MULTIPLE INTEGRATION NOTES AND EXERCISES
by
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Overview

In this discussion we consider the integral of a function of two variables $z = f(x,y)$ over a region $\mathscr{D}$ in the xy-plane (2-space) and the integral of a function of three variables $w = F(x,y,z)$ over a solid region $\mathscr{E}$ in 3-space. These integrals are called multiple integrals and are defined as the limit of approximating Riemann sums, much like the single-variable integrals you learned about in earlier calculus courses. We can use multiple integrals to calculate quantities that vary over two or three dimensions, such as the total mass or the angular momentum of an object of varying density and the volumes of solids with general curved boundaries.

DOUBLE INTEGRATION

In your earlier studies of calculus we defined the definite integral of a continuous function $y = f(x)$ over a closed interval $[a,b]$ as a limit of Riemann sums. In this discussion we extend this idea to define the integral of a continuous function of two variables $z = f(x,y)$ over a bounded region $\mathscr{D}$ in the plane. In both cases the integrals are limits of approximating Riemann sums. The Riemann sums for the integral of a single-variable function $y = (x)$ are obtained by:

1. Partitioning a finite interval $[a,b]$ into thin subintervals,

$$[x_0,\ x_1],\ [x_1,\ x_2],\ [x_2,\ x_3],\ldots,[x_{i-1},\ x_i],\ \ldots,[x_{n-1},\ x_n]\ \text{ where } a = x_0 \text{ and } x_n = b$$

multiplying the width of each subinterval, Δx_i, by the value of f at a point x_i^* inside that subinterval, and then

2. Forming a Riemann Sum by adding together all the products of the form $f(x_i^*)\,\Delta x_i$.

A similar method of partitioning, multiplying, and summing is used to construct double integrals. However, this time we pack a planar region $\mathscr{D}$ with small rectangles, rather than small subintervals. We then take the product of each small rectangles area with the value of $f(x,y)$ at a point inside that rectangle, and finally sum together all these products. When $f(x,y)$ is continuous, these sums converge to a single number as each of the small rectangles shrinks in both width and height. The limit is the double integral of $f(x,y)$ over $\mathscr{D}$. As with single integrals, we can evaluate multiple integrals via antiderivatives, which frees us from the formidable task of calculating a double integral directly from its definition as a limit of Riemann sums. The major practical problem that arises in evaluating multiple integrals lies in determining the limits of integration. While the integrals of your earlier studies of calculus were evaluated over an interval, which is determined by its two endpoints, multiple integrals are evaluated over a region in the plane or in space. This gives rise to limits of integration which often involve variables, not just constants. Describing the regions of integration is the main new issue that arises in the calculation of multiple integrals.

DOUBLE INTEGRALS OVER RECTANGULAR REGIONS OF THE xy-PLANE

We begin our investigation of double integrals by considering the simplest type of planar region, a rectangle. We consider a function $z = f(x,y)$ defined on a rectangular region, $\mathscr{D}$, defined as

$$\mathscr{D} = \{(x,y) :\ a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}.$$

IDEA: We subdivide the plane region $\mathscr{D}$ into small rectangles using a network of lines parallel to the x- and y-axes. The lines divide $\mathscr{D}$ into n rectangular pieces, where the number of such pieces n gets large as the width and height of each piece gets small. These rectangles form a partition of $\mathscr{D}$. A small rectangular piece of width, Δx and height Δy has area $\Delta A = \Delta x\ \Delta y$. If we number the small pieces partitioning $\mathscr{D}$ in some order, then their areas are given by numbers $\Delta A_1,\ \Delta A_2,\ldots,\Delta A_n$ where ΔA_k is the area of the kth small rectangle.

To form a Riemann sum over $\mathscr{D}$, we choose a point (x_k, y_k) in the kth small rectangle, multiply the number $f(x_k,y_k)$ by the area ΔA_k and add together the products $f(x_k,y_k)\ \Delta A_k$:

$$S_n = \sum_{k=1}^{n} f(x_k,y_k)\ \Delta A_k$$

Depending on how we pick (x_k,y_k) in the kth small rectangle, we may get different values for S_n.

We are interested in what happens to these Riemann sums as the widths and heights of all the small rectangles in the partition of $\mathscr{D}$ approach zero. The **norm** of a partition $\mathscr{P}$, written $\parallel \mathscr{P} \parallel$ is the largest width or height of any rectangle in the partition. For example, if $\parallel \mathscr{P} \parallel = 0.1$, then all the rectangles in the partition of $\mathscr{D}$ have width at most 0.1 and height at most 0.1. Sometimes the Riemann sums converge as the norm of $\mathscr{P}$ goes to zero, written $\parallel \mathscr{P} \parallel \to 0$. The resulting limit is then written as

$$\lim_{\parallel \mathscr{P} \parallel \to 0} \sum_{k=1}^{n} f(x_k, y_k)\ \Delta A_k.$$

As $\parallel \mathscr{P} \parallel \to 0$ and the rectangles get narrow and short, their number n increases, so we can also write this limit as

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k)\ \Delta A_k.$$

with the understanding that $\Delta A_k \to 0$ as $n \to \infty$ and $\parallel \mathscr{P} \parallel \to 0$. There are many choices involved in a limit of this kind. The collection of small rectangles is determined by the grid of vertical and horizontal lines that determine a rectangular partition of $\mathscr{P}$. In each of the resulting small rectangles there is a choice of an arbitrary point at which f is evaluated. These choices together determine a single Riemann sum. To form a limit, we repeat the whole process again and again, choosing partitions whose rectangle widths and heights both go to zero and whose number goes to infinity. When a limit of the sums exists, giving the same limiting value no matter what choices are made, then the function $f(x, y)$ is said to be integrable over $\mathscr{D}$ and the limit is called the **double integral of** $f(x, y)$ **over** $\mathscr{D}$, written as

$$\iint\limits_{\mathscr{D}} f(x, y)\ dA \text{ or } \iint\limits_{\mathscr{D}} f(x, y)\ dx\ dy.$$

It can be shown that if $f(x, y)$ is a continuous function throughout $\mathscr{D}$, then f is integrable, as in the single-variable case discussed in your earlier studies of calculus. Many discontinuous functions are also integrable, including functions which are discontinuous only on a finite number of points or smooth curves. The proof of these facts can be found in more advanced textbooks about calculus.

Interpreting Double Integrals as Volumes of Solids in 3-space

When $f(x, y)$ is a positive function over a plane region $\mathscr{D}$ in the xy-plane, we may interpret the positive number $\iint\limits_{\mathscr{D}} f(x, y)\ dA$ as **the volume of the 3-dimensional solid region** $\mathscr{E}$ over the xy-plane bounded below by $\mathscr{D}$ and bounded above by the graph of the surface given as $z = f(x, y)$. Each term $f(x_k, y_k)\ \Delta A_k$ (a positive number) in the sum $S_n = \sum_{k=1}^{n} f(x_k, y_k)\ \Delta A_k$ is the volume of a vertical rectangular box that approximates the volume of the portion of the solid region $\mathscr{E}$ in space that stands directly above the base ΔA_k. The sum S_n thus approximates what we want to call the total volume of the solid $\mathscr{E}$.

Definition {Volume as a double Integral} If $f(x, y) \geqslant 0$ for all $(x, y) \in \mathscr{D}$, then we define the Volume of the solid $\mathscr{E}$ in space whose base is $\mathscr{D}$ and bounded above by the surface $z = f(x, y)$ as $V(\mathscr{E})$, and

$$V(\mathscr{E}) = \lim_{n \to \infty} \sum_{k=1}^{n} S_n = \iint\limits_{\mathscr{D}} f(x, y)\ dA$$

where $\Delta A_k \to 0$ as $n \to \infty$. As you might expect, this more general method of calculating volume agrees with the Solids of Revolution methods you encountered in your earlier studies of calculus, but we do not prove this here.

Fubini's Theorem for Calculating Double Integrals

Suppose that we wish to calculate the volume, V, under the plane $z = f(x,y) = 4 - x - y$ over the rectangular region $\mathscr{D} = \{(x,y) : 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 1\}$ in the xy-plane. If we apply the method of slicing you learned in working with volumes in your earlier studies of calculus, with slices perpendicular to the x-axis, then the volume is

$$\int_{x=0}^{x=2} A(x)\, dx \tag{1}$$

where $A(x)$ is the cross-sectional area at x. For each value of x, we may calculate $A(x)$ as the integral

$$A(x) = \int_{y=0}^{y=1} (4 - x - y)\, dy \tag{2}$$

which is the area under the curve in the plane of the cross-section at x. In calculating $A(x)$, x is held fixed and the integration takes place with respect to y. Combining Equations (1) and (2), we see that the volume of the entire solid is

$$\begin{aligned}
\text{Volume} &= \int_{x=0}^{x=2} A(x)\, dx = \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4 - x - y)\, dy \right) dx \\
&= \int_{x=0}^{x=2} \left[4y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1} dx \\
&= \int_{x=0}^{x=2} \left(\frac{7}{2} - x \right) dx \\
&= \left[\frac{7}{2}x - \frac{x^2}{2} \right]_{x=0}^{x=2} = 5 \ \text{ cubic units}
\end{aligned}$$

To obtain the cross-sectional area $A(x)$, we hold x fixed and integrate with respect to y.

If we just wanted to write a formula for the volume, without carrying out any of the integrations, we could write

$$\text{Volume} = \int_0^2 \int_0^1 (4 - x - y)\, dy\, dx$$

The expression on the right, called an **iterated** or **repeated integral**, says that

> *the volume is obtained by integrating* $4 - x - y$ *with respect to* y *from* $y = 0$ *to* $y = 1$, *holding* x *fixed, and then integrating the resulting expression in* x *with respect to* x *from* $x = 0$ *to* $x = 2$. *The limits of integration* 0 *and* 1 *are associated with* y, *so they are placed on the integral closest to* dy. *The other limits of integration,* 0 *and* 2, *are associated with the variable* x, *so they are placed on the outside integral symbol that is paired with* dx.

What would have happened if we had calculated the volume by slicing with planes perpendicular to the y-axis? As a function of y, the typical cross-sectional area is

$$A(y) = \int_{x=0}^{x=2} (4 - x - y)\, dx = \left[4x - \frac{x^2}{2} - xy \right]_{x=0}^{x=2} = 6 - 2y. \tag{4}$$

The volume of the entire solid is therefore

$$\text{Volume} = \int_{y=0}^{y=1} A(y)\, dy = \int_{y=0}^{y=1} (6 - 2y)\, dy = \left[6y - y^2 \right]_{y=0}^{y=1} = 5 \ \text{ cubic units,}$$

in agreement with our earlier calculation.

Again, we may give a formula for the volume as an iterated integral by writing

$$\text{Volume} = \int_0^1 \int_0^2 (4 - x - y)\, dx\, dy.$$

The expression on the right says we can find the volume by integrating with respect to x from $x = 0$ to $x = 2$ as in Equation (4) and then integrating the result with respect to y from $y = 0$ to $y = 1$. In this iterated integral, the order of integration is first x and then y, the reverse of the order in Equation (3).

What does the two volume calculations above with iterated integrals have to do with the double integral

$$\iint_{\mathscr{D}} (4 - x - y)\, dA$$

over the rectangle $\mathscr{D} = \{(x, y) :\ 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 1\}$? The answer is that both iterated integrals give the value of the double integral. This is what we would reasonably expect, since the double integral measures the volume of the same region as the two iterated integrals. A theorem published in 1907 by Guido Fubini says that:

> The double integral of any continuous function $f(x, y)$ over a rectangle $\mathscr{D}$ can be calculated as an iterated integral in either order of integration.

Guido Fubini proved his theorem in greater generality, but this is what it says in our setting:

Theorem 1. Fubini's Theorem (Weak Form) If $f(x, y)$ is continuous throughout the rectangular region $\mathscr{D} = \{(x, y) :\ a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$, then

$$\iint_{\mathscr{D}} f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

Fubini's Theorem says that: *Double integrals over rectangles can be calculated as iterated integrals*. Thus, we can evaluate a double integral by integrating with respect to one variable at a time. Fubini's Theorem also says that: *We may calculate the double integral by integrating in either order.*

When we calculate a volume by slicing, we may use, either planes perpendicular to the x-axis or planes perpendicular to the y-axis.

Example 1. Calculate $\iint_{\mathscr{D}} f(x, y)\, dA$ for $f(x, y) = 1 - 6x^2 y$ and $\mathscr{D} = \{(x, y) :\ 0 \leqslant x \leqslant 2,\ -1 \leqslant y \leqslant 1\}$.

Solution. By Fubini's Theorem

$$\begin{aligned}\iint_{\mathscr{D}} f(x, y)\, dA &= \int_{-1}^{1} \int_0^2 (1 - 6x^2 y)\, dx\, dy \\ &= \int_{-1}^{1} \left[x - 2x^3 y\right]_{x=0}^{x=2} dy \\ &= \int_{-1}^{1} (2 - 16y)\, dy \\ &= \left[2y - 8y^2\right]_{y=-1}^{y=1} = 4\end{aligned}$$

Reversing the order of integration gives the same answer:

$$\begin{aligned}\int_0^2 \int_{-1}^{1} (1 - 6x^2 y)\, dy\, dx &= \int_{-1}^{1} \left[y - 3x^2 y^2\right]_{y=-1}^{y=1}\, dx \\ &= \int_0^2 \left[(1 - 3x^2) - (-1 - 3x^2)\right]\, dx \\ &= \int_0^2 2\, dy = 4\end{aligned}$$

□

Double Integrals over bounded Non Rectangular Regions

To define the double integral of a function $f(x, y)$ over a bounded, nonrectangular region $\mathscr{D}$, we again begin by covering $\mathscr{D}$ with a grid of small rectangular cells whose union contains all points of $\mathscr{D}$. This time, however, we cannot exactly fill $\mathscr{D}$ with a finite number of rectangles lying inside $\mathscr{D}$, since its boundary is curved, and some of the small rectangles in the grid lie partly outside $\mathscr{D}$. A partition of $\mathscr{D}$ is formed by taking the rectangles that lie completely inside it, not using any that are either partly or completely outside. For commonly arising regions, more and more of $\mathscr{D}$ is included as the norm of a partition (the largest width or height of any rectangle used) approaches zero.

Once we have a partition of $\mathscr{D}$, we number the rectangles in some order from 1 to n and let ΔA_k be the area of the k-th rectangle. We then choose a point (x_k, y_k) in the k-th rectangle and form the Riemann sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$$

As the norm of the partition forming goes to zero, i.e., $\| \mathscr{P} \| \to 0$, the width and height of each enclosed rectangle goes to zero and their number goes to infinity. If $f(x, y)$ is a continuous function, then these Riemann sums converge to a limiting value, not dependent on any of the choices we made. This limit is called the double integral of $f(x, y)$ over $\mathscr{D}$:

$$\lim_{\| \mathscr{P} \| \to 0} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta\, A_k = \iint_{\mathscr{D}} f(x, y)\, dA$$

The nature of the boundary of $\mathscr{D}$ introduces issues not found in integrals over an interval. When $\mathscr{D}$ has a curved boundary, the n rectangles of a partition lie inside $\mathscr{D}$ but do not cover all of $\mathscr{D}$. In order for a partition to approximate $\mathscr{D}$ well, the parts of $\mathscr{D}$ covered by small rectangles lying partly outside $\mathscr{D}$ must become negligible as the norm of the partition approaches zero. This property of being nearly filled in by a partition of small norm is satisfied by all the regions that we will encounter. There is no problem with boundaries made from polygons, circles, ellipses, and from continuous graphs over an interval, joined end to end. A curve with a fractal type of shape would be problematic, but such curves are not relevant for most applications. A careful discussion of which type of regions $\mathscr{D}$ can be used for computing double integrals is left to a more advanced text.

Double integrals of continuous functions over nonrectangular regions have the same algebraic properties (summarized further on) as integrals over rectangular regions. The domain Additivity Property says that if $\mathscr{D}$ is decomposed into nonoverlapping regions $\mathscr{D}_1$ and $\mathscr{D}_2$ with boundaries that are again made of a finite number of line segments or smooth curves, then

$$\iint_{\mathscr{D}} f(x, y)\, dA = \iint_{\mathscr{D}_1} f(x, y)\, dA + \iint_{\mathscr{D}_2} f(x, y)\, dA$$

If $f(x, y)$ is positive and continuous over $\mathscr{D}$ we define the volume of the solid region, $\mathscr{E}$, between $\mathscr{D}$ and the surface $z = f(x, y)$ to be $\iint_{\mathscr{D}} f(x, y)\, dA$ as before.

A Procedure for evaluating Double Integrals over x-simple bounded nonrectangular Regions

If $\mathscr{D} = \{(x, y) :\ a \leqslant x \leqslant b,\ \ g_1(x) \leqslant y \leqslant g_2(x)\}$ is an **x-simple** region in the xy-plane that is, $\mathscr{D}$ is **bounded above** and **bounded below** by the plane curves $y = g_2(x)$ and $y = g_1(x)$ and on the sides by the lines $x = a$ and $x = b$ we may again calculate the volume $\iint_{\mathscr{D}} f(x, y)\, dA$ by the **method of slicing**.

STEP 1. We first calculate the cross-sectional area

$$A(x) = \int_{y = g_1(x)}^{y = g_2(x)} f(x, y)\, dy$$

and then

STEP 2. integrate $A(x)$ from $x = a$ to $x = b$ to get the volume as an iterated integral:

$$\text{Volume} = V(\mathscr{E}) = \iint_{\mathscr{D}} f(x, y)\, dA = \int_a^b A(x)\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx \qquad (5)$$

A Procedure for Evaluating Double Integrals over y-simple bounded nonrectangular Regions

Similarly, if $\mathscr{D} = \{(x, y) : c \leqslant x \leqslant d, \quad h_1(y) \leqslant x \leqslant h_2(y)\}$ is an **y-simple** region in the xy-plane that is, $\mathscr{D}$ is **bounded above** and **bounded below** by the lines $y = c$ and $y = d$ and on the sides by the plane curves $x = h_2(y)$ and $x = h_1(y)$, then the volume calculated by slicing as follows:

Step 1. We first calculate the cross-sectional area

$$A(y) = \int_{x=h_1(y)}^{x=h_2(y)} f(x, y)\, dx$$

and then

Step 2. integrate $A(y)$ from $y = c$ to $y = d$ to get the volume as an iterated integral:

$$\text{Volume} = V(\mathscr{E}) = \iint_{\mathscr{D}} f(x, y)\, dA = \int_c^d A(y)\, dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy \tag{6}$$

The fact that the iterated integrals (5) and (6) both give the volume that we defined to be the double integral of $f(x, y)$ over $\mathscr{D}$ is a consequence of the following stronger form of Fubini's Theorem.

Theorem 2. Fubini's Theorem (Strong Form) Let $f(x, y)$ be a continuous function throughout a region $\mathscr{D}$.

1. If $\mathscr{D}$ is defined by $a \leqslant x \leqslant b$, $g_1(x) \leqslant y \leqslant g_2(x)$ with $g_1(x)$ and $g_2(x)$ continuous on $[a, \ b]$, that is, $\mathscr{D}$ is an x-simple region, then

$$\iint_{\mathscr{D}} f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If $\mathscr{D}$ is defined by $c \leqslant y \leqslant d$, $h_1(y) \leqslant x \leqslant h_2(y)$ with $h_1(y)$ and $h_2(y)$ continuous on $[c, \ d]$, that is, $\mathscr{D}$ is a y-simple region, then

$$\iint_{\mathscr{D}} f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Example 2. Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis (i.e., the line $y = 0$) and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = f(x, y) = 3 - x - y$.

Solution. First notice that the region $\mathscr{D}$ is bounded below by the curve $g_1(x) = 0$ and bounded above the the curve $g_2(x) = x$ and on the sides by the lines $x = 0$ and $x = 1$, i.e, $\mathscr{D}$ is an x-simple region. Thus by Fubini's Theorem (Strong Form):

$$\begin{aligned}
\text{Volume} &= \iint_{\mathscr{D}} f(x, y)\, dA \\
&= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dA \\
&= \int_0^1 \int_0^x (3 - x - y)\, dy\, dx \\
&= \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx \\
&= \int_0^1 \left(3x - \frac{3}{2}x^2 \right) dx = \left[\frac{3}{2}x^2 - \frac{1}{2}x^3 \right]_{x=0}^{x=1} = 1 \text{ cubic unit}
\end{aligned}$$

When the order of the integration is reversed, i.e., we consider $\mathscr{D}$ as a y-simple region, then the volume is

$$\begin{aligned}
\text{Volume} &= \iint_{\mathscr{D}} f(x,y)\, dA \\
&= \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dA \\
&= \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy \\
&= \int_0^1 \left[3x - \frac{1}{2}x^2 - xy\right]_{x=y}^{x=1} dy \\
&= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{1}{2}y^2 + y^2\right) dy \\
&= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy \\
&= \left[\frac{5}{2}y - 2y^2 + \frac{1}{2}y^3\right]_{y=0}^{y=1} = 1 \text{ cubic unit.}
\end{aligned}$$

The two integrals are equal as they should be. □

Although Fubini's Theorem assures us that: *A double integral may be calculated as an iterated integral in either order of integration*, the value of one integral may be easier to find than the value of the other. The next example shows how this can happen.

Example 3. Calculate $\iint_{\mathscr{D}} f(x,y)\, dA$ where $f(x,y) = \frac{\sin(x)}{x}$ and $\mathscr{D}$ is the triangular region in the xy-plane bounded by the x-axis, the line $y = x$ and the line $x = 1$.

Solution. If we first integrate with respect to y and then with respect to x, we find

$$\int_0^1 \left[\int_0^x \frac{\sin(x)}{x}\, dy\right] dx = \int_0^1 \left[y\, \frac{\sin(x)}{x}\right]_{y=0}^{y=x} dx = \int_0^1 \sin(x)\, dx = [-\cos(x)]_{x=0}^{x=1} = 1 - \cos(1) \approx 0.46.$$

If we reversed the order of integration and attempt to calculate

$$\int_0^1 \int_y^1 \frac{\sin(x)}{x}\, dx\, dy,$$

we run into a problem because the integral

$$\int \frac{\sin(x)}{x}\, dx$$

cannot be expressed in terms of elementary functions (there is no simple antiderivative!) □

When calculating double Integrals over bounded nonrectangular regions, there is no general rule for predicting which order of integration will be the easiest or the good one in circumstances like these. If the order you choose first, doesnt work, then try the other. Sometimes neither order will work, and then we need to use **numerical approximation methods**.

A Procedure for Finding the limits of integration for a Double Integrals

We now give a procedure for finding limits of integration that applies for many regions in the plane. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

When faced with evaluating $\iint\limits_{\mathscr{D}} f(x,y)\;dA$, *integrating first with respect to* y *and then with respect to* x, *take the following steps:*

Step 1. Sketch: *Sketch the region* $\mathscr{D}$ *of integration and its bounding curves.*

Step 2. Find the y-limits of integration: *Find the* y*-limits of integration imagining a vertical line, say* L *passing through the region* $\mathscr{D}$ *in the direction of increasing* y*. Mark the* y *values where* L *enters and leaves* $\mathscr{D}$*. These are your* y*-limits of integration and are usually functions of* x *(instead of constants, unless* $\mathscr{D}$ *is a rectangle).*

Step 3. Find the x-limits of integration: *we find the* x*-limits of integration by choosing* x*-limits that include all vertical lines passing through* $\mathscr{D}$*.*

Exercise 1. `Write out a 3-step procedure similar to that above, when faced with evaluating`

$$\iint\limits_{\mathscr{D}} f(x,y)\;dA,$$

`integrating first with respect to` x `and then with respect to` y`.`

Example 4. {Reversing the order of integration} Reverse the order of integration for the double integral

$$\iint\limits_{\mathscr{D}} f(x,y)\;dA = \int_0^2 \int_{x^2}^{2x} (4x+2)\;dy\;dx$$

using the above 3 Step procedure.

Solution.

$$\iint\limits_{\mathscr{D}} f(x,y)\;dA = \int_0^2 \int_{x^2}^{2x} (4x+2)\;dy\;dx = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\;dx\;dy$$

□

Properties of Double Integrals

Like single integrals, double integrals of continuous functions have algebraic properties that are useful in computations and applications.

If $f(x,y)$ and $g(x,y)$ are continuous function on a plane region $\mathscr{D}$, then

1. Constant Multiple: $\iint\limits_{\mathscr{D}} c\, f(x,y)\, dA = c \iint\limits_{\mathscr{D}} f(x,y)\, dA$ for any c.

2. Sum and Difference: $\iint\limits_{\mathscr{D}} (f(x,y) \pm g(x,y))\, dA = \iint\limits_{\mathscr{D}} f(x,y)\, dA \pm \iint\limits_{\mathscr{D}} g(x,y)\, dA.$

3. Domination:

a) $\iint\limits_{\mathscr{D}} f(x,y)\, dA \geqslant 0$ if $f(x,y) \geqslant 0$ on $\mathscr{D}$.

b) $\iint\limits_{\mathscr{D}} f(x,y)\, dA \geqslant \iint\limits_{\mathscr{D}} g(x,y)\, dA$ if $f(x,y) \geqslant g(x,y)$ on $\mathscr{D}$.

4. Additivity:
$$\iint\limits_{\mathscr{D}} f(x,y) = \iint\limits_{\mathscr{D}_1} f(x,y)\, dA + \iint\limits_{\mathscr{D}_2} g(x,y)\, dA$$
if $\mathscr{D}$ is the union of two nonoverlapping regions $\mathscr{D}_1$ and $\mathscr{D}_2$.

Areas of bounded regions in the Plane

If we take $f(x,y) = 1$ in the definition of the double integral over a region $\mathscr{D}$ in the preceding section, the Riemann sums reduce to

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k = \sum_{k=1}^{n} \Delta A_k \tag{1}$$

This is simply the sum of the areas of the small rectangles in the partition of $\mathscr{D}$, and approximates what we would like to call the **area of the region** $\mathscr{D}$. As the norm of a partition of $\mathscr{D}$ approaches zero, the height and width of all rectangles in the partition approach zero, and the coverage of $\mathscr{D}$ becomes increasingly complete. We define the area of the region $\mathscr{D}$, which we denote by $A(\mathscr{D})$ be the limit

$$\text{Area of the region } \mathscr{D} = A(\mathscr{D}) = \lim_{\|\mathscr{P}\| \to 0} \sum_{k=1}^{n} \Delta A_k = \iint\limits_{\mathscr{D}} dA.$$

Definition {area of planar region} The **area** of a closed, bounded region $\mathscr{D}$ of the plane is

$$A(\mathscr{D}) = \iint\limits_{\mathscr{D}} dA.$$

Average Value

The average value of an integrable function of one variable on a closed interval is the integral of the function over the interval divided by the length of the interval. For an integrable function of two variables defined on a bounded region $\mathscr{D}$ in the plane, the average value is **the integral of f over the region $\mathscr{D}$ divided by the area of the region $\mathscr{D}$**. This can be visualized by thinking of the function as giving the height at one instant of some water sloshing around in a tank whose vertical walls lie over the boundary of the region. The average height of the water in the tank can be found by letting the water settle down to a constant height. The height is then equal to the volume of water in the tank divided by the area of $\mathscr{D}$. We are led to define the average value of an integrable function f over a region $\mathscr{D}$ to be

Definition {Average value of $f(x,y)$} The **average value** of $f(x,y)$ on a closed, bounded region $\mathscr{D}$ of the plane is

$$\text{Avg}(f) = \frac{1}{A(\mathscr{D})} \iint\limits_{\mathscr{D}} f(x,y)\, dA.$$

Moments, Mass and Centers-of-Mass for Thin Flat Plates (or Laminas)

Using multiple integrals we can calculate the mass, Moment and center-of-mass (the balance point) of laminas with varying density. We first consider the problem of finding the center of mass of a thin flat plate or lamina: A disk of Aluminum, say, or a triangular sheet of metal. We assume the distribution of mass in such a plate to be continuous. A materials density function, denoted by $\delta(x, y)$ is its mass per unit area. The mass of a thin plate is obtained by integrating the density function $\delta(x, y)$ over the region $\mathscr{D}$ forming the thin plate. The first moment about an axis is calculated by integrating over $\mathscr{D}$ the distance from the axis times the density. The center of mass is found from the first moments. Below we give the double integral formulas for mass, first moments, and center of mass.

Mass and first moment formulas for thin plates (also called laminas) covering a region $\mathscr{D}$ in the xy-plane.

MASS: $M = \iint_{\mathscr{D}} \delta(x, y)\, dA$

FIRST MOMENTS: $M_x = \iint_{\mathscr{D}} y\, \delta(x, y)\, dA, \quad M_y = \iint_{\mathscr{D}} x\, \delta(x, y)\, dA.$

CENTER-OF-MASS: $\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$

Example 5. A thin plate covers a triangular region $\mathscr{D}$ of the xy-plane bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the first quadrant. The plate's density at the point $(x, y) \in \mathscr{D}$ is $\delta(x, y) = 6x + 6y + 6$. Find the plate's Mass, First Moments and Center-of-Mass about the coordinate axes.

Solution.

1. Make a sketch of $\mathscr{D} = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 2x\}$ (Not shown).

Calculating

$$M = \iint_{\mathscr{D}} \delta(x, y)\, dA$$

by integrating first with respect to y and then with respect to x.

2. Finding the y-limits of integration: Any vertical line L passing through $\mathscr{D}$ in the direction of increasing y must enter$\mathscr{D}$ at $y = 0$ and leaves $\mathscr{D}$ at $y = 2x$ and so

$$A(x) = \int_0^{2x} (6x + 6y + 6)dy = \left[6xy + 3y^2 + 6y\right]_{y=0}^{y=2x} = 24x^2 + 12x$$

3. Finding the x-limits of integration: We include all vertical lines cutting through $\mathscr{D}$ starting with $x = 0$ and ending with $x = 1$. Thus

$$M = \iint_{\mathscr{D}} \delta(x, y)\, dA \int_0^1 \left[\int_0^{2x} (6x + 6y + 6)\, dy\right] dx = \int_0^1 (24x^2 + 12x)\, dx$$

$$= \left[8x^3 + 6x^2\right]_{x=0}^{x=1} = 14 \text{ mass Units}$$

First Moment about the x-axis

$$M_x = \iint_{\mathscr{D}} y\, \delta(x, y)\, dA = \int_0^1 \left[\int_0^{2x} (6xy + 6y^2 + 6y)\, dy\right] dx = \int_0^1 \left[3xy^2 + 2y^3 + 3y^2\right]_{y=0}^{y=2x} dx$$

$$= \int_0^1 (28x^3 + 12x^2)\, dx = \left[7x^4 + 4x^3\right]_{x=0}^{x=1} = 11$$

First Moment about the y-axis

$$M_y = \iint_{\mathscr{D}} x\, \delta(x, y)\, dA = \int_0^1 \left[\int_0^{2x} (6x^2 + 6xy + 6x)\, dy\right] dx = \int_0^1 \left[6x^2y + 3xy^2 + 6xy\right]_{y=0}^{y=2x} dx$$

$$= \int_0^1 (24x^3 + 12x^2)\, dx = \left[6x^4 + 4x^3\right]_{x=0}^{x=1} = 10$$

Finally, the coordinates of the Center-Of-Mass are:

$$\bar{x} = \frac{M_y}{M} = \frac{10}{14} \text{ and } \bar{y} = \frac{M_x}{M} = \frac{11}{14}.$$

In other words, the triangular plate balances at the point $(5/7,\ 11/14)$ in $\mathscr{D}$. □

Double Integrals in Polar Form

Integrals are sometimes easier to evaluate if we change to polar coordinates. This discussion shows how to accomplish the change and how to evaluate integrals over regions whose boundaries are given by polar equations.

Integrals in Polar Coordinates

When we defined the double integral of a function over a region $\mathscr{D}$ in rectangular coordinates of the xy-plane, we began by cutting $\mathscr{D}$ into rectangles whose sides were parallel to the x and y coordinate axes. These were the natural shapes to use because their sides have either constant x-values or constant y-values. In polar coordinates, the natural shape is a "polar rectangle" whose sides have constant r- and θ-values. Suppose that a function $F(r, \theta)$ is defined over a region $\mathscr{D}$ that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = g_1(\theta)$ and $r = g_2(\theta)$. Suppose also that $0 \leqslant g_1(\theta) \leqslant g_2(\theta)$ for every value of θ between α and β. Then $\mathscr{D}$ lies in a fan-shaped region $\mathscr{D}_p$ defined by the inequalities $0 \leqslant r \leqslant a$ and $\alpha \leqslant \theta \leqslant \beta$.

We cover $\mathscr{D}_p$ by a grid of *circular arcs and rays.* The **arcs** are cut from circles centered at the origin, with radii,

$$\Delta r,\ 2\,\Delta r,\ \ldots,\ m\,\Delta r \quad \text{where } \Delta r = \frac{a}{m}.$$

The **rays** are given by:

$$\alpha,\ \alpha + \Delta\theta,\ \alpha + 2\Delta\theta,\ \ldots,\ \alpha + \tilde{m}\Delta\theta$$

where $\Delta\theta = \frac{(\beta - \alpha)}{\tilde{m}}$. The arcs and rays partition $\mathscr{D}_p$ into small patches called "polar rectangles."

We number the polar rectangles that lie inside $\mathscr{D}$ (the order does not matter), calling their areas:

$$\Delta A_1,\ \Delta A_2,\ \ldots,\ \Delta A_n.$$

We let $(r_k,\ \theta_k)$ be any sample point in the polar rectangle whose area is ΔA_k. We then form the sum

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\,\Delta A_k.$$

If $F(r, \theta)$ is continuous throughout $\mathscr{D}$, the above sum will approach a limit as we refine the grid to make Δr and $\Delta\theta$ go to zero. The limit is called **the double integral of** $F(r, \theta)$ **over** $\mathscr{D}$. In symbols,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(r_k, \theta_k)\,\Delta A_k = \iint_{\mathscr{D}} F(r, \theta)\,dA.$$

To evaluate the above limit, we first have to write the sum S_n in a way that expresses ΔA_k in terms of Δr and $\Delta\theta$. For convenience we choose r_k to be the average of the radii of the inner and outer arcs bounding the k-th polar rectangle ΔA_k. The radius of the inner arc bounding is then $r_k - (\Delta r)/2$. The radius of the outer arc is $r_k + (\Delta r)/2$.

The area of a wedge-shaped sector of a circle having radius r and angle θ is

$$\text{Area of Sector} = \frac{1}{2}\,r^2\,\theta \quad \text{(A formula you can derive using integration)},$$

as can be seen by multiplying $\pi\,r^2$, the area of the circle, by $\theta/2\pi$, the fraction of the circles area contained in the wedge. So the areas of the circular sectors subtended by these arcs at the origin are:

$$\text{Inner Radius:} = \frac{1}{2}\left(r_k - \frac{\Delta r}{2}\right)\Delta\theta$$

$$\text{Outer Radius:} = \frac{1}{2}\left(r_k + \frac{\Delta r}{2}\right)\Delta\theta$$

Therefore,

$$\begin{aligned}\Delta A_k &= \text{area of large sector} - \text{area of small sector}\\ &= \frac{\Delta\theta}{2}\left[\left(r_k + \frac{\Delta r}{2}\right)^2 - \left(r_k - \frac{\Delta r}{2}\right)^2\right] = \frac{\Delta\theta}{2}(2\,r_k\,\Delta r) = r_k\,\Delta r\,\Delta\theta\end{aligned}$$

Combining this result with the sum defining S_n gives

$$S_n = \sum_{k=1}^{n} F(r_k, \theta_k)\, \Delta A_k = \sum_{k=1}^{n} F(r_k, \theta_k)\, r_k\, \Delta r\, \Delta\theta$$

As $n \to \infty$, and the values Δr and $\Delta\theta$ approach zero, these sums converge to the double integral

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} F(r_k, \theta_k)\, r_k\, \Delta r\, \Delta\theta = \iint_{\mathscr{D}_p} F(r, \theta)\, r\, dr\, d\theta$$

A version of Fubini's Theorem says that the limit approached by these sums can be evaluated by repeated single integrations with respect to r and θ as

$$\iint_{\mathscr{D}_p} F(r, \theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} F(r, \theta)\, r\, dr\, d\theta.$$

A Procedure For Finding Limits of Integration for Integrals in Polar coordinates

The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates. When faced with evaluating $\iint_{\mathscr{D}_p} F(r, \theta)\, dA$ over a region $\mathscr{D}_p$ in polar coordinates, integrating first with respect to r and then with respect to θ, take the following steps.

Step 1. Sketch: *Sketch the region $\mathscr{D}$ of integration and its bounding curves.*

Step 2. Find the r-limits of integration: *Find the r-limits of integration imagining a ray, say M passing through the region $\mathscr{D}_p$ in the direction of increasing r. Mark the r values where M enters and leaves $\mathscr{D}_p$. These are your r-limits of integration. They usually depend on the angle θ that the ray M makes with the positive x-axis (that is, the r-limits are usually functions of θ (instead of constants, unless $\mathscr{D}_p$ is a circle))*

Step 3. Find the θ-limits of integration: *we find the θ-limits of integration by choosing θ-limits that include all rays passing through $\mathscr{D}_p$.*

If $F(r, \theta)$ is the constant function whose value is 1 on $\mathscr{D}_p$, then the integral of F over $\mathscr{D}_p$ is the area of $\mathscr{D}_p$ which we denote as $A(\mathscr{D}_p)$.

Definition {Area in Polar Coordinates} The area of a closed and bounded polar region

$$\mathscr{D}_p = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ g_1(\theta) \leqslant r \leqslant g_2(\theta)\}$$

in the polar coordinate plane is given by

$$A(\mathscr{D}_p) = \iint_{\mathscr{D}_p} r\, dr\, d\theta.$$

or

Changing Cartesian (i.e., Integrals in Rectangular Coordinates) Integrals into Polar Integrals

The procedure for changing a Cartesian integral $\iint_{\mathscr{D}} f(x, y)\, dA$ of the form $\iint_{\mathscr{D}} f(x, y)\, dx\, dy$, into a polar integral $\iint_{\mathscr{D}_p} F(r, \theta)\, r\, dr\, d\theta$ requires two steps.

Step 1. *First substitute $r\cos\theta$ for x and $r\sin\theta$ for y in the function $f(x, y)$, and replace $dA = dx\, dy$ by $r\, dr\, d\theta$ in the Cartesian integral.*

Step 2. *Then supply polar limits of integration for the boundary of $\mathscr{D}$.*

The Rectangular Coordinates Integral then becomes

$$\iint_{\mathscr{D}} f(x, y)\, dx\, dy = \iint_{\mathscr{D}_p} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta = \iint_{\mathscr{D}_p} F(r, \theta)\, r\, dr\, d\theta$$

where $F(r, \theta) = f(r\cos\theta, r\sin\theta)$ and $\mathscr{D}_p$ is the region $\mathscr{D}$ of integration in Polar Coordinates.

This is like the substitution method you encountered in your earlier studies of calculus, except that there are now two variables to substitute for instead of one. Notice that $dx\, dy$ is not replaced by $dr\, d\theta$ but by $r\, dr\, d\theta$ instead. A more general discussion of changes of variables (substitutions) in multiple integrals is given in Section 16.9 of your textbook.

Example 6. Find the polar moment of Inertia, I_0 of a thin plate of density $\delta(x, y) = 1$ about the origin of a region $\mathscr{D}$ bounded by the quarter circle $x^2 + y^2 = 1$ in the first quadrant.

Solution. The formula for I_0 is

$$I_0 = \iint_{\mathscr{D}} (x^2 + y^2)\, \delta(x, y)\, dA = I_x + I_y.$$

In rectangular coordinates the polar moment of Inertia is the value of the Double integral

$$\iint_{\mathscr{D}} (x^2 + y^2)\delta(x, y)\, dA = \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx$$

integrating first with respect to y and then with respect to x gives

$$\int_0^1 \left(x^2\sqrt{1 - x^2} + \frac{(1 - x^2)^{3/2}}{3} \right) dx,$$

an integral difficult to evaluate without tables. Things go better if we change the original integral to polar coordinates. Substituting $r\cos\theta$ for x and $r\sin\theta$ for y and $rdrd\theta$ for $dy\, dx$, we get

$$\begin{aligned}\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx &= \int_0^{\pi/2} \int_0^1 (r^2)\, r\, dr\, d\theta \\ &= \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{8}\end{aligned}$$

□

Why is the polar coordinate transformation so effective here? One reason is that $x^2 + y^2$ simplifies to r^2. Another is that the limits of integration become constants.

Surface Area as a Double Integral

Idea: Let S be a surface whose equation is given by $z = f(x,y)$ with domain $\mathscr{D}$. We assume that $f(x,y)$ is differentiable with continuous partial derivatives on a plane region $\mathscr{D}$ of the xy-plane. We further assume that $z = f(x,y) \geqslant 0$ on $\mathscr{D}$. Next, we partition $\mathscr{D}$ into sub-rectangles R_{ij} where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ and whose area is $A(R_{ij}) = \Delta A = \Delta x\,\Delta y$. If we choose a sample point $(x_i^*, y_j^*) \in R_{ij}$ at the corner of R_{ij} closest to the origin $(0,0)$, then let $P_{ij} = P_{ij}(x_i^*, y_j^*, f(x_i^*, y_j^*))$ be the point on the surface S directly above the sample point. The tangent plane to the surface S at the point P_{ij} is an approximation to S near P_{ij}. So the area ΔT_{ij} of the part of the tangent plane that lies directly above R_{ij} is an approximation to the surface area ΔS_{ij} of the part of S that lies directly above R_{ij}. Now the total area of S (the surface area) is approximated by the Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \Delta T_{ij}.$$

Since the approximation to the total area of S improves as the number of sub-rectangles R_{ij} increases, we therefore define the **surface area** of S to be

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta T_{ij}.$$

For computational purposes, we need a more convenient formula for surface area than the one give above. In particular, we need a nice formula for ΔT_{ij}.

The idea for a formula for ΔT_{ij} lies in our ability to parameterize the surface S whose equation is $z = f(x,y)$. Of course, we can parameterize $z = f(x,y)$ as follows:

$$x = x(x,y)\,, \quad y = y(x,y), \quad z = f(x,y) \ \text{ for } \ (x,y) \in \mathscr{D}.$$

Now we can write a vector function $\mathbf{r}(x,y) = x(x,y)\mathbf{i} + y(x,y)\mathbf{j} + f(x,y)\mathbf{k}$ with two parameters x and y as:

$$\mathbf{r}(x,y) = \langle x,\ y,\ f(x,y)\rangle \ \text{ for } (x,y) \in \mathscr{D} \ \ \{\text{A parametric representation of } z = f(x,y)\}$$

We now have the vectors $\mathbf{r}_x(x,y)$ and $\mathbf{r}_y(x,y)$ that form the part of the tangent plane to the surface S at the point $(x,y) \in \mathscr{D}$ and is given by

$$\mathbf{r}_x(x,y) = \langle 1,\ 0, f_x(x,y)\rangle \ \text{ and } \ \mathbf{r}_y(x,y) = \langle 0,\ 1,\ f_y(x,y)\rangle.$$

Now

$$\mathbf{r}_x(x,y) \times \mathbf{r}_y(x,y) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(x,y) \\ 0 & 1 & f_y(x,y) \end{vmatrix} = \langle -f_x(x,y), -f_y(x,y), 1\rangle$$

and the area of the tangent plane to the surface S at point $(x,y) \in D$ is given by

$$\|\mathbf{r}_x(x,y) \times \mathbf{r}_y(x,y)\| = \sqrt{(-f_x(x,y))^2 + (-f_y(x,y))^2 + 1^2} = \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}.$$

Certainly, we see that the area of the surface S with equation $z = f(x,y)$ with $(x,y) \in D$, where $f_x(x,y)$ and $f_y(x,y)$ exists and are continuous is given by

$$A(S) = \iint\limits_D \|\mathbf{r}_x(x,y) \times \mathbf{r}_y(x,y)\|\, dA = \iint\limits_D \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\, dA \ \ \{\text{see Page 1056 in your Textbook}\}.$$

Clearly, based on our discussion above we see that a formula for ΔT_{ij} can be given as:

$$\begin{aligned} \Delta T_{ij} &= \|\mathbf{r}_x(x_j^*, y_j^*)\Delta x \times \mathbf{r}_y(x_i^*, y^*)\Delta y\| = \|\mathbf{r}_x(x_i^*, y_j^*) \times \mathbf{r}_y(x_i^*, y_j^*)\|\ \Delta A \\ &= \sqrt{f_x(x_i^*, y_j^*)^2 + f_y(x_i^*, y_j^*)^2 + 1}\ \ \Delta A. \end{aligned}$$

This is what the author was saying on page 1056 near the bottom of the page {My hope is that this derivation is clear}.

$$\begin{aligned} A(S) &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta T_{ij} \\ &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \|\mathbf{r}_x(x_i^*, y_j^*) \times \mathbf{r}_y(x_i^*, y_j^*)\|\ \Delta A \\ &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \sqrt{f_x(x_i^*, y_j^*)^2 + f_y(x_i^*, y_j^*)^2 + 1}\ \Delta A \\ A(S) &= \iint\limits_{\mathscr{D}} \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\, dA. \end{aligned}$$

Let us use our Surface Area formula above or **see page 1056** to solve some Double Integration problems:

{STUDY THIS WORKED OUT EXERCISE CAREFULLY}

EXERCISE 0: Let us use double integrals to find a formula for the **surface area of a sphere with radius** R **centered at the origin** $(0,0,0)$. Recall that the rectangular equation of such a sphere is given as $x^2+y^2+z^2 = R^2$ where R is a positive constant.

SOLUTION. Solve for z to get $z = f(x,y) = \sqrt{R^2-x^2-y^2}$ {the upper hemisphere} and $z = g(x,y) = -f(x,y) = -\sqrt{R^2-x^2-y^2}$ {the lower hemisphere}. Now to solve the problem we use only the upper hemisphere. Thus, $f_x(x,y) = -\frac{x}{\sqrt{R^2-x^2-y^2}}$ and $f_y(x,y) = -\frac{y}{\sqrt{R^2-x^2-y^2}}$ so that

$$\sqrt{f_x(x,y)^2+f_y(x,y)^2\,+1} = \sqrt{\frac{x^2}{R^2-x^2-y^2}+\frac{y^2}{R^2-x^2-y^2}+\frac{R^2-x^2-y^2}{R^2-x^2-y^2}} = \frac{R}{\sqrt{R^2-x^2-y^2}}.$$

Now

$$A(S) = 2\iint_D \frac{R}{\sqrt{R^2-x^2-y^2}}\,dA$$

where $D = \{(x,y)|\, x^2+y^2 \leqslant R^2\}$ is the projection of the upper hemisphere on the xy-plane. There is a subtle issue with this integral. The integrand in $A(S)$ above {blows up! i.e., it goes to infinity} for points (x,y) on the boundary circle $x^2+y^2 = R^2$; because of this, the double integral $A(S)$ is an improper integral. Nevertheless, we can still compute $A(S)$ provided that we approach the boundary circle as a limit from the interior towards the boundary. In addition, notice that $f_x(x,y)$ and $f_y(x,y)$ are not continuous on $\partial\mathscr{D}$ (the boundary of $\mathscr{D}$) but are continuous on the interior of $\mathscr{D}$.

USING POLAR COORDINATES WE CAN EVALUATE THE INTEGRAL $A(S)$ AS FOLLOWS:

$$\begin{aligned}
A(S) &= \lim_{\alpha\to R^-}\int_0^{2\pi}\int_0^{\alpha}\frac{2R}{\sqrt{R^2-r^2}}\,r\,dr\,d\theta\\
&= \left(-R\int_0^{2\pi}d\theta\right)\left(\lim_{\alpha\to R^-}\int_0^{\alpha}\frac{-2r}{\sqrt{R^2-r^2}}\,dr\right)\\
&= (-2\pi R)\left(\lim_{\alpha\to R^-}\left[2\sqrt{R^2-r^2}\right]_{r=0}^{r=\alpha}\right)\\
&= (-2\pi R)(2\lim_{\alpha\to R^-}\left[\sqrt{R^2\,-\,\alpha^2}\,-\,R\right])\\
&= 4\pi R^2 \;\;\text{\{This formula should appear somewhere in your textbook\}}
\end{aligned}$$

YOUR EXAMINATION PAPER BEGINS HERE

Instructions For solving the problems:

0. It is important to Read these Notes First before solving any of the problems.
1. Draw a labeled picture of the region of integration for each Double Integral that you do.
2. Give a set description of the solid region of integration for each triple integral that you do. Also state whether the region of integration is x-simple, y-simple, r-simple or θ-simple.
3. Give a set description of the region of integration for each double integral that you do. Also state whether the region of integration is a Type I or II or III solid region.
4. Each Integral (Double or Triple) require a 3 step or 4 step procedure described above. Show how you are using these steps. Draw a sketch (of surfaces and regions of integration) where possible.
5. All problems are to be done in a neat and readable way!
6. This is an Exam. The Rules Apply (Don't forget that). Do all the problems!

PROBLEM 1 {To be handed in}: Consider a right circular cylinder of radius R and height H (which includes the top, bottom and the sides) whose surface area $A(S)$ is given as

$$2\pi R^2 + 2\pi RH.$$

where (S_1 {the top}, S_2 {the bottom}, and S_3 {the side}) denotes various pieces of your cylinder $S = S_1 \cup S_2 \cup S_3$

Use double integrals

$$A(S) = 2A(S_1) + A(S_3) = 2 \iint_{\mathscr{D}_1} dA + 4 \iint_{\mathscr{D}_2} \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\, dA$$

to derive the above formula. You must give a description and a sketch of the plane regions $\mathscr{D}_1$ and $\mathscr{D}_2$). $\mathscr{D}_2$ is the projection of the graph of $f(x,y)$ (which you must supply!) in the xy-plane.

Hint: Sketch the right circular cylinder S so the either the x or y-axis acts as a central axis of rotation (or symmetry). Consider only the portion of the surface S_3 in the first octant. This should give you some idea of how to define the surface S_3 as a function f of two variables. Furthermore, you should carefully read through Exercise 0 above before taking on this problem!.

PROBLEM 2 {To be handed in}: Use double integrals to derive the given formula for the surface area of a right circular cone of radius R and height H. The surface area of a cone is given by the formula

$$\pi R^2 + \pi R\sqrt{R^2 + H^2}.$$

Hint: Consider the right circular cone surface with equation $z = f(x,y) = \frac{H}{R}\sqrt{x^2 + y^2}$

PROBLEM 3 {To be handed in}: Use double integrals to derive the given formula for the surface area of a cap of a sphere of radius R and height H where $0 < H < R$. (The cap of a sphere is the portion of the sphere bounded below by the plane $z = R - H$ and bounded above by the plane $z = R$). The surface area of a cap of a sphere with radius R is given by the formula

$$2\pi RH.$$

Question: Consider what happens to the above formula as H approaches R (Explain). Is the result of this consideration what you expected?

PROBLEM 4 {To be handed in}: Use a double integral of the form

$$\iint_{\mathscr{D}} f(x,y)\, dA$$

derive the given formula for the volume of the sphere $x^2 + y^2 + z^2 = R^2$ described in Exercise 0. The volume of such a sphere is given by the formula

$$\frac{4}{3}\pi R^3$$

PROBLEM 5 {To be handed in}: Use double integrals to derive the given formula for the volume of a right circular cone of radius R and height H. The volume of a cone is given by the formula

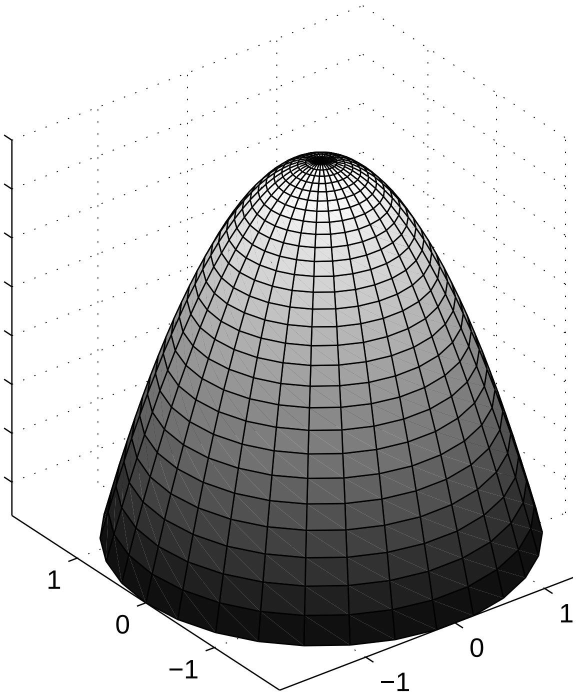
$$\frac{1}{3}\pi R^2 H.$$

PROBLEM 6 {To be handed in}: Use a double integral of the form $\iint_{\mathscr{D}} f(x, y)\ dA$ (Pay close attention to how you write $f(x, y)$) to derive the given formula for the volume a of a cap of a sphere of radius R and height H where $0 < H < R$. (The cap of a sphere is the portion of the sphere bounded below by the plane $z = R - H$ and bounded above by the plane $z = R$). The volume of a cap of a sphere with radius R is given by the formula

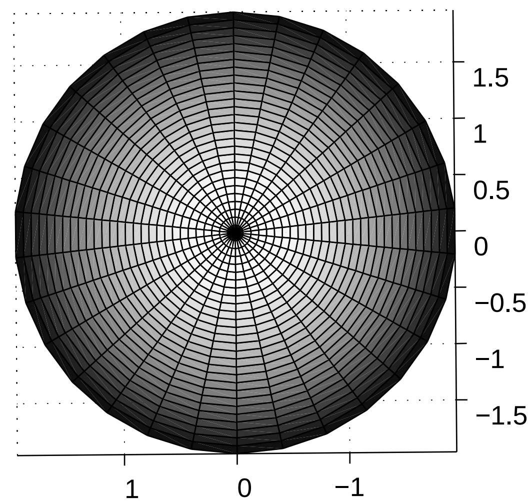
$$\frac{1}{3}\pi H^2(3R - H).$$

QUESTION: CONSIDER WHAT HAPPENS TO THE ABOVE FORMULA AS H APPROACHES R (EXPLAIN?). IS THE RESULT OF THIS CONSIDERATION WHAT YOU EXPECTED?

Worked out Exercise 4 (on page 1058) Find the area of the part of the paraboloid $z = f(x,y) = 4 - x^2 - y^2$ that lies above the plane $z = 0$ (i.e., the xy-plane) { See the picture below}



Courtesy of MatLab 7.1



Solution. We need to compute $G(x,y) = \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}$. So, $f_x(x,y) = -2x$ and $f_y(x,y) = -2y$ so that $\sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1} = \sqrt{4x^2 + 4y^2 + 1}$. Now $\mathscr{D}$ is the projection of the surface S (i.e., the graph of $z = f(x,y)$) on the xy-plane ($z = 0$), thus $\mathscr{D} = \{(x,y) |\, x^2 + y^2 \leqslant 4\}$ this a circular region centered at $(0,0)$ in $\mathbb{R}^2$ of radius r where $0 \leqslant r \leqslant 2$. We can now compute our surface area $A(S)$ as follows:

$$
\begin{aligned}
A(S) &= \iint_D G(x,y)\,dA = \iint_D \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\,dA \\
&= \iint_D \sqrt{4(x^2 + y^2) + 1}\,dA \\
&= \int_0^{2\pi}\int_0^2 \sqrt{4r^2 + 1}\; r\,dr\,d\theta \quad \text{where } D = \{(r,\theta) |\, 0 \leqslant r \leqslant 2;\ 0 \leqslant \theta \leqslant 2\pi\} \ \text{ in polar coordinates} \\
&= \frac{1}{8}\int_0^{2\pi}\int_0^2 \sqrt{4r^2 + 1}\; 8r\,dr\,d\theta \\
&= \frac{1}{8}\int_0^{2\pi} d\theta \int_1^{17} \sqrt{u}\,du = \frac{1}{8}\cdot 2\pi \cdot \left[\frac{2}{3}(\sqrt{u}\,)^3\right]_{u=1}^{u=17} \\
&= \frac{\pi}{6}\cdot\left[(\sqrt{17}\,)^3 - (\sqrt{1}\,)^3\right] \\
&= \frac{\pi}{6}\cdot(17\sqrt{17} - 1) \text{ sq units.}
\end{aligned}
$$

WORKED OUT EXERCISE 12 { PAGE 1058} FIND THE SURFACE AREA OF THE PART OF THE SPHERE WHOSE EQUATION IS $x^2 + y^2 + z^2 = 4z$ THAT LIES INSIDE THE PARABOLOID $z = x^2 + y^2$ (BE CAREFUL! THIS IS NOT A CONE!)

SOLUTION. Rewrite the equation of the sphere as

$$(x-0)^2 + (y-0)^2 + (z-2)^2 = 2^2.$$

We now have to find out how the paraboloid and the sphere intersect; i.e., we must solve the equation

$$0 = x^2 + y^2 + (z-2)^2 - 4 = z - x^2 - y^2$$

for z. So we obtain $(z-2)^2 - 4 = z$ or $z^2 - 3z = 0$ and so $z = 0$ and $z = 3$. This tells us that the two surfaces meet in the planes $z = 0$ (the bottom of the paraboloid) and $z = 3$. The upper hemisphere (the top half of the sphere) whose equation is $z = f(x,y) = 2 + \sqrt{4 - x^2 - y^2}$ is the part of the sphere that lies inside the paraboloid and they intersect in the plane $z = 3$. The trace of the upper hemisphere in the plane $z = 3$ is the boundary curve of the region $\mathscr{D}$ in the $z = 3$-plane. This boundary curve has equation $3 = 2 + \sqrt{4 - x^2 - y^2}$ or $x^2 + y^2 = 3$ {This is a circle of radius $\sqrt{3}$}. Now $\mathscr{D} = \{(x,y)|\, x^2 + y^2 \leqslant 3\}$ is the projection of the Surface on the plane $z = 3$. Next, $f_x(x,y) = -\frac{x}{\sqrt{4-x^2-y^2}}$ and $f_y(x,y) = -\frac{y}{\sqrt{4-x^2-y^2}}$ are continuous on $\mathscr{D}$ and so our double integral below will not be an improper integral as it was in the case of Exercise 1. Now

$$G(x,y) = \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1} = \frac{2}{\sqrt{4 - x^2 - y^2}}$$

$$\begin{aligned}
A(S) &= \iint_{\mathscr{D}} G(x,y)\, dA \\
&= \iint_{\mathscr{D}} \frac{2}{\sqrt{4 - x^2 - y^2}}\, dA \\
&= -\int_0^{2\pi} \int_0^{\sqrt{3}} \frac{-2r}{\sqrt{4 - r^2}}\, dr\, d\theta \\
&= -2\pi \cdot 2 \cdot (1 - 2) \\
&= 4\pi \text{ sq. units}
\end{aligned}$$

Triple Integrals in Rectangular Coordinates

Just as double integrals allow us to deal with more general situations than could be handled by single integrals, triple integrals enable us to solve still more general problems. We use triple integrals to calculate the volumes of three-dimensional shapes, the masses and moments of solids of varying density, and the average value of a function over a threedimensional region. Triple integrals also arise in the study of vector fields and fluid flow in three dimensions, as is discussed in Chapter 17.

Triple Integrals

IDEA: We extended the notion of a double integral over a region $\mathscr{D} \subseteq \mathbb{R}^2$ to 3 dimensions ($\mathbb{R}^3$). Let us consider the following simple set $\mathscr{B} \subseteq \mathbb{R}^3$ defined as:

$$\mathscr{B} = \{(x, y, z) |\ \alpha \leqslant x \leqslant \beta,\ \gamma \leqslant y \leqslant \delta,\ \sigma \leqslant z \leqslant \tau\}\ \ \{\text{called a } \textbf{solid rectangular box}\}.$$

We next form a partition of the box $\mathscr{B}$ by dividing it into little sub-boxes by partitioning the interval $[\alpha, \beta]$ into l sub-intervals of equal width Δx, and dividing the interval $[\gamma, \delta]$ into m sub-intervals of equal width Δy and similarly dividing the interval $[\sigma, \tau]$ into n sub-interval of equal width Δz. The planes through the endpoints of these sub-intervals parallel to the coordinate planes divides $\mathscr{B}$ into lmn sub-boxes

$$B_{ijk} = [x_{i-1},\, x_i] \times [y_{j-1},\, y_j] \times [z_{k-1},\, z_k] \ \text{ with } 1 \leqslant i \leqslant l,\, 1 \leqslant j \leqslant m,\, 1 \leqslant k \leqslant n.$$

Each sub-box B_{ijk} has a volume $\Delta V_{ijk} = \Delta x_i\, \Delta y_j\, \Delta z_k$. If $F(x, y, z)$ is a function of three variables defined on $\mathscr{B}$, then we form the **Riemann triple sum**

$$\sum_{k=1}^{n}\sum_{j=1}^{m}\sum_{i=1}^{l} F(x^*_{ijk},\, y^*ijk,\, z_{ijk})\, \Delta V_{ijk}$$

where $(x^*_{ijk},\, y^*_{ijk},\, z^*_{ijk})$ is a sample point in the sub-box B_{ijk} in the ijk-th position of the partition of $\mathscr{B}$. Now we define the **Triple integral over the box** $\mathscr{B}$ as the limiting value (if it exists) of the Riemann triple sums above; i.e., The **triple integral** of F over the box $\mathscr{B}$ is given by

$$\iiint\limits_{\mathscr{B}} F(x, y, z)\, dV = \lim_{l,m,n\to\infty} \sum_{k=1}^{n}\sum_{j=1}^{m}\sum_{i=1}^{l} F(x^*_{ijk},\, y^*ijk,\, z_{ijk})\, \Delta V_{ijk}.$$

If $F(x, y, z)$ is continuous on the box $\mathscr{B}$ then the sample point can be chosen to be any point (x_k, y_k, z_k) in the k-th sub-box B_k (assuming we have n sub-boxes) and so the integral above can be simply written as

$$\iiint\limits_{\mathscr{B}} F(x, y, z)\, dV = \lim_{n\to\infty} \sum_{k=1}^{n} F(x_k,\, y_k,\, z_k)\, \Delta V_k \qquad (1).$$

Fubini's Theorem Extended to $\mathbb{R}^3$: If $w = F(x, y, z)$ is continuous on the solid rectangular box

$$\mathscr{B} = \{(x, y, z) |\ \alpha \leqslant x \leqslant \beta,\ \gamma \leqslant y \leqslant \delta,\ \sigma \leqslant z \leqslant \tau\},$$

then

$$\begin{aligned}\iiint\limits_{\mathscr{B}} F(x, y, z)\, dV &= \int_{\sigma}^{\tau}\int_{\gamma}^{\delta}\int_{\alpha}^{\beta} F(x,\, y,\, z)\, dx\, dy\, dz \\ &= \int_{\alpha}^{\beta}\int_{\sigma}^{\tau}\int_{\gamma}^{\delta} F(x,\, y,\, z)\, dy\, dz\, dx\, \cdot \\ &= \int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\int_{\sigma}^{\tau} F(x,\, y,\, z)\, dz\, dy\, dx\end{aligned}$$

There is a total of six possible orders of the above iterated triple integral (I wrote out 3 of them for you)

PROBLEM 7 {To be handed in}: Write out the other 3 orders of integration for the triple integral given above.

INTEGRATION OVER MORE GENERAL BOUNDED REGIONS (SOLIDS) $\mathscr{E}$ OF $\mathbb{R}^3$

IDEA: We proceed in much the same way as we did above by enclosing the solid region $\mathscr{E}$ inside the solid box $\mathscr{B}$, thus $\mathscr{E} \subseteq \mathscr{B}$. We then define a new function $G(x, y, z)$ which agrees with $F(x, y, z)$ over the solid region $\mathscr{E}$ but is 0 for points over $\mathscr{B}$ that are not on $\mathscr{E}$; that is,

$$G(x,y,z) = \begin{cases} F(x,y,z) & \text{if } (x,y,z) \in \mathscr{E} \\ \\ 0 & \text{if } (x,y,z) \in \mathscr{B} \setminus \mathscr{E} \end{cases}$$

The new function $G(x, y, z)$ is called the extension of $F(x, y, z)$ on $\mathscr{E}$ to $\mathscr{B}$. Thus, we have

$$\iiint\limits_{\mathscr{E}} F(x,y,z)\,dV = \iiint\limits_{\mathscr{B}} G(x,y,z)\,dV.$$

Such an integral exists only if $F(x, y, z)$ is a continuous function and $\partial\mathscr{E}$ (the boundary of $\mathscr{E}$) is "reasonably smooth." By this, we mean that we can parametrically represent the boundary surface of $\mathscr{E}$. The triple integral above now have the usual properties similar to those of double integrals.

Naturally, of course, these solid regions $\mathscr{E}$ of 3-space can have a variety of shapes, so we restrict our attention to continuous functions $w = F(x, y, z)$ and to certain types of simple regions.

INTEGRATION OVER **Type I** SOLID REGIONS IN SPACE

We say a solid region $\mathscr{E}$ is of **Type I** if it lies between the graphs of two continuous functions of x and y, that is,

$$\mathscr{E} = \{(x,y,z) |\ (x,y) \in \mathscr{D}_{xy},\ u_1(x,y) \leqslant z \leqslant u_2(x,y)\}$$

where $\mathscr{D}_{xy}$ is the projection of $\mathscr{E}$ in the xy-plane $(z = 0)$. It follows immediately, that if $\mathscr{E}$ is a **Type I**, then we can express the triple integral $\iiint\limits_{\mathscr{E}} F(x,y,z)\ dV$ as

$$\iiint\limits_{\mathscr{E}} F(x,y,z)\,dV = \iint\limits_{\mathscr{D}_{xy}} \left[\int_{u_1(x,y)}^{u_2(x,y)} F(x,y,z)\ dz\right] dA.$$

In particular, if $\mathscr{D}_{xy}$ is the projection of $\mathscr{E}$ onto the xy-plane is an x-**Simple** plane region, then

$$\mathscr{E} = \{(x,y,z) |\ \alpha \leqslant x \leqslant \beta,\ g_1(x) \leqslant y \leqslant g_2(x),\ u_1(x,y) \leqslant z \leqslant u_2(x,y)\}$$

and the above integral becomes

$$\iiint\limits_{\mathscr{E}} F(x,y,z)\,dV = \int_\alpha^\beta \int_{g_1(x)}^{g_2(x)} \left[\int_{u_1(x,y)}^{u_2(x,y)} F(x,y,z)\ dz\right] dy\,dx \quad \{\text{if } \mathscr{D}_{xy} \text{ is an } x\textbf{-Simple} \text{ plane region}\}$$

$$= \int_\gamma^\delta \int_{h_1(y)}^{h_2(y)} \left[\int_{u_1(x,y)}^{u_2(x,y)} F(x,y,z)\ dz\right] dx\,dy \quad \{\text{if } \mathscr{D}_{xy} \text{ is a } y\textbf{-Simple} \text{ plane region}\}.$$

INTEGRATION OVER **Type II** SOLID REGIONS IN SPACE

We say that a solid region $\mathscr{E}$ is of **Type II** if it lies between two continuous functions of y and z, that is,

$$\mathscr{E} = \{(x,y,z) |\ (y,z) \in \mathscr{D}_{yz},\ u_1(y,z) \leqslant x \leqslant u_2(y,z)\}$$

where the plane region $\mathscr{D}_{yz}$ is the projection of $\mathscr{E}$ in the yz-plane $(x = 0)$ and so we can express the triple integral $\iiint\limits_{\mathscr{E}} F(x,y,z)\ dV$ as

$$\iiint\limits_{\mathscr{E}} F(x,y,z)\,dV = \iint\limits_{\mathscr{D}_{yz}} \left[\int_{u_1(y,z)}^{u_2(y,z)} F(x,y,z)\ dx\right] dA.$$

PROBLEM 8 {To be handed in}: Suppose $\mathscr{E}$ is a **Type II solid region in space**. In particular, if the plane region $\mathscr{D}_{yz}$, the projection of the solid region $\mathscr{E}$ on the yz-plane is a y**-Simple** plane region or an z**-Simple** plane region, then write out two formulas {similar in form to the red formulas above} for

$$\iiint_{\mathscr{E}} F(x,y,z)\,dV = \iint_{\mathscr{D}_{yz}} \left[\int_{u_1(y,z)}^{u_2(y,z)} F(x,y,z)\ dx\right] dA.$$

INTEGRATION OVER **Type III** SOLID REGIONS IN SPACE

PROBLEM 9 {To be handed in}: Define a **Type III** solid region $\mathscr{E}$. {Hint: See the blue and red sections above}.

PROBLEM 10 {To be handed in}: Suppose $\mathscr{E}$ is a **Type III solid region in space** and if your plane region, $\mathscr{D}_{xz}$, the projection of the solid region $\mathscr{E}$ on the xz-plane is an x**-Simple** plane region or a z**-Simple** plane region, then write out two formulas again for

$$\iiint_{\mathscr{E}} F(x,y,z)\,dV = \iint_{\mathscr{D}_{xz}} \left[\int_{u_1(x,z)}^{u_2(x,z)} F(x,y,z)\ dy\right] dA.$$

Volume of a Bounded Solid Region in Space

If $F(x, y, z)$ is the constant function whose value is 1 on $\mathscr{E}$, then the Riemann sums in Equation (1) above reduce to

$$S_n = \sum_{k=1}^{n} \Delta V_k.$$

As Δx_k, Δy_k and Δz_k approach 0, the cells ΔV_k becomes smaller and more numerous and fill up more and more of $\mathscr{E}$. We therefore define the volume of $\mathscr{E}$, denoted as $V(\mathscr{E})$ to be the triple integral

$$V(\mathscr{E}) = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta V_k = \iiint_{\mathscr{E}} dV.$$

If $\mathscr{E}$ is a solid Type I region and $w = F(x, y, z) = 1$ everywhere on $\mathscr{E}$, then we write the volume of the solid $\mathscr{E}$ as

$$V(\mathscr{E}) = \iiint_{\mathscr{E}} dV = \iint_{\mathscr{D}_{xy}} \left[\int_{u_1(x,y)}^{u_2(x,y)} dz\right] dA = \iint_{\mathscr{D}_{xy}} \left[u_2(x, y) - u_1(x, y)\right] dA$$

A Procedure For finding Limits of Integration For a TYPE 1 Triple Integral

We evaluate a triple integral by applying a three-dimensional version of Fubinis Theorem to evaluate it by three repeated single integrations. As with double integrals, there is a geometric procedure for finding the limits of integration for these single integrals.

When faced with evaluating a Type 1 triple integral

$$\iiint_{\mathscr{E}} F(x, y, z)\, dV$$

over a solid region $\mathscr{E}$ in space, integrating first with respect to z, then with respect to y and finally with respect to x, take the following steps:

Step 1. Sketch: *Sketch the region $\mathscr{E}$ of integration along with its shadow $\mathscr{D}_{xy}$ (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of $\mathscr{E}$ and the upper and lower bounding curves of $\mathscr{D}_{xy}$.*

Step 2. Find the z-limits of integration: *Find the z-limits of integration imagining a vertical line, say M passing through a typical point (x, y) in $\mathscr{D}_{xy}$ in the direction of increasing z. Mark the z values where the line M enters and leaves $\mathscr{E}$. These are your z-limits of integration. They usually depend on the point (x, y) where the line M enters (at $z = u_1(x, y)$)and leaves at $z = u_2(x, y)$) $\mathscr{E}$ (that is, the z-limits are usually functions of x and y (instead of constants, unless $\mathscr{E}$ is a Box))*

Step 3. Find the remaining limits of integration: *we find the remaining limits of integration by using STEPS 2 and 3 of the Procedure for finding the limits of integration for the Double Integral $\iint_{\mathscr{D}_{xy}} [u_2(x, y) - u_1(x, y)]\, dA$. This double integral is over an x-simple region $\mathscr{D}_{xy}$, thus*

$$\iiint_{\mathscr{E}} F(x, y, z)\, dV = \iint_{\mathscr{D}_{xy}} [u_2(x, y) - u_1(x, y)]\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} [u_2(x, y) - u_1(x, y)]\, dy\, dx$$

Follow similar procedures if you change the order of integration. The "shadow" of region $\mathscr{E}$ lies in the plane of the last two variables with respect to which the iterated integration takes place.

Exercise. Write out procedures similar to above when faced with Evaluating Triple Integrals over Type II and Type III regions in space (Note: You must first define Type III regions (see Problem 9))

Average Value of a Function in Space

The average value of a function $F(x, y, z)$ over a region solid region $\mathscr{E}$ in space is defined by the formula

$$\textbf{Average Value of } F(x, y, z) \textbf{ over } \mathscr{E} = \frac{1}{V(\mathscr{E})} \iiint_{\mathscr{E}} F(x, y, z)\, dV \qquad (2)$$

For example, if $F(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, then the average value of $F(x, y, z)$ over $\mathscr{E}$ is the average distance of points in $\mathscr{E}$ from the origin. If $F(x, y, z)$ is the temperature at point (x, y, z) on a solid that occupies a region $\mathscr{E}$ in space, then the average value of $F(x, y, z)$ over $\mathscr{E}$ is the average temperature of the solid.

Properties of Triple Integrals

Triple integrals have the same algebraic properties as double and single integrals.

If $F(x, y, z)$ and $G(x, y, z)$ are continuous function on a solid bounded region $\mathscr{E}$ in space, then

1. Constant Multiple: $\iint_{\mathscr{E}} c\, F(x, y, z)\, dV = c \iint_{\mathscr{E}} F(x, y, z)\, dV$ for any c.

2. Sum and Difference: $\iint_{\mathscr{E}} (F(x, y, z) \pm G(x, y, z))\, dV = \iint_{\mathscr{E}} F(x, y, z)\, dV \pm \iint_{\mathscr{E}} G(x, y, z)\, dV.$

3. Domination:
   a) $\iint_{\mathscr{E}} F(x, y, z)\, dV \geqslant 0$ if $F(x, y, z) \geqslant 0$ on $\mathscr{E}$.

   b) $\iint_{\mathscr{E}} F(x, y, z)\, dV \geqslant \iint_{\mathscr{E}} G(x, y, z)\, dV$ if $F(x, y, z) \geqslant G(x, y, z)$ on $\mathscr{E}$.

4. Additivity: $$\iint_{\mathscr{E}} F(x, y, z) = \iint_{\mathscr{E}_1} F(x, y, z)\, dV + \iint_{\mathscr{E}_2} G(x, y, z)\, dV$$
   if $\mathscr{E}$ is the union of two nonoverlapping regions $\mathscr{E}_1$ and $\mathscr{E}_2$.

Masses and Moments of solids in Three Dimensions

If $\delta(x, y, z)$ is the density of an object occupying a bounded solid region $\mathscr{E}$ in space (mass per unit volume), the integral of over $\mathscr{E}$ gives the **mass** of the object. To see why, imagine partitioning the object into n mass elements $\Delta m_k = \delta(x_k, y_k, z_k)\, \Delta V_k$. The objects mass is the limit

$$M = \lim_{n \to \infty} \sum_{k=1}^{n} \Delta m_k = \lim_{n \to \infty} \sum_{k=1}^{n} \delta(x_k, y_k, z_k)\, \Delta V_k = \iiint_{\mathscr{E}} \delta(x, y, z)\, dV$$

We now derive a formula for the moment of inertia. If $r(x, y, z)$ is the distance from the point (x, y, z) in $\mathscr{E}$ to a line L, then the moment of inertia of the mass $\Delta m_k = \delta(x_k, y_k, z_k)\, \Delta V_k$ about the line L is approximately $\Delta I_k = r^2(x_k, y_k, z_k)\, \Delta m_k$. The **moment of inertia about** L of the entire object is

$$I_L = \lim_{n \to \infty} \sum_{k=1}^{n} \Delta I_k = \lim_{n \to \infty} \sum_{k=1}^{n} r^2(x_k, y_k, z_k)\, \delta(x_k, y_k, z_k)\, \Delta V_k = \iiint_{\mathscr{E}} r^2\, \delta(x, y, z)\, dV$$

If L is the x-axis, then $r^2 = y^2 + z^2$ and

$$I_x = \iiint_{\mathscr{E}} (y^2 + z^2)\, \delta\, dV.$$

Similarly, if L is the y-axis or the z-axis, then we have

$$I_y = \iiint_{\mathscr{E}} (x^2 + z^2)\, \delta\, dV \quad \text{and} \quad I_z = \iiint_{\mathscr{E}} (x^2 + y^2)\, \delta\, dV.$$

Likewise, we can obtain *the first moments about the coordinate planes*. For example,

$$\iiint_{\mathscr{E}} x\, \delta(x, y, z)\, dV$$

gives **the first moment about the yz-plane**.

The mass and moment formulas in space analogous to those discussed for planar regions in our earlier discussions above are summarized below.

Mass and first moment formulas for solid objects in space covering a region $\mathscr{E}$ in $\mathbb{R}^3$.

MASS: $M = \iint_{\mathscr{E}} \delta(x, y, z)\, dV$

FIRST MOMENTS ABOUT THE COORDINATE PLANES:

$$M_{yz} = \iiint_{\mathscr{E}} x\, \delta(x, y, z)\, dV, \quad M_{xz} = \iiint_{\mathscr{E}} y\, \delta(x, y, z)\, dV \quad M_{xy} = \iiint_{\mathscr{E}} z\, \delta(x, y, z)\, dV.$$

CENTER-OF-MASS:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}.$$

MOMENTS OF INERTIA (SECOND MOMENTS) ABOUT THE COORDINATE AXES:

$$I_x = \iiint_{\mathscr{E}} (y^2 + z^2)\, \delta(x, y, z)\, dV$$

$$I_y = \iiint_{\mathscr{E}} (x^2 + z^2)\, \delta(x, y, z)\, dV$$

$$I_z = \iiint_{\mathscr{E}} (x^2 + y^2)\, \delta(x, y, z)\, dV$$

Moments of inertia about a line L:

$$I_L = \iiint_{\mathscr{E}} r^2\, \delta(x, y, z)\, dV \quad \text{where } r(x, y, z) \text{ is the distance from } (x, y, z) \text{ to } L.$$

Radius of gyration about a line L:

$$R_L = \sqrt{\frac{I_L}{M}}$$

When a calculation in physics, engineering, or geometry involves a cylinder, cone, or sphere, we can often simplify our work by using cylindrical or spherical coordinates, which are introduced in this discussion. The procedure for transforming to these coordinates and evaluating the resulting triple integrals is similar to the transformation to polar coordinates in the plane studied earlier above.

Triple Integration in Cylindrical Coordinates

We obtain cylindrical coordinates for space by combining polar coordinates in the xy-plane with the usual z-axis. This assigns to every point in space one or more coordinate triples of the form $(r,\ \theta,\ z)$.

DEFINITION {Cylindrical Coordinates} Cylindrical coordinates represent a point P in space by ordered triples $(r,\ \theta,\ z)$ in which

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane

2. z is the rectangular vertical coordinate.

The values of x, y, r, and θ in rectangular and cylindrical coordinates are related by the usual equations.

Equations Relating Rectangular (x, y, z) and Cylindrical $(r,\ \theta,\ z)$ Coordinates

$$x = r\ \cos\theta, \qquad y = r\ \sin\theta, \qquad z = z, \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}.$$

In cylindrical coordinates, the equation $r = a$ describes not just a circle in the xy-plane but an entire cylinder about the z-axis. The z-axis is given by $r = 0$. The equation $\theta = \theta_0$ describes the plane that contains the z-axis and makes an angle θ_0 with the positive x-axis. And, just as in rectangular coordinates, the equation $z = z_0$ describes a plane perpendicular to the z-axis.

Cylindrical coordinates are good for describing cylinders whose axes run along the z-axis and planes that either contain the z-axis or lie perpendicular to the z-axis. Surfaces like these have equations of constant coordinate value:

$r = 4.$	Cylinder, radius 4, axis the z-axis
$\theta = \frac{\pi}{3}.$	Plane containing the z-axis
$z = 2.$	Plane perpendicular to the z-axis

When computing triple integrals over a region $\mathscr{E}$ in cylindrical coordinates, we partition the region into n **small cylindrical wedges**, rather than into rectangular boxes as before. In the kth cylindrical wedge, r, θ and z change by Δr_k, $\Delta\theta_k$ and Δz_k, and the largest of these numbers among all the cylindrical wedges is called the **norm of the partition**. We define the triple integral as a limit of Riemann sums using these wedges. The volume of such a cylindrical wedge is ΔV_k obtained by taking the area ΔA_k of its base in the $r\theta$-plane and multiplying by the height Δz.

For a point $(r_k,\ \theta_k,\ z_k)$ in the center of the k-th wedge, we calculated in polar coordinates that $\Delta A_k = r_k\ \Delta r_k\ \Delta\theta_k$. So $\Delta V_k = \Delta z_k\ r_k\ \Delta r_k\ \Delta\theta_k$ and a Riemann sum for f over $\mathscr{E}$ has the form

$$S_n = \sum_{k=1}^{n} f(r_k,\ \theta_k,\ z_k)\ \Delta z_k\ r_k\ \Delta r_k\ \Delta\theta_k$$

The triple integral of a function f over $\mathscr{E}$ is obtained by taking a limit of such Riemann sums with partitions whose norms approach zero

$$\lim_{n\to\infty} S_n = \iiint_{\mathscr{E}} f(r,\theta,z)\ dV = \iiint_{\mathscr{E}} f(r,\theta,z)\ dz\ r\ dr\ d\theta.$$

Triple integrals in cylindrical coordinates are then evaluated as iterated integrals, as in the following example.

EXAMPLE 7. {Finding Limits of Integration in Cylindrical Coordinates} Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region $\mathscr{E}$ bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y-1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

SOLUTION. The base of $\mathscr{E}$ is also the region's projection $\mathscr{D}_{xy}$ on the xy-plane. The boundary of $\mathscr{D}_{xy}$ is the circle $x^2 + (y-1)^2 = 1$. Its polar coordinate equation is

$$\begin{aligned} x^2 + (y-1) &= 1 \\ x^2 + y^2 - 2y + 1 &= 1 \\ r^2 - 2r\sin\theta &= 0 \\ r &= 2\sin\theta. \end{aligned}$$

We find the limits of integration, starting with the z-limits. A line M through a typical point $(r, \theta$ in $\mathscr{D}_{xy}$ parallel to the z-axis enters $\mathscr{E}$ at $z = 0$ and leaves at $z = x^2 + y^2 = r^2$.

Next we find the r-limits of integration. A ray L through (r, θ) from the origin enters $\mathscr{D}_{xy}$ at $r = 0$ and leaves at $r = 2\sin\theta$.

Finally we find the θ-limits of integration. As the ray L sweeps across $\mathscr{D}_{xy}$, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = \pi$. The integral is

$$\iiint_{\mathscr{E}} f(r, \theta, z)\, dV = \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r^2} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$

□

Example above illustrates a good procedure for finding limits of integration in cylindrical coordinates. The procedure is summarized below as follows:

A PROCEDURE FOR FINDING THE LIMITS OF INTEGRATION FOR TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES

We now give a procedure for finding limits of integration that applies for many regions in space. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

When faced with evaluating a triple integral of the form

$$\iiint_{\mathscr{E}} f(r, \theta, z)\, dV$$

over a region $\mathscr{E}$ in space in cylindrical coordinates, integrating first with respect to z, then with respect to r, and finally with respect to θ take the following steps:

STEP 1 SKETCH: *Sketch the region $\mathscr{E}$ along with its projection $\mathscr{D}_{xy}$ on the xy-plane. Label the surfaces and curves that bound $\mathscr{E}$ and $\mathscr{D}_{xy}$.*

STEP 2. FIND THE z-LIMITS OF INTEGRATION: *Draw a line M through a typical point (r, θ) of $\mathscr{D}_{xy}$ parallel to the z-axis. As z increases, M enters $\mathscr{E}$ at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z-limits of integration.*

STEP 3. FIND THE r-LIMITS OF INTEGRATION: *Draw a ray L through (r, θ) from the origin. The ray enters $\mathscr{D}_{xy}$ at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.*

STEP 4. FIND THE θ-LIMITS OF INTEGRATION: *As the ray L sweeps across $\mathscr{D}_{xy}$, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration.*The integral is

$$\iiint_{\mathscr{E}} f(r, \theta, z)\, dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$

In this example I evaluate a Triple Integral in cylindrical coordinates using the procedures above. You too, should use these procedures when evaluating your integrals.

EXAMPLE 8. Consider the problem of finding the **centroid** (i.e., the center of mass when the solid has constant density, δ in this case we consider $\delta = 1$) of the solid $\mathscr{E}$ in space enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the plane $z = 0$.

SOLUTION.

STEP 1 SKETCH: Sketch the region $\mathscr{E}$ along with its projection R on the xy-plane. Label the surfaces and curves that bound $\mathscr{E}$ and $\mathscr{D}_{xy}$.

No Sketch provided here. See if you can make a sketch of the solid.

STEP 2. FIND THE z-LIMITS OF INTEGRATION: Draw a line M through a typical point (r, θ) of $\mathscr{D}_{xy}$ parallel to the z-axis. As z increases, M enters $\mathscr{E}$ at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z-limits of integration.

$\mathscr{D}_{xy}$ is the vertical projection of the solid $\mathscr{E}$ on the xy-plane, and it is a closed disk of radius 2 centered at the origin. Any line M passing through $\mathscr{D}_{xy}$ at a typical point (r, θ) enters the solid at $z = 0$ and leaves at $z = x^2 + y^2 = r^2$. So our z-limits of integration goes from $z = 0$ to $z = r^2$.

STEP 3. FIND THE r-LIMITS OF INTEGRATION: Draw a ray L through (r, θ) from the origin. The ray enters $\mathscr{D}_{xy}$ at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.

Any ray L through $\mathscr{D}_{xy}$ through (r, θ) from the origin, must enter $\mathscr{D}_{xy}$ at $r = 0$ and leaves at $r = 2$. So our r-limits of integration goes from $r = 0$ to $r = 2$.

STEP 4. FIND THE θ-LIMITS OF INTEGRATION: As the ray L sweeps across $\mathscr{D}_{xy}$, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration.

The ray L sweeps across $\mathscr{D}_{xy}$, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = 2\pi$. So our θ-limits of integration goes from $\theta = 0$ to $\theta = 2\pi$.

Now we compute M_{xy} (First moment about the xy-plane) which is given by

$$M_{xy} = \iiint\limits_{\mathscr{E}} z\,\delta(x,y,z)\,dV = \iiint\limits_{\mathscr{E}} z\,dV = \int_0^{2\pi}\int_0^2\int_0^{r^2} z\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \frac{r^5}{2}\,dr\,d\theta = \frac{32\pi}{3}.$$

Next, we must compute the mass M which is given by

$$M = \iiint\limits_{\mathscr{E}} \delta(x,y,z)\,dV = \iiint\limits_{\mathscr{E}} dV = \int_0^{2\pi}\int_0^2\int_0^{r^2} dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = 8\pi.$$

Finally, we have the centroid of the solid $\mathscr{E}$ is

$$\bar{x} = \bar{y} = 0 \text{ and } \bar{z} = \frac{M_{xy}}{M} = \frac{\frac{32\pi}{3}}{8\pi} = \frac{4}{3}.$$

Thus the centroid of the solid has coordinates $(0, 0, 4/3)$. It is important to understand why $\bar{x}$ and $\bar{y}$ are 0. This is because the z-axis the axis of symmetry of the solid and the centroid lies on this axis. This explains why the $\bar{x}$ and $\bar{y}$ are 0 without having to calculating it directly. Notice also that the centroid lies outside of the solid. □

Triple Integration in Spherical Coordinates

Spherical coordinates locate points P in space with two angles and one distance. The first coordinate, ρ is the distance from the origin O to the point P (i.e., $\rho = \| OP \|$ see section 13 of textbook). Unlike r (in cylindrical coordinates), the variable ρ **is never negative**. The second coordinate, ϕ, is the angle the position vector $\vec{OP}$ makes with the positive z-axis. It is required to satisfy the inequality $0 \leqslant \phi \leqslant \pi$. The third coordinate is the angle θ as measured in cylindrical coordinates.

DEFINITION {Spherical Coordinates} Spherical coordinates represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from the point P to the origin.
2. ϕ is the angle the position vector $\vec{OP}$ makes with the positive z-axis $(0 \leqslant \phi \leqslant \pi)$
3. θ is the angle from cylindrical coordinates.

On maps of the Earth, θ is related to the meridian of a point on the Earth and ϕ to its latitude, while ρ is related to elevation above the Earths surface.

The equation $\rho = a$ describes the sphere of radius a centered at the origin. The equation $\phi = \phi_0$ describes a single cone whose vertex lies at the origin and whose axis lies along the z-axis. (We broaden our interpretation to include the xy-plane as the cone $\phi = \frac{\pi}{2}$.) If ϕ_0 is greater than $\pi/2$, the cone $\phi = \phi_0$ opens downward. The equation $\theta = \theta_0$ describes the half-plane that contains the z-axis and makes an angle θ_0 with the positive x-axis.

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$\begin{aligned} r &= \rho \sin\phi, & x &= r\cos\theta = \rho\sin\phi\cos\theta, \\ z &= \rho\cos\phi, & y &= r\sin\theta = \rho\sin\phi\sin\theta \\ \rho &= \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}. \end{aligned} \tag{1}$$

Example 9. {converting rectangular to spherical} Find a spherical coordinate equation for the sphere

$$x^2 + y^2 + (z-1)^2 = 1.$$

Solution. We use Equations (1) to substitute for x, y, and z as follows:

$$\begin{aligned} x^2 + y^2 + (z-1)^2 &= 1 \\ (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 + (\rho\cos\phi - 1)^2 &= 1 \\ \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 &= 1 \\ \rho^2\sin^2\phi\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} + \rho^2\cos^2\phi &= 2\rho\cos\phi \\ \rho^2\underbrace{(\sin^2\phi + \cos^2\phi)}_{=1} &= 2\rho\cos\phi \\ \rho^2 &= 2\rho\cos\phi \\ \rho = 2\cos\phi \quad \{\text{finally!}\} \end{aligned}$$

□

Spherical coordinates are good for describing spheres centered at the origin, half-planes hinged along the z-axis, and cones whose vertices lie at the origin and whose axes lie along the z-axis. Surfaces like these have equations of constant coordinate value:

$\rho = 4.$ Sphere, radius 4, centered at the origin

$\phi = \dfrac{\pi}{3}.$ Cone opening up from the origin, making an angle of $\pi/3$ radians with the positive z-axis

$\theta = \dfrac{\pi}{3}.$ Half-plane, hinged along the z-axis, making an angle of $\pi/3$ radians with the positive x-axis

When computing triple integrals over a region $\mathscr{E}$ in spherical coordinates, we partition the region into n **spherical wedges**. The size of the k-th spherical wedge, which contains a point $(\rho_k, \phi_k, \theta_k)$ is given by changes by $\Delta\rho_k$, $\Delta\theta_k$, and $\Delta\phi_k$ in ρ, θ and ϕ. Such a spherical wedge has one edge a circular arc of length $\rho_k\Delta\phi_k$, another edge a circular arc of length $\rho_k \sin\phi_k \Delta\theta_k$, and thickness $\Delta\rho_k$. The spherical wedge closely approximates a cube of these dimensions when $\Delta\rho_k$, $\Delta\theta_k$ and $\Delta\phi_k$ are all small. It can be shown that the volume of this spherical wedge ΔV_k is $\Delta V_k = \rho_k^2 \sin\phi_k \Delta\phi_k \Delta\theta_k$ for a point $(\rho_k, \phi_k, \theta_k)$ chosen inside the wedge. The corresponding Riemann sum for a function $F(\rho, \phi, \theta)$ is

$$S_n = \sum_{k=1}^{n} F(\rho_k, \phi_k, \theta_k)\ \rho_k^2 \sin\phi_k\ \Delta\rho_k\ \Delta\phi_k \Delta\theta_k.$$

As the norm of the partition approaches zero, and the spherical wedges get smaller, the Riemann sums have a limit when F is continuous:

$$\lim_{n\to\infty} S_n = \iiint_{\mathscr{E}} F(\rho, \phi, \theta)\ dV = \iiint_{\mathscr{E}} F(\rho, \phi, \theta)\ \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta.$$

In spherical coordinates we have

$$dV = \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta.$$

To evaluate integrals in spherical coordinates, we usually integrate first with respect to ρ. The procedure for finding the limits of integration is shown below. We restrict our attention to integrating over domains that are solids of revolution about the z-axis (or portions thereof) and for which the limits for θ and ϕ are constant.

A Procedure for Finding the limits of integration for Triple Integrals in Spherical Coordinates

We now give a procedure for finding limits of integration that applies for many regions in space. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

When faced with evaluating

$$\iiint_{\mathscr{E}} F(\rho, \phi, \theta)\ dV$$

over a region $\mathscr{E}$ in space in spherical coordinates, integrating first with respect to ρ, then with respect to ϕ, and finally with respect to θ take the following steps:

STEP 1 Sketch: *Sketch the region $\mathscr{E}$ along with its projection $\mathscr{D}_{xy}$ on the xy-plane. Label the surfaces and curves that bound $\mathscr{E}$ and $\mathscr{D}_{xy}$.*

STEP 2. Find the ρ-limits of integration: *Draw a ray M from the origin through $\mathscr{E}$ making an angle ϕ with the positive z-axis. Also draw the projection M on the xy-plane (call such a projection L). The ray L makes an angle θ with the positive x-axis. As ρ increases, M enter $\mathscr{E}$ at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the ρ-limits of integration.*

STEP 3. Find the ϕ-limits of integration: *For any given θ, the angle ϕ that M makes with the positive z-axis runs from $\phi = \phi_{min}$ to $\phi = \phi_{max}$. These are the ϕ-limits of integration.*

STEP 4. Find the θ-limits of integration: *As the ray L sweeps across $\mathscr{D}_{xy}$, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration. The integral is*

$$\iiint_{\mathscr{E}} F(\rho, \phi, \theta)\ dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{min}}^{\phi=\phi_{max}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} F(\rho, \phi, \theta)\ \rho^2 \sin\phi\ d\rho\ dr\phi\ d\theta.$$

EXERCISE {DON'T HAND IN}: Use the procedure just above to calculate the volume of the "ice-cream cone" $\mathscr{E}$ cut from the solid sphere $\rho \leqslant 1$ by the cone $\phi = \pi/3$.

PROBLEM 11 {To be handed in}: Use triple integrals to find the volume of a **right-circular cone** of radius R and height H. {Hint: $z = f(x, y) = \frac{H}{R}\sqrt{x^2 + y^2}$ is a cone surface}.

PROBLEM 12 {To be handed in}: Use triple integrals to find the volume of a **right-circular cylinder** of radius R and height H. {Hint: $x^2 + y^2 = R^2$ is the surface of a cylinder}.

STUDY THIS WORKED OUT EXERCISE

EXERCISE 19 {ON PAGE 1067}: Use a triple integral to find the volume of the solid enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $y + z = 5$ and $z = 1$.

SOLUTION. Let $\mathscr{E}$ denote this Type I solid enclosed by the given cylinder and the given planes. Then $w = F(x, y, z) = 1$ everywhere on $\mathscr{E}$ so that

$$V(\mathscr{E}) = \iiint\limits_{\mathscr{E}} dV$$

where $z = u_1(x, y) = 1$ (the plane $z = 1$) and $z = u_2(x, y) = 5 - y$ and $\mathscr{D}_{xy} = \{(x, y) | \; x^2 + y^2 \leqslant 9\}$. Now

$$\begin{aligned}
V(\mathscr{E}) &= \iint_{\mathscr{D}_{xy}} [u_2(x, y) - u_1(x, y)] \, dA \\
&= \iint\limits_{\mathscr{D}_{xy}} (4 - y) \, dA \\
&= \int_{-3}^{3} \int_{-\sqrt{9 - x^2}}^{\sqrt{9 - x^2}} (4 - y) \, dy \, dx \quad \text{\{I believe that is integral can be done using polar coordinates\}} \\
&= \int_{-3}^{3} \left[4y - \frac{y^2}{2} \right]_{y = -\sqrt{9 - x^2}}^{y = \sqrt{9 - x^2}} dx \\
&= \int_{-3}^{3} \left[4\sqrt{9 - x^2} - \frac{9 - x^2}{2} + 4\sqrt{9 - x^2} + \frac{9 - x^2}{2} \right] dx \\
&= 8 \int_{-3}^{3} \sqrt{9 - x^2} \, dx \quad \text{\{USE THE METHOD OF TRIGONOMETRIC SUBSTITUTION\}} \\
&= 8 \left[\frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2} \arcsin\left(\frac{x}{3}\right) \right]_{x = -3}^{x = 3} \\
&= 36 \, (\arcsin(1) - \arcsin(-1)) = 36 \cdot \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) \\
&= 36\,\pi \quad \text{cubic units.}
\end{aligned}$$

TRY THIS EXERCISE IF YOU DARE: Consider a sphere $x^2 + y^2 + z^2 = R^2$ of radius $R > 0$ centered at the origin. Let $\mathscr{E}$ be the solid region in space bounded by the xy-plane, the xz-plane, the plane $y = x$ and the above sphere. Now let $F(x, y, z) = 5z$ be a continuous function over $\mathscr{E}$. Can you write (Do not evaluate)

$$\iiint\limits_{\mathscr{E}} F(x, y, z) \, dV. \quad \text{\{This is not a volume integral!\}}$$

using our 3 main coordinate systems (Rectangular, cylindrical and spherical coordinates)?

PROBLEM 13 {To be handed in}: Use polar coordinates to evaluate the Red integral in the above Exercise 19.

PROBLEM 14 {To be handed in}: Setup and evaluate a Double Integral to calculate the area for the region in xy-plane bounded by $y = H$, $y = 0$, $x = 0$ and the line containing the points $(a, 0)$ and (b, H) where $a, b, H > 0$ and $b < a$.

PROBLEM 15 {To be handed in}: Setup and evaluate a Double Integral to calculate the area for the sector in the polar plane bounded by the rays $\theta = 0$ and $\theta = \theta_0$ and the circle $x^2 + y^2 = R^2$ where θ_0 is a number satisfying the inequality $0 < \theta_0 < 2\pi$.

PROBLEM 16 {To be handed in}: Setup only (but do not evaluate) a triple integral using Rectangular, cylindrical and spherical coordinates for the volumes of the four 3-dimensional solids mentioned in PROBLEMS 1 through 6

THIS COMPLETES YOUR INTEGRATION PROJECT, ENJOY!!!!