1. $(x\cos y + 1)y' + \sin y = e^x$ may be written in the form $M + Ny' = (-e^x + \sin y) + (x\cos y + 1)y' = 0$. This is an exact equation, since $M_y = N_x = \cos y$. An implicit solution $F(x,y) = 0$ is found by letting $F = \int -e^x + \sin y\,dx = -e^x + x\sin y + g(y)$. Set $F_y = N$: $x\cos y + g'(y) = x\cos y + 1$. Therefore, $g' = 1$, $g = y + c$, and an implicit solution to the differential equation is $-e^x + x\sin y + y + c = 0$.

2. (a) Separate variables to solve $xy^3y' - 2x = 3$, $x > 0$, $y(1) = -1$: $\int y^3\,dy = \int 2 + 3/x\,dx$; $y^4/4 = 2x + 3\ln x + c_1$, and $y = \pm\sqrt[4]{8x + 12\ln x + c}$. Substitute $x = 1$: $-1 = -\sqrt[4]{8 + c}$, so $c = -7$ and $y = -\sqrt[4]{8x + 12\ln x - 7}$.

(b) The equation $xtu_{xx} + 2u_{tt} + x^2tu = 0$ may be separated into two ordinary differential equations by subtsituting a solution of the form $u(x,t) = X(x)T(t)$ into the original equation:

$$xtX''T + 2XT'' + x^2tXT = 0 \implies tT(xX'' + x^2X) = -2XT'' \implies (xX'' + x^2X)/X = -2T''/tT.$$

The two sides of the last equation give two descriptions of a function depending on x and t. From the left side, we see there is no dependence on t, and, from right side, we see there is no dependence on x. Thus, both sides must represent a constant, call it λ. That is, $(xX'' + x^2X)/X = \lambda$ and $-2T''/tT = \lambda$. The last pair of equations can also be written in the form $xX'' + (x^2 - \lambda)X = 0$ and $2T'' + \lambda tT = 0$.

3. (a) Divide by t and use an integrating factor to solve $ty' + (t-1)y = t^3$, $t > 0$: $y' + py = g$, with $\mu = e^{\int p}$; in this example is $y' + (1 - 1/t)y = t^2$, $\mu = e^{\int 1 - 1/t} = e^t/t$. Thus, $y = (1/\mu)\int \mu g = te^{-t}\int (e^t/t)t^2 = te^{-t}(te^t - e^t + c) = t^2 - t + cte^{-t}$.

(b) The Wronskian of two solutions to the equation $2ty'' + 4y' + (t\sin t)y = 0$ is obtained by dividing by $2t$ and applying Abel's theorem: $W = e^{-\int p} = e^{-\int 4/2t} = c/t^2$.

4. To find the general solution to $L = y'' + 6y' - 7y = 1 + 2e^t$, add the sum of particular solutions to $L = 1$ and $L = 2e^t$ to the general solution to $L = 0$. Since the characteristic equation $r^2 + 6r - 7 = 0$ has solutions $r = 1, -7$, the general homogeneous solution is $y = c_1e^t + c_2e^{-7t}$. Substitute $y = A$ (A constant) into $L = 1$, to obtain $y = -1/7$ is a particular solution. Substitute $y = Bte^t$ to obtain a solution to $L = 2e^t$ (note that Be^t had to be multiplied by t because Be^t is a solution to the homogeneous equation):

$$\begin{aligned} y'' &= \quad Bte^t + 2Be^t \\ 6y' &= \quad 6Bte^t + 6Be^t \\ -7y &= -7Bte^t, \end{aligned}$$

so $L = 8Be^t = 2e^t$ and $B = 1/4$. The general solution is then $y = c_1e^t + c_2e^{-7t} + (1/4)te^t - 1/7$.

5. (a) The function

$$f(x) = \begin{cases} -1, & -1 < x < 0; \\ \ \ 1, & 0 \le x < 1. \end{cases}$$

would be odd if we redefined $f(0) = 0$. Therefore the constant coefficient (and also each cos-term coefficient) is zero, and the sin-term coefficient is $b_5 = (2/L)\int_0^L f(x)\sin(5\pi x/L)\,dx = 2\int_0^1 \sin 5\pi x\,dx = 4/(5\pi)$.

(b) At 0 the series converges to $(f(0^-) + f(0^+))/2 = (-1 + 1)/2 = 0$.

6. The point $x = 0$ is a regular point of $y'' - xy' + 2y = 0$, so we substitute a series solution of form $y = \sum_0^\infty a_nx^n$ into the equation and solve for the a_n. In the first sum below, observe the terms for $n = 0$ and $n = 1$ are zero and may be omitted; then replace n by $n+2$ to obtain the second line below from the first.

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} - \sum_{n=0}^{\infty} na_nx^n + \sum_{n=0}^{\infty} 2a_nx^n = 0;$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} na_nx^n + \sum_{n=0}^{\infty} 2a_nx^n = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - (n-2)a_n]x^n = 0.$$

For last series to be identically equal to 0, each coefficient must be 0:

$$(n+2)(n+1)a_{n+2}-(n-2)a_n=0; \quad \text{or} \quad a_{n+2}=\frac{(n-2)a_n}{(n+2)(n+1)},$$

for all nonnegative integers n. Thus, a_0 and a_1 may be chosen arbitrarily, and a_n, for n even, may be found by substituting $n=0,2,4,\ldots$ into the recursion relation: $a_2=-2a_0/2\cdot 1=-a_0$, $a_4=0a_2/4\cdot 3=0$, $a_6=2a_4/6\cdot 5=0$, and continuing in this fashion we see $a_n=0$ for all even values of n greater than 2. So that answer to part (b) is $a_{38}=0$. Similarly, by substituting $n=1$ and $n=3$, we obtain $a_3=-a_1/6$ and $a_5=a_3/20=-a_1/120$. Substituting these values into the original series for y and factoring a_0 and a_1 out where possible, we obtain the that the answer to (a) may be written $y=a_0(1-x^2)+a_1(x-x^3/6-x^5/120+\ldots)$.

7. To solve the equation $y''+y=4$, with $y(0)=0$ and $y'(0)=2$, we denote the Laplace transform of the solution y by Y, equate the Laplace transforms of the two sides of the differential equation, solve for Y, and find a partial fraction expansion : $(s^2Y-0s-2)+Y=4/s$ and

$$Y=(4/s+2)/(s^2+1)=\frac{2s+4}{s(s^2+1)}=\frac{A}{s}+\frac{Bs+C}{s^2+1}=\frac{A(s^2+1)+s(Bs+C)}{s(s^2+1)}=\frac{(A+B)s^2+Cs+A}{s(s^2+1)}.$$

Equate coefficients: $A+B=0$; $C=2$; and $A=4$, so that $A=4$, $B=-4$ and $C=2$. Thus, $Y=(4/s)-4s/(s^2+1)+2/(s^2+1)$ and $y=4-4\cos t+2\sin t$.

8. If 1 gal/min of water with 20 mg/gal of impurities flows into a 10 gallon tank filled with pure water, then the rate impurities flow in is $Q_{\text{in}}=\left(1\frac{\text{gal}}{\text{min}}\right)\left(20\frac{\text{mg}}{\text{gal}}\right)=20\frac{\text{mg}}{\text{min}}$. If the mixed solution exits the tank at 2 gal/min, impurities flow out at a rate of $Q_{\text{out}}=\left(\frac{Q(t)}{10-t}\frac{\text{mg}}{\text{gal}}\right)\left(2\frac{\text{gal}}{\text{min}}\right)=\frac{2Q}{10-t}\frac{\text{mg}}{\text{min}}$. Then the net rate at which impurities flow into the tank is $Q'=Q_{\text{in}}-Q_{\text{out}}=20-\frac{2Q}{10-t}$. This differential equation, which may be written as $Q'+\frac{2}{10-t}Q=20$, can be solved using the integrating factor $\mu=e^{\int 2/(10-t)}=1/(10-t)^2$. Then

$$Q(t)=(10-t)^2\int\frac{20}{(10-t)^2}\,dt=(10-t)^2\left(\frac{20}{10-t}+c\right)=(10-t)[20+c(10-t)].$$

Now, $Q(0)=0=10(20+10c)$ and $c=-2$, so $Q(t)=(10-t)2t$.

9. Since a 4 lb weight stretches the spring by 1/2 a foot, we obtain from $F=kx$ that $4=k(1/2)$ or $k=8$. The mass of the weight is found by substituting into $F=mg$: $4=m\cdot 32$ and $m=1/8$. For air resistance of 8 lbs at a velocity of 8 feet/sec, we obtain from $F=\gamma v$ that $8=\gamma\cdot 8$ or $\gamma=1$. Thus, the equation $mu''+\gamma u'+ku=0$ becomes $(1/8)u''+u'+8u=0$, whose characteristic equation has solution $r=(-8\pm\sqrt{64-4\cdot 64})/2=-4\pm 4\sqrt{3}i$. Thus, u and, by differentiating, u' have form

$$u=e^{-4t}(c_1\cos 4\sqrt{3}t+c_2\sin 4\sqrt{3}t);$$
$$u'=-4e^{-4t}(c_1\cos 4\sqrt{3}t+c_2\sin 4\sqrt{3}t)+e^{-4t}(4\sqrt{3}c_2\cos 4\sqrt{3}t-4\sqrt{3}c_1\sin 4\sqrt{3}t).$$

Substituting the initial conditions $u(0)=0$ and $u'(0)=4$ into the equations above yields $u(0)=c_1=0$ and $u'(0)=-4\sqrt{3}c_1+4\sqrt{3}c_2=4$, so that $c_2=\sqrt{3}/3$. Thus, $u(t)=(\sqrt{3}/3)e^{-4t}\sin 4\sqrt{3}t$.

10. Since 0 is a regular singular point of the equation $x^2y''-(x+2)y=0$ we substitute a solution of the form $y=\sum_{n=0}^{\infty}a_nx^{n+r}$ into the differential equation; then we replace n by $n-1$ in the second sum and combine the sums:

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}-\sum_{n=0}^{\infty}a_nx^{n+r+1}-\sum_{n=0}^{\infty}2a_nx^{n+r}=0;$$
$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}-\sum_{n=1}^{\infty}a_{n-1}x^{n+r}-\sum_{n=0}^{\infty}2a_nx^{n+r}=0;$$
$$[(0+r)(0+r-1)-2]a_0x^r+\sum_{n=1}^{\infty}\left([(n+r)(n+r-1)-2]a_n-a_{n-1}\right)x^{n+r}=0.$$

The indicial equation is obtained by setting the constant coefficient equal to zero: $[r(r-1)-2] = r^2 - r - 2 = 0$; thus, $r = -1, 2$. The recursion relation for $r = 2$ is obtained by substituting $r = 2$ into the last displayed equation above and setting the coefficients of x^{n+r}, $n \geq 1$, equal to zero: $[(n+2)(n+1) - 2]a_n - a_{n-1} = 0$, so that $a_n = a_{n-1}/(n^2 + 3n)$, $n \geq 1$. Choose a_0 arbitrarily, and substitute successive values of n into the recursion relation to obtain $a_1 = a_0/4$, $a_2 = a_0/(4 \cdot 10)$, $a_3 = a_0/(4 \cdot 10 \cdot 18)$, and $a_4 = a_0/(4 \cdot 10 \cdot 18 \cdot 28)$. The solution to the differential equation is

$$a_0 x^2 \left[1 + \frac{1}{4}x + \frac{1}{40}x^2 + \frac{1}{720}x^3 + \frac{1}{720 \cdot 28}x^4 + \dots \right].$$

11. (a) The independent solutions to the Euler equation $ty'' - 4ty' + 6y = 0$, $x >)$, are of the form t^r, where r is a solution to $r(r-1) - 4r + 6 = 0$. Thus, $r = 2, 3$; and the general solution is $c_1 t^2 + c_2 t^3$.

(b) Rewrite the equation $t^2 y'' - 4ty' + y = t^4 e^t$ as an equivalent equation with y'' coefficient one: $y'' - (4/t)y + (1/t^2)y = t^2 e^t$. Using the method of variation of parameters, we obtain a particular solution to the last equation in the form $Y = -y_1 \int \frac{y_2 g}{W}\, dy + y_2 \int \frac{y_1 g}{W}\, dy$, where g is the inhomogeneous term $t^2 e^t$. W is the Wronskian, and $y_1 = t^2$ and $y_2 = t^3$ are independent solutions to the homogeneous equation in part (a). Calculate

$$W = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} = t^4; \qquad \frac{g}{W} = \frac{e^t}{t^2};$$

$$\int y_2 \frac{g}{W} = \int te^t = te^t - e^t; \qquad \int y_1 \frac{g}{W} = \int e^t = e^t;$$

$$Y = -t^2(te^t - e^t) + t^3(e^t) = t^2 e^t; \qquad y = c_1 t^2 + c_2 t^3 + t^2 e^t,$$

where the general solution y to the inhomogeneous equation was obtained by adding the particular inhomogeneous solution Y to the general homogeneous solution found in (a).

12. (a) The equation $y' = \frac{xy - y^2}{x^2}$ is homogeneous in the sense that numerator and denominator of the right side are both homogeneous polynomials of the same degree. We may substitute $v = y/x$ to obtain an equation which may be solved by separating variables. Calculate $y' = (vx)' = v + xv'$ and write the original equation as $y' = (y/x) - (y/x)^2$. Thus, $v + xv' = v - v^2$. Separate variables to obtain $\int 1/v^2\, dv = -\int 1/x\, dx$; $-1/v \ (= -x/y) = -\ln x + c$; and $y = x/(\ln x + c)$.

(b) Divide by the y'' coefficient in the equation $(x^3 - 4x^2)y'' + (x+1)y' - xy = 0$ to obtain

$$y'' + \frac{x+1}{x^2(x-4)}y' - \frac{1}{x(x-4)}y = 0,$$

which has singular points $x = 0, 4$. A solution including $x_0 = 5$ is guaranteed on any interval not containing either of the singular points. The largest such interval is $(4, \infty)$.

(c) The singular point $x = 0$ is not regular because the y' coefficient has more than one power of x in the denominator, but the singular point $x = 4$ is regular, since the denominator of the y' coefficient has at most one power of $x - 4$ and the y coefficient has at most two (actually only one) power of $x - 4$ in the denominator.