M203 Fall 2007 Exam solutions

Q1a: find a normal to the plane by taking cross product of two vectors in the plane.
vector along line: <2,3,4> point on line: (1,2,3), displacement vector from line to point:
<1,2,3>-<1,2,0>= <0,0,3> Cross product: <2,3,4> x <0,0,3> = <9,-6,0> giving an equation: 9 (x-1) - 6 (y-2) + 0 (z-3) = 0

Q1b: Find a normal vector to the plane by putting into a standard form:
x+2y-z=4 gives normal vector <1,2,-1>. That gives parameterization:
x=1+1t
y=2+2t
z=3-t

Q2a: find tangent vector by computing the derivative and evaluating it at the proper time

$$\mathbf{r = \left\{\sqrt{t}, \frac{4}{t}, \frac{t^2}{2}\right\}}$$

$$\left\{\sqrt{t}, \frac{4}{t}, \frac{t^2}{2}\right\}$$

$$\mathbf{rp = D[r, t]}$$

$$\left\{\frac{1}{2\sqrt{t}}, -\frac{4}{t^2}, t\right\}$$

$$\mathbf{rp\ /.\ t \to 4}$$

$$\left\{\frac{1}{4}, -\frac{1}{4}, 4\right\}$$

gives the parameterization:
$x = 2 + \frac{t}{4}$
$y = 1 - \frac{t}{4}$
$z = 8 + 4\,t$

Q2b: find the gradient of the implict description of the surface:

$$\mathbf{\left\{D\left[\frac{x^3}{z^2} + 4\,(y-1)^2, x\right], D\left[\frac{x^3}{z^2} + 4\,(y-1)^2, y\right], D\left[\frac{x^3}{z^2} + 4\,(y-1)^2, z\right]\right\}}$$

$$\left\{\frac{3\,x^2}{z^2}, 8\,(-1+y), -\frac{2\,x^3}{z^3}\right\}$$

Evaluate at the point to get a normal vector:

$$\mathbf{\left\{\frac{3\,x^2}{z^2}, 8\,(-1+y), -\frac{2\,x^3}{z^3}\right\}\ /.\ \{x \to 1,\ y \to 2,\ z \to -1\}}$$

$$\{3, 8, 2\}$$

gives the equation $3\,(x-1) + 8\,(y-2) + 2\,(z+1) = 0$

Q3a: find the gradient at the point and use the method $D_u\, f = \nabla f \cdot u$

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f = x/(2 x + 3 y)
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$\frac{x}{2x+3y}$

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grad = {D[f, x], D[f, y]}
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$\left\{-\frac{2x}{(2x+3y)^2}+\frac{1}{2x+3y},\ -\frac{3x}{(2x+3y)^2}\right\}$

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grad /. {x → 2, y → -1}
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{-3, -6}

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u = {-1 - 2, 1 + 1}/Sqrt[(-3)^2 + 2^2]
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$\left\{-\frac{3}{\sqrt{13}},\ \frac{2}{\sqrt{13}}\right\}$

So the resulting directional derivative is:

$$\{-3, -6\} . \left\{-\frac{3}{\sqrt{13}},\ \frac{2}{\sqrt{13}}\right\}$$

$-\frac{3}{\sqrt{13}}$

For the maximum, we take the length of the gradient at P:

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Norm[grad /. {x → 2, y → -1}]
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$3\sqrt{5}$

Q4a: easier as a type II region: $\int_0^6 \int_{y/2}^{y} f(x, y)\, dx\, dy$

Q4b: a parabolic sliver. The other order of integration gives: $\int_0^4 \int_{y/2}^{\sqrt{y}} f(x, y)\, dx\, dy$

Q5: region outside a cylinder, inside a sphere. above the x-y plane. Volume via cylindrical coordinates:

$$\int_0^{2\pi} \int_1^2 \int_0^{\sqrt{4-r^2}} r\, dz\, dr\, dt$$

$2\sqrt{3}\,\pi$

Q6a: diverges by the nth term test, as the limit is not zero:

$$\text{Limit}\left[\left(\frac{n}{n+1}\right)^n, \{n \to \infty\}\right]$$

$\left\{\frac{1}{e}\right\}$

Q6b: converges by the alternating series test. For the absolute value of the terms, it diverges by the integral test, so this series converges conditionally.

$$\int_2^\infty \frac{1}{x\,\mathrm{Log}[x]}\,dx$$

Integrate::idiv : Integral of $\frac{1}{x\,\mathrm{Log}[x]}$ does not converge on {2, ∞}. More…

$$\int_2^\infty \frac{1}{x\,\mathrm{Log}[x]}\,dx$$

Q6c: Taking the absolute value and comparing with $\Sigma \frac{1}{n^2}$ the convergent p-series gives that this converges absolutely. (The alternating series test cannot be used as the signs do not alternate.)

Q7: Using the ratio test, we get

$$\mathrm{Limit}\left[\frac{\frac{(x+1)^{n+1}}{(n+1)^2\,2^{n+1}}}{\frac{(x+1)^n}{(n)^2\,2^n}},\ \{n\to\infty\}\right]$$

$$\left\{\frac{1+x}{2}\right\}$$

which we want to have absolute value less than 1, giving that the series converges for -3<x<1. For the endpoints, we compare with a p-series with p=2 and get that it converges at both endpoints, which gives the interval of convergence as [-3,1].

Q8: setting both derivatives equal to zero and solving gives three critical points:

$$f = x\,y^2 - 2\,x\,y + x^2$$

$$x^2 - 2\,x\,y + x\,y^2$$

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Solve[{D[f, x] == 0, D[f, y] == 0}]
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$$\left\{\{x\to 0,\ y\to 0\},\ \{x\to 0,\ y\to 2\},\ \left\{x\to \frac{1}{2},\ y\to 1\right\}\right\}$$

$$\mathrm{disc} = D[f, x, x]\,D[f, y, y] - D[f, x, y]^2$$

$$4\,x - (-2 + 2\,y)^2$$

$$\mathrm{disc}\ /.\ \left\{\{x\to 0,\ y\to 0\},\ \{x\to 0,\ y\to 2\},\ \left\{x\to \frac{1}{2},\ y\to 1\right\}\right\}$$

{-4, -4, 2}

$$D[f, x, x]\ /.\ \left\{x\to \frac{1}{2},\ y\to 1\right\}$$

2

which gives saddles at (0,0) and (0,2) and a min at $(\frac{1}{2}, 1)$.

Q9: Setting up the triple integral gives:

$$\int_0^{2\pi}\int_1^2\int_{r\,Cos[t]}^4 z\,r\,dz\,dr\,dt$$

$$\frac{177\,\pi}{8}$$

Q10: We start with the power series for the geometric series for $\frac{1}{1-- x^2} = 1 - x^2 + x^4$ ... and integrate to get $\int \arctan(x) = \int 1 - x^2 + x^4 \ldots = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$ which has the general form for the n^{th} term as $a_n = (-1)^{n+1}\,\frac{x^{2n-1}}{2n-1}$, starting at n=1.

To control the error, we note since the series is alternating, the error is no more than the subsequent term, so we need the absolute value of $a_{n+1} < \frac{1}{10^4}$. The second term is $\frac{10^{-3}}{3} = \frac{1}{3000}$ which is not small enough, and the third term is $\frac{1}{5\,10^5}$ which is small enough, so the first two terms suffice.

Q11a: we approach along two lines. The horizontal line y=0 gives limit $\frac{x^2}{x^2}$ which has limit 1, and the vertical line x=0 gives limit $\frac{0}{y^2}$ gives limit 0, so the limit cannot exist.

Q11b: completing the square twice to put this into standard form gives the equation:

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-(x - 1)^2 - y^2 + (z + 1)^2 == 1
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which is a hyperboloid of two sheets, opening in directions parallel to the z-axis,
with vertices at (1,0,0) and (1,0,-2).

Q12: finding the surface area, we compute the integral of $\sqrt{1 + {f_x}^2 + {f_y}^2}$ over the shadow of the region to get:

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fx = D[4 - x^2 - y^2, x]
fy = D[4 - x^2 - y^2, y]
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$-2\,x$

$-2\,y$

$$\sqrt{1 + {f_x}^2 + {f_y}^2}$$

$$\sqrt{1 + 4\,x^2 + 4\,y^2}$$

converting to polar, we get for the surface area:

$$\int_0^{2\pi}\left(\int_0^{\sqrt{3}} \sqrt{1 + 4\,r^2}\; r\,dr\right) d\theta$$

$$-\frac{\pi}{6} + \frac{13\sqrt{13}\,\pi}{6}$$