Math 391: Methods of Differential Equations
Spring 2011 Final Exam Solutions

These solutions were written hastily by Pat Hooper <whooper@ccny.cuny.edu>. If you see any mistakes, please send an email so they can be corrected.

1. Compute the general solution of each of the following:

   (a) $t\frac{dy}{dt} + 4y = e^t/t^2$ $(t > 0)$.

Solution: This is a first order linear ODE. We write it in standard form as

$$\frac{dy}{dt} + \frac{4}{t}y = \frac{e^t}{t^3}.$$

We have $p(t) = \frac{4}{t}$ and $g(t) = \frac{e^t}{t^3}$. The integrating factor is $\mu = \exp \int p\ dt = \exp \int \frac{4}{t}\ dt = \exp(4 \ln t) = t^4$. The general solution is

$$\begin{aligned} y &= \tfrac{1}{\mu}\big(\int \mu g dt + c\big) = t^{-4}(\int te^t\ dt + c) \\ &= t^{-4}(te^t - \int e^t\ dt + c) = \tfrac{te^t - e^t + c}{t^4}. \end{aligned}$$

Our final answer is $\boxed{y = \frac{te^t - e^t + c}{t^4}.}$

   (b) $\big(e^x \cos(y) + (1-x)\sin y\big)\dfrac{dy}{dx} + e^x(1 + \sin y) + \cos y = 0.$

Solution: This equations looks likely to be exact. We define

$$M = e^x(1 + \sin y) + \cos y \quad \text{and} \quad N = e^x \cos(y) + (1-x)\sin y.$$

It is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. We compute

$$\frac{\partial M}{\partial y} = e^x \cos y - \sin y \quad \text{and} \quad \frac{\partial N}{\partial x} = e^x \cos y - \sin y.$$

So it is exact. Now our solution is $\psi(x,y) = c$ where ψ is a function satisfying $\frac{\partial \psi}{\partial x} = M$ and $\frac{\partial \psi}{\partial y} = N$. Since $\frac{\partial \psi}{\partial y} = N$, we have

$$\psi = \int N\ dy = \int e^x \cos(y) + (1-x)\sin y\ dy = e^x \sin y - (1-x)\cos y + h(x).$$

We must also have

$$M = \frac{\partial \psi}{\partial x} = e^x \sin y - \cos y + h'(x).$$

Therefore, we have $h'(x) = e^x$. We can choose $h(x) = e^x$. This makes

$$\psi(x,y) = e^x \sin y - (1-x)\cos y + e^x.$$

And our final answer is $\boxed{e^x \sin y - (1-x)\cos y + e^x = c.}$

Remark: We could have done the integrals in the other order (i.e. first integrate M with respect to x to find ψ up to a function depending only on y). We chose this order to make the integrals easier.

(c) $(x-y)dy = ydx$.

Solution: When we rewrite the equation as

$$\frac{dy}{dx} = \frac{y}{x-y},$$

we see that the equation is homogeneous (i.e., the function $h(x,y) = \frac{y}{x-y}$ satisfies $h(\alpha x, \alpha y) = h(x,y)$ for every non-zero α).

We let $v(x) = y(x)/x$. Then $y = xv(x)$ and $y' = v(x) + xv'(x)$. So by a change of variables our equation becomes

$$v + xv' = \frac{xv}{x - xv} = \frac{v}{1-v}.$$

We simplify this equation:

$$xv' = \frac{v^2}{1-v}.$$

$$\frac{1-v}{v^2}v' = \frac{1}{x}.$$

We have shown the equation is separable. So, we integrate.

$$\int \frac{1-v}{v^2}\, dv = \int \frac{1}{x}\, dx.$$

We have $\int \frac{1}{x}\, dx = \ln x + c$. We evaluate the integral and then substitute y/x for v:

$$\begin{aligned} \int \tfrac{1-v}{v^2}\, dv &= \int v^{-2} - \tfrac{1}{v}\, dv = \tfrac{-1}{v} - \ln v \\ &= \tfrac{-x}{y} - \ln \tfrac{y}{x} = \tfrac{-x}{y} - \ln y + \ln x. \end{aligned}$$

So our answer will be of the form:

$$\frac{-x}{y} - \ln y + \ln x = \ln x + c.$$

We can further simplify this to obtain:

$$\boxed{\tfrac{-x}{y} - \ln y = c.}$$

I don't believe that this equation can be simplified much beyond this, though you could solve for x in terms of y.

2. Solve the following initial value problems:

(a) $y\ dx + e^{-x}dy = 0, \quad y(0) = e^{-1}$.

Solution: This equation is separable. The equation is equivalent to $e^{-x}dy = -ydx$ or $\frac{-1}{y}dy = e^{x}dx$. We integrate both sides

$$\int \frac{-1}{y}dy = \int e^{x}dx + c.$$

$$-\ln y = e^{x} + c.$$

So the general solution to the ODE is $y = \exp(-e^{x} - c)$. We plug in $x = 0$ and $y = e^{-1}$ to find the constant. We get that $\exp(-1) = \exp(-e^{0} - c)$, so that $c = 0$. So our final answer is

$$\boxed{y = \exp(-e^{x}).}$$

(b) $y'' - 4y' + 5y = 16\cos t; \quad y(0) = 0,\ y'(0) = 0$.

Solution: This problem is best attacked by the method of undetermined coefficients. First we solve the homogeneous equation $y'' - 4y' + 5y = 0$. The characteristic polynomial is $r^2 - 4r + 5$. We find the roots with the quadratic equation:

$$r = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i.$$

So a fundamental set of solutions to the homogeneous equation is given by

$$y_1 = e^{2t}\cos t \quad \text{and} \quad y_2 = e^{2t}\sin t.$$

Now we will find a particular solution to the equation $y'' - 4y' + 5y = 16\cos t$. Since $\cos t$ is not a solution to the homogeneous equation, we have a particular solution of the form

$$Y = A\cos t + B\sin t.$$

We compute the first two derivatives:

$$Y' = -A\sin t + B\cos t \quad \text{and} \quad Y'' = -A\cos t - B\sin t.$$

Let $LHS = Y'' - 4Y' + 5Y$. We have,

$$\begin{aligned} LHS &= (-A\cos t - B\sin t) - 4(-A\sin t + B\cos t) + 5(A\cos t + B\sin t) \\ &= (4A - 4B)\cos t + (4A + 4B)\sin t. \end{aligned}$$

To have a solution, we have $4A - 4B = 16$ and $4A + 4B = 0$. So $A = 2$ and $B = -2$. Thus our particular solution is

$$Y = 2\cos t - 2\sin t.$$

The general solution to $y'' - 4y' + 5y = 16\cos t$ is

$$y = c_1 y_1 + c_2 y_2 + Y = c_1 e^{2t}\cos t + c_2 e^{2t}\sin t + 2\cos t - 2\sin t.$$

Since $y(0) = 0$, we have

$$0 = y(0) = c_1 + 2.$$

Therefore $c_1 = -2$. We have

$$y' = 2c_1 e^{2t}\cos t - c_1 e^{2t}\sin t + 2c_2 e^{2t}\sin t + c_2 e^{2t}\cos t - 2\sin t - 2\cos t.$$

Since $0 = y'(0)$ we have

$$0 = y'(0) = 2c_1 + c_2 - 2 = -4 + c_2 - 2.$$

So $c_2 = 6$. This gives the solution:

$$\boxed{y = -2e^{2t}\cos t + 6e^{2t}\sin t + 2\cos t - 2\sin t.}$$

3. Compute the general solution of each of the following:

(a) $x^2y'' - 3xy' + 4y = 0$.

Solution: This is an Euler equation. The characteristic polynomial is $r(r-1) - 3r + 4 = 0$, which simplifies to $r^2 - 4r + 4$. We can factor this as $(r-2)^2$. So the general solution is

$$\boxed{y = c_1 x^2 + c_2 x^2 \ln|x|.}$$

(b) $y'' + y = \sec t$.

Solution: We begin by observing that a fundamental set of solutions to the homogeneous ODE $y'' + y = 0$ is given by

$$y_1 = \cos t \quad \text{and} \quad y_2 = \sin t.$$

We can use the following formula from the method of variation of parameters to find a particular solution to the non-homogeneous equation $y'' + y = \sec t$.

$$Y(t) = -y_1 \int \frac{y_2 g}{W}\, dt + y_2 \int \frac{y_1 g}{W}\, dt.$$

Here $g = \sec t$. We also have

$$W = y_1 y_2' - y_1' y_2 = \cos^2 t + \sin^2 t = 1.$$

Substituting these into the equation, we see that

$$Y = -\cos t \int \sin t \sec t \; dt + \sin t \int \cos t \sec t \; dt.$$

Recall that $\sec t = \frac{1}{\cos t}$. Therefore, the second integral evaluates easily:

$$\int \cos t \sec t \; dt = \int 1dt = t.$$

The first integral can be evaluated by substitution:

$$\int \sin t \sec t \; dt = \int \frac{\sin t}{\cos t} \; dt = -\ln|\cos t|.$$

We have found a particular solution:

$$Y = \cos t \ln|\cos t| + t \sin t.$$

The general solution is

$$\boxed{c_1 \cos t + c_2 \sin t + \cos t \ln|\cos t| + t \sin t.}$$

4. For the differential equation:

$$(1 + 2x^3)y'' - xy' + 2y = 0$$

Compute the recursion formula for the coefficients of the power series solution centered at $x_0 = 0$ and use it to compute the first four nonzero terms of the solution with $y(0) = -6$ and $y'(0) = 0$.

Solution: The point $x_0 = 0$ is an ordinary point (i.e., the function $P(x) = 1 + 2x^3$ is nonzero at $x_0 = 0$). So, we are looking for solutions of the form $y = \sum_{n=0}^{\infty} a_n x^n$. We have

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

We will now simplify the terms of $(1 + 2x^3)y'' - xy' + 2y$.

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

$$2x^3 y'' = 2x^3 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} 2n(n-1) a_n x^{n+1} = \sum_{n=3}^{\infty} 2(n-1)(n-2) a_{n-1} x^n.$$

$$-xy' = -x\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=1}^{\infty} -na_n x^n.$$

$$2y = \sum_{n=0}^{\infty} 2a_n x^n.$$

We sum these terms to obtain our series. Terms with a common factor of x^n have been grouped together. We use LHS to abbreviate $(1+2x^3)y'' - xy' + 2y$.

$$\begin{aligned} LHS &= (2a_2 + 2a_0) + (6a_3 + a_1)x + (12a_4)x^2 \\ &\quad + \textstyle\sum_{n=3}^{\infty} \big((n+2)(n+1)a_{n+2} + 2(n-1)(n-2)a_{n-1} + (2-n)a_n\big)x^n. \end{aligned}$$

All coefficients of x^n on the right must be zero. This gives several equations:

$$2a_2 + 2a_0 = 0, \quad 6a_3 + a_1 = 0, \quad 12a_4 = 0, \quad \text{and}$$

$$(n+2)(n+1)a_{n+2} + 2(n-1)(n-2)a_{n-1} + (2-n)a_n = 0 \quad \text{for } n \geq 3.$$

Solving each equation for the largest root yields the recurrence relation:

$$a_2 = -a_0, \quad a_3 = \frac{-a_1}{6}, \quad a_4 = 0, \quad \text{and}$$

$$a_{n+2} = \frac{-2(n-1)(n-2)a_{n-1} + (n-2)a_n}{(n+2)(n+1)} \quad \text{for } n \geq 3.$$

Since we are looking for solutions with $y(0) = -6$ and $y'(0) = 0$, we have $a_0 = -6$ and $a_1 = 0$. Then we evaluate the first several coefficients using the recurrence relation:

$$a_2 = -a_0 = 6, \quad a_3 = \frac{-a_1}{6} = 0, \quad a_4 = 0.$$

We also have

$$a_5 = a_{3+2} = \frac{-2(2)(1)a_2 + (1)a_3}{(5)(4)} = \frac{-4(6) + (0)}{(5)(4)} = \frac{-24}{20} = \frac{-6}{5}.$$

$$a_6 = a_{4+2} = \frac{-2(3)(2)a_3 + (2)a_4}{(6)(5)} = \frac{-12(0) + 2(0)}{(6)(5)} = 0.$$

$$a_7 = a_{5+2} = \frac{-2(4)(3)a_4 + (3)a_5}{(7)(6)} = \frac{-24(0) + 3(\frac{-6}{5})}{(7)(6)} = \frac{-3}{35}.$$

This gives a final answer of

$$\boxed{y = -6 + 6x^2 - \tfrac{6}{5}x^5 - \tfrac{3}{35}x^7 - \ldots.}$$

5. A tank with a capacity of 100 gallons initially contains 50 gallons of water with 10 pounds of salt in solution. Fresh water enters at a rate of 2 gallons per minute and a well-stirred mixture is pumped out at the same rate. Compute the amount of salt (in pounds) in the tank 10 minutes after the process begins.

Solution: Let $Q(t)$ denote the amount of salt (in pounds) in the tank t minutes after the process begins. We have $Q(0) = 10$. So the concentration of salt in the tank is $Q(t)/50$ pounds per gallon. Since the fluid pours out at 2 gallons per minute, the salt is flowing out at a rate of $Q(t)/25$ pounds per minute. Thus,

$$Q' = -Q/25.$$

We will solve this differential equation by separation of variables. We have $\frac{Q'}{Q} = \frac{-1}{25}$. So, $\ln Q = \frac{-t}{25} + c$. Thus

$$Q = \exp(\frac{-t}{25} + c) = Ae^{\frac{-t}{25}},$$

where $A = e^c$. To determine A we consider the initial condition, $Q(0) = 10$. Thus, $A = 10$. So, $Q = 10e^{\frac{-t}{25}}$. We evaluate at $t = 10$ to see that:

After 10 minutes, the tank has $10e^{\frac{-2}{5}}$ pounds of salt in solution.

6. With y_1 and y_2 two solutions of the equation

$$y'' + py' + qy = 0,$$

show that the Wronskian of y_1 and y_2 is either never zero or identically zero (Hint: derive and solve a differential equation which is satisfied by the Wronskian of y_1 and y_2). Include the definition of the Wronskian.

Solution: Recall that the *Wronskian* of y_1 and y_2 is

$$W = y_1y_2' - y_1'y_2.$$

Observe that the derivative of W is given by

$$W' = y_1y_2'' - y_1''y_2.$$

Suppose that y_1 and y_2 both solve this equation. Then we know that

$$y_1'' + py_1' + qy_1 = 0 \quad \text{and} \quad y_2'' + py_2' + qy_2 = 0.$$

We multiply through by y_2 and y_1 respectively obtaining:

$$y_1''y_2 + py_1'y_2 + qy_1y_2 = 0 \quad \text{and} \quad y_2''y_1 + py_2'y_1 + qy_2y_1 = 0.$$

Taking a difference yields

$$(y_2''y_1 - y_1''y_2) + p(y_2'y_1 - y_1'y_2) + q(y_2y_1 - y_1y_2) = 0.$$

This simplifies to $W' + pW = 0$. This is a linear first order ODE for W. This is separable, because we can write it as $W'/W = -p$. So we have that $\ln W = \int -p(t)\ dt + c$ for some constant c. Taking the exponential of both sides yields $W = A \exp \int -p(t)\ dt$ where $A = e^c$. Observe that $\exp \int -p(t)\ dt$ is always positive. So, if $W(t_0) = 0$ for some t_0, we must have $A = 0$, in which case the Wronskian is the constant zero function.

7. (a) Compute the general solution of the differential equation

$$y^{vi} + 8y''' = 0.$$

Solution: The characteristic polynomial is $r^6 + 8r^3$. This factors as

$$r^6 + 8r^3 = r^3(r^3 + 8) = r^3(r + 2)(r^2 - 2r + 4).$$

We see that $r = 0$ is a root of multiplicity 3, and $r = 2$ is a root. We get two more from the quadratic formula:

$$r = \frac{2 \pm \sqrt{4 - 16}}{2} = 1 \pm i\sqrt{3}.$$

This gives the general solution:

$$\boxed{y = c_1 + c_2t + c_3t^2 + c_4e^{-2t} + c_5e^t\cos(\sqrt{3}\ t) + c_6e^t\sin(\sqrt{3}\ t).}$$

(b) Determine the test function $Y(t)$ with the fewest terms to be used to obtain a particular solution of the following equation via the method of undetermined coefficients. Do not attempt to determine the coefficients.

$$y^{vi} + 8y''' = t^2 + e^{-2t} - e^{-2t}\cos(\sqrt{3}\ t).$$

Solution: The following test function is optimal:

$$\boxed{Y = At^3 + Bt^4 + Ct^5 + Dte^{-2t} + Ee^{-2t}\cos(\sqrt{3}\ t) + Fe^{-2t}\sin(\sqrt{3}\ t).}$$

8. (a) A 2 pound weight stretches a spring 6 inches to a position $y = 0$. Set up an initial value problem (differential equation and initial conditions) for the position $y(t)$ (t seconds later) if the spring has been stretched down another 6 inches and is released.

Solution: Let $w = 2$ be the weight of the spring in pounds. Then $w = ma$ where m is the mass of the weight and a is the acceleration due to gravity, 32 ft/sec^2. We see that $m = \frac{1}{16}\frac{\text{lb}\cdot\text{sec}^2}{\text{ft}}$. By Hooke's law, we have $F = kl$ where, F is the force of the spring, k is the spring constant, and l is the amount of elongation of the spring. Since the force is 2 pounds when the spring is elongated 6 inches, we know that $2 = k\frac{1}{2}$. So $k = 4\text{ft}\cdot\text{lb}$. Since there are the only forces on the mass, the differential equation satisfied by the spring mass system is $my'' + ky = 0$. The initial conditions are given by the statement that the spring is stretched down another 6 inches and released (presumably with zero initial velocity). So we have $y(0) = \frac{1}{2}$ and $y'(0) = 0$. This gives us our initial value problem:

$$\boxed{\tfrac{1}{16}y'' + 4y = 0; \quad y(0) = \tfrac{1}{2}, y'(0) = 0.}$$

(b) Compute the Laplace transform $\mathcal{L}(y)(s)$ where y is the solution of the initial value problem:

$$y'' + 5y' + 3y = 0 \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = -1.$$

Solution: Let $Y = \mathcal{L}\{y\}$. Then we have

$$\mathcal{L}\{y'\} = sY - y(0) = sY - 1 \quad \text{and} \quad \mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0) = s^2Y - s + 1.$$

So the Laplace transform of the equation is

$$(s^2Y - s + 1) + 5(sY - 1) + 3Y = 0.$$

This can be simplified to $(s^2 + 5s + 3)Y = s + 4$. So we have

$$\boxed{\mathcal{L}\{Y\}(s) = \tfrac{s+4}{s^2+5s+3}.}$$

9. For the differential equation:

$$3xy'' + 2y' + (x+5)y = 0,$$

show that $x_0 = 0$ is a *regular singular point.* Compute the recursion formula for the coefficients of the series solution centered at $x_0 = 0$ which is associated with the larger root of the indicial equation and compute a_1 and a_2 when $a_0 = 1$.

Solution: Let $P(x) = 3x$, $Q(x) = 2$ and $R(x) = x+5$. We see that this is a singular point because $P(0) = 0$ and $Q(0) \neq 0$. To see it is a regular singular point, observe that the following two limits are finite:

$$\lim_{x\to 0} \frac{xQ(x)}{P(x)} = \lim_{x\to 0} \frac{2x}{3x} = \frac{2}{3}.$$

$$\lim_{x\to 0} \frac{x^2R(x)}{P(x)} = \lim_{x\to 0} \frac{x^2(x+5)}{3x} = \lim_{x\to 0} \frac{x(x+5)}{3} = 0.$$

This proves that $x_0 = 0$ is regular singular.

We will look for a solution of the form $y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n}$. We have

$$y' = \sum_{n=0}^{\infty} a_n(r+n)x^{r+n-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{r+n-2}.$$

The terms of the left hand side of the ODE are given by:

$$5y = \sum_{n=0}^{\infty} 5a_n x^{r+n}.$$

$$xy = \sum_{n=0}^{\infty} a_n x^{r+n+1} = \sum_{n=1}^{\infty} a_{n-1}x^{r+n}.$$

$$2y' = \sum_{n=0}^{\infty} 2a_n(r+n)x^{r+n-1} = \sum_{n=-1}^{\infty} 2a_{n+1}(r+n+1)x^{r+n}.$$

$$3xy'' = \sum_{n=0}^{\infty} 3a_n(r+n)(r+n-1)x^{r+n-1} = \sum_{n=-1}^{\infty} 3a_{n+1}(r+n+1)(r+n)x^{r+n}.$$

Let $LHS = 3xy'' + 2y' + (x+5)y$. Summing these terms and grouping yields:

$$\begin{aligned} LHS = &\big(2a_0r + 3a_0r(r-1)\big)x^{r-1} + \big(5a_0 + 2(r+1)a_1 + 3a_1(r+1)r\big)x^r \\ &+ \textstyle\sum_{n=1}^{\infty} \big(5a_n + a_{n-1} + 2a_{n+1}(r+n+1) + 3a_{n+1}(r+n+1)(r+n)\big)x^{r+n}. \end{aligned}$$

All the coefficients must be zero above yielding several equations:

$$2a_0r + 3a_0r(r-1) = 0, \quad 5a_0 + 2(r+1)a_1 + 3a_1(r+1)r = 0, \quad \text{and}$$

$$5a_n + a_{n-1} + 2a_{n+1}(r+n+1) + 3a_{n+1}(r+n+1)(r+n) = 0 \quad \text{for } n \geq 1.$$

Since $a_0 \neq 0$, we know that $2r + 3r(r-1) = 0$. This is the indicial equation. It simplifies to $3r^2 - r = 0$. So the roots are $r = 0$ and $r = \frac{1}{3}$.

We will now compute the recursion formula for the coefficients associated to the larger root of $r = \frac{1}{3}$. Ignoring the indicial equation, our equations become:

$$5a_0 + \frac{8}{3}a_1 + \frac{4}{3}a_1 = 0, \quad \text{and}$$

$$5a_n + a_{n-1} + \frac{6n+8}{3}a_{n+1} + \frac{(3n+4)(3n+1)}{3}a_{n+1} = 0 \quad \text{for } n \geq 1.$$

Simplifying yields $5a_0 + 4a_1 = 0$ and

$$5a_n + a_{n-1} + (3n+4)(n+1)a_{n+1} = 0 \quad \text{for } n \geq 1.$$

To get the recurrence relations we solve for the larger indices. $\boxed{a_1 = \frac{-5a_0}{4}}$ and

$$\boxed{a_{n+1} = \frac{-5a_n - a_{n-1}}{(3n+4)(n+1)} \quad \text{for } n \geq 1.}$$

Now suppose that $a_0 = 1$. We see that $\boxed{a_1 = \frac{-5}{4}}$ and

$$a_2 = \frac{-5a_1 - a_0}{14} = \frac{\frac{25}{4} - 1}{14} = \frac{21}{4 \cdot 14} = \frac{3}{8}.$$

This gives $\boxed{a_2 = \frac{3}{8}.}$

10. (a) Compute the sine series for the function $f(x) = 1$ on the interval $[0, \pi]$.

Solution: We have $L = \pi$. Our solution will be of the form

$$F = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{L}) = \sum_{n=1}^{\infty} a_n \sin(nx).$$

The coefficients are given by

$$\begin{aligned} a_n &= \tfrac{2}{L}\int_0^L f(x)\sin(nx)\;dx = \tfrac{2}{\pi}\int_0^\pi \sin(nx)\;dx \\ &= \tfrac{2}{\pi}[\tfrac{-1}{n}\cos(nx)]_0^\pi = \tfrac{-2}{n\pi}[\cos(n\pi) - 1] = \begin{cases} 0 & \text{if } n \text{ is even.} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Letting $n = 2k - 1$ to iterate over the odds, the sine series is

$$F = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \sin\big((2k-1)x\big).$$

(b) Use your answer to part (a) to obtain a series solution $u(t, x)$ for the partial differential equation with x in the interval $[0, \pi]$ and $t > 0$:

$$u_t = 64u_{xx} \quad \textit{with}$$
$$u(t,0) = u(t,\pi) = 0 \quad \text{for } t > 0 \quad \text{(boundary conditions)}$$
$$u(0,x) = 1 \quad \text{for } 0 < x < \pi \quad \text{(initial conditions).}$$

Solution: We treat this equation with separation of variables. Suppose that $u = T(t)X(x)$. Then, $u_t = T'(t)X(x)$ and $u_{xx} = T(t)X''(x)$. So our differential equation becomes $T'(t)X(x) = 64T(t)X''(x)$. We separate obtaining

$$\frac{T'}{64T} = \frac{X''}{X}.$$

This quantity is constant in both x and t. Let λ denote this constant. We have

$$\frac{T'}{64T} = \lambda \quad \text{and} \quad X'' - \lambda X = 0.$$

Consider the eigenvalue problem

$$X'' - \lambda X = 0 \quad \text{and} \quad X(0) = 0, X(\pi) = 0.$$

All non-trivial solutions occur when $\lambda < 0$. In this case, the characteristic equation is $r^2 - \lambda = 0$. So the roots are $r = \pm\sqrt{\lambda} = \pm i\sqrt{-\lambda}$. Let μ be the positive real constant $\mu = \sqrt{-\lambda}$, so that the roots are $r = \pm i\mu$. Then, the general solution to the differential equation is

$$X = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

Since $X(0) = 0$, we see that $c_1 = 0$. We also have

$$0 = X(\pi) = c_2 \sin(\mu\pi).$$

We are looking for non-trivial solutions, so $c_2 \neq 0$. Thus, $0 = \sin(\mu\pi)$. This means that μ is an integer. Since $\mu > 0$, we find solutions parametrized by a positive integer n:

$$\mu_n = n \quad \text{and} \quad X_n = \sin(nx).$$

Our eigenvalues are λ_n where $\mu_n = \sqrt{-\lambda_n}$. Thus

$$\lambda_n = -\mu_n^2 = -n^2.$$

Now lets solve the first order ODE, $\frac{T'}{64T} = \lambda$. By integration, we have $\frac{1}{64}\ln T = \lambda t$. So we get $T = e^{64\lambda t}$. We only care about the answer when $\lambda = \lambda_n$. So we get associated solutions:

$$T_n = e^{-64n^2 t}.$$

We use this define the solution to the PDE,

$$u_n = T_n X_n = e^{-64n^2 t}\sin(nx).$$

This satisfies the PDE, but has initial conditions

$$u_n(0, x) = X_x(x) = \sin(nx).$$

For $0 < x < \pi$, we have

$$1 = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \sin\big((2k-1)x\big)$$

So, our solution is given by

$$\boxed{u(t, x) = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} e^{-64(2k-1)^2 t} \sin\big((2k-1)x\big).}$$