CCNY 203 Fall 2011 Final Solutions

1a: Take cross products of the displacement vectors to get a normal to plane P:

```
Cross[{0, 1, 2} - {1, 0, -1}, {1, 2, 3} - {0, 1, 2}]
```

```
{-2, 4, -2}
```

That gives an equation of the plane from the normal vector and a point:

```
-2 (x) + 4 (y - 1) - 2 (z - 2) == 0
```

```
-2 x + 4 (-1 + y) - 2 (-2 + z) == 0
```

1b: The direction of the line is the same as the normal vector above, so parameterizing the line as $P + t V$ gives

```
l = {0, 1, 2} + t {-2, 4, -2}
```

```
{-2 t, 1 + 4 t, 2 - 2 t}
```

If we write this in standard form, we get $x = -2t$, $y = 1 + 4t$, $z = 2 - 2t$.

1c: We use dot product to measure the angle:

```
{0, 2, -2}.{-2, 4, -2}
```

```
12
```

The dot product is not zero and the vectors are not scalar multiples, so the vectors are neither perpendicular nor parallel.

2a: We take the gradient:

$$f = z + z\,\mathrm{Log}\left[x^2 + y^2\right]$$

$$z + z\,\mathrm{Log}\left[x^2 + y^2\right]$$

```
gradf = {D[f, x], D[f, y], D[f, z]}
```

$$\left\{\frac{2\,x\,z}{x^2 + y^2}, \frac{2\,y\,z}{x^2 + y^2}, 1 + \mathrm{Log}\left[x^2 + y^2\right]\right\}$$

At the desired point, we substitute to get:

$$\mathrm{gradf}\ /.\ \{x \to 1, y \to 0, z \to 2\}$$

```
{4, 0, 1}
```

The directional derivative is obtained by dotting the gradient with a unit vector in the desired direction:

$$u = \frac{\{4, 4, 7\} - \{1, 0, 2\}}{\sqrt{3^2 + 4^2 + 5^2}}$$

$$\left\{\frac{3}{5\sqrt{2}}, \frac{2\sqrt{2}}{5}, \frac{1}{\sqrt{2}}\right\}$$

```
Together[{4, 0, 1}. u]
```

$$\frac{17}{5\sqrt{2}}$$

2b: This the direction of the gradient: <4,0,1>.

2c: The appropriate version of the chain rule here is $\frac{\partial E}{\partial s} = \frac{\partial E}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial E}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial E}{\partial z}\frac{\partial z}{\partial s}$ which at the relevant points gives $2 \times 1 + 0 \times 1 + (1 + \ln(4))\ 2 = 4{+}2\ \ln(4)$. Or you can substitute but that is more tedious and prone to error.

3: Taking the first partials and setting them to zero and solving gives:

```
f = 3 x^2 - 12 x y + 8 y^3
```

```
3 x^2 - 12 x y + 8 y^3
```

```
Solve[{D[f, x] == 0, D[f, y] == 0}]
```

```
{{x → 0, y → 0}, {x → 2, y → 1}}
```

At the first point, we evalute the discriminant to get:

```
D[f, x, x] D[f, y, y] - D[f, x, y]^2 /. {{x → 0, y → 0}}
```

```
{-144}
```

So there is a saddle at (0,0).

At the second point, we evalute the discriminant to get:

```
D[f, x, x] D[f, y, y] - D[f, x, y]^2 /. {{x → 2, y → 1}}
```

```
{144}
```

Since it is positive, we look at f_{xx} :

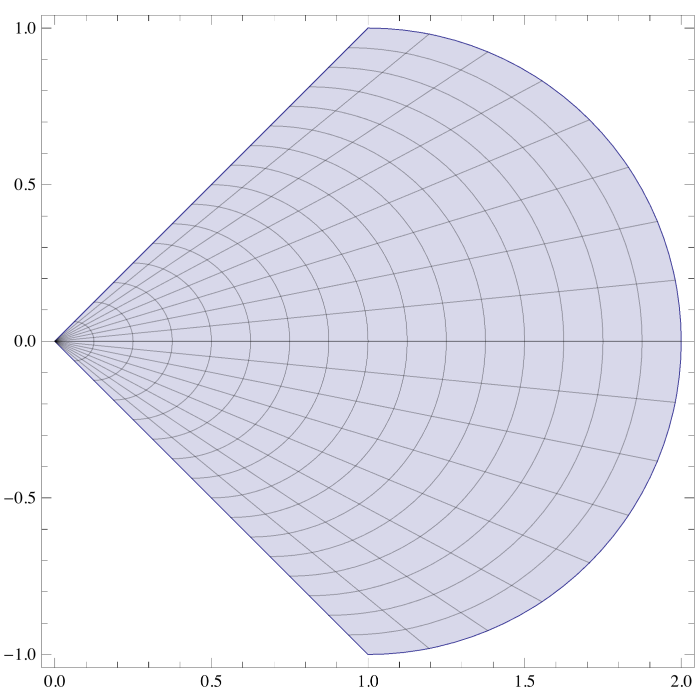
```
D[f, x, x] /. {{x → 2, y → 1}}
```

```
{6}
```

So there is a relative minimum at (2,1).

4a: We sketch the region in the plane (it's a circle tangent to the y-axis at the origin) and convert to polar:

```
ParametricPlot[{t 2 Cos[θ] Cos[θ], t 2 Cos[θ] Sin[θ] }, {θ, -π/4, π/4}, {t, 0, 1}]
```



$$\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\int_0^{2\,\mathrm{Cos}[\theta]} r\, r\, dr\, d\theta$$

$$\frac{20\sqrt{2}}{9}$$

4b: We differentiate and evaluate at the point and get:

```
f = x (y^2 + 2)
D[f, x]
D[f, y]
D[f, x] /. {x → 3, y → -1}
D[f, y] /. {x → 3, y → -1}
```

$x\,(2 + y^2)$

$2 + y^2$

$2\,x\,y$

3

-6

which gives an equation of the tangent plane as:

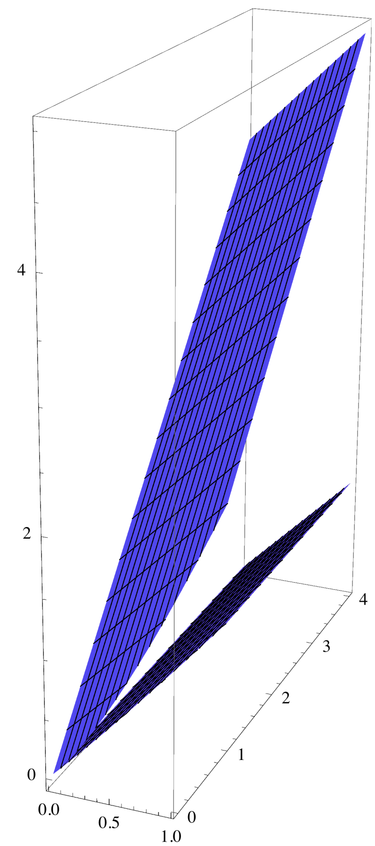
```
z - 9 == 3 (x - 3) - 6 (y + 1)
```

5: For the volume via a double integral, taking the integral of the (top surface - bottom surface) over the shadow to get:

$$\int_0^1\int_{x^2}^4 ((x + y) - x)\, dy\, dx$$

$$\frac{79}{10}$$

```
ParametricPlot3D[{{x, y, x}, {x, y, x + y}}, {x, 0, 1}, {y, x^2, 4}]
```



5: Alternatively, volume via a triple integral:

$$\int_0^1 \int_{x^2}^4 \int_x^{x+y} 1 \, dz \, dy \, dx$$

$$\frac{79}{10}$$

6a: Divergent by comparison with the harmonic series. Limit comparison does not work as the limit of the ratio does not exist.
6b: Convergent by alternating series test, and the absolute values are divergent by comparison with the harmonic series, so the series is conditionally convergent.
6c: Absolutely convergent by the ratio test.

7: We use the ratio test to get convergence on the interval -9<x< 3. For x=3, divergent by comparison with the harmonic series. For x=-9, converent by the alternating series test. So the power series convergenes on the interval [-9,3).

8a: We use spherical coordinates to find the center of mass, and the fact that the relevant part of the cone has equation $\phi = \frac{\pi}{4}$:

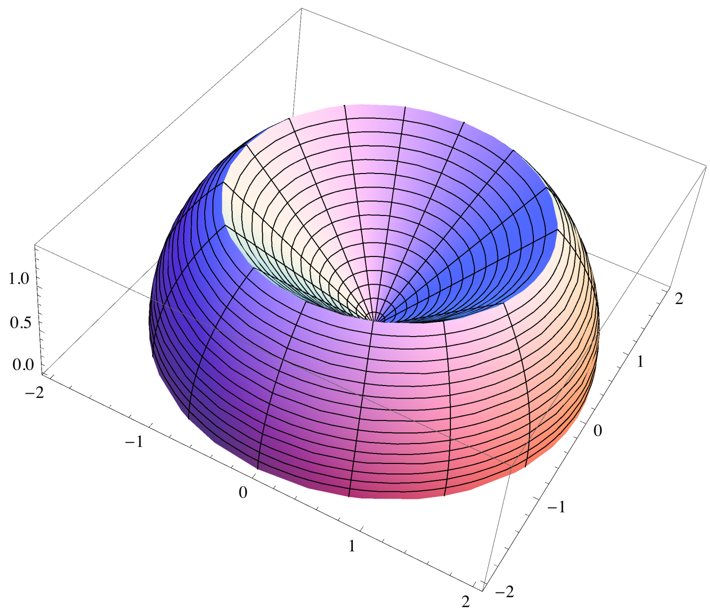
$$\text{mass} = \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 \rho \, \rho^2 \, \text{Sin}[\phi] \, d\rho \, d\phi \, d\theta$$

$$4\sqrt{2}\,\pi$$

```
zbar = 1/mass ∫_0^{2π} ∫_{π/4}^{π/2} ∫_0^2 ρ ρ Cos[ϕ] ρ^2 Sin[ϕ] dρ dϕ dθ
```

$$\frac{2\sqrt{2}}{5}$$

```
Show[{ParametricPlot3D[{ 2 Cos[θ] Sin[ϕ],  2 Sin[θ] Sin[ϕ],  2 Cos[ϕ] },
   {θ, 0, 2 π}, {ϕ, π/4, π/2}], ParametricPlot3D[
   { t Cos[θ] Sin[π/4],  t Sin[θ] Sin[π/4],  t Cos[π/4] }, {θ, 0, 2 π}, {t, 0, 2}]}]
```



8b: We consider horizontal and vertical paths approaching the origin, which have limits of 0,

```
x = t;
y = 0;
Limit[2 x Sin[y] / (x^2 + y^2), {t → 0}]
```

{0}

```
x = 0;
y = t;
Limit[2 x Sin[y] / (x^2 + y^2), {t → 0}]
```

{0}

Those were the same, so either we should try to prove that limit exists or we should look at another direction. We consider coming in along the line y=x, which gives a limit of 1 using that the limit of $\frac{\sin(x)}{x}$ is 1 as x approaches 0, so therefore the limit does not exist:

```
x = t;
y = t;
Limit[2 x Sin[y]/(x^2 + y^2), {t → 0}]
```

```
{1}
```

Converting to polar coordinates is another approach- we can see that there are different limits for different values of θ as r approaches zero, which means the limit cannot exist:

```
x = r Cos[θ];
y = r Sin[θ];
Simplify[2 x Sin[y]/(x^2 + y^2)]
```

$$\frac{2\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[r\,\mathrm{Sin}[\theta]]}{r}$$

```
Limit[2 Cos[θ] Sin[r Sin[θ]]/r /. {θ → π/4}, {r → 0}]
```

```
{1}
```

```
Limit[2 Cos[θ] Sin[r Sin[θ]]/r /. {θ → 0}, {r → 0}]
```

```
{0}
```

9a: we take the nearby point of (1,3) and use differentials from there:

```
f = e^(3 x-y) Cos[x - 1]
```

$e^{3x-y}\,\mathrm{Cos}[1-x]$

```
f /. {x → 1, y → 3}
```

```
1
```

```
D[f, x] /. {x → 1, y → 3}
```

```
3
```

```
D[f, y] /. {x → 1, y → 3}
```

```
-1
```

which gives us the approximation of f at that point as 1+3(-.02) + -1(.01) = .93

9b: we take the gradient of the function since the surface is given implicily:

```
f = 2 Sin[x - y] + e^(4 y^2-z^2)
```

$e^{4y^2-z^2} + 2\,\mathrm{Sin}[x-y]$

```
f /. {x → 1, y → 1, z → 2}
```

```
1
```

```
gradf = {D[f, x], D[f, y], D[f, z]} /. {x → 1, y → 1, z → 2}
```

```
{2, 6, -4}
```

which gives an equation of $2(x-1) + 6(y-1) - 4(z-2) = 0$ for the tangent plane there.

10a: For surface area, we integrate $\sqrt{1 + f_x^2 + f_y^2}$ over the shadow down on the $x\,y$ plane, then switch to polar, to get

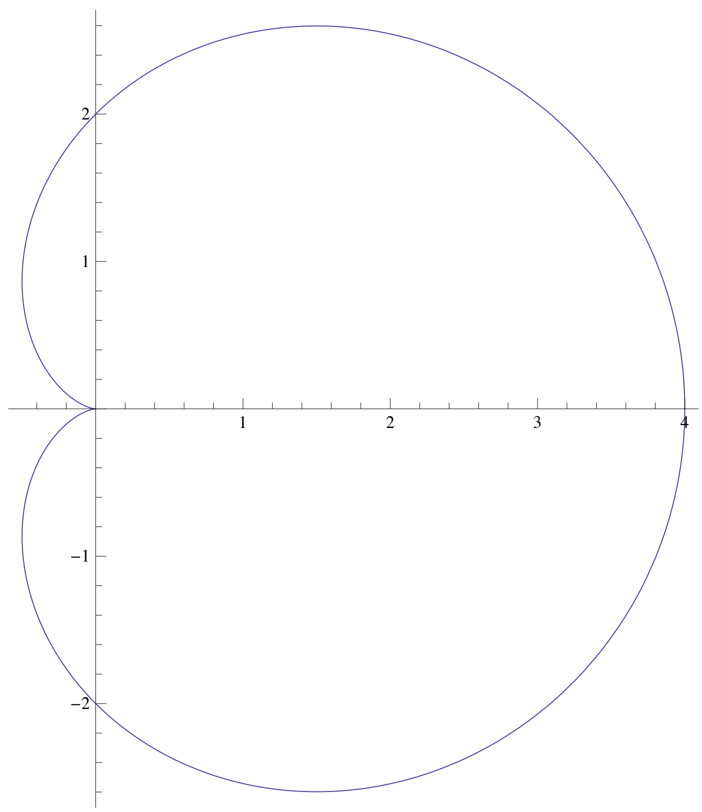
$$\int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{1+4r^2}\; r\, dr\, d\theta$$

$$\frac{1}{6}\left(-1+13\sqrt{13}\right)\pi$$

10b: this is easier to recognize when we complete the square for x to get into the form $(x-2)^2 - 4y^2 - 4z^2 = 4$, where we can regonize this as a hyperboloid of 2 sheets opening up along the $+x$-axis and $-x$ axis The vertices are (0,0,0) and (4,0,0).

11a: this is a sideways heart-shaped region, and we set up a mass integral in cylindrical coordinates

```
PolarPlot[2 + 2 Cos[t], {t, 0, 2 π}]
```



$$\text{mass} = \int_0^{2\pi}\int_0^{2+2\,\mathrm{Cos}[\theta]}\int_0^{2} z\; r\, dz\, dr\, d\theta$$

$$12\pi$$

12: the correct series is $\sum_{n=0}^{\infty} \frac{1+(-2)^n}{n!} x^n$. To approximate $f\left(\frac{1}{4}\right)$, we take the first few terms as $2 + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{-1}{2!}\right)\left(\frac{1}{4}\right)^2 + \left(\frac{3}{3!}\right)\left(\frac{1}{4}\right)^3 - \left(\frac{7}{4!}\right)\left(\frac{1}{4}\right)^4 + \ldots$ By the alternating series test, the series converges and the error is no more than the first omitted term. So since the second term is $\frac{1}{8}$, the first term is not guarranteed to be enough. But since the third term is $\frac{-1}{32}$, the error in using the first two terms is less than the desired .1. So the approximation is $2 - \frac{1}{8}$.

For the series of the derivative, since the series converges absolutely, we can differentiate each term to get $\sum_{n=1}^{\infty} n\, \frac{1+(-2)^n}{n!} x^{n-1}$ as a description of the series for the derivative, which could be re-indexed if desired. The first three non-zero terms are:

$$\sum_{n=1}^{3} n \frac{1 + (-2)^n}{n!} x^{n-1}$$

$$-1 + 5x - \frac{7x^2}{2}$$