1) To facilitate this section draw a sketch of the graph of the functions given.
2) If the intervals are not given, calculate the points of intersection of the functions.
3) Take a value between the interval to find out which function is Upper and Lower.
4) The height function *h* is the difference of upper function minus lower function.
5) Integrate the function *h* over the interval to find the area between curves.

Note: If there are more than 2 function or more than 2 points of intersections, the Upper and Lower functions may be different depending on regions. Remember in these cases, you may need to do step 3 multiple times.

Additional examples:

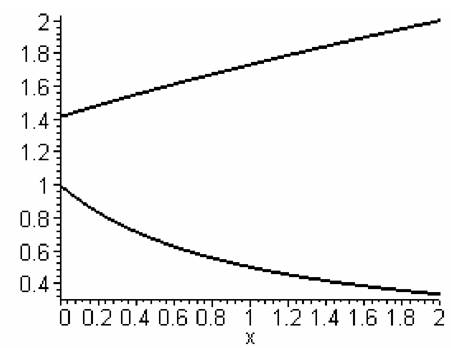
2) $y=\sqrt{x+2} \quad y=\dfrac{1}{x+1}$

interval: $0 \le x \le 2$

test at $x=1$

$y=\sqrt{x+2} \quad y=\dfrac{1}{x+1}$

Upper *Lower*



$$h(x)=(\sqrt{x+2})-\left(\frac{1}{x+1}\right)=\left(\sqrt{x+2}-\frac{1}{x+1}\right)$$

$$A=\int_0^2\left(\sqrt{x+2}-\frac{1}{x+1}\right)dx=\left[\frac{2}{3}\left(\sqrt{x+2}\right)^3-\ln|x+1|+C\right]_0^2$$

$$=\left[\frac{2}{3}\left(\sqrt{(2)+2}\right)^3-\ln|(2)+1|+C\right]-\left[\frac{2}{3}\left(\sqrt{(0)+2}\right)^3-\ln|(0)+1|+C\right]=\left[\frac{2}{3}(2)^3-\ln|3|\right]-\left[\frac{2}{3}\left(\sqrt{2}\right)^3-\ln|1|\right]$$

$$=\frac{2}{3}(8)-\ln 3-\frac{2}{3}\left(\sqrt{2}\right)^3=\left(\frac{16-4\sqrt{2}}{3}-\ln 3\right)\text{units}^2$$

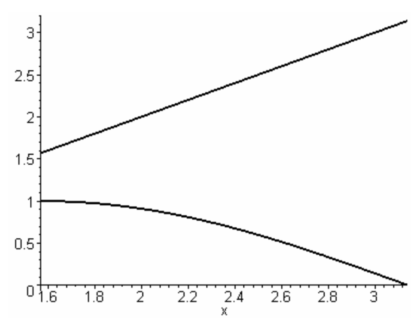
6) $y=\sin x \quad y=x \quad x=\dfrac{\pi}{2} \quad x=\pi$

interval: $\dfrac{\pi}{2} \le x \le \pi$

test at $x=\dfrac{2\pi}{3}$

$y=\sin x \quad y=x$

Lower *Upper*



$$h(x)=(x)-(\sin x)=(x-\sin x)$$

$$A=\int_{\frac{\pi}{2}}^{\pi}(x-\sin x)\,dx=\left[\frac{1}{2}x^2+\cos x+C\right]_{\frac{\pi}{2}}^{\pi}=\left[\frac{1}{2}(\pi)^2+\cos(\pi)+C\right]-\left[\frac{1}{2}\left(\frac{\pi}{2}\right)^2+\cos\left(\frac{\pi}{2}\right)+C\right]$$

$$=\left[\frac{\pi^2}{2}+(-1)\right]-\left[\frac{\pi^2}{8}+(0)\right]=\frac{4\pi^2}{8}-1\frac{\pi^2}{8}=\left(\frac{3\pi^2}{8}-1\right)\text{units}^2$$

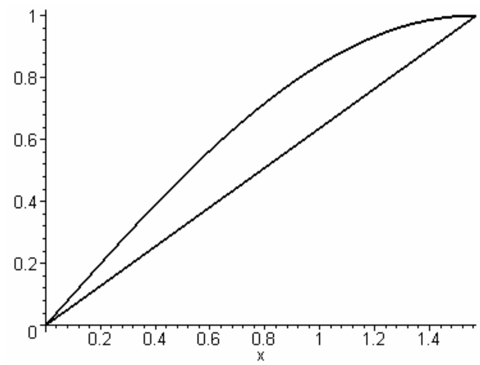
8) $y = \sin x \quad y = \dfrac{2x}{\pi} \quad x \ge 0$

interval: $0 \le x \le \dfrac{\pi}{2}$

test at $x = \dfrac{\pi}{6}$

$y = \sin x \quad y = \dfrac{2x}{\pi}$

Upper *Lower*



$$h(x) = (\sin x) - \left(\frac{2x}{\pi}\right) = \left(\sin x - \frac{2}{\pi}x\right)$$

$$A = \int_0^{\frac{\pi}{2}} \left(\sin x - \frac{2}{\pi}x\right) dx = \left[-\cos x - \frac{1}{\pi}x^2 + C\right]_0^{\frac{\pi}{2}} = \left[-\cos\left(\frac{\pi}{2}\right) - \frac{1}{\pi}\left(\frac{\pi}{2}\right)^2 + C\right] - \left[-\cos(0) - \frac{1}{\pi}(0)^2 + C\right]$$

$$= \left[(0) - \frac{\pi}{4}\right] - \left[-(1) - (0)\right] = \left(1 - \frac{\pi}{4}\right) \text{units}^2$$

10) $4x + y^2 = 12 \quad x = y$

$4(y) + y^2 = 12$

$y^2 + 4y - 12 = 0$

$(y+6)(y-2) = 0$

$y = -6 \quad y = 2$

interval: $-6 \le y \le 2$

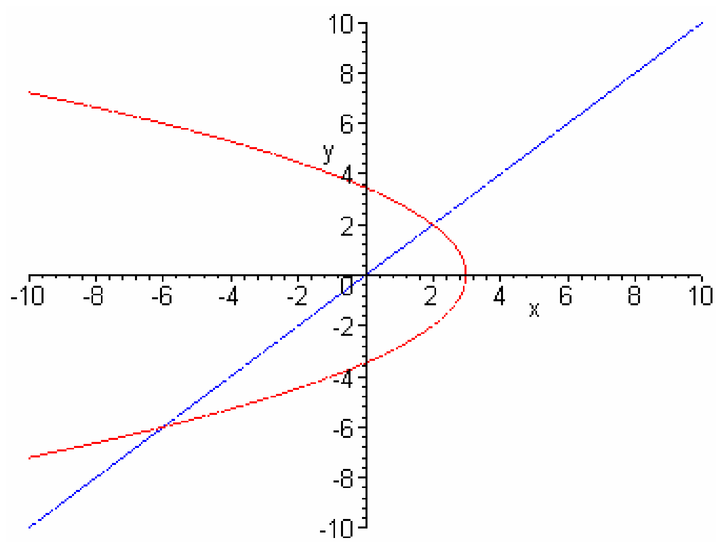
test at $y = 0$

$4x + y^2 = 12$

$4x = 12 - y^2 \qquad x = y$

$x = 3 - \dfrac{1}{4}y^2$

Upper *Lower*



$$h(y) = \left(3 - \frac{1}{4}y^2\right) - (y) = \left(3 - y - \frac{1}{4}y^2\right)$$

$$A = \int_{-6}^{2} \left(3 - y - \frac{1}{4}y^2\right) dy = \left[3y - \frac{1}{2}y^2 - \frac{1}{12}y^3 + C\right]_{-6}^{2}$$

$$= \left[3(2) - \frac{1}{2}(2)^2 - \frac{1}{12}(2)^3 + C\right] - \left[3(-6) - \frac{1}{2}(-6)^2 - \frac{1}{12}(-6)^3 + C\right]$$

$$= \left[6 - 2 - \frac{2}{3}\right] - \left[-18 - 18 + 18\right] = 22 - \frac{2}{3} = \frac{66}{3} - \frac{2}{3} = \left(\frac{64}{3}\right) \text{units}^2$$

12) $y = x^2 \quad y = 4x - x^2$

$x^2 = 4x - x^2$

$2x^2 - 4x = 0$

$2x(x-2) = 0$

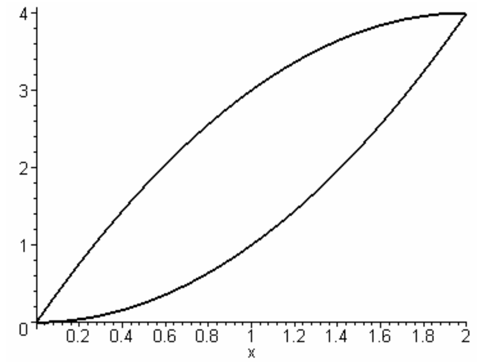
$x = 0 \quad x = 2$

interval: $0 \le x \le 2$

test at $x = 1$

$y = x^2 \quad y = 4x - x^2$

Lower *Upper*



$h(x) = (4x - x^2) - (x^2) = (4x - 2x^2)$

$$A = \int_0^2 (4x - 2x^2)\,dx = \left[2x^2 - \frac{2}{3}x^3 + C\right]_0^2 = \left[2(2)^2 - \frac{2}{3}(2)^3 + C\right] - \left[2(0)^2 - \frac{2}{3}(0)^3 + C\right]$$

$$= \left[8 - \frac{16}{3}\right] - [0] = \frac{24}{3} - \frac{16}{3} = \left(\frac{8}{3}\right)\text{units}^2$$

14) $y = \cos x \quad y = 2 - \cos x \quad 0 \le x \le 2\pi$

$\cos x = 2 - \cos x$

$2\cos x - 2 = 0$

$2(\cos x - 1) = 0$

$\cos x = 1$

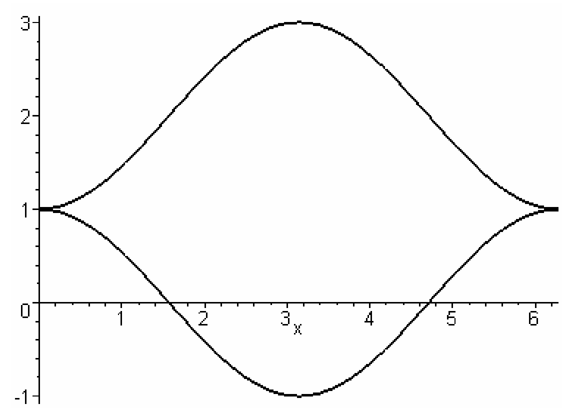
$x = 0, 2\pi$

There is no other intersection in the interval

test at $x = \pi$

$y = \cos x \quad y = 2 - \cos x$

Lower *Upper*



$h(x) = (2 - \cos x) - (\cos x) = (2 - 2\cos x)$

$$A = \int_0^{2\pi} (2 - 2\cos x)\,dx = \left[2x - 2\sin x + C\right]_0^{2\pi} = \left[2(2\pi) - 2\sin(2\pi) + C\right] - \left[2(0) - 2\sin(0) + C\right]$$

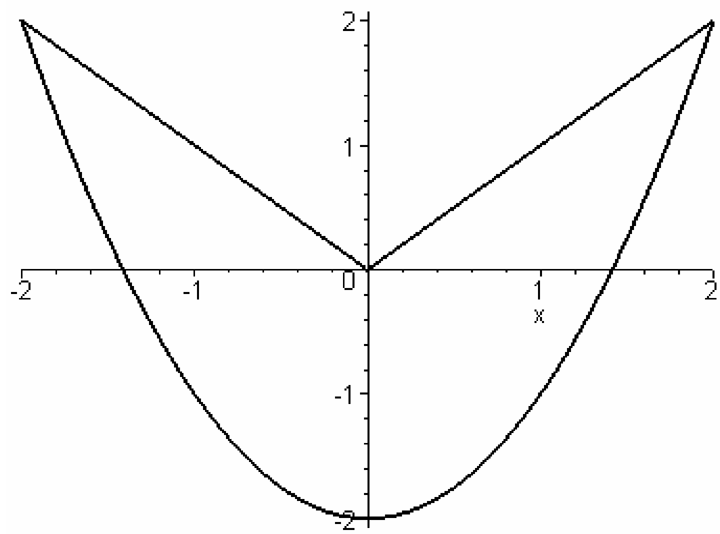
$$= \left[4\pi - 2(0)\right] - \left[0 - 2(0)\right] = (4\pi)\text{ units}^2$$

18) $y = |x| \quad y = x^2 - 2$

For $y = |x|$, there are 2 cases:

(Right side) One for $y = x$ for $x \ge 0$ and

(Left side) One for $y = -x$ for $x \le 0$



intersect R ($x \ge 0$)

$$x = x^2 - 2 \qquad\qquad 0 = (x+1)(x-2)$$
$$0 = x^2 - x - 2 \quad \Rightarrow \quad \underset{discard}{x = -1} \qquad x = 2$$

interval R: $0 \le x \le 2$

test at $x = 1$

$y = x \quad y = x^2 - 2$

Upper *Lower*

$$h_R(x) = (x) - (x^2 - 2) = (2 + x - x^2)$$

$$A_R = \int_0^2 (2 + x - x^2)\, dx = \left[2x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + C \right]_0^2$$

$$= \left[2(2) + \frac{1}{2}(2)^2 - \frac{1}{3}(2)^3 + C \right] - \left[2(0) + \frac{1}{2}(0)^2 - \frac{1}{3}(0)^3 + C \right]$$

$$= \left[4 + 2 - \frac{8}{3} \right] - [0] = 6 - \frac{8}{3} = \frac{18}{3} - \frac{8}{3} = \frac{10}{3}$$

intersect L ($x \le 0$)

$$-x = x^2 - 2 \qquad\qquad 0 = (x+2)(x-1)$$
$$0 = x^2 + x - 2 \quad \Rightarrow \quad x = -2 \qquad \underset{discard}{x = 1}$$

interval L: $-2 \le x \le 0$

test at $x = -1$

$y = -x \quad y = x^2 - 2$

Upper *Lower*

$$h_L(x) = (-x) - (x^2 - 2) = (2 - x - x^2)$$

$$A_L = \int_{-2}^0 (2 - x - x^2)\, dx = \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + C \right]_{-2}^0$$

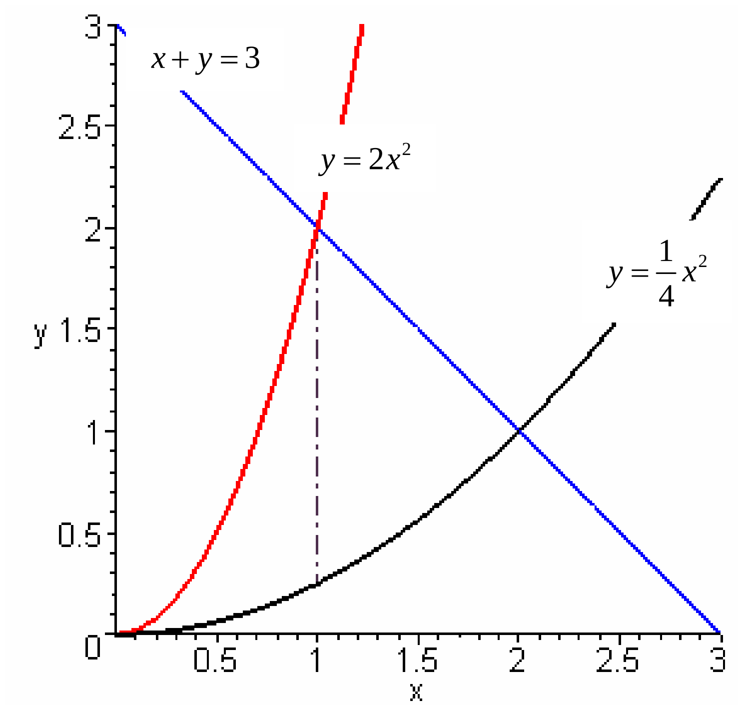
$$= \left[2(0) + \frac{1}{2}(0)^2 - \frac{1}{3}(0)^3 + C \right] - \left[2(-2) - \frac{1}{2}(-2)^2 - \frac{1}{3}(-2)^3 + C \right]$$

$$= [0] - \left[-4 - 2 + \frac{8}{3} \right] = 6 - \frac{8}{3} = \frac{18}{3} - \frac{8}{3} = \frac{10}{3}$$

$$A = A_L + A_R = \frac{10}{3} + \frac{10}{3} = \left(\frac{20}{3} \right) \text{units}^2$$

20) $y=\frac{1}{4}x^2,\quad y=2x^2,\quad x+y=3,\quad x\geq 0$

These graphs generates 2 different regions (see figure to the right)
The Upper curve changes while the Lower curve stays the same.



$y=\frac{1}{4}x^2\quad y=2x^2\quad x+y=3$
$y=3-x$

Left region: $y=\frac{1}{4}x^2\quad y=2x^2$

Interval L: $0\leq x\leq 1$

Test at $x=\frac{1}{2}$

$y=\frac{1}{4}x^2$ *Lower* $\quad y=2x^2$ *Upper*

$h_L(x)=(2x^2)-\left(\frac{1}{4}x^2\right)=\frac{7}{4}x^2$

$$A_L=\int_0^1\frac{7}{4}x^2\,dx=\frac{7}{4}\int_0^1 x^2\,dx=\frac{7}{4}\left[\frac{x^3}{3}+C\right]_0^1=\frac{7}{4}\left\{\left[\frac{(1)^3}{3}+C\right]-\left[\frac{(0)^3}{3}+C\right]\right\}=\frac{7}{4}\left\{\frac{1}{3}\right\}=\frac{7}{12}$$

Right region: $y=\frac{1}{4}x^2\quad y=3-x$

Interval R: $1\leq x\leq 2$

Test at $x=\frac{3}{2}$

$y=\frac{1}{4}x^2$ *Lower* $\quad y=3-x$ *Upper*

$h_R(x)=(3-x)-\left(\frac{1}{4}x^2\right)=\left(3-x-\frac{1}{4}x^2\right)$

$$A_R=\int_1^2\left(3-x-\frac{1}{4}x^2\right)dx=\left[3x-\frac{1}{2}x^2-\frac{1}{4}\left(\frac{1}{3}x^3\right)+C\right]_1^2$$

$$=\left[3(2)-\frac{1}{2}(2)^2-\frac{1}{4}\left(\frac{1}{3}(2)^3\right)+C\right]-\left[3(1)-\frac{1}{2}(1)^2-\frac{1}{4}\left(\frac{1}{3}(1)^3\right)+C\right]=\left[6-2-\frac{2}{3}\right]-\left[3-\frac{1}{2}-\frac{1}{12}\right]$$

$$=1-\frac{2}{3}+\frac{1}{2}+\frac{1}{12}=\frac{12}{12}-\frac{8}{12}+\frac{6}{12}+\frac{1}{12}=\frac{11}{12}$$

$$A=A_L+A_R=\frac{7}{12}+\frac{11}{12}=\frac{18}{12}=\left(\frac{3}{2}\right)\text{units}^2$$