PART I: Answer ALL questions in this part (10 points each)

Show all work and simplify all answers. NO CALCULATORS!!!

1. Find all solutions: \((x \cos y + 1)y' + \sin y = e^x\)

2. (a) Solve explicitly: \(xy^2y' = 2x - 3, x > 0,\) and \(y(1) = -1\)
   (b) Replace the partial differential equation, in the function \(u(x,t),\)
   \(xu_{xx} + 2u_{tt} + x^2u_t = 0\)
   with two ordinary differential equations. Do NOT solve the equations.

3. (a) Find the general solution: \(ty' + (t-1)y = t^3, t > 0\)
   (b) Find, to within a multiplicative constant, the Wronskian of two solutions to the equation \(2ty'' + 4y' + (t+1)y = 0\)

4. Find the general solution: \(y'' + 6y' + 7y = 1 + 2e^t\)

5. (a) For the function
   \[f(x) = \begin{cases} -1, & -1 < x < 0; \\ 1, & 0 \leq x < 1. \end{cases}\]
   find the constant coefficient and the coefficient of \(\sin x\) of the Fourier series representation of the function of period 2 which coincides at all points of continuity with \(f(x)\). You DO NOT have to compute the general coefficient.
   (b) Find the value to which the Fourier series converges at \(x = 0\).

6. (a) Find the terms through \(x^5\) of the power series expansion centered at 0 for the general solution to \(y'' - xy' + 2y = 0.\)
   (b) Find the coefficient of \(x^5\) in this series.

7. Use Laplace transforms to solve the initial value problem \(y'' + y = 4, y(0) = 0\) and \(y'(0) = 2.\) No credit will be given for any other method.

CONTINUED ON BACK
Part II: Answer any THREE COMPLETE questions (10 points each).

Write the numbers of the TWO OMITTED questions on the cover of your answer booklet.

8. A 10 gallon tank is filled to the top with pure water. Water which has 20 mg of impurity per gallon pours into the tank at the rate of 1 gallon per minute and water (which is well mixed in the tank) flows out of the tank at the rate of 2 gallons per minute. Find the function Q(t), the number of milligrams of impurities in the tank at time t.

9. A 4 lb weight in its equilibrium position stretches the spring from which it is suspended by 1/2 foot. Beginning in its equilibrium position, the weight is given an initial velocity of 4 ft/sec downwards. Air resistance which is proportional to velocity in 8 lbs when the weight has a velocity of 8 feet/sec. Find the function s(t) which describes the displacement in feet of the weight t seconds after it is released. [Assume the acceleration due to gravity is 32 feet/sec^2.]

10. (a) For the equation \( x^2 y'' + (x - 2)y = 0 \), find and solve the indicial equation.
   (b) For the larger solution \( r \) to the indicial equation, find the recursion relation for a solution to the differential equation of the form \( y = x^r \sum_{n=0}^{\infty} a_n x^n \) with \( a_0 \neq 0 \).
   (c) Find the terms of the series described in (b) through the \( x^4 \) term.

11. (a) Find the general solution: \( x^2 y'' + 4ty' + 6y = 0 \)
   (b) Use the answer to (a) above to find the general solution to \( t^2 y'' + 4ty' + 6y = t^4 e^t \)

12. (a) Use the fact that the equation is homogeneous to find, in explicit form, the general solution of \( y' = \frac{xy - y^2}{x^2}, x > 0 \).
   (b) What is the largest interval containing the point \( x_0 = 5 \) on which the equation \( (x^2 - 4x^2)y'' + (x + 1)y' - xy = 0 \) is guaranteed to have a continuous solution? Find all singular points for this equation and state which of those, if any, are regular.