

## Definition of Improper integrals

Type 1:

$$(a) \quad \int_a^\infty f(x) dx = \lim_{U \rightarrow \infty} \int_a^U f(x) dx \qquad (b) \quad \int_{-\infty}^b f(x) dx = \lim_{L \rightarrow -\infty} \int_L^b f(x) dx$$

$$(c) \quad \int_{-\infty}^\infty f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^\infty f(x) dx \quad \text{for any real number } c$$

Type 2:

$$(a) \quad \text{If } f \text{ is continuous on } [a, b) \text{ and is discontinuous at } b, \text{ then } \int_a^b f(x) dx = \lim_{U \rightarrow b^-} \int_a^U f(x) dx$$

$$(b) \quad \text{If } f \text{ is continuous on } (a, b] \text{ and is discontinuous at } a, \text{ then } \int_a^b f(x) dx = \lim_{L \rightarrow a^+} \int_L^b f(x) dx$$

$$(c) \quad \text{If } f \text{ has a discontinuity at } c, \text{ where } a < c < b, \text{ then } \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

Comparison Theorem: Suppose that  $f$  and  $g$  are continuous functions with  $f(x) \geq g(x) \geq 0$  for  $x \geq a$ .

$$(a) \quad \text{If } \int_a^\infty f(x) dx \text{ is convergent, then } \int_a^\infty g(x) dx \text{ is convergent.}$$

$$(b) \quad \text{If } \int_a^\infty g(x) dx \text{ is divergent, then } \int_a^\infty f(x) dx \text{ is divergent.}$$

Reference for Comparison Theorem:

$$\int_1^\infty \frac{1}{x^p} dx = \begin{cases} \text{convergent if } p > 1 \\ \text{divergent if } p \leq 1 \end{cases} \qquad \int_0^1 \frac{1}{x^p} dx = \begin{cases} \text{divergent if } p \geq 1 \\ \text{convergent if } p < 1 \end{cases}$$

Additional examples:

$$10) \quad \int_{-\infty}^0 2^r dr$$

$$\int 2^r dr = \frac{1}{\ln 2} 2^r + C$$

$$\int_{-\infty}^0 2^r dr = \lim_{L \rightarrow -\infty} \int_L^0 2^r dr = \lim_{L \rightarrow -\infty} \left[ \frac{1}{\ln 2} 2^r + C \right]_L^0 = \lim_{L \rightarrow -\infty} \left\{ \left[ \frac{1}{\ln 2} 2^{(0)} + C \right] - \left[ \frac{1}{\ln 2} 2^L + C \right] \right\}$$

$$= \left[ \frac{1}{\ln 2} (1) \right] - \left[ \frac{1}{\ln 2} (0) \right] = \frac{1}{\ln 2}$$

Convergent or Converges

14)  $\int_{-\infty}^{\infty} x^2 e^{-x^3} dx$

$$p = -x^3 \quad dp = -3x^2 dx \quad \frac{-1}{3} dp = x^2 dx$$

$$\int x^2 e^{-x^3} dx = \int e^p \left( \frac{-1}{3} dp \right) = \frac{-1}{3} e^p + C = \frac{-1}{3} e^{-x^3} + C = \frac{-1}{3e^{x^3}} + C$$

$$\int_{-\infty}^{\infty} x^2 e^{-x^3} dx = \int_{-\infty}^0 x^2 e^{-x^3} dx + \int_0^{\infty} x^2 e^{-x^3} dx$$

$$\int_0^{\infty} x^2 e^{-x^3} dx = \lim_{U \rightarrow \infty} \int_0^U x^2 e^{-x^3} dx = \lim_{U \rightarrow \infty} \left[ \frac{-1}{3e^{x^3}} + C \right]_0^U = \lim_{U \rightarrow \infty} \left\{ \left[ \frac{-1}{3e^{U^3}} + C \right] - \left[ \frac{-1}{3e^{(0)^3}} + C \right] \right\} = \frac{1}{3}$$

$$\int_{-\infty}^0 x^2 e^{-x^3} dx = \lim_{L \rightarrow -\infty} \int_L^0 x^2 e^{-x^3} dx = \lim_{L \rightarrow -\infty} \left[ \frac{-1}{3e^{x^3}} + C \right]_L^0 = \lim_{L \rightarrow -\infty} \left\{ \left[ \frac{-1}{3e^{(0)^3}} + C \right] - \left[ \frac{-1}{3e^{L^3}} + C \right] \right\} = \infty$$

Divergent or Diverges

16)  $\int_{-\infty}^{\infty} \cos \pi t dt$

$$\int \cos \pi t dt = \frac{1}{\pi} \sin \pi t + C$$

$$\int_{-\infty}^{\infty} \cos \pi t dt = \int_{-\infty}^0 \cos \pi t dt + \int_0^{\infty} \cos \pi t dt$$

$$\int_0^{\infty} \cos \pi t dt = \lim_{U \rightarrow \infty} \int_0^U \cos \pi t dt = \lim_{U \rightarrow \infty} \left[ \frac{1}{\pi} \sin \pi t + C \right]_0^U = \lim_{U \rightarrow \infty} \left\{ \left[ \frac{1}{\pi} \sin \pi U + C \right] - \left[ \frac{1}{\pi} \sin \pi(0) + C \right] \right\} = \text{DNE}$$

Limit does not exist for  $\sin(\pi U)$  as  $U \rightarrow \infty$ . Therefore, the limit is divergent, thus making

$$\int_{-\infty}^{\infty} \cos \pi t dt \text{ divergent.}$$

20)  $\int_1^{\infty} \frac{\ln x}{x^3} dx$

$$\int \frac{\ln x}{x^3} dx = (\ln x) \left( \frac{-1}{2x^2} \right) - \int \left( \frac{-1}{2x^2} \right) \left( \frac{1}{x} dx \right) \leftarrow \begin{matrix} u_1 = \ln x & dv_1 = \frac{1}{x^3} dx \\ du_1 = \frac{1}{x} dx & v_1 = \frac{-1}{2x^2} \end{matrix}$$

$$= \frac{-\ln x}{2x^2} + \int \frac{1}{2x^3} dx = \frac{-\ln x}{2x^2} - \frac{1}{4x^2} + C$$

$$\int_1^{\infty} \frac{\ln x}{x^3} dx = \lim_{U \rightarrow \infty} \int_1^U \frac{\ln x}{x^3} dx = \lim_{U \rightarrow \infty} \left[ \frac{-\ln x}{2x^2} - \frac{1}{4x^2} + C \right]_1^U$$

$$= \lim_{U \rightarrow \infty} \left\{ \left[ \frac{-\ln U}{2U^2} - \frac{1}{4U^2} + C \right] - \left[ \frac{-\ln(1)}{2(1)^2} - \frac{1}{4(1)^2} + C \right] \right\} = \frac{1}{4}$$

Converges

because:  $\lim_{U \rightarrow \infty} \frac{-\ln U}{2U^2} = \lim_{U \rightarrow \infty} \frac{-1}{4U} = \lim_{U \rightarrow \infty} \frac{-1}{4U^2} = 0$

$$22) \int_0^{\infty} \frac{e^x}{e^{2x} + 3} dx$$

$$p = e^x \quad dp = e^x dx$$

$$\int \frac{e^x}{e^{2x} + 3} dx = \int \frac{e^x}{(e^x)^2 + 3} dx = \int \frac{p}{p^2 + (\sqrt{3})^2} dx = \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{p}{\sqrt{3}} \right) + C = \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{e^x}{\sqrt{3}} \right) + C$$

$$\begin{aligned} \int_0^{\infty} \frac{e^x}{e^{2x} + 3} dx &= \lim_{U \rightarrow \infty} \int_0^U \frac{e^x}{e^{2x} + 3} dx = \lim_{U \rightarrow \infty} \left[ \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{e^x}{\sqrt{3}} \right) + C \right]_0^U \\ &= \lim_{U \rightarrow \infty} \left\{ \left[ \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{e^U}{\sqrt{3}} \right) + C \right] - \left[ \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{e^{(0)}}{\sqrt{3}} \right) + C \right] \right\} = \left[ \frac{1}{\sqrt{3}} \left( \frac{\pi}{2} \right) \right] - \left[ \frac{1}{\sqrt{3}} \left( \frac{\pi}{6} \right) \right] \\ &= \frac{3\pi}{6\sqrt{3}} - \frac{\pi}{6\sqrt{3}} = \frac{\pi}{3\sqrt{3}} \end{aligned}$$

Converges

$$26) \int_6^8 \frac{4}{(x-6)^3} dx$$

$$\int \frac{4}{(x-6)^3} dx = 4 \left( \frac{-1}{2(x-6)^2} \right) + C = \frac{-2}{(x-6)^2} + C$$

$$\int_6^8 \frac{4}{(x-6)^3} dx = \lim_{L \rightarrow 6^+} \int_L^8 \frac{4}{(x-6)^3} dx = \lim_{L \rightarrow 6^+} \left[ \frac{-2}{(x-6)^2} + C \right]_L^8 = \lim_{L \rightarrow 6^+} \left\{ \left[ \frac{-2}{((8)-6)^2} + C \right] - \left[ \frac{-2}{(L-6)^2} + C \right] \right\} = \infty$$

Diverges

$$30) \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{1}{\sqrt{1^2-x^2}} dx = \int \frac{1}{(\cos \theta)} (\cos \theta d\theta) = \int 1 d\theta = \theta + C = \sin^{-1} \left( \frac{x}{1} \right) + C = \sin^{-1} x + C$$

$$\frac{x}{1} = \sin \theta \quad \frac{\sqrt{1^2-x^2}}{1} = \cos \theta \quad \theta = \sin^{-1} \left( \frac{x}{1} \right)$$

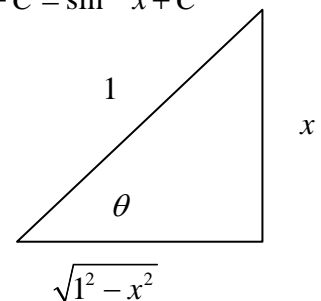
$$x = \sin \theta \quad \sqrt{1^2-x^2} = \cos \theta$$

$$dx = \cos \theta d\theta$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{U \rightarrow 1^-} \int_0^U \frac{dx}{\sqrt{1-x^2}} = \lim_{U \rightarrow 1^-} \left[ \sin^{-1} x + C \right]_0^U$$

$$= \lim_{U \rightarrow 1^-} \left\{ \left[ \sin^{-1} U + C \right] - \left[ \sin^{-1}(0) + C \right] \right\} = \frac{\pi}{2}$$

Converges



$$\begin{aligned}
 32) \quad & \int_0^1 \frac{\ln x}{\sqrt{x}} dx \\
 & \int \frac{\ln x}{\sqrt{x}} dx = (\ln x)(2\sqrt{x}) - \int (2\sqrt{x}) \left( \frac{1}{x} dx \right) \leftarrow \begin{array}{l} u_1 = \ln x \quad dv_1 = \frac{1}{\sqrt{x}} dx \\ du_1 = \frac{1}{x} dx \quad v_1 = 2\sqrt{x} \end{array} \\
 & = 2\sqrt{x} \ln x - \int \frac{2}{\sqrt{x}} dx = 2\sqrt{x} \ln x - 4\sqrt{x} + C \\
 & \int_0^1 \frac{\ln x}{\sqrt{x}} dx = \lim_{L \rightarrow 0^+} \int_L^1 \frac{\ln x}{\sqrt{x}} dx = \lim_{L \rightarrow 0^+} \left[ 2\sqrt{x} \ln x - 4\sqrt{x} + C \right]_L^1 \\
 & = \lim_{L \rightarrow 0^+} \left\{ \left[ 2\sqrt{(1)} \ln(1) - 4\sqrt{(1)} + C \right] - \left[ 2\sqrt{L} \ln L - 4\sqrt{L} + C + C \right] \right\} = -4
 \end{aligned}$$

Converges

$$\text{because } \lim_{L \rightarrow 0^+} 2\sqrt{L} \ln L = \lim_{L \rightarrow 0^+} \frac{2 \ln L}{\frac{1}{\sqrt{L}}} = \lim_{L \rightarrow 0^+} \frac{2 \ln L}{\frac{1}{\sqrt{L}}} = \lim_{L \rightarrow 0^+} \frac{2}{-1} \frac{L^{-3}}{L^2} = \lim_{L \rightarrow 0^+} \frac{2}{-1} = \lim_{L \rightarrow 0^+} \left( \frac{2}{L} \right) \left( \frac{2(\sqrt{L})^3}{-1} \right) = \lim_{L \rightarrow 0^+} -4\sqrt{L} = 0$$

$$\begin{aligned}
 46) \quad & \int_0^\pi \frac{\sin^2 x}{\sqrt{x}} dx \\
 & \text{For } 0 \leq x \leq 1, 0 < \frac{\sin^2 x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}} \\
 & \int_0^\pi \frac{1}{\sqrt{x}} dx = \lim_{L \rightarrow 0^+} \int_L^\pi \frac{1}{\sqrt{x}} dx = \lim_{L \rightarrow 0^+} \left[ 2\sqrt{x} + C \right]_L^\pi = \lim_{L \rightarrow 0^+} \left\{ \left[ 2\sqrt{(\pi)} + C \right] - \left[ 2\sqrt{L} + C \right] \right\} \\
 & = \left[ 2\sqrt{\pi} \right] - \left[ 2\sqrt{(0^+)} \right] = 2\sqrt{\pi}
 \end{aligned}$$

Converges and thus  $\int_0^\pi \frac{\sin^2 x}{\sqrt{x}} dx$  converges