Basic Integration Formulas

Formulas needed to be memorized: (*=absolutely needed)

* $\int p^n\,dp = \dfrac{p^{(n+1)}}{(n+1)} + C \quad n \neq -1$	* $\int \dfrac{1}{p}\,dp = \ln\lvert p\rvert + C$
* $\int e^p\,dp = e^p + C$	* $\int a^p\,dp = \dfrac{a^p}{\ln a} + C$
* $\int \sin p\,dp = -\cos p + C$	* $\int \cos p\,dp = \sin p + C$
* $\int \sec^2 p\,dp = \tan p + C$	* $\int \csc^2 p\,dp = -\cot p + C$
* $\int \sec p \tan p\,dp = \sec p + C$	* $\int \csc p \cot p\,dp = -\csc p + C$
* $\int \sec p\,dp = \ln\lvert \sec p + \tan p\rvert + C$	* $\int \csc p\,dp = -\ln\lvert \csc p + \cot p\rvert + C = \ln\lvert \csc p - \cot p\rvert + C$
* $\int \tan p\,dp = \ln\lvert \sec p\rvert + C$	* $\int \cot p\,dp = \ln\lvert \sin p\rvert + C$
* $\int \sinh p\,dp = \cosh p + C$	* $\int \cosh p\,dp = \sinh p + C$
* $\int \dfrac{1}{p^2 + a^2}\,dp = \dfrac{1}{a}\tan^{-1}\left(\dfrac{p}{a}\right) + C$	$\int \dfrac{1}{\sqrt{a^2 - p^2}}\,dp = \sin^{-1}\left(\dfrac{p}{a}\right) + C$
* $\int \sec^3 p\,dp = \dfrac{1}{2}\left(\sec p \tan p + \ln\lvert \sec p + \tan p\rvert\right) + C$	$\int \csc^3 p\,dp = \dfrac{1}{2}\left(-\csc p \cot p + \ln\lvert \csc p - \cot p\rvert\right) + C$

Section 7.1: Integration by Parts

The formula for integration by parts:

$$\int u\,dv = uv - \int v\,du$$

Make sure to determine how you want to choose u and dv. Remember that your 1st choice might not be the optimized method of solving by this technique.

There are problems that require you to apply integration by parts more than once.

Section 7.2: Trigonometric Integrals

Strategy for evaluating $\int \sin^m x \cos^n x\,dx$

a) If the power of cosine is odd ($n = 2k+1$), then use $\cos^2 x = 1 - \sin^2 x$:
$\int \sin^m x \cos^{2k+1} x\,dx = \int \sin^m x(\cos^2 x)^k \cos x\,dx = \int \sin^m x(1-\sin^2 x)^k \cos x\,dx$ then let $u = \sin x$

b) If the power of sine is odd ($m = 2k+1$), then use $\sin^2 x = 1 - \cos^2 x$:
$\int \sin^{2k+1} x \cos^n x\,dx = \int (\sin^2 x)^k \sin x \cos^n x\,dx = \int (1-\cos^2 x)^k \cos^n x \sin x\,dx$ then let $u = \cos x$

c) If both powers of sine and cosine are even, then use half angle identities:
$\sin^2 x = \tfrac{1}{2}(1-\cos 2x) \quad \cos^2 x = \tfrac{1}{2}(1+\cos 2x)$.
Sometimes this identity is helpful: $\sin x \cos x = \tfrac{1}{2}\sin 2x$

Strategy for evaluating $\int \tan^m x \sec^n x\, dx$

a) If the power of secant is even ($n = 2k$), use $\sec^2 x = 1 + \tan^2 x$:
$\int \tan^m x \sec^{2k} x\, dx = \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x\, dx = \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x\, dx$ then let $u = \tan x$

b) If the power of tangent is odd ($m = 2k+1$), use $\tan^2 x = \sec^2 x - 1$:
$\int \tan^{2k+1} x \sec^n x\, dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x\, dx = \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x\, dx$
then let $u = \sec x$

Recall $\int \tan x\, dx = \ln|\sec x| + C$ $\int \sec x\, dx = \ln|\sec x + \tan x| + C$

To evaluate the following:

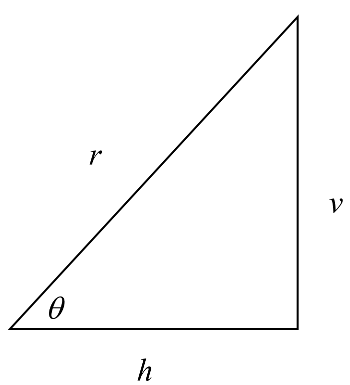
a) $\int \sin mx \cos nx\, dx$ use $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

b) $\int \sin mx \sin nx\, dx$ use $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

c) $\int \cos mx \cos nx\, dx$ use $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Section 7.3: Trigonometric Substitution (TRIANGULATION)

For this section, it will be easier to recall the basic trigonometry of a right triangle. Given triangle below:



Recall: SOH CAH TOA

$\sin\theta = \frac{opp}{hyp} = \frac{v}{r}$ $\cos\theta = \frac{adj}{hyp} = \frac{h}{r}$ $\tan\theta = \frac{opp}{adj} = \frac{v}{h}$

By Pythagorean theorem: $r^2 = v^2 + h^2$

If we solve for hypotenuse and each of the legs, we get:

$r = \sqrt{v^2 + h^2}$ $v = \sqrt{r^2 - h^2}$ $h = \sqrt{r^2 - v^2}$

The trick to this section is to recognize which of the following is present in the problem:

$$\sqrt{v^2 + h^2} \quad \sqrt{r^2 - h^2} \quad \sqrt{r^2 - v^2}$$
$$v^2 + h^2 \quad r^2 - h^2 \quad r^2 - v^2$$

If this expression ($\sqrt{v^2 + h^2}$ or $v^2 + h^2$) is present then this part represents the hypotenuse of the triangle; therefore, each part represent the legs of the triangle.

If these expressions ($\sqrt{r^2 - h^2}$ or $r^2 - h^2$) or ($\sqrt{r^2 - v^2}$ or $r^2 - v^2$) are present then this part represents the leg of the triangle; therefore, the fist part is the hypotenuse and second the other leg of the triangle.

Now use this triangle to pick out 2 trigonometric relationships that involve the pairs given below:

$\sin^2\theta + \cos^2\theta = 1$ $\tan^2\theta + 1 = \sec^2\theta$ $1 + \cot^2\theta = \csc^2\theta$

This is the encoding step. Then use techniques of trigonometric integration to solve the problem. After solving, use the triangle we set up for encoding to decode our solution.

Section 7.4: Integration of Rational Function by Partial Fractions

If $f(x)=\dfrac{P(x)}{Q(x)}$ such that $\deg(P(x))\geq\deg(Q(x))$, then use the long division to obtain
$f(x)=\dfrac{P(x)}{Q(x)}=S(x)+\dfrac{R(x)}{Q(x)}$ where $S(x)$ and $R(x)$ are polynomials.

Case 1: The denominator $Q(x)$ is a product of distinct linear factors.
This means that we can write $Q(x)=(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$ where no factor is repeated. In this case the partial fraction theorem states that there exist constants $A_1,A_2,\ldots,A_k$ such that
$\dfrac{R(x)}{Q(x)}=\dfrac{A_1}{a_1x+b_1}+\dfrac{A_2}{a_2x+b_2}+\cdots+\dfrac{A_k}{a_kx+b_k}$.

Case 2: $Q(x)$ is a product of linear factors, some which are repeated.
Suppose the first linear factor (a_1x+b_1) is repeated r times; that is, $(a_1x+b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $\dfrac{A_1}{(a_1x+b_1)}$ in previous case 1, we would use
$\dfrac{A_1}{(a_1x+b_1)}+\dfrac{A_2}{(a_2x+b_2)^2}+\cdots+\dfrac{A_r}{(a_rx+b_r)^r}$.

Case 3: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.
If $Q(x)$ has the factor ax^2+bx+c, where $b^2-4ac<0$, then, in addition to the partial fractions in equations from case 1 and 2, the expression for $\dfrac{R(x)}{Q(x)}$ will have a term of the form $\dfrac{Ax+B}{ax^2+bx+c}$ where A and B are constants to be determined.
The term $\dfrac{Ax+B}{ax^2+bx+c}$ can be integrated by completing the square and using the formula
$\displaystyle\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$.

Case 4: $Q(x)$ contains a repeated irreducible quadratic factor.
If $Q(x)$ has the factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then instead of the single partial fraction in case 3, the sum $\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+\cdots+\dfrac{A_rx+B_r}{\left(ax^2+bx+c\right)^r}$ occurs in the partial fraction decomposition of $\dfrac{R(x)}{Q(x)}$. Each of the terms above can be integrated by first completing the square.