203 Spring 2009 Group Final Solutions

Q1a: unit vectors are:

```
v1 = {4, 0, 1} / Sqrt[{4, 0, 1} . {4, 0, 1}]
```

$$\left\{\frac{4}{\sqrt{17}}, 0, \frac{1}{\sqrt{17}}\right\}$$

```
v2 = -{4, 0, 1} / Sqrt[{4, 0, 1} . {4, 0, 1}]
```

$$\left\{-\frac{4}{\sqrt{17}}, 0, -\frac{1}{\sqrt{17}}\right\}$$

Q1b: substitute and solve to get the point of intersection as (-8, 4, 0):

```
Solve[4 t + 2 (4) + 3 (2 + t) == 0]
```

{{t → -2}}

```
{4 t, 4, 2 + t} /. t → -2
```

{-8, 4, 0}

Q1c: Not perpendicular as the normal vector to the plane {1,2,3} is not parallel to the direction of the line {4,0,1}

Q2a: Find the gradient at the point and its length:

```
f = x^2 - Cos[x y] + Exp[y z]
```

$$e^{y z} + x^2 - \text{Cos}[x\, y]$$

```
v = {D[f, x], D[f, y], D[f, z]} /. {x → 3, y → 0, z → 1}
```

{6, 1, 0}

a unit vector in the direction of greatest increase:

```
v / Sqrt[v.v]
```

$$\left\{\frac{6}{\sqrt{37}}, \frac{1}{\sqrt{37}}, 0\right\}$$

the rate of increase is the length of the gradient:

```
Sqrt[v.v]
```

$$\sqrt{37}$$

Q2b: take the dot product to get the desired directional derivative:

$$\frac{\mathbf{v}\,.\,\{-5,\,12,\,0\}}{\sqrt{\{-5,\,12,\,0\}\,.\,\{-5,\,12,\,0\}}}\ /.\ \{x\to 3,\ y\to 0,\ z\to 1\}$$

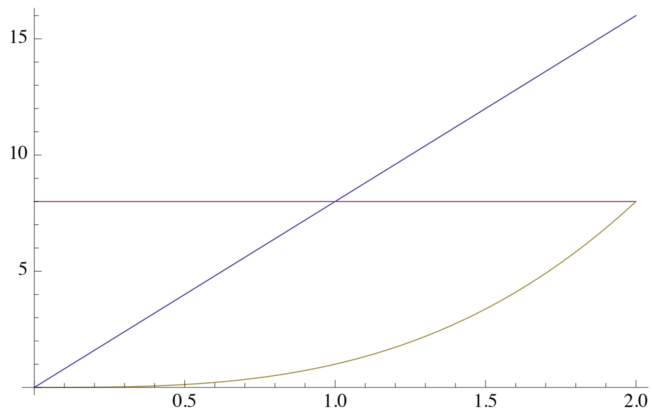
$$-\frac{18}{13}$$

Q2c: gradient vector gives the normal to tangent plane:

```
6 (x - 3) + 1 (y - 0) + 0 (z - 1) == 0
```

Q3: the region is given by:

```
Plot[{8 x, 8, x^3}, {x, 0, 2}]
```



To find its mass, integrate to get:

$$\int_0^8 \int_{y/8}^{\sqrt[3]{y}} \left(3\,x^2\right) \,dx\,dy$$

30

Q4: Set the first derivatives equal to zero and solve to get:

$$f = 16\,y^2 + x^4\,y + 4\,x^2 + 4$$

$$4 + 4\,x^2 + x^4\,y + 16\,y^2$$

```
D[f, x]
```

$$8\,x + 4\,x^3\,y$$

```
D[f, y]
```

$$x^4 + 32\,y$$

```
Solve[{D[f, x] == 0, D[f, y] == 0}]
```

$$\left\{\left\{y\to-\frac{1}{2},\ x\to-2\right\},\ \left\{y\to-\frac{1}{2},\ x\to 2\right\},\ \{y\to 0,\ x\to 0\},\ \left\{y\to\frac{1}{2}\,(-1)^{1/3},\ x\to-2\,(-1)^{1/3}\right\},\right.$$
$$\left.\left\{y\to\frac{1}{2}\,(-1)^{1/3},\ x\to 2\,(-1)^{1/3}\right\},\ \left\{y\to-\frac{1}{2}\,(-1)^{2/3},\ x\to-2\,(-1)^{2/3}\right\},\ \left\{y\to-\frac{1}{2}\,(-1)^{2/3},\ x\to 2\,(-1)^{2/3}\right\}\right\}$$

real solutions are the first 3, evaluating the discriminant gives:

```
disc = D[f, x, x] D[f, y, y] - D[f, x, y]^2
```

$-16\,x^6 + 32\,(8 + 12\,x^2\,y)$

```
disc /. {y → -1/2, x → -2}
```

-1536

saddle at ($-\frac{1}{2}$, -2)

```
disc /. {y → -1/2, x → 2}
```

-1536

saddle at ($-\frac{1}{2}$, 2)

```
disc /. {y → 0, x → 0}
```

256

```
D[f, x, x] /. {y → 0, x → 0}
```

8

relative min at (0,0)

Q5: integrate $\sqrt{1 + f_x{}^2 + f_y{}^2}$ by switching to polar to get:

$$\int_0^{2\pi}\int_0^2 \sqrt{1 + 4\,r^2}\; r\, dr\, dt$$

$$\frac{1}{6}\left(-1 + 17\sqrt{17}\right)\pi$$

Q6a: converges by the alternating series test, but its absolute value diverges by comparison with the harmonic series, so it is conditionally convergent.

Q6b: diverges by comparison with the alternating series

Q6c: diverges by the k-th term test for divergence

Q7: The first few terms:

$$\sum_{n=1}^{4} (-2)^n\,(x-1)^n$$

$-2\,(-1 + x) + 4\,(-1 + x)^2 - 8\,(-1 + x)^3 + 16\,(-1 + x)^4$

Using the ratio test:

```
Limit[Abs[((-2)^(n+1) (x - 1)^(n+1))/((-2)^n (x - 1)^n)], {n -> ∞}]
```

{2 Abs[-1 + x]}

converges for x between 1/2 and 3/2. Endpoints: both diverge.

Q8:

```
T = x^3 + y^3 - 3 x y
```

$x^3 - 3xy + y^3$

interior points:

```
Solve[{D[T, x] == 0, D[T, y] == 0}]
```

$\{\{x \to 0, y \to 0\}, \{x \to 1, y \to 1\}, \{x \to -(-1)^{1/3}, y \to (-1)^{2/3}\}, \{x \to (-1)^{2/3}, y \to -(-1)^{1/3}\}\}$

consider the discriminant, if desired:

```
D[T, x, x] D[T, y, y] - D[T, x, y]^2
```

$-9 + 36xy$

```
D[T, x, x]
```

$6x$

gives saddle at origin, rel min at (1,1)

Now considering the edges of square:

```
Solve[D[T /. {x -> t, y -> 0}, t] == 0]
```

$\{\{t \to 0\}, \{t \to 0\}\}$

```
Solve[D[T /. {x -> 2, y -> t}, t] == 0]
```

$\{\{t \to -\sqrt{2}\}, \{t \to \sqrt{2}\}\}$

```
Solve[D[T /. {x -> t, y -> 2}, t] == 0]
```

$\{\{t \to -\sqrt{2}\}, \{t \to \sqrt{2}\}\}$

```
Solve[D[T /. {x -> 0, y -> t}, t] == 0]
```

$\{\{t \to 0\}, \{t \to 0\}\}$

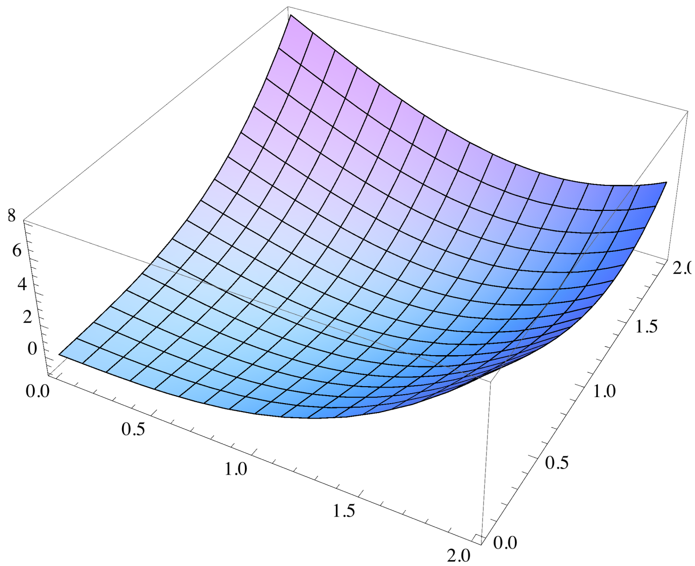
which gives points to consider at (0,0), (1,1), the corners, $(\sqrt{2}, 2)$ and $(2, \sqrt{2})$.Plugging in, we get:

```
T /. {{x -> 0, y -> 0}, {x -> 0, y -> 2},
  {x -> 2, y -> 2}, {x -> 2, y -> 0}, {x -> 2, y -> Sqrt[2]}, {x -> Sqrt[2], y -> 2}}
```

$\{0, 8, 4, 8, 8 - 4\sqrt{2}, 8 - 4\sqrt{2}\}$

So the hottest points on the plate are the corners (0,2) and (2,0).

```
Plot3D[T, {x, 0, 2}, {y, 0, 2}]
```



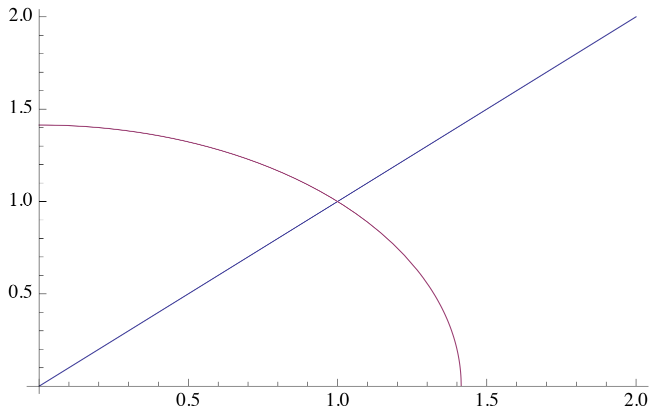
Q9: find a normal by taking cross products of displacement vectors:

```
p = {1, 0, 1}
{1, 0, 1}
q = {1, 0, 2}
{1, 0, 2}
r = {2, 2, 1}
{2, 2, 1}
Cross[q - p, r - p]
{-2, 1, 0}
```

9b: approaching the origin along x=0, the limit is 0. Approaching along the line x=y the limit is 1, so the limit doesn't exist.

10a: looks like an quartered ice cream cone, slightly licked down some. The intersection with the xz plane is:

```
Plot[{x, Sqrt[2 - x^2]}, {x, 0, 2}]
```



volume in rectangular:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} 1 \, dz \, dy \, dx$$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \mathbf{1} \, dz \, dy \, dx$$

$$\frac{1}{3}\left(-1+\sqrt{2}\right)\pi$$

volume in cylindrical:

$$\int_0^{\pi/2} \int_0^1 \int_r^{\sqrt{2-r^2}} \mathbf{1}\, r \, dz \, dr \, d\theta$$

$$\frac{1}{3}\left(-1+\sqrt{2}\right)\pi$$

volume in spherical:

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho^2 \, \mathrm{Sin}[\phi] \, d\rho \, d\phi \, d\theta$$

$$\frac{1}{3}\left(-1+\sqrt{2}\right)\pi$$

probably easiest in spherical

Q10: We use geometric series to get the first and then integrate:

```
Series[1/(1 + x^2), {x, 0, 15}]
```

$$1 - x^2 + x^4 - x^6 + x^8 - x^{10} + x^{12} - x^{14} + O[x]^{16}$$

```
∫Series[1/(1 + x^2), {x, 0, 15}] dx
```

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \frac{x^{13}}{13} - \frac{x^{15}}{15} + O[x]^{17}$$

```
Series[ArcTan[x], {x, 0, 15}]
```

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \frac{x^{13}}{13} - \frac{x^{15}}{15} + O[x]^{16}$$

Q11: take the cross products of r_u and r_v to get a normal vector to the tangent plane:

```
r = {u^2 - v^2, 4 u v, (3 u - 2 v)^2}
```

$\{u^2 - v^2, 4\,u\,v, (3\,u - 2\,v)^2\}$

```
r1 = D[r, u] /. {u → 1, v → 1}
```

{2, 4, 6}

```
r2 = D[r, v] /. {u → 1, v → 1}
```

{-2, 4, -4}

```
n = Cross[r1, r2]
```

{-40, -4, 16}

The point is:

```
r /. {u → 1, v → 1}
```

{0, 4, 1}

which gives the tangent plane equation as:

```
-40 (x - 0) + -4 (y - 4) + 16 (z - 1) == 0
```