1. \((\cos y + 1)y' + \sin y - x^2\) may be written in the form \(M + N y' = (-x^2 + \sin y) + (\cos y + 1)y' = 0\). This is an exact equation, since \(M_y = N_x = \cos y\). An implicit solution \(F(x, y) = 0\) is found by letting \(F = -x^2 + \cos y + x^2 + \cos y = g(x, y)\). Set \(P = N = \cos y + 1\), \(Q = M = -x^2 + \cos y + x^2\). Therefore, \(q' = 1\), \(g = y\), and an implicit solution to the differential equation \(\Lambda = x^2 + \cos y + x^2 = y\). We have \(y = 2x\) and \(y = x\).

2. (a) Separate variables to solve \(2xy' = x^2 - 3\), \(x > 0\). \(y(1) = 1\) \(\Rightarrow y' = 2x/x - 3/y = 2x + \sqrt{3x}/x\), \(y'/4 = 3\sqrt{3}/x\), \(y' = 1\). \(\Rightarrow y = x + c\), so \(c = -3\) and \(y = -3x\). (b) The equation \(2xy' = 2x^2 + x^2 y^2 = 0\) may be separated into two ordinary differential equations by substituting a solution of the form \(u(x, y) = x(\sqrt{y})\) into the original equation:

\[
\pi X^2 + 2X Y' + x^2 Y = 0 \Rightarrow \pi x^2 (X' + xY' = -2XY' \Rightarrow x^2 X + e Y = x^2 X / X - 2x^2 / x^2 \]
For last series to be identically equal to 0, each coefficient must be 0:

\[(n+2)(n+1)a_{n+2} - (n+2)a_{n} = 0, \quad \text{or} \quad a_{n+2} = \frac{(n+2)a_{n}}{(n+1)}\]

for all nonnegative integers \(n\). Thus, \(a_{0}\) and \(a_{1}\) may be chosen arbitrarily, and \(a_{n}\), for \(n\) even, may be found by substituting \(n = 0, 2, \ldots\) into the recursion relation:

\[a_{n} = -2a_{n}/2 = -a_{n}/2, \quad a_{0} = a_{0}/2, \quad a_{2} = a_{2}/2, \quad \ldots\]

and continuing in this fashion we see \(a_{n} = 0\) for all even values of \(n\) greater than 2. So that answer to part (b) is \(\sum a_{n} = 0\). Similarly, by interchanging \(n = 1, 3, \ldots\) we obtain \(a_{n} = -a_{n}/6\) and \(a_{0} = a_{0}/6 = a_{0}/60\). Substituting these values into the original series \(y = \sum a_{n}\) and factoring \(a_{0}\) and \(n\) out, where possible, we obtain the that the answer to (a) may be written as \(y = a_{0}(1 - z^2) + a_{2}(z^2 - 2z^2/3 - 2z^2/5 - \ldots)\).

7. To solve the equation \(y' = -y\), with \(y(0) = 0\) and \(y'(0) = 1\), we denote the Laplace transforms of the solution \(y\) by \(Y\), equate the Laplace transforms of the two sides of the differential equation, solve for \(Y\), and find a partial fraction expansion: \(sY - y(0) - y'(0) + Y = 1/s\) and

\[Y = \frac{1}{s^2 + 1}\]

Equation coefficients: \(A = B = 0\), \(C = 1\), and \(A = 1\), so that \(A = 4\), \(B = -4\) and \(C = 2\). Thus, \(Y = 4/(s^2 + 1)\) and \(y = 4\cos t + 2\sin t\).

8. If 1 gal/min of water with 20 mg/gal of impurities flows into a 10 gallon tank filled with pure water, then the rate of impurities flow in is \(Q_{in} = 20(10 - t)\) gal/min. If the mixed solution exits the tank at 2 gal/min, impurities flow out at a rate of \(Q_{out} = 2Q_{out} = 20 - 10 = 10\). Thus, differential equation, which may be written as \(Q_{in} + 2 = 0\), can be solved using the integrating factor \(\mu = e^{\int (10 - t) dt} = 1/(10 - t)^2\).

Then

\[Q(t) = \int \frac{20}{(10 - t)^2} dt = -\frac{20}{10 - t} + C\]

Now, \(Q(0) = 0\) and \(s = 2\), so \(Q(t) = \frac{20}{10 - t}\).

9. Since a 4 lb weight stretches the spring by 1/2 a foot, we obtain from \(F = ks\) that \(4 = k(1/2) = k/2\). The force of the weight is found by substituting into \(F = mg\): \(4 = 32 - m\) and \(m = 28\). For air resistance of \(8\) lb at a velocity of \(8\) feet/sec, we obtain from \(F = mg + \frac{1}{2}CV^2\) that \(m = 7\) or \(G = 1\). Thus, the equation \(m \ddot{y} + 2y = 0\) becomes \((1/2)\ddot{y} + \dot{y} + y = 0\), whose characteristic equation has solution \(r^2 + 2 = 0\) or \(r = \pm \sqrt{2}\). Then, \(u(t) = e^{-t}\), and by differentiation, \(u(t) = -e^{-t}\).

10. If \(x\) is a regular singular point of the equation \(x^2 \ddot{y} + \alpha x + \beta = 0\), we substitute a solution of the form \(y = \sum_{n=0}^{\infty} a_n x^{n-r}\) into the differential equation, then we replace \(r\) by \(n - 1\) in the second sum and combine the terms:

\[\sum_{n=0}^{\infty} \sum_{r=0}^{n+r} \frac{r(r+1)\ldots(r+n-1)}{r!} a_{n-r} x^{n-r+1} = x y'' + \alpha x y' + \beta y = 0\]

\[\sum_{n=0}^{\infty} \sum_{r=0}^{n+r} \frac{r(r+1)\ldots(r+n-1)}{r!} a_{n-r} x^{n-r+1} = 0\]

\[x y'' + (\alpha + 1)x y' + \beta y = 0\]
The indicial equation is obtained by setting the constant coefficient equal to zero: \( r^2 - 2 = r^2 - 2 = 0 \); thus, \( r = 1, 2 \). The recursion relation for \( r = 2 \) is obtained by substituting \( r = 2 \) into the last displayed equation above and collecting the coefficients of \( x^{n+1}, n \geq 2 \), equal to zero: \( [n+2][n+1] - 2] a_n - \alpha_1 a_{n-1} = 0 \), so that \( a_n = \alpha_1 a_{n-1} \), \( n \geq 1 \). Choose \( \alpha_1 \) arbitrarily, and substitute successive values of \( n \) into the recursion relation to obtain \( a_1 = a_0, a_2 = a_0/4, a_3 = a_0/(4 \cdot 10) \), \( a_4 = a_0/(4 \cdot 10 \cdot 10 - 25) \), and \( a_5 = a_0/(4 \cdot 10 \cdot 10 - 25) \).

The solution to the differential equation is

\[
a_k x^k \left[ \frac{1}{4} + \frac{1}{40} \frac{1}{x^2} + \frac{1}{720} \frac{1}{x^4} + \frac{1}{7200} \frac{1}{x^6} + \ldots \right].
\]

11. (a) The independent solutions to the Euler equation \( t^2 y'' - 4ty' + 6y = 0 \), are of the form \( t^r \), where \( r \) is a solution to \( r(r - 1) - 6r + 6 = 0 \). Thus, \( r = 3, 1 \), and the general solution is \( c_1 t^3 + c_2 t^1 \).

(b) Rewriting the equation \( t^2 y'' - 4ty' + y = t^1 \) as an equivalent equation with \( y'' \) coefficient one:

\[
y'' - \frac{4}{t} y' + \frac{1}{t^2} y = \frac{t}{t^2}.
\]

Using the method of variation of parameters, we obtain a particular solution to the last equation in the form \( Y = y_1 \int W y_2 dy + y_2 \int W y_1 dy \), where \( y \) is the inhomogeneous term \( t \), \( W \) is the Wronskian, and \( y_1 = t^3 \) and \( y_2 = t^1 \) are independent solutions to the homogeneous equation in part (a).

(c) Calculating:

\[
W = \frac{t^2}{t^2 - t^2} = t^2 - t^2,
\]

\[
\int W \frac{y_2}{W} = \int t^2 - t^2 = t^2 - 1,
\]

\[
\int W \frac{y_1}{W} = \int t^2 - t^2 = t^2 - t^2,
\]

\[
Y = t^2(t^3 - t^3) + t^3(t^1) = t^2 t^3 - t^2 t^1,
\]

where the general solution to the inhomogeneous equation was obtained by adding the particular inhomogeneous solution \( Y \) to the general homogeneous solution found in (a).

12. (a) The equation \( y'' + \frac{a_2 - \frac{y}{a_2}}{a_2} \) is inhomogeneous in the sense that numerator and denominator of the right side are both homogeneous polynomials of the same degree. We may substitute \( u = y/x^n \) to obtain an equation which may be solved by separating variables. Calculate \( y'' = (ux^2)/x + y'' \) and write the original equations as \( y'' = (ux^2)/x + y'' \).

Thus \( y + 2y = u = y'' \). Separate variables to obtain \( \int 1/y^2 dy = \int 1/x^2 dx \), \( -1/u = \ln u, u = e^{-1}, u = 1 \), and \( y'' = y(x) \).

(b) Divide by the \( y'' \) coefficient in the equation \( (x^2 - a_2)x^2 + (x + 1)y' - xy = 0 \) to obtain:

\[
y'' + \frac{x + 1}{x^2 - a_2} y' - \frac{x}{x^2 - a_2} y = 0,
\]

which has singular points \( x = 0 \). A solution including \( x_0 = 0 \) is guaranteed on any interval not containing either of the singular points. The largest such interval is \( [0, \infty) \).

(c) The singular point \( x = 0 \) is irregular since the \( y'' \) coefficient has more than one power of \( x \) in the denominator, but the singular point \( x = 3 \) is regular, since the denominator of the \( y'' \) coefficient has at most one power of \( x - a_2 \) and the \( y \) coefficient has at most two (actually only one) power of \( x - a_2 \) in the denominator.