4) $y = x^3 \quad y = 0 \quad x = 1 \quad x = 2$; about y-axis

interval: $1 \le x \le 2$

$r = (x) + 0 = (x)$

$l = 2\pi r = 2\pi(x)$

$h(x) = (x^3) - (0) = (x^3)$

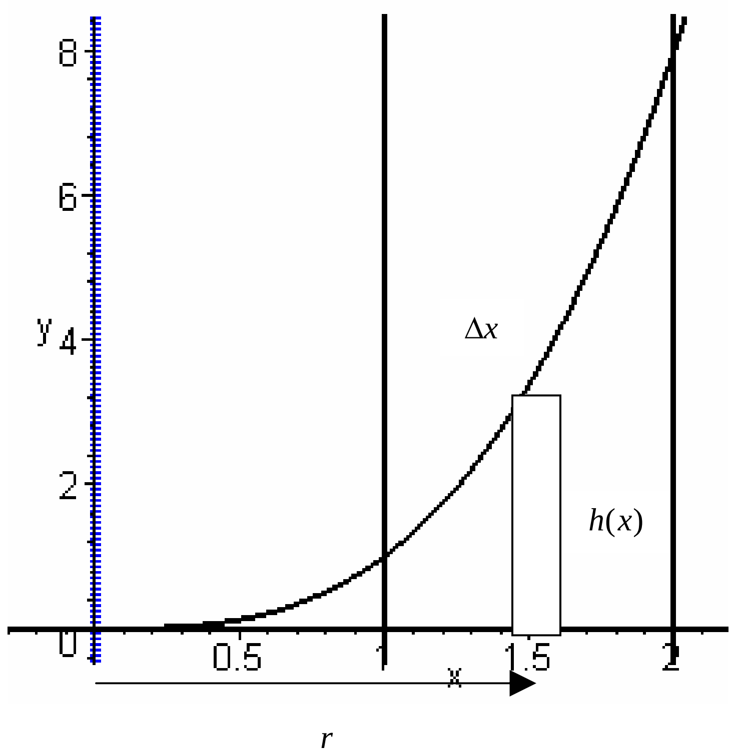
$A = lh(x) = 2\pi(x)(x^3) = 2\pi(x^4)$

$\Delta V = A\Delta x = 2\pi(x^4)\Delta x$

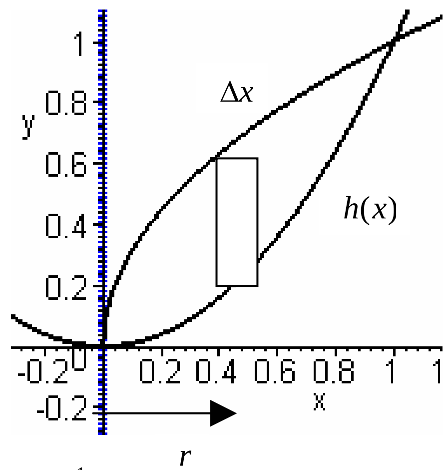
$$V = \int_1^2 2\pi(x^4)\,dx = 2\pi\left[\frac{1}{5}x^5 + C\right]_1^2$$

$$= 2\pi\left\{\left[\frac{1}{5}(2)^5 + C\right] - \left[\frac{1}{5}(1)^5 + C\right]\right\}$$

$$= 2\pi\left\{\left[\frac{32}{5}\right] - \left[\frac{1}{5}\right]\right\} = 2\pi\left\{\frac{31}{5}\right\} = \frac{62\pi}{5}\ units^3$$



8) $y = \sqrt{x} \quad y = x^2$; about y-axis



By Shells

interval: $0 \le x \le 1$

$r = (x) + 0 = (x)$

$l = 2\pi r = 2\pi(x)$

$h(x) = (\sqrt{x}) - (x^2) = (\sqrt{x} - x^2)$

$A = lh(x) = 2\pi(x)(\sqrt{x} - x^2) = 2\pi(x^{\frac{3}{2}} - x^3)$

$\Delta V = A\Delta x = 2\pi(x^{\frac{3}{2}} - x^3)\Delta x$

$$V = \int_0^1 2\pi(x^{\frac{3}{2}} - x^3)\,dx = 2\pi\left[\frac{2}{5}x^{\frac{5}{2}} - \frac{1}{4}x^4 + C\right]_0^1 = 2\pi\left[\frac{2}{5}(\sqrt{x})^5 - \frac{1}{4}x^4 + C\right]_0^1$$

$$= 2\pi\left\{\left[\frac{2}{5}(\sqrt{(1)})^5 - \frac{1}{4}(1)^4 + C\right] - \left[\frac{2}{5}(\sqrt{(0)})^5 - \frac{1}{4}(0)^4 + C\right]\right\} = 2\pi\left\{\frac{2}{5} - \frac{1}{4}\right\}$$

$$= 2\pi\left\{\frac{8}{20} - \frac{5}{20}\right\} = 2\pi\left\{\frac{3}{20}\right\} = \frac{3\pi}{10}\ units^3$$

slicing (washer method)

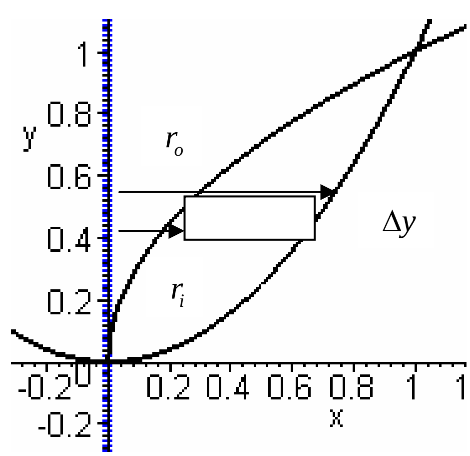
interval: $0 \le y \le 1$

$y = \sqrt{x} \quad y = x^2$

$x = y^2 \quad x = \sqrt{y}$

$r_o = (\sqrt{y}) + 0 = (\sqrt{y}) \quad r_i = (y^2) + 0 = (y^2)$

$A_o = \pi(\sqrt{y})^2 = \pi(y) \quad A_i = \pi(y^2)^2 = \pi(y^4)$

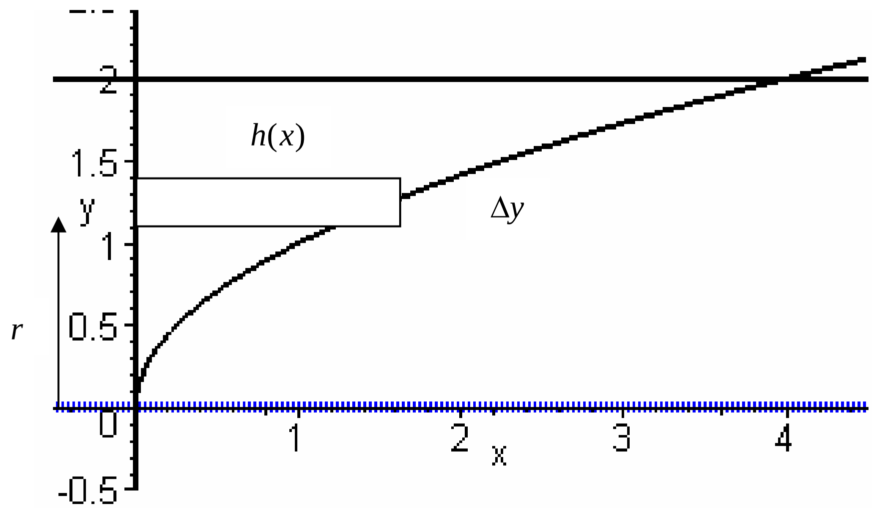


$$A = A_o - A_i = \pi(y) - \pi(y^4) = \pi(y - y^4) \qquad \Delta V = A\Delta y = \pi(y - y^4)\Delta y$$

$$V = \int_0^1 \pi(y - y^4)\,dy = \pi\left[\frac{1}{2}y^2 - \frac{1}{5}y^5 + C\right]_0^1 = \pi\left\{\left[\frac{1}{2}(1)^2 - \frac{1}{5}(1)^5 + C\right] - \left[\frac{1}{2}(0)^2 - \frac{1}{5}(0)^5 + C\right]\right\}$$

$$= \pi\left\{\frac{1}{2} - \frac{1}{5}\right\} = \pi\left\{\frac{5}{10} - \frac{2}{10}\right\} = \pi\left\{\frac{3}{10}\right\} = \frac{3\pi}{10}\ units^3$$

10) $y = \sqrt{x} \quad x = 0 \quad y = 2$; about x-axis



$y = \sqrt{x} \quad \Rightarrow \quad x = y^2$

interval: $0 \le y \le 2$

$r = (y) + 0 = (y)$

$l = 2\pi r = 2\pi(y)$

$h(y) = (y^2) - (0) = (y^2)$

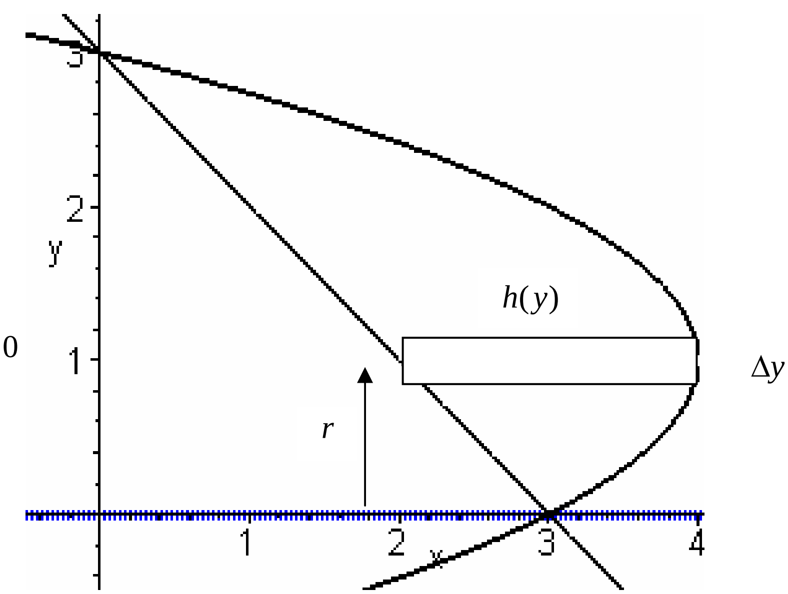
$A = lh(y) = 2\pi(y)(y^2) = 2\pi(y^3)$

$\Delta V = A\Delta y = 2\pi(y^3)\Delta y$

$$V = \int_0^2 2\pi(y^3)\,dy = 2\pi\left[\frac{1}{4}y^4 + C\right]_0^2$$

$$= 2\pi\left\{\left[\frac{1}{4}(2)^4 + C\right] - \left[\frac{1}{4}(0)^4 + C\right]\right\} = 2\pi\{4\} = 8\pi\ units^3$$

14) $x + y = 3 \quad x = 4 - (y-1)^2$; about x-axis



$x + y = 3$

$x = 3 - y$

$x = 4 - (y-1)^2$

intersect:

$3 - y = 4 - (y-1)^2$

$3 - y = 4 - (y^2 - 2y + 1)$

$3 - y = 3 - y^2 + 2y$

$y^2 - 3y = 0$

$\Rightarrow \quad y(y-3) = 0$

$y = 0 \qquad y - 3 = 0$

$y = 3$

interval: $0 \le y \le 3$

$r = (y) + 0 = (y)$

$l = 2\pi r = 2\pi(y)$

$h(y) = (4 - (y-1)^2) - (3 - y)$

$\quad = (4 - (y^2 - 2y + 1)) - (3 - y)$

$\quad = (3 - y^2 + 2y) - (3 - y) = (3y - y^2)$

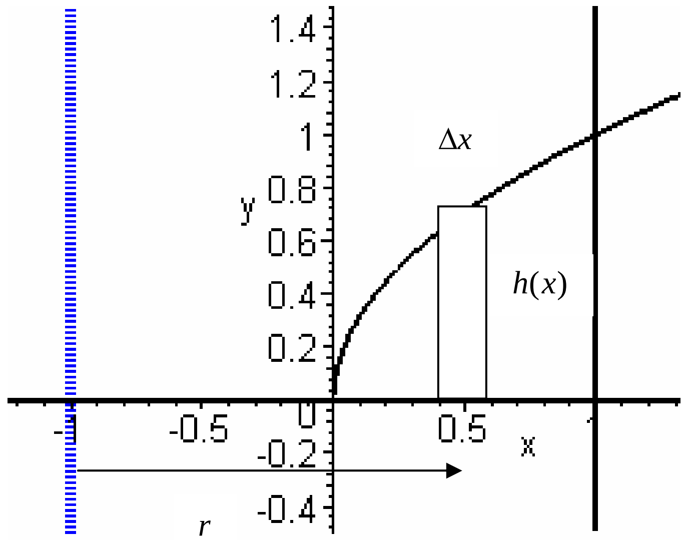
$A = lh(y) = 2\pi(y)(3y - y^2) = 2\pi(3y^2 - y^3)$

$\Delta V = A\Delta y = 2\pi(3y^2 - y^3)\Delta y$

$$V=\int_0^3 2\pi(3y^2-y^3)\,dy=2\pi\left[y^3-\frac{1}{4}y^4+C\right]_0^3=2\pi\left\{\left[(3)^3-\frac{1}{4}(3)^4+C\right]-\left[(0)^3-\frac{1}{4}(0)^4+C\right]\right\}$$

$$=2\pi\left\{(3)^3\left[1-\frac{3}{4}\right]-[0]\right\}=2\pi\left\{27\left[\frac{1}{4}\right]\right\}=\frac{27\pi}{2}\ units^3$$

16) $y=\sqrt{x}\quad y=0\quad x=1$; about $x=-1$



interval: $0\le x\le 1$

$r=(x)+1=(x+1)$

$l=2\pi r=2\pi(x+1)$

$h(x)=(\sqrt{x})-(0)=(\sqrt{x})$

$A=lh(x)=2\pi(x+1)(\sqrt{x})=2\pi(x^{\frac{3}{2}}+\sqrt{x})$

$\Delta V=A\Delta x=2\pi(x^{\frac{3}{2}}+\sqrt{x})\Delta x$

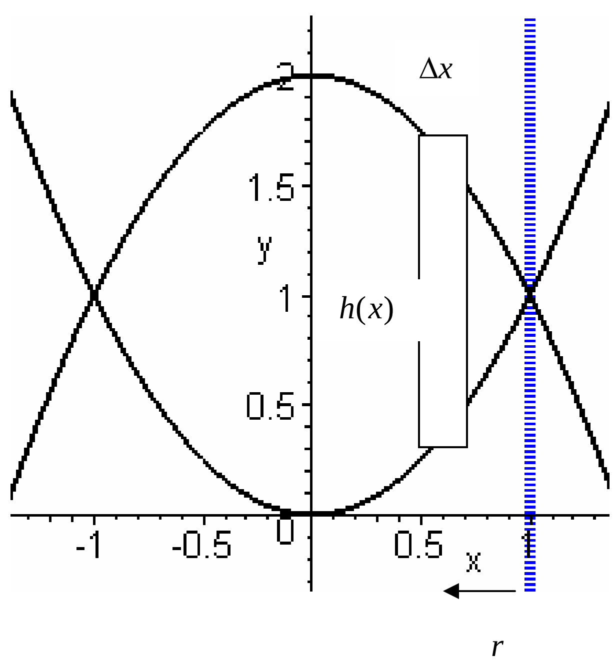
$$V=\int_0^1 2\pi(x^{\frac{3}{2}}+\sqrt{x})\,dx=2\pi\left[\frac{2}{5}x^{\frac{5}{2}}+\frac{2}{3}x^{\frac{3}{2}}+C\right]_0^1$$

$$=2\pi\left[\frac{2}{5}\left(\sqrt{x}\right)^5+\frac{2}{3}\left(\sqrt{x}\right)^3+C\right]_0^1$$

$$=2\pi\left\{\left[\frac{2}{5}\left(\sqrt{(1)}\right)^5+\frac{2}{3}\left(\sqrt{(1)}\right)^3+C\right]-\left[\frac{2}{5}\left(\sqrt{(0)}\right)^5+\frac{2}{3}\left(\sqrt{(0)}\right)^3+C\right]\right\}$$

$$=2\pi\left\{\left[\frac{2}{5}+\frac{2}{3}\right]-[0]\right\}=2\pi\left\{\frac{6}{15}+\frac{10}{15}\right\}=2\pi\left\{\frac{16}{15}\right\}=\frac{32\pi}{15}\ units^3$$

18) $y=x^2\quad y=2-x^2$; about $x=1$



intersect:

$x^2=2-x^2$ $\qquad 2(x+1)(x-1)=0$

$2x^2-2=0\quad\Rightarrow\quad x+1=0\quad x-1=0$

$2(x^2-1)=0\qquad x=-1\qquad x=1$

interval: $-1\le x\le 1$

$r=1-(x)=(1-x)$

$l=2\pi r=2\pi(1-x)$

$h(x)=(2-x^2)-(x^2)=(2-2x^2)$

$A=lh(x)=2\pi(1-x)(2-2x^2)$

$=2\pi(2-2x-2x^2+2x^3)$

$\Delta V=A\Delta x=2\pi(2-2x-2x^2+2x^3)\Delta x$

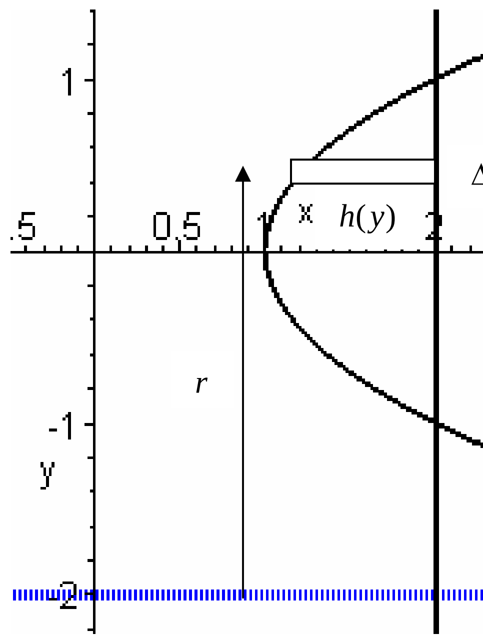
$$V=\int_{-1}^{1} 2\pi(2-2x-2x^2+2x^3)\,dx=2\pi\left[2x-x^2-\frac{2}{3}x^3+\frac{1}{2}x^4+C\right]_{-1}^{1}$$

$$=2\pi\left\{\left[2(1)-(1)^2-\frac{2}{3}(1)^3+\frac{1}{2}(1)^4+C\right]-\left[2(-1)-(-1)^2-\frac{2}{3}(-1)^3+\frac{1}{2}(-1)^4+C\right]\right\}$$

$$=2\pi\left\{\left[2-1-\frac{2}{3}+\frac{1}{2}\right]-\left[-2-1+\frac{2}{3}+\frac{1}{2}\right]\right\}=2\pi\left\{\left[\frac{3}{2}-\frac{2}{3}\right]-\left[\frac{-5}{2}+\frac{2}{3}\right]\right\}=2\pi\left\{\frac{3}{2}-\frac{2}{3}+\frac{5}{2}-\frac{2}{3}\right\}$$

$$=2\pi\left\{4-\frac{4}{3}\right\}=2\pi\left\{\frac{12}{3}-\frac{4}{3}\right\}=2\pi\left\{\frac{8}{3}\right\}=\frac{16\pi}{3}\ units^3$$

20) $x=y^2+1 \quad x=2$; about $y=-2$



intersect: $2=y^2+1$, $0=y^2-1$; $0=(y+1)(y-1)$; $y+1=0 \quad y-1=0$; $y=-1 \quad y=1$

interval: $-1\le y\le 1$

$r=(y)+2=(y+2)$

$l=2\pi r=2\pi(y+2)$

$h(y)=(2)-(y^2+1)=(1-y^2)$

$A=lh(y)=2\pi(y+2)(1-y^2)=2\pi(2+y-2y^2-y^3)$

$\Delta V=A\Delta y=2\pi(2+y-2y^2-y^3)\Delta y$

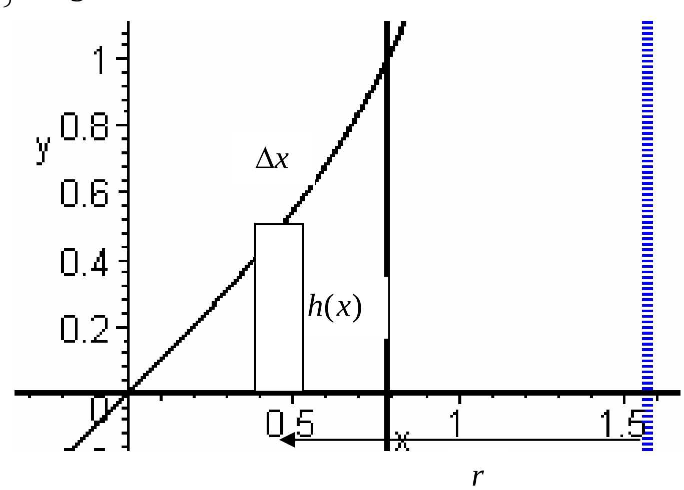
$$V=\int_{-1}^{1} 2\pi(2+y-2y^2-y^3)\,dy=2\pi\left[2y+\frac{1}{2}y^2-\frac{2}{3}y^3-\frac{1}{4}y^4+C\right]_{-1}^{1}$$

$$=2\pi\left\{\left[2(1)+\frac{1}{2}(1)^2-\frac{2}{3}(1)^3-\frac{1}{4}(1)^4+C\right]-\left[2(-1)+\frac{1}{2}(-1)^2-\frac{2}{3}(-1)^3-\frac{1}{4}(-1)^4+C\right]\right\}$$

$$=2\pi\left\{\left[2+\frac{1}{2}-\frac{2}{3}-\frac{1}{4}\right]-\left[-2+\frac{1}{2}+\frac{2}{3}-\frac{1}{4}\right]\right\}=2\pi\left\{\left[\frac{9}{4}-\frac{2}{3}\right]-\left[\frac{2}{3}-\frac{7}{4}\right]\right\}$$

$$=2\pi\left\{\frac{16}{4}-\frac{4}{3}\right\}=2\pi\left\{4-\frac{4}{3}\right\}=2\pi\left\{\frac{13}{3}-\frac{4}{3}\right\}=2\pi\left\{\frac{8}{3}\right\}=\frac{16\pi}{3}\ units^3$$

22-a) $y=\tan x \quad y=0 \quad x=\frac{\pi}{4}$; about $x=\frac{\pi}{2}$



interval: $0\le x\le\frac{\pi}{4}$

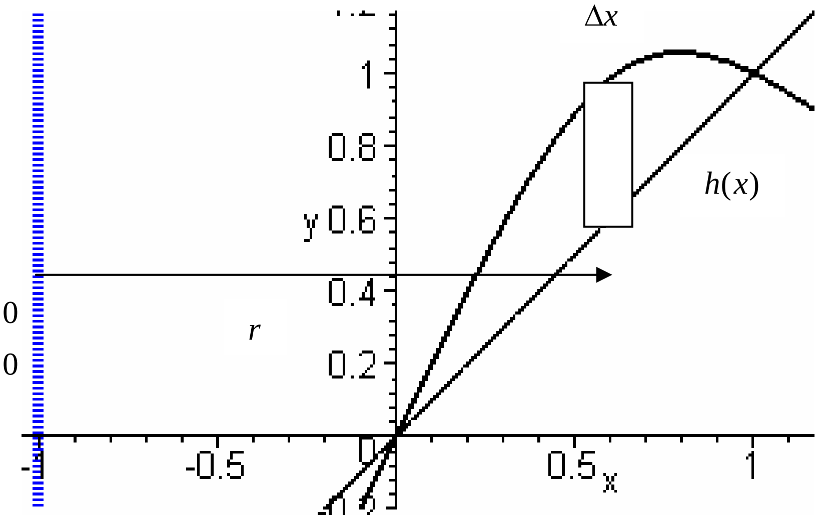
$r=\frac{\pi}{2}-(x)=\left(\frac{\pi}{2}-x\right) \qquad l=2\pi r=2\pi\left(\frac{\pi}{2}-x\right)$

$h(x)=(\tan x)-(0)=(\tan x)$

$A = lh(x) = 2\pi\left(\frac{\pi}{2} - x\right)(\tan x) = 2\pi\left(\frac{\pi}{2}\tan x - x\tan x\right)$ $\Delta V = A\Delta x = 2\pi\left(\frac{\pi}{2}\tan x - x\tan x\right)\Delta x$

$V = \int_0^{\frac{\pi}{4}} 2\pi\left(\frac{\pi}{2}\tan x - x\tan x\right) dx$

24-a) $y = x$ $y = \frac{2x}{1+x^3}$; about $x = -1$



intersection:

$x = \frac{2x}{1+x^3}$ $x(x^3 - 1) = 0$

$x + x^4 = 2x$ $\Rightarrow$ $x(x-1)(x^2+x+1) = 0$

$x^4 - x = 0$ $x = 0$ $x - 1 = 0$

$x = 1$

interval: $0 \le x \le 1$

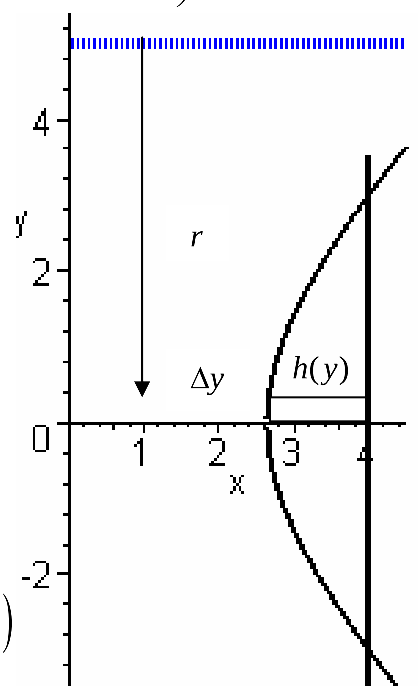
$r = (x) + 1 = (x+1)$ $l = 2\pi r = 2\pi(x+1)$

$h(x) = \left(\frac{2x}{1+x^3}\right) - (x) = \left(\frac{2x}{1+x^3} - x\right)$

$A = lh(x) = 2\pi(x+1)\left(\frac{2x}{1+x^3} - x\right) = 2\pi\left(\frac{2x(x+1)}{1+x^3} - x(x+1)\right) = 2\pi\left(\frac{2x^2}{1+x^3} + \frac{2x}{1+x^3} - x^2 - x\right)$

$\Delta V = A\Delta x = 2\pi\left(\frac{2x^2}{1+x^3} + \frac{2x}{1+x^3} - x^2 - x\right)\Delta x$ $V = \int_0^1 2\pi\left(\frac{2x^2}{1+x^3} + \frac{2x}{1+x^3} - x^2 - x\right) dx$

26-a) $x^2 - y^2 = 7$ $x = 4$; about $y = 5$



$(4)^2 - y^2 = 7$ $0 = (y+3)(y-3)$

intersection: $16 - y^2 = 7$ $\Rightarrow$ $y + 3 = 0$ $y - 3 = 0$

$0 = y^2 - 9$ $y = -3$ $y = 3$

$x^2 - y^2 = 7$ $x^2 = y^2 + 7$ $x = \pm\sqrt{y^2+7}$ $x = \sqrt{y^2+7}$

interval: $-3 \le y \le 3$

$r = 5 - (y) = (5 - y)$

$l = 2\pi r = 2\pi(5 - y)$

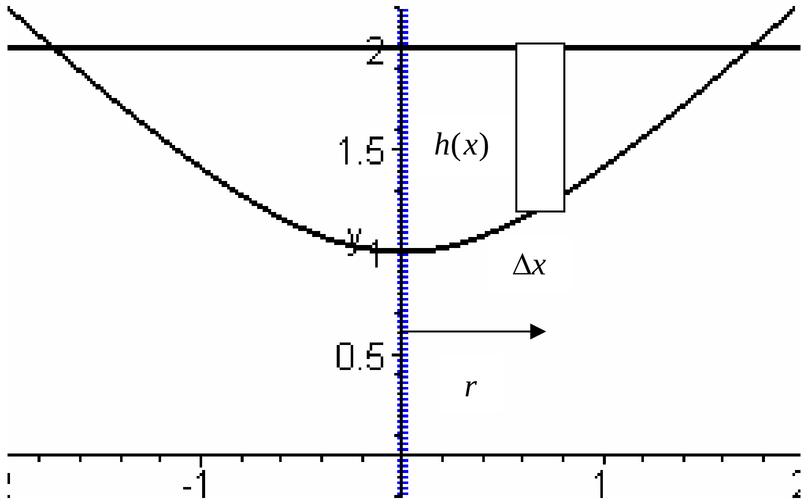
$h(y) = 4 - \left(\sqrt{y^2+7}\right) = \left(4 - \sqrt{y^2+7}\right)$

$A = lh(y) = 2\pi(5-y)\left(4 - \sqrt{y^2+7}\right) = 2\pi\left(20 - 4y - 5\sqrt{y^2+7} + y\sqrt{y^2+7}\right)$

$\Delta V = A\Delta y = 2\pi\left(20 - 4y - 5\sqrt{y^2+7} + y\sqrt{y^2+7}\right)\Delta y$

$V = \int_{-3}^{3} 2\pi\left(20 - 4y - 5\sqrt{y^2+7} + y\sqrt{y^2+7}\right) dy$

36) $y^2 - x^2 = 1 \quad y = 2$; about y-axis



intersect:
$$(2)^2 - x^2 = 1$$
$$4 - x^2 = 1$$
$$0 = x^2 - 3$$
$$0 = \left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)$$
$$x+\sqrt{3}=0 \quad x-\sqrt{3}=0$$
$$x=-\sqrt{3} \qquad x=\sqrt{3}$$

interval: $0 \le x \le \sqrt{3}$

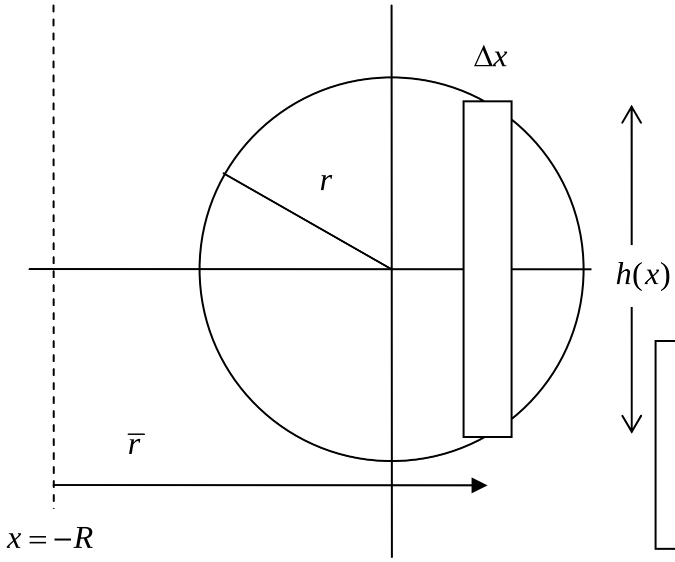
This is from 0 to $\sqrt{3}$ because the graph is symmetric and we don't want to double count the volume.

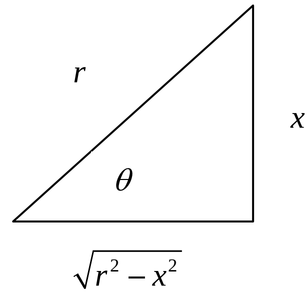
$$y^2 - x^2 = 1$$
$$y^2 = 1 + x^2 \quad \Rightarrow \quad y = \sqrt{1+x^2} \quad y = 2$$
$$y = \pm\sqrt{1+x^2}$$
$$r = (x) + 0 = (x) \qquad l = 2\pi r = 2\pi(x)$$
$$h(x) = (2) - \left(\sqrt{1+x^2}\right) = \left(2 - \sqrt{1+x^2}\right)$$
$$A = lh(x) = 2\pi(x)\left(2-\sqrt{1+x^2}\right) = 2\pi\left(2x - x\sqrt{1+x^2}\right)$$
$$\Delta V = A\Delta x = 2\pi\left(2x - x\sqrt{1+x^2}\right)\Delta x$$
$$V = \int_0^{\sqrt{3}} 2\pi\left(2x - x\sqrt{1+x^2}\right)dx = 2\pi\left[x^2 - \frac{1}{3}\left(\sqrt{1+x^2}\right)^3 + C\right]_0^{\sqrt{3}}$$
$$= 2\pi\left\{\left[\left(\sqrt{3}\right)^2 - \frac{1}{3}\left(\sqrt{1+\left(\sqrt{3}\right)^2}\right)^3 + C\right] - \left[(0)^2 - \frac{1}{3}\left(\sqrt{1+(0)^2}\right)^3 + C\right]\right\}$$
$$= 2\pi\left\{\left[3 - \frac{1}{3}(2)^3\right] - \left[0 - \frac{1}{3}(1)^3\right]\right\} = 2\pi\left\{3 - \frac{8}{3} + \frac{1}{3}\right\} = 2\pi\left\{\frac{9}{3} - \frac{7}{3}\right\} = 2\pi\left\{\frac{2}{3}\right\} = \frac{4\pi}{3}\ units^3$$

because:
$$p = 1 + x^2 \quad dp = 2x\,dx \quad \frac{1}{2}dp = x\,dx$$
$$\int x\sqrt{1+x^2}\,dx = \int \sqrt{p}\left(\frac{1}{2}dp\right) = \frac{1}{2}\left(\frac{2}{3}p^{\frac{3}{2}} + C\right) = \frac{1}{3}\left(\sqrt{1+x^2}\right)^3 + C$$

40) For this, set the origin in the middle of the tube of torus with center of torus (middle of donut hole) being the rotational axis at $x = -R$

Now the equation of the cross section of torus becomes $x^2 + y^2 = r^2$. Also when we solve this equation for y, we get
$$\begin{matrix} x^2 + y^2 = r^2 \\ y = \pm\sqrt{r^2 - x^2} \end{matrix} \quad \Rightarrow \quad \begin{matrix} y = +\sqrt{r^2 - x^2} \\ y = -\sqrt{r^2 - x^2} \end{matrix}$$
where positive is upper curve and negative lower one.





$$\frac{x}{r} = \sin\theta \qquad \frac{\sqrt{r^2-x^2}}{r} = \cos\theta \qquad \theta = \sin^{-1}\left(\frac{x}{r}\right)$$
$$x = r\sin\theta$$
$$dx = r\cos\theta\, d\theta \qquad \sqrt{r^2-x^2} = r\cos\theta$$

interval: $-r \le x \le r$ This range is from $-r$ to r because we have a circle of radius r.

$\bar{r} = (x) + R = (x+R)$ $\quad l = 2\pi\bar{r} = 2\pi(x+R)$

$$h(x) = \left(+\sqrt{r^2-x^2}\right) - \left(-\sqrt{r^2-x^2}\right) = \left(2\sqrt{r^2-x^2}\right)$$

$$A = lh(x) = 2\pi(x+R)\left(2\sqrt{r^2-x^2}\right) = 2\pi\left(2x\sqrt{r^2-x^2} + 2R\sqrt{r^2-x^2}\right)$$

$$\Delta V = A\Delta x = 2\pi\left(2x\sqrt{r^2-x^2} + 2R\sqrt{r^2-x^2}\right)\Delta x$$

$$\int\sqrt{r^2-x^2}\,dx = \int(r\cos\theta)(r\cos\theta\,d\theta) = \int r^2\cos^2\theta\,d\theta = \int r^2\left(\frac{1}{2}(1+\cos 2\theta)\right)d\theta = \frac{r^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{r^2}{2}(\theta + \sin\theta\cos\theta) + C = \frac{r^2}{2}\left(\sin^{-1}\left(\frac{x}{r}\right) + \left(\frac{x}{r}\right)\left(\frac{\sqrt{r^2-x^2}}{r}\right)\right) + C = \frac{r^2}{2}\sin^{-1}\left(\frac{x}{r}\right) + \frac{1}{2}x\sqrt{r^2-x^2} + C$$

$$V = \int_{-r}^{r} 2\pi\left(2x\sqrt{r^2-x^2} + 2R\sqrt{r^2-x^2}\right)dx = 2\pi\left\{\int_{-r}^{r} 2x\sqrt{r^2-x^2}\,dx + \int_{-r}^{r} 2R\sqrt{r^2-x^2}\,dx\right\}$$

$$= 2\pi\left\{\left[-\left(\sqrt{r^2-x^2}\right)^3 + C\right]_{-r}^{r} + 2R\left[\frac{r^2}{2}\sin^{-1}\left(\frac{x}{r}\right) + \frac{1}{2}x\sqrt{r^2-x^2} + C\right]_{-r}^{r}\right\}$$

$$= 2\pi\left\{\begin{array}{l}\left\{\left[-\left(\sqrt{r^2-(r)^2}\right)^3 + C\right] - \left[-\left(\sqrt{r^2-(-r)^2}\right)^3 + C\right]\right\} \\ +2R\left\{\left[\frac{r^2}{2}\sin^{-1}\left(\frac{(r)}{r}\right) + \frac{1}{2}(r)\sqrt{r^2-(r)^2} + C\right] - \left[\frac{r^2}{2}\sin^{-1}\left(\frac{(-r)}{r}\right) + \frac{1}{2}(-r)\sqrt{r^2-(-r)^2} + C\right]\right\}\end{array}\right\}$$

$$= 2\pi\left\{\{[0]-[0]\} + 2R\left\{\left[\frac{r^2}{2}\sin^{-1}(1) + 0\right] - \left[\frac{r^2}{2}\sin^{-1}(-1) + 0\right]\right\}\right\}$$

$$= 2\pi\left\{\{0\} + 2R\left\{\left[\frac{r^2}{2}\left(\frac{\pi}{2}\right)\right] - \left[\frac{r^2}{2}\left(\frac{-\pi}{2}\right)\right]\right\}\right\}$$

$$= 2\pi\left\{2R\left\{\left[\frac{\pi r^2}{4}\right] - \left[\frac{-\pi r^2}{4}\right]\right\}\right\} = 2\pi\left\{2R\left\{\frac{\pi r^2}{2}\right\}\right\} = 2\pi^2 r^r R \; units^3$$