CCNY 203 Fall 2010 Final Solutions

1a: We take the cross product of the vector along the line and a displacement vector from the point to a point on the line to get a normal to plane P:

```
Cross[{3, -1, -2}, {2, -2, 1} - {2, -3, 1}]
```

```
{2, 0, 3}
```

That gives an equation of the plane:

$$2\,(x-2) + 0\,(y+3) + 3\,(z-1) == 0$$

1b: The distance from the plane to the origin we can get by projecting a vector from the origin to a point on the plane onto the normal to the plane:

$$d = \frac{\{2, 0, 3\}}{\sqrt{4+0+9}} . \{2, -3, 1\}$$

$$\frac{7}{\sqrt{13}}$$

2a: We take the gradient:

$$f = x^2\, y^3 + e^{x+z} - 2\, y\, \text{Sin}[z]$$

$$e^{x+z} + x^2\, y^3 - 2\, y\, \text{Sin}[z]$$

```
gradf = {D[f, x], D[f, y], D[f, z]}
```

$$\left\{e^{x+z} + 2\, x\, y^3,\ 3\, x^2\, y^2 - 2\, \text{Sin}[z],\ e^{x+z} - 2\, y\, \text{Cos}[z]\right\}$$

At the desired point, we substitute to get:

$$\text{gradf} \;/.\; \{x \to 0,\ y \to 1,\ z \to 0\}$$

```
{1, 0, -1}
```

The directional derivative is obtained by dotting the gradient with a unit vector in the desired direction:

$$\{1, 0, -1\} . \frac{\{3, 4, -12\}}{\sqrt{3^2 + 4^2 + 12^2}}$$

$$\frac{15}{13}$$

2b: Implicit differentiation with respect to x gives:

$$\text{eqn} = 2\, x\, y^3 + e^{(x+z)}\, (1 + \partial_x z[x]) - 2\, y\, \text{Cos}[z]\, \partial_x z[x] == 0$$

$$2\, x\, y^3 - 2\, y\, \text{Cos}[z]\, z'[x] + e^{x+z}\, (1 + z'[x]) == 0$$

Solving gives:

```
Solve[eqn, z'[x]]
```

$$\left\{\left\{z'[x] \to \frac{-e^{x+z} - 2\,x\,y^3}{e^{x+z} - 2\,y\,\mathrm{Cos}[z]}\right\}\right\}$$

More simply, since we already have the gradient from 2a, we can divide to use: $\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial x}}$ in this case.

2c: Using the gradient evaluated at the point from 2a gives:

```
1 (x - 0) + 0 (y - 1) + -1 (z - 0) == 0
```

3: Taking the first partials and setting them to zero and solving gives:

```
f = 3 x^2 - 6 x y + y^3 - 9 y
```

$3\,x^2 - 9\,y - 6\,x\,y + y^3$

```
Solve[{D[f, x] == 0, D[f, y] == 0}]
```

{{x → -1, y → -1}, {x → 3, y → 3}}

At the first point, we evalute the discriminant to get:

```
D[f, x, x] D[f, y, y] - D[f, x, y]^2 /. {{x → -1, y → -1}}
```

{-72}

So there is a saddle at (-1,-1).

At the second point, we evalute the discriminant to get:

```
D[f, x, x] D[f, y, y] - D[f, x, y]^2 /. {{x → 3, y → 3}}
```

{72}

Since it is positive, we look at f_{xx} :

```
D[f, x, x] /. {{x → 3, y → 3}}
```

{6}

So there is a relative minumum at (3,3).

4: We sketch the region in the plane (a quarter disk in the first quadrant) and convert to polar:

$$\int_0^{\pi/2}\int_0^2 e^{-r^2}\, r\, dr\, d\theta$$

$$\frac{1}{4}\left(1 - \frac{1}{e^4}\right)\pi$$

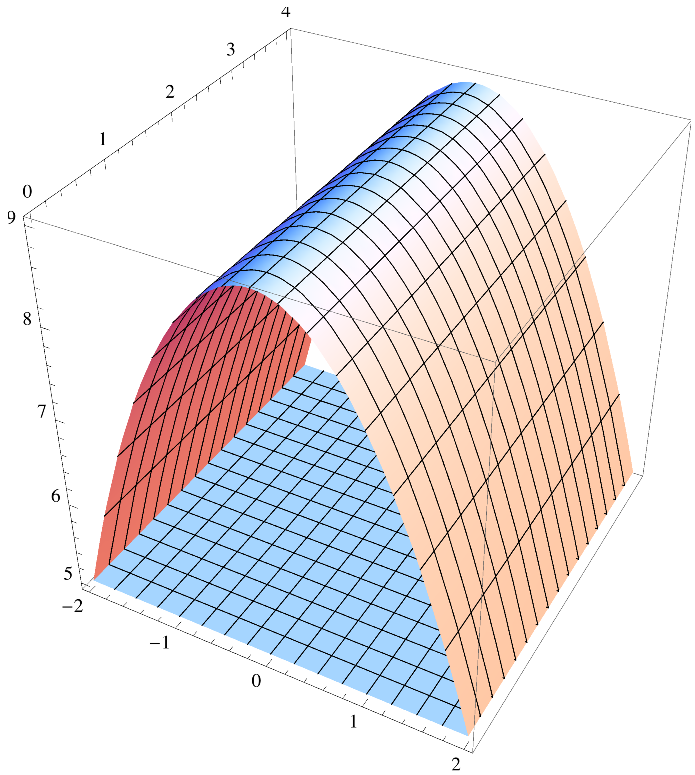
5: For the volume via a double integral, taking the top surface - the bottom surface to get:

$$\int_{-2}^{2}\int_0^4 \left(\left(9 - x^2\right) - 5\right) dy\, dx$$

$$\frac{128}{3}$$

For the sketch, note that this is bounded below by the plane z=5, not the xy-plane:

```
ParametricPlot3D[{{x, y, 9 - x^2}, {x, y, 5}}, {x, -2, 2}, {y, 0, 4}]
```



5: Alternatively, volume via a triple integral:

$$\int_{-2}^{2}\int_{0}^{4}\int_{5}^{9-x^2} 1\, dz\, dy\, dx$$

$$\frac{128}{3}$$

6a: Convergent by the alternating series test, absolute values diverge by limit comparison with $\sum \frac{1}{n}$ which gives conditional convergence

6b: Absolutely convergent by limit comparison with $\sum \frac{1}{n^2}$

6c: Divergent by the integral test, integrating by substitution $u = \ln(x)$

$$\int \frac{1}{x\sqrt{\mathrm{Log}[x]}}\, dx$$

$$2\sqrt{\mathrm{Log}[x]}$$

7: We use the ratio test to get convergence on the interval -1<x< 5. For x=-1, convergent by the alternating series test. For x=5, converent by the comparison test with $\sum \frac{1}{k^4}$.

8a: The chain rule gives: $\frac{\partial u}{\partial s} = \frac{\partial f}{\partial x} 3\, t^2 + \frac{\partial f}{\partial y} 2\, s\, t$

8b: The limit along the +x direction is 1 and the limit along the +y direction is -2, so the limit doesn't exist.

9: We take the first partials and setting them to zero and solving gives:

```
f = x^2 - 2 x  + y^2 + 2 y
```

$-2x + x^2 + 2y + y^2$

```
Solve[{D[f, x] == 0, D[f, y] == 0}]
```

{{x → 1, y → -1}}

Since this lies inside the disk, we evalute the function to get

```
f /. {x → 1, y → -1}
```

-2

On the boundary, we parameterize and consider

```
fb = f /. {x → 3 Cos[t], y → 3 Sin[t]}
```

-6 Cos[t] + 9 Cos[t]^2 + 6 Sin[t] + 9 Sin[t]^2

notice that $9\sin^2(x) + 9\cos^2(x) = 9$, and differentiating and solving gives:

```
Solve[D[-6 Cos[t] + 9 + 6 Sin[t], t] == 0, t]
```

$\left\{\left\{t \to -\frac{\pi}{4}\right\}, \left\{t \to \frac{3\pi}{4}\right\}\right\}$

Evaluating gives:

$f \;/.\; \left\{x \to 3\,\mathrm{Cos}\left[-\frac{\pi}{4}\right],\ y \to 3\,\mathrm{Sin}\left[-\frac{\pi}{4}\right]\right\}$

$9 - 6\sqrt{2}$

$f \;/.\; \left\{x \to 3\,\mathrm{Cos}\left[\frac{3\pi}{4}\right],\ y \to 3\,\mathrm{Sin}\left[\frac{3\pi}{4}\right]\right\}$

$9 + 6\sqrt{2}$

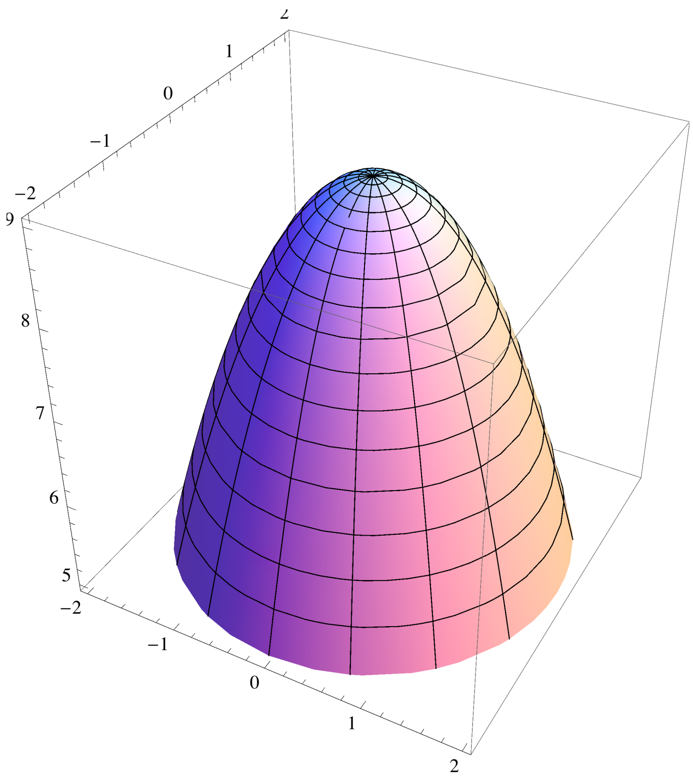
So the absolute max is $9 + 6\sqrt{2}$ at $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$ and the absolute min is -2 at (1,-1)

10: this looks like the top part of a cap, with the intersection with the plane $z = 5$ being a circle of radius 2. For surface area, we integrate $\sqrt{1 + f_x^2 + f_y^2}$ over the shadow down on the $x\,y$ plane, then switch to polar, to get

$$\int_0^{2\pi}\int_0^2 \sqrt{1 + 4r^2}\; r\, dr\, d\theta$$

$$\frac{1}{6}\left(-1 + 17\sqrt{17}\right)\pi$$

```
ParametricPlot3D[{r Cos[θ], r Sin[θ], 9 - r^2}, {r, 0, 2}, {θ, 0, 2 π}]
```



11. We use spherical coordinates to find the center of mass, taking the constant density to be just δ=1. We know the volume of a hemisphere as half of the volume of a sphere so using zbar = $\frac{M_{xy}}{M} = \frac{1}{\text{vol}} \iiint z\, dV$ we get:

$$\text{zbar} = \frac{1}{\frac{2\pi 3^3}{3}} \int_0^{2\pi} \int_{\pi/2}^{\pi} \int_0^3 \rho \operatorname{Cos}[\phi]\, \rho^2 \operatorname{Sin}[\phi]\, d\rho\, d\phi\, d\theta$$

$$-\frac{9}{8}$$

so the center of mass is 1.125 inches below the top, in the middle layer, which is vanilla.

12:a) use the geometric series and substitute $x = -t^3$ then multiply to get $3\,t^2(\,1 - t^3 + t^6 - t^9 + ...)$ which converges for $-1 < t < 1$. In summation notation, that will be $\sum_{k=0}^{\infty} 3\,(-1)^k\, t^{3k+2}$

b) integrate part a) to get $\sum_{k=0}^{\infty} \frac{3\,(-1)^k\, t^{3k+3}}{3k+3}$

$$\sum_{k=0}^{\infty} \frac{3\,(-1)^k\, t^{3k+3}}{3k+3}$$

$$\operatorname{Log}[1 + t^3]$$