq 20$. Thus the x-axis (the line $y = 0$) and the x-axis (the line $y = 0$) are not visible. In such a case, write street numbers to the left of the grid and avenue numbers below the grid.

Avenues (South-North) | Streets (East-West) | combine to form grid







Street Map: 16th Avenue and 13th Street meet at the corner of 16th and 13th

Math: Vertical line x = 16 and Horizontal line y = 13 meet at Point(x,y) =(16,13)

Procedure

To **plot point** (c, d) **by starting at the origin**

- *If* $c \geq 0$, *move* c *units right. If* $c \leq 0$, *move c units left. You are now at point* $(c, 0)$.
- *If* $d \geq 0$, *move* d *units up. If* $d \geq 0$, *move* d *units down. This brings you to* (c, d).

- **Horizontal line facts:**
  - If two points have the same y-coordinate, the line through them is horizontal.
  - If a line is horizontal, all of its points have the same y-coordinate.
  - The length of a horizontal line segment is the x-coordinate of the right point minus the x-coordinate of the left point.
- **Vertical line facts:**
  - If two points have the same x-coordinate, the line through them is vertical.
  - If a line is vertical, all of its points have the same x-coordinate.
  - The length of a vertical line segment is the y-coordinate of the top point minus the y-coordinate of the bottom point.

For any two points P and Q, the line segment joining them is written as PQ.

Definition

The **distance between two points** $P(x_1, y_1)$ *and* $Q(x_2, y_2)$, *also called the length of line segment* PQ, *is written as* $d(P, Q)$ *or* $\overline{PQ}$ *and is given by the distance formula*

$$d(P,Q) = \overline{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1: Find the distance betwen points $A(-3, 2)$ and $D(3, 3)$.

Solution: The distance is $\sqrt{(3-(-3))^2 + (3-2)^2} = \sqrt{6^2 + 1^2} = \sqrt{37}$.

Definition

The **midpoint of the line segment** *joining* $P(x_1, y_1)$ *and* $Q(x_2, y_2)$ *is the point* $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. *It is halfway between* P *and* Q*: thus* $\overline{PM} = \overline{MQ} = \frac{1}{2}\overline{PQ}$.

Example 2: Find the midpoint M of the line segment PQ joining the points $P(-3,2)$ and $Q(3,3)$. Show that your point M is indeed the midpoint by checking that $\overline{PM} = \overline{MQ} = \frac{1}{2}\overline{PQ}$.

Solution: The midpoint M of the line segment PQ is $\left(\frac{-3+3}{2}, \frac{3+2}{2}\right) = (0, \frac{5}{2})$. Now compute the lengths of the requested line segments:

- $\overline{PQ}$ = distance from P to Q = $\sqrt{(3-(-3))^2 + (3-2)^2} = \sqrt{36+1} = \sqrt{37}$
- $\overline{MQ}$ = distance from M to Q = $\sqrt{(3-0)^2 + (3-\frac{5}{2})^2} = \sqrt{9+\frac{1}{4}} = \sqrt{\frac{37}{4}} = \frac{\sqrt{37}}{\sqrt{4}} = \frac{1}{2}\sqrt{37}$
- $\overline{MP}$ = distance from M to P = $\sqrt{(-3-0)^2 + (2-\frac{5}{2})^2} = \sqrt{9+\frac{1}{4}} = \sqrt{\frac{37}{4}} = \frac{\sqrt{37}}{\sqrt{4}} = \frac{1}{2}\sqrt{37}$

We have found the midpoint since $\overline{MQ} = \overline{MP} = \frac{1}{2}\sqrt{37} = \frac{1}{2}\overline{PQ}$.

Equations and their graphs

Every pair of numbers (x, y) corresponds to a point in the x,y-plane. A point is typically labeled by a letter such as $P, Q, R \ldots$ The order of the two numbers in the pair matters: $P(3,4)$ and $Q(4,3)$ are different points.

Definition

An **equation in variables** x **and** y *is a statement that two expressions involving* x *and* y *are equal.* **A number pair** (c, d) **satisfies the equation** *provided the equation becomes a true numerical statement when you substitute* c *for* x *and* d *for* y*.*

Example 3: $x^2 + y^2 = 25$ is an equation.

- The pair $(3,4)$ satisfies the equation because substituting 3 for x and 4 for y yields the true statement $3^3 + 4^2 = 25$, whereas
- The pair $(4,2)$ doesn't satisfy the equation because substituting 4 for x and 2 for y yields the false statement $4^2 + 2^2 = 25$.

The graph of the equation $x^2 + y^2 = 4$ consists of all points (x, y) in the x,y-plane with $x^2 + y^2 = 4$. Based on the last paragraph, the point $(3, 4)$ lies on the graph but $(4, 2)$ does not. We restate this as the

Fundamental Principle of analytic geometry

A point lies on the graph of an equation if and only if the point's coordinates satisfy the equation.

Example 4: The equation $x^2 + y^2 + 4 = 0$ is not satisfied by any pair of numbers (x, y), since the equation can be rewritten $x^2 + y^2 = -4$. Since the square of any real number is non-negative, the graph is empty: it contains no points at all.

If any point on a vertical line has x-coordinate c, then every point on that line has x-coordinate c, and the equation of the line is $x = c$. For any number d, the vertical line through point (c, d) has equation $x = c$.

Example 5: Find the equations of the horizontal and vertical lines through point $(3, 4)$.

Solution: The horizontal line through $(3, 4)$ has equation $y = 4$. The vertical line through $(3, 4)$ has equation $x = 3$.

The last example illustrates an important point: If a variable is missing from an equation, the variable is arbitrary. *Don't make the mistake of thinking that the missing variable is zero.* For example , the equation $x^2 = 1$ can be rewritten as $x = 1$ or $x = -1$. Since y is missing, it is arbitrary, and the graph consists of the two vertical lines $x = 1$ and $x = -1$.

Make sure to read Examples 4,5, and 6 in the text. However, do not be misled by the examples given. Please keep in mind the following important points when you draw a graph.

- Don't assume that you should choose only x-values that are whole numbers.
- Don't assume that you should choose only x-values near $x = 0$.
- Don't assume that every box in a graph measures one unit by one unit.
- If you are drawing a graph with an axis that runs from -10 to 10, you are wasting time if you label the axis with all the whole numbers from -10 to 10. Labels $-10, -5, 0, 5, 10$ are just fine.

Intercepts of the graph of an equation

A good way to start drawing the graph of an equation is to find the points where

- the graph meets the x-axis (which consists of all points with y-coordinate 0) or
- the graph meets the y-axis (which consists of all points with x-coordinate 0).

Definition

- *The* **x-intercepts of the graph** *of an equation in x and y are the x-coordinates of all points where the graph meets the x-axis.*
- *The* **y-intercepts of the graph** *of an equation x and y are the y-coordinates of all points where the graph meets the y-axis.*

Procedure

To **find the intercepts of the graph of an equation:**

- *set $x = 0$ and solve the equation for y to find the y-intercepts*
- *set $y = 0$ and solve the equation for x to find the x-intercepts.*

We have defined the intercepts to be numbers, not points. For example, a graph has x-intercept 17 if an only if the point $(17, 0)$ is on the graph. However, some books may refer to the point $(17, 0)$ as an x-intercept. You may do so as well.

Example 6: Find all x- and y- intercepts of the graph of the equation $x^2 + y^2 = 4$. At what points does the graph meet the x-axis? the y-axis?

We will see in the next section that the graph of $x^2 + y^2 = 4$ is a circle with center $(0, 0)$ and radius 2.

Solution: Setting $y = 0$ gives $x^2 = 4$ .This equation has two solutions: $x = 2$ and $x = -2$. The x-intercepts are 2 and -2, and the graph meets the x-axis at points $(2, 0)$ and $(-2, 0)$.

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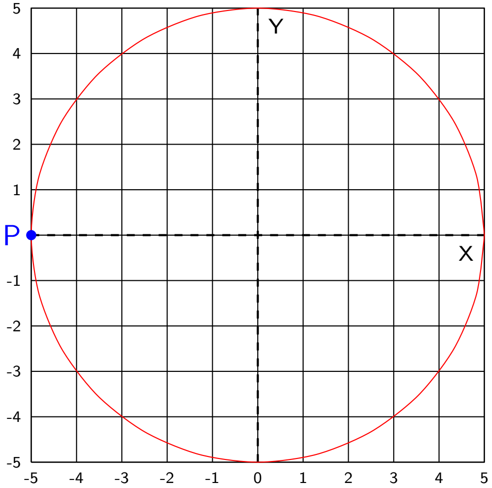
Finding points on graphs



This is the graph of the radius 5 circle with center at the origin. The simplest equation for this circle is $x^2 + y^2 = 25$.
As each point appears in the upper semicircle, figure out its coordinates, make sure they satisfy the equation $x^2 + y^2 = 25$, then click or roll the mouse wheel to check your answer.

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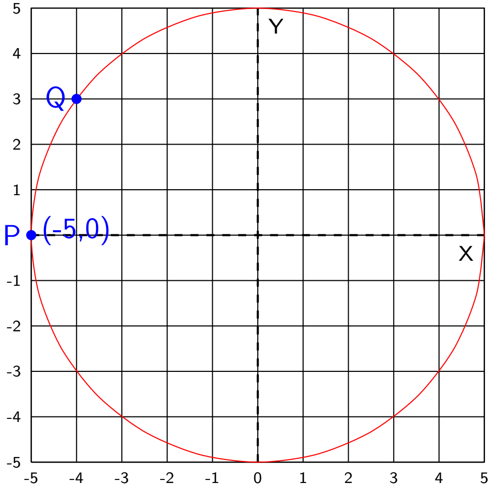

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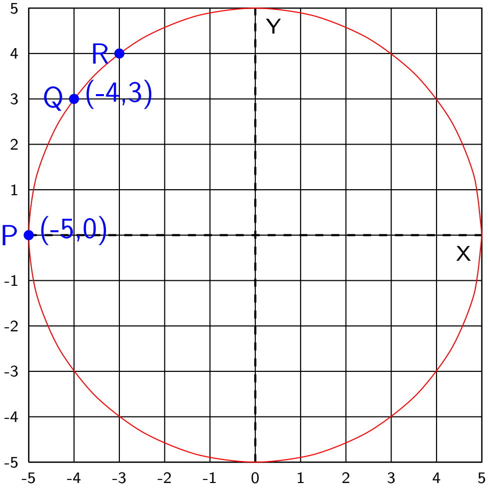

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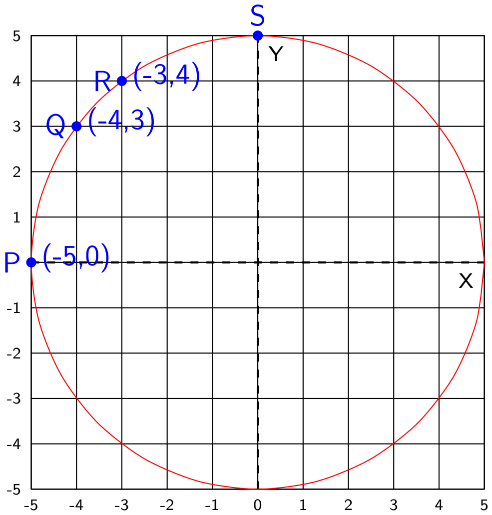

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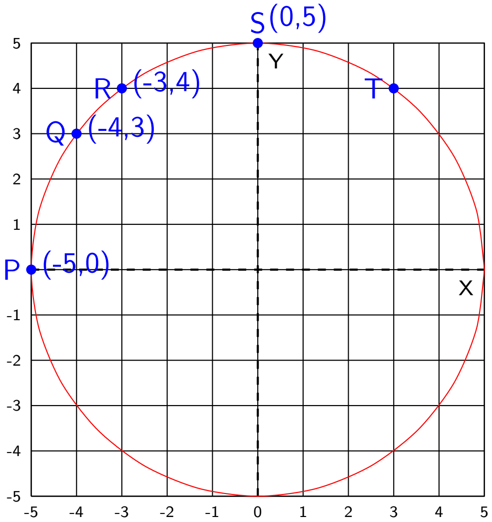

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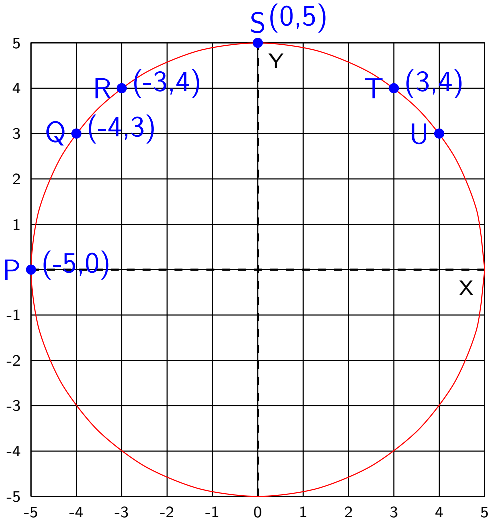

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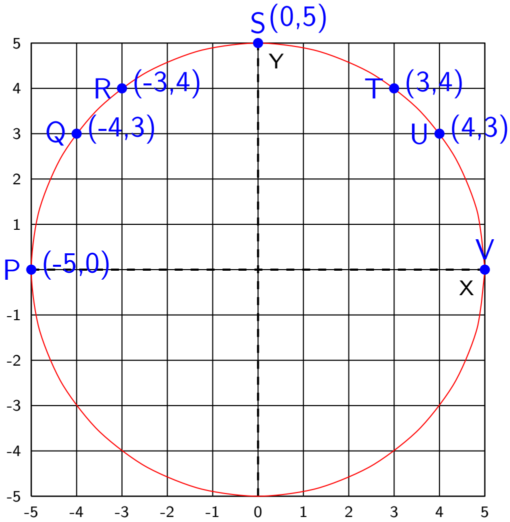

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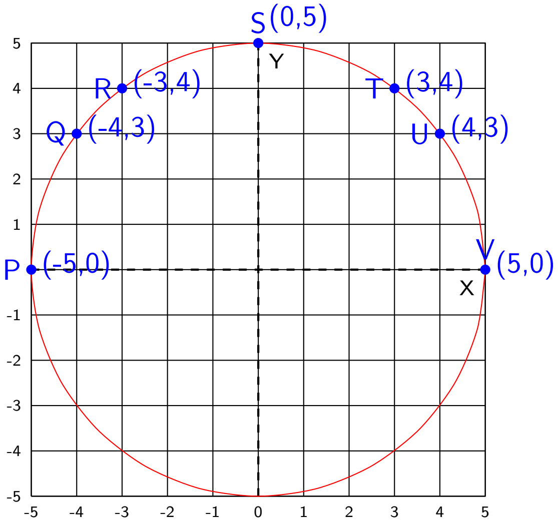

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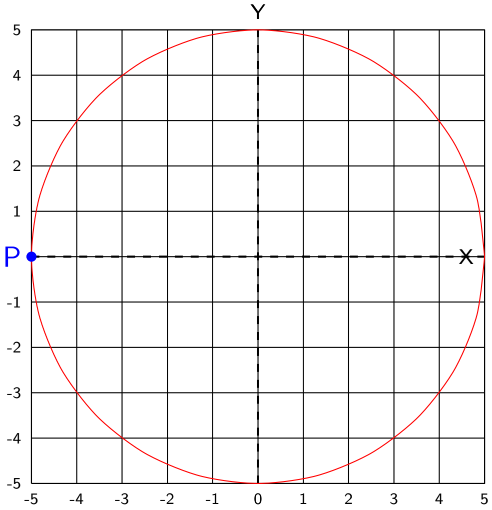

Example 7: This is the graph of the radius 5 circle with equation $x^2 + y^2 = 25$.
As each point appears in the lower semicircle, figure out its coordinates, make sure they satisfy the equation $x^2 + y^2 = 25$, then click or roll the mouse wheel to check your answer.

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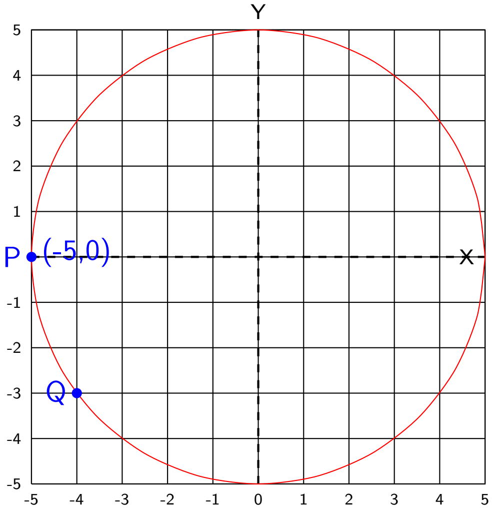

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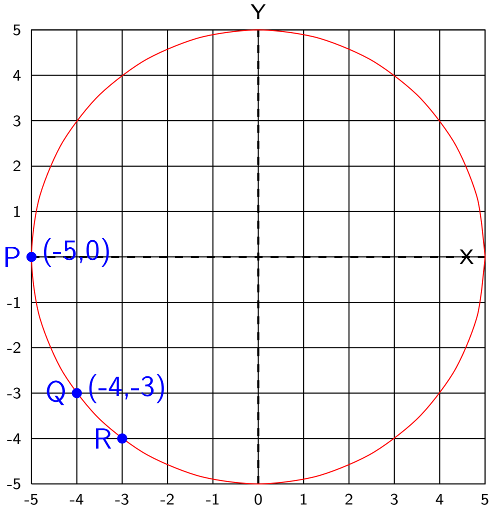

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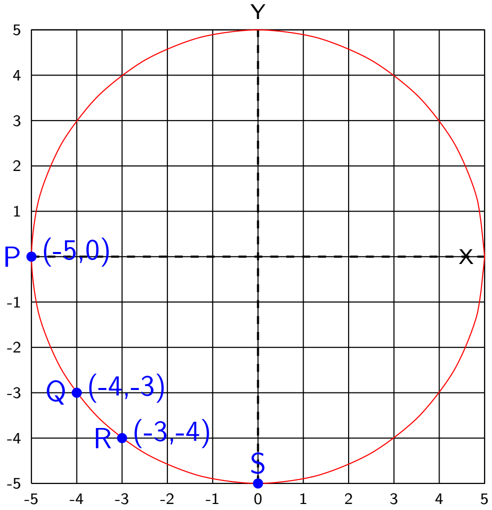

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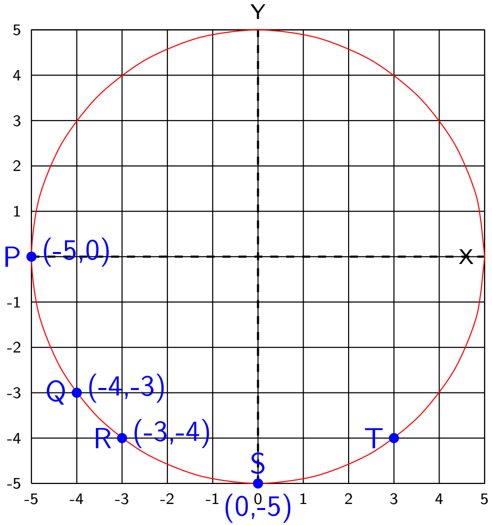

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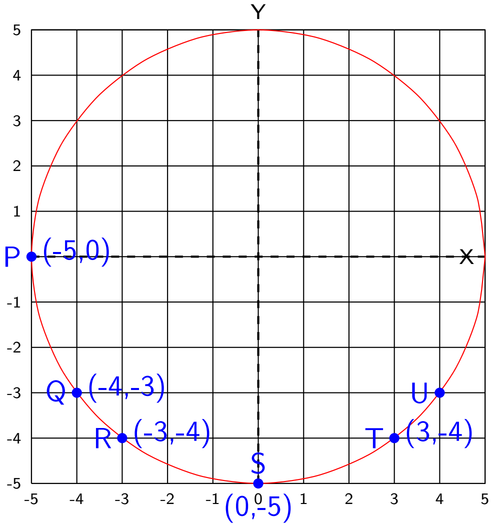

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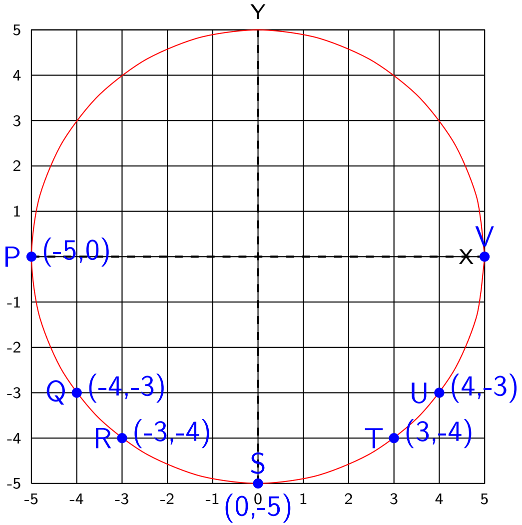

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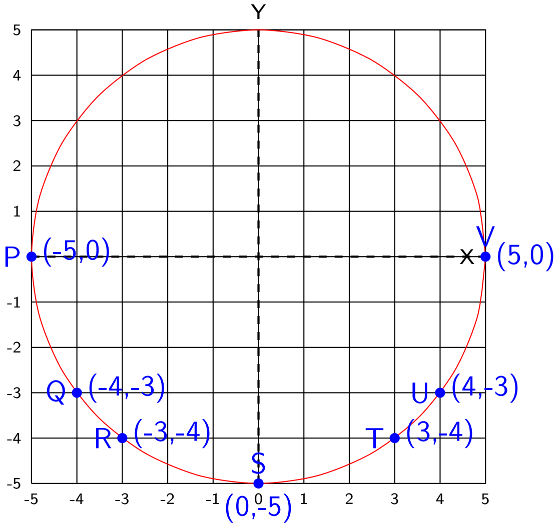

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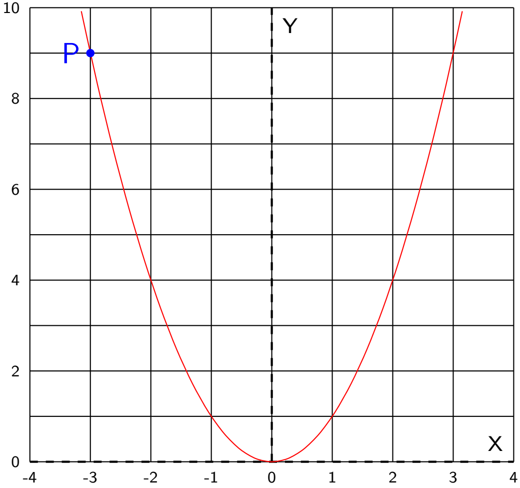

Example 8: This is the graph of the parabola with equation $y = x^2$.
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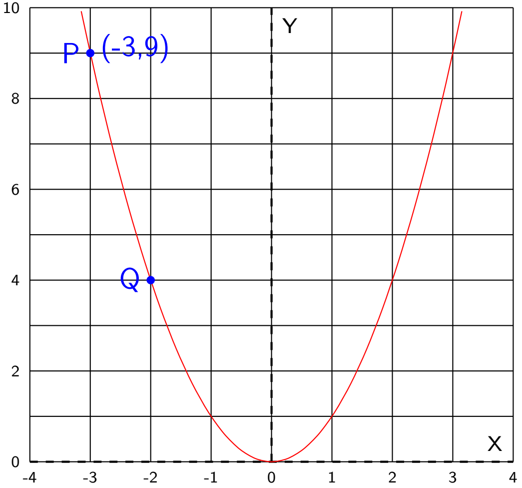

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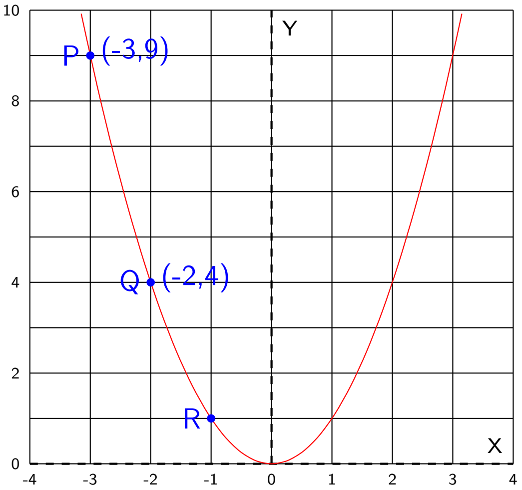

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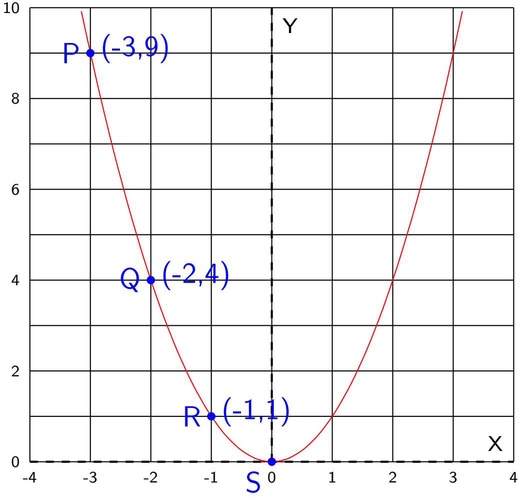

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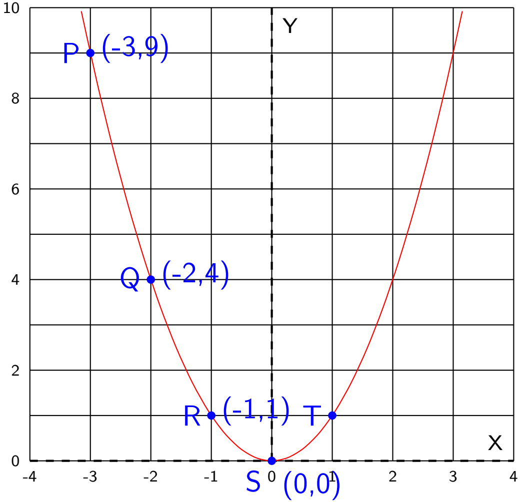
Finding points on graphs



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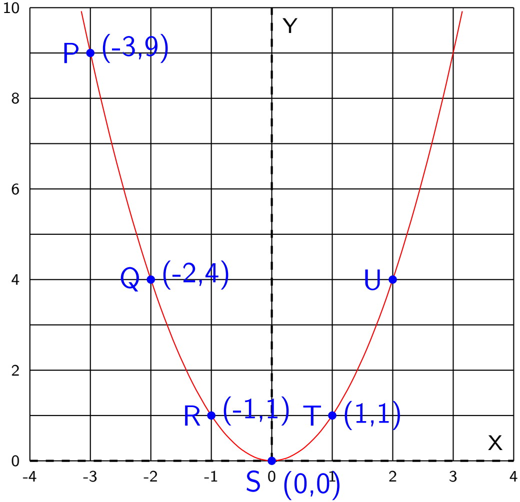

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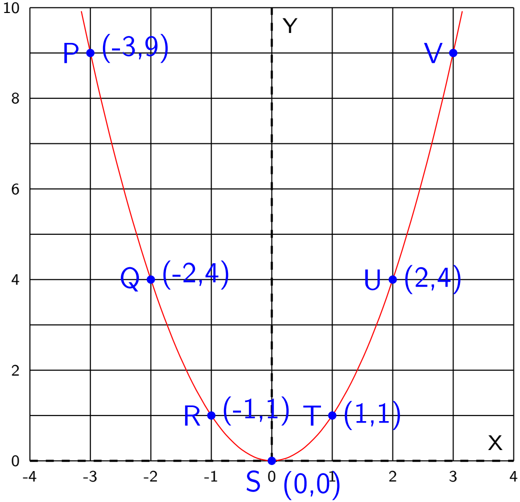

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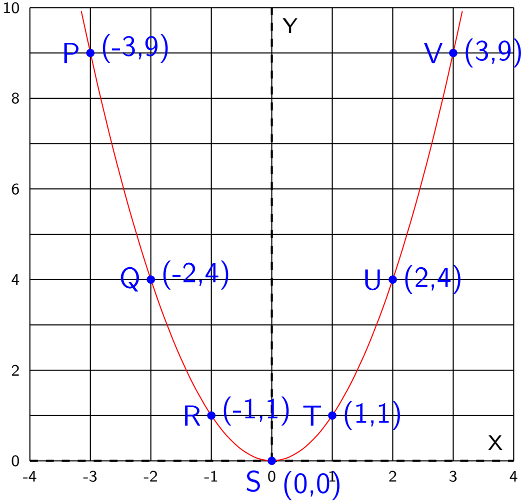

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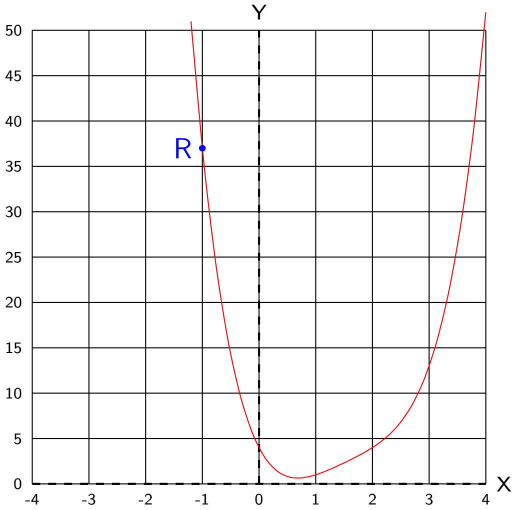

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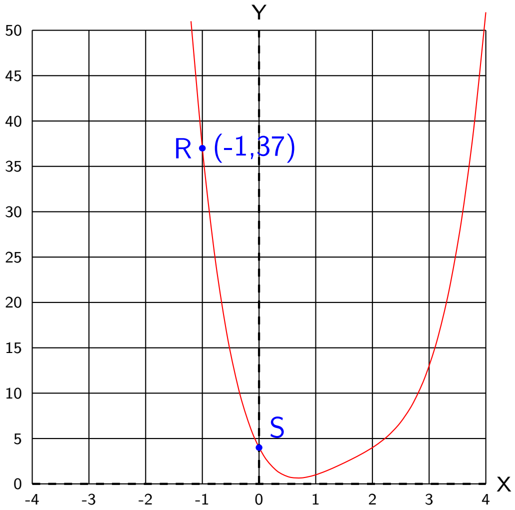
As each point appears, figure out its coordinates, make sure they satisfy the equation, then click or roll the mouse wheel to check your answer.

Finding points on graphs



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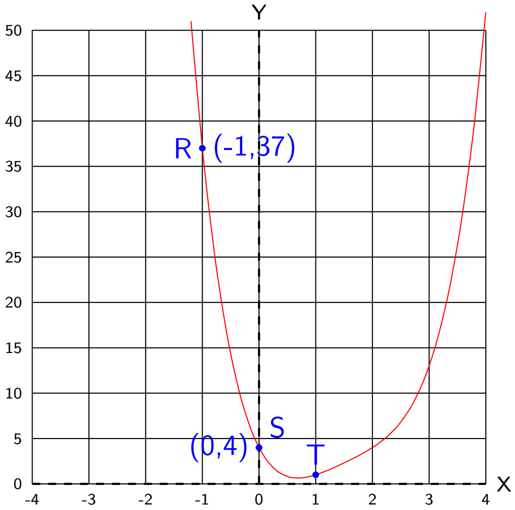

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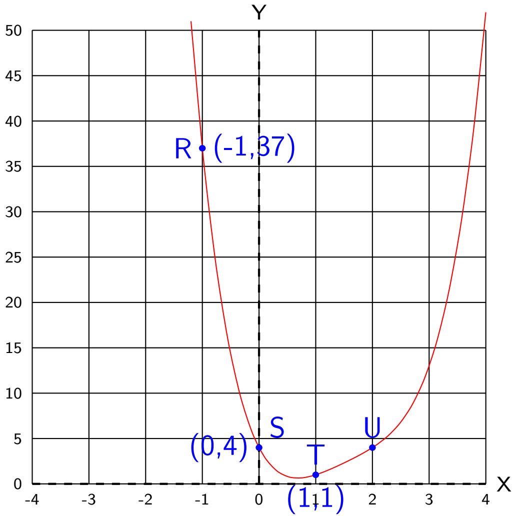

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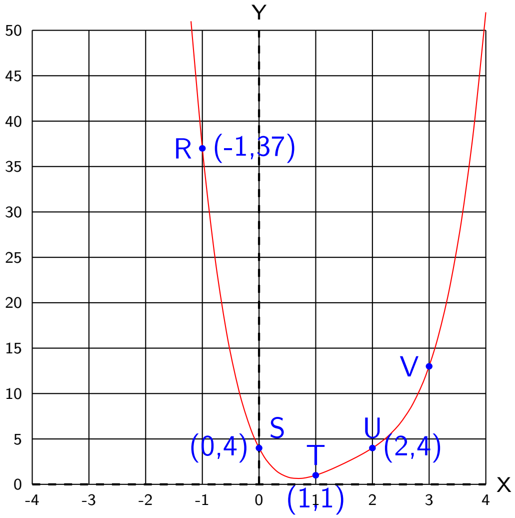

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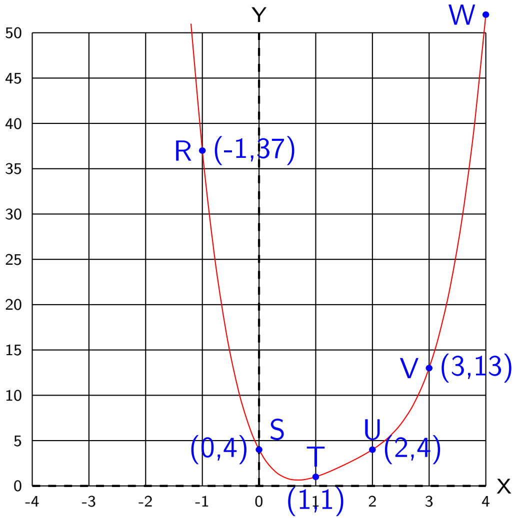

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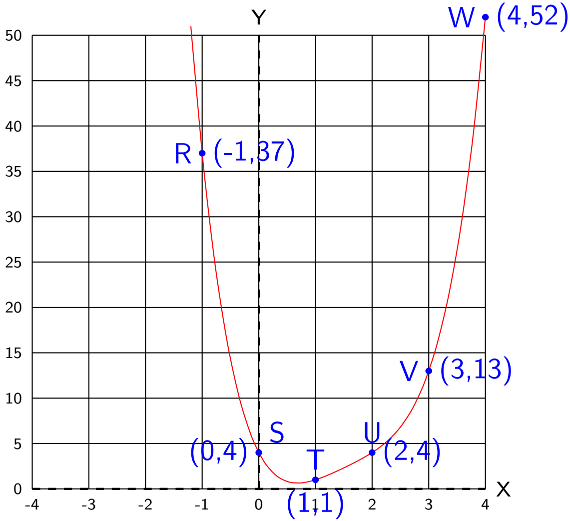

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Circle graphs and equations

The circle with radius 2 and center $(4,5)$ consists of all points (x,y) whose distance from $(4,5)$ is 2. Using the distance formula, this says that $\sqrt{(x-4)^2+(y-5)^2}=2$. Squaring both sides yields $(x-4)^2+(y-5)^2=4$.

Definition

The **standard form equation of the circle** *with center* (h,k) *and radius* r *is* $(x-h)^2+(y-k)^2=r^2$.

Example 10 : Find the center and radius of the circle with equation $(x-4)^2+(y+5)^2=49$.

Solution: We want the given equation $(x-4)^2+(y+5)^2=49$ to match the general formula $(x-h)^2+(y-k)^2=r^2$. To make the equations match, we take $h=4$ and $k=-5$ and $r=7$.

Warning: the coordinates h and k are the *negatives* of the numbers following x and y.

Answer: the center is $(h,k)=(4,-5)$ and the radius is 7.

There is no such thing as *the* equation of a graph. The language 'standard form equation' singles out a particular equation that is useful, since it displays in a convenient way the center and radius of the circle. But there are many other forms that are also of interest.

For example, the graph of the standard form circle equation $(x-4)^2+(y-5)^2=4$ can be multiplied out to obtain another equation of that circle: $x^2-8x+16+y^2-10y+25=4$. Collecting terms gives yet another equation: $x^2-8x+y^2-10y=-37$. This suggests an obvious question.

Question: How do we rewrite any given circle equation in standard form?
Answer: Use an algebra process called completing the square, as follows.

Apply the identity $(A+B)^2=A^2+2AB+B^2$ with $A=x$ and $B=h/2$ to rewrite $(x+h/2)^2$ as $x^2+hx+(h/2)^2$. Subtract $(h/2)^2$ from both sides to get the *completing the square identity*

$$x^2+hx=\left(x+\frac{h}{2}\right)^2-\left(\frac{h}{2}\right)^2$$

Example using $h = 8$ and $h/2 = 4$: $x^2 + 8x = (x+4)^2 - 4^2$.
Example using $h = -8$ and $h/2 = -4$: $x^2 - 8x = (x-4)^2 - 4^2$.

The same idea works for a similar expression involving y.
Example using $h = -10$ and $h/2 = -5$: $y^2 - 10y = (y-5)^2 - 5^2$

Lets assemble these ideas to do a complete problem.

Example 11: Find the standard form equation, center, and radius of the circle with equation $x^2 - 8x + y^2 + 10y = 8$.

Solution: as above, use $-8/2 = -4$ and $10/2 = 5$ to obtain
$x^2 - 8x = (x-4)^2 - 4^2 = (x-4)^2 - 16$ and
$y^2 + 10y = (y+5)^2 - 5^2 = (y+5)^2 - 25$. Therefore
$x^2 - 8x + y^2 + 10y = 50$ can be rewritten as
$(x-4)^2 - 16 + (y+5)^2 - 25 = 8$. Move constants to the right side to get
$(x-4)^2 + (y+5)^2 = 8 + 16 + 25 = 49 = 7^2$
Answer: The standard form equation is $(x-4)^2 + (y+5)^2 = 7^2$.
The circle's center is $(4, -5)$ and its radius is 7.

The method that you may have seen in high school is a bit longer but it involves exactly the same ideas. If you prefer to use it, see the following

Example 12: Find the standard form equation, center, and radius of the circle with equation $x^2 + y^2 = 8 + 8x - 10y$

Solution: $x^2 + y^2 = 8 + 8x - 10y$ is the original equation. Rewrite it as $x^2 - 8x + y^2 + 10y = 8$. Use completing the square to obtain
$x^2 - 8x + (\quad) + y^2 + 10y + (\quad) = 8 + (\quad) + (\quad)$
To fill in the first blank, use $h = -8$ and so $(h/2)^2 = (-8/2)^2 = 16$.
To fill in the second blank, use $h = 10$ and so $(h/2)^2 = (10/2)^2 = 25$.
Now fill in the blanks on both sides to get
$x^2 - 8x + (16) + y^2 + 10y + (25) = 8 + (16) + (25) = 49$. Rewrite this as
$(x-4)^2 + (y+5)^2 = 7^2$, exactly as before.

Answer: The standard form equation is $(x-4)^2 + (y+5)^2 = 7^2$. The center is $(4, -5)$ and the radius is 7.

Symmetry

Start at either of the two points (x, y) and $(-x, y)$. If you draw a horizontal line to the y-axis, and then continue that line an equal distance past the y-axis, you arrive at the other point.

We say that the two points are located symmetrically with respect to the y-axis.
We also say that you have reflected the original point across the y-axis to obtain the second point.

Definition

A **graph is symmetric with respect** *to the y-axis if reflecting any point on the graph across the y-axis yields another point on the graph.*

This will be the case provided replacing y by $-y$ in that graph's equation yields an equivalent equation (one with the same solutions).

If you start with two points (x, y) and $(x, -y)$, copy over the last paragraph to see that each point is obtained from the other by reflection across the x-axis.

Definition

A **graph is symmetric with respect** *to the x-axis if reflecting any point on the graph across the x-axis yields another point on the graph.*

This will be the case provided replacing y by $-y$ in that graph's equation yields an equivalent equation (one with the same solutions).

From these statements its easy to see the following:

Procedure

- *To reflect the graph of an equation across the y-axis, substitute $-x$ for x to obtain a new equation and draw the graph of that new equation.*
- *To reflect the graph of an equation across the x-axis, substitute $-y$ for y to obtain a new equation and draw the graph of that new equation.*

Example 13: Is the graph of $y = x + 7x^3 + 8x^5$ x-axis symmetric?
Solution: Substitute $-y$ for y in

- $y = x + 7x^3 + 8x^5$ to get $-y = x + 7x^3 + 8x^5$, which is the same as
- $y = -x - 7x^3 - 8x^5$

Clearly the two bulleted equations are not equivalent, since letting $x = 1$ yields $y = 16$ in the first equation and $y = -16$ in the second equation. Therefore
$\boxed{\text{the graph is not x-axis symmetric.}}$

Example 14: Is the graph of $y = 7x^4 + 8x^2$ y-axis symmetric?
Solution: Substitute $-x$ for x in

- $y = 7x^4 + 8x^2$ to get $y = 7(-x)^4 + 8(-x)^2$ which is the same as
- $y = 7x^4 + 8x^2$. Since the two equations are identical

$\boxed{\text{the graph is y-axis symmetric.}}$

Definition

A **graph is origin symmetric** *if reflecting a point on the graph through the origin yields another point on the graph. This is the case if substituting $-x$ for x and $-y$ for y in the equation yields an equivalent equation.*

Example 15: Is the graph of $y = x^3 + x$ origin symmetric?

Solution: Substitute $-x$ for x and $-y$ for y in

- $y = x^3 + x$ to get $-y = (-x)^3 + (-x) = -x^3 - x$ Multiply by -1 to get
- $y = x^3 + x$

Since the original and new equations are identical, $\boxed{\text{the graph is origin-symmetric.}}$

Quiz Review

Example 1: Find the distance betwen points $A(-3,2)$ and $D(3,3)$.

Example 2: Find the midpoint M of the line segment PQ joining the points $P(-3,2)$ and $Q(3,3)$. Show that your point M is indeed the midpoint by checking that $\overline{PM} = \overline{MQ} = \frac{1}{2}\overline{PQ}$.

Example 3: Is the point $(3,4)$ on the graph of the equation $x^2 + y^2 = 25$? What about the point $(4,2)$?

Example 4: Explain why the graph of $x^2 + y^2 + 4 = 0$ is empty.

Example 5: Find the equations of the vertical and horizontal lines through $(4,5)$.

Example 6: Find all x- and y- intercepts of the graph of the equation $x^2 + y^2 = 4$.

At what points does the graph meet the x-axis? the y-axis?

Examples 7,8,9: Go back in these notes and do these examples interactively.

Example 10 : Find the center and radius of the circle with equation $(x-4)^2 + (y+5)^2 = 49$.

Example 11: Find the standard form equation, center, and radius of the circle with equation $x^2 - 8x + y^2 + 10y = 8$.

Example 12: Find the standard form equation, center, and radius of the circle with equation $x^2 + y^2 = 8 + 8x - 10y$.

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Example 14: Is the graph of $y = 7x^4 + 8x^2$ y-axis symmetric?

Example 15: Is the graph of $y = x^3 + x$ origin symmetric?