Decoding the rules: $\Delta x = \dfrac{b-a}{n}$ $\quad x_i = a + i\Delta x \qquad \overline{x}_i = \dfrac{1}{2}(x_{i-1} + x_i)$

Midpoint Rule: $\int_a^b f(x)\,dx \approx M_n = \dfrac{\Delta x}{1}\left[f(\overline{x}_1) + f(\overline{x}_2) + f(\overline{x}_3) + \cdots + f(\overline{x}_{n-1}) + f(\overline{x}_n) \right]$

Trapezoidal Rule: $\int_a^b f(x)\,dx \approx T_n = \dfrac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n) \right]$

Simpson's Rule: $\int_a^b f(x)\,dx \approx S_n = \dfrac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]$

The value of n must be even for Simpson's rule to work.

Error Bounds: For $a \le x \le b$

Midpoint error: $|E_M| \le \dfrac{K(b-a)^3}{24n^2} \qquad |f''(x)| \le K$

Trapezoidal error: $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \qquad |f''(x)| \le K$

Simpson error: $|E_S| \le \dfrac{K(b-a)^5}{180n^4} \qquad |f^{(4)}(x)| \le K$

Additional examples:

8) $\int_0^2 \dfrac{1}{1+x^6}\,dx \qquad n = 8$

$\Delta x = \dfrac{(2)-(0)}{(8)} = \dfrac{2}{8} = \dfrac{1}{4} \qquad \dfrac{\Delta x}{2} = \dfrac{1}{8}$

$\frac{0}{8}$	$\frac{1}{8}$		$\frac{3}{8}$		$\frac{5}{8}$		$\frac{7}{8}$		$\frac{9}{8}$		$\frac{11}{8}$		$\frac{13}{8}$		$\frac{15}{8}$	
x_0	$\overline{x}_1$	x_1	$\overline{x}_2$	x_2	$\overline{x}_3$	x_3	$\overline{x}_4$	x_4	$\overline{x}_5$	x_5	$\overline{x}_6$	x_6	$\overline{x}_7$	x_7	$\overline{x}_8$	x_8
$0 = \frac{0}{4}$		$\frac{1}{4}$		$\frac{2}{4}$		$\frac{3}{4}$		$\frac{4}{4}$		$\frac{5}{4}$		$\frac{6}{4}$		$\frac{7}{4}$		$\frac{8}{4}$

$$M_8 = \frac{\left(\frac{1}{4}\right)}{1}\left[\frac{1}{1+\left(\frac{1}{8}\right)^6} + \frac{1}{1+\left(\frac{3}{8}\right)^6} + \frac{1}{1+\left(\frac{5}{8}\right)^6} + \frac{1}{1+\left(\frac{7}{8}\right)^6} + \frac{1}{1+\left(\frac{9}{8}\right)^6} + \frac{1}{1+\left(\frac{11}{8}\right)^6} + \frac{1}{1+\left(\frac{13}{8}\right)^6} + \frac{1}{1+\left(\frac{15}{8}\right)^6} \right]$$

$$T_8=\frac{\left(\frac{1}{4}\right)}{2}\left[\frac{1}{1+\left(\frac{1}{4}\right)^6}+2\left(\frac{1}{1+\left(\frac{2}{4}\right)^6}\right)+2\left(\frac{1}{1+\left(\frac{3}{4}\right)^6}\right)+2\left(\frac{1}{1+\left(\frac{4}{4}\right)^6}\right)\right.$$
$$\left.+2\left(\frac{1}{1+\left(\frac{5}{4}\right)^6}\right)+2\left(\frac{1}{1+\left(\frac{6}{4}\right)^6}\right)+2\left(\frac{1}{1+\left(\frac{7}{4}\right)^6}\right)+\frac{1}{1+\left(\frac{8}{4}\right)^6}\right]$$

$$S_8=\frac{\left(\frac{1}{4}\right)}{3}\left[\frac{1}{1+\left(\frac{1}{4}\right)^6}+4\left(\frac{1}{1+\left(\frac{2}{4}\right)^6}\right)+2\left(\frac{1}{1+\left(\frac{3}{4}\right)^6}\right)+4\left(\frac{1}{1+\left(\frac{4}{4}\right)^6}\right)\right.$$
$$\left.+2\left(\frac{1}{1+\left(\frac{5}{4}\right)^6}\right)+4\left(\frac{1}{1+\left(\frac{6}{4}\right)^6}\right)+2\left(\frac{1}{1+\left(\frac{7}{4}\right)^6}\right)+\frac{1}{1+\left(\frac{8}{4}\right)^6}\right]$$

12) $\int_0^1 \sin(x^3)\,dx \qquad n=10$

$$\Delta x=\frac{(1)-(0)}{(10)}=\frac{1}{10}=\frac{2}{20} \qquad \frac{\Delta x}{2}=\frac{1}{20}$$

x_0	$\bar{x}_1$	x_1	$\bar{x}_2$	x_2	$\bar{x}_3$	x_3	$\bar{x}_4$	x_4	$\bar{x}_5$	x_5
$\frac{0}{20}$, $0=\frac{0}{10}$	$\frac{1}{20}$	$\frac{1}{10}$	$\frac{3}{20}$	$\frac{2}{10}$	$\frac{5}{20}$	$\frac{3}{10}$	$\frac{7}{20}$	$\frac{4}{10}$	$\frac{9}{20}$	$\frac{5}{10}$

$\bar{x}_6$	x_6	$\bar{x}_7$	x_7	$\bar{x}_8$	x_8	$\bar{x}_9$	x_9	$\bar{x}_{10}$	x_{10}
$\frac{11}{20}$	$\frac{6}{10}$	$\frac{13}{20}$	$\frac{7}{10}$	$\frac{15}{20}$	$\frac{8}{10}$	$\frac{17}{20}$	$\frac{9}{10}$	$\frac{19}{20}$	$\frac{10}{10}$

$$M_{10}=\frac{\left(\frac{1}{10}\right)}{1}\left[\sin\left(\frac{1}{20}\right)+\sin\left(\frac{3}{20}\right)+\sin\left(\frac{5}{20}\right)+\sin\left(\frac{7}{20}\right)+\sin\left(\frac{9}{20}\right)\right.$$
$$\left.+\sin\left(\frac{11}{20}\right)+\sin\left(\frac{13}{20}\right)+\sin\left(\frac{15}{20}\right)+\sin\left(\frac{17}{20}\right)+\sin\left(\frac{19}{20}\right)\right]$$

$$T_{10} = \frac{\left(\frac{1}{10}\right)}{2}\left[\sin\left(\frac{0}{10}\right) + 2\left(\sin\left(\frac{1}{10}\right)\right) + 2\left(\sin\left(\frac{2}{10}\right)\right) + 2\left(\sin\left(\frac{3}{10}\right)\right) + 2\left(\sin\left(\frac{4}{10}\right)\right) + 2\left(\sin\left(\frac{5}{10}\right)\right) + 2\left(\sin\left(\frac{6}{10}\right)\right) + 2\left(\sin\left(\frac{7}{10}\right)\right) + 2\left(\sin\left(\frac{8}{10}\right)\right) + 2\left(\sin\left(\frac{9}{10}\right)\right) + \sin\left(\frac{10}{10}\right)\right]$$

$$S_{10} = \frac{\left(\frac{1}{10}\right)}{2}\left[\sin\left(\frac{0}{10}\right) + 4\left(\sin\left(\frac{1}{10}\right)\right) + 2\left(\sin\left(\frac{2}{10}\right)\right) + 4\left(\sin\left(\frac{3}{10}\right)\right) + 2\left(\sin\left(\frac{4}{10}\right)\right) + 4\left(\sin\left(\frac{5}{10}\right)\right) + 2\left(\sin\left(\frac{6}{10}\right)\right) + 4\left(\sin\left(\frac{7}{10}\right)\right) + 2\left(\sin\left(\frac{8}{10}\right)\right) + 4\left(\sin\left(\frac{9}{10}\right)\right) + \sin\left(\frac{10}{10}\right)\right]$$

14) $\int_4^6 \ln(x^3+2)\,dx \qquad n = 10$

$$\Delta x = \frac{(6)-(4)}{(10)} = \frac{2}{10} = \frac{1}{5} \qquad \frac{\Delta x}{2} = \frac{1}{10}$$

$\frac{40}{10}$	$\frac{41}{10}$		$\frac{43}{10}$		$\frac{45}{10}$		$\frac{47}{10}$		$\frac{49}{10}$		$\frac{51}{10}$		$\frac{53}{10}$		$\frac{55}{10}$		$\frac{57}{10}$		$\frac{59}{10}$	
x_0	$\bar{x}_1$	x_1	$\bar{x}_2$	x_2	$\bar{x}_3$	x_3	$\bar{x}_4$	x_4	$\bar{x}_5$	x_5	$\bar{x}_6$	x_6	$\bar{x}_7$	x_7	$\bar{x}_8$	x_8	$\bar{x}_9$	x_9	$\bar{x}_{10}$	x_{10}
$4 = \frac{20}{5}$		$\frac{21}{5}$		$\frac{22}{5}$		$\frac{23}{5}$		$\frac{24}{5}$		$\frac{25}{5}$		$\frac{26}{5}$		$\frac{27}{5}$		$\frac{28}{5}$		$\frac{29}{5}$		$\frac{30}{5}$

$$M_{10} = \frac{\left(\frac{1}{5}\right)}{1}\left[\ln\left(\left(\frac{41}{10}\right)^3 + 2\right) + \ln\left(\left(\frac{43}{10}\right)^3 + 2\right) + \ln\left(\left(\frac{45}{10}\right)^3 + 2\right) + \ln\left(\left(\frac{47}{10}\right)^3 + 2\right) + \ln\left(\left(\frac{49}{10}\right)^3 + 2\right) + \ln\left(\left(\frac{51}{10}\right)^3 + 2\right) + \ln\left(\left(\frac{53}{10}\right)^3 + 2\right) + \ln\left(\left(\frac{55}{10}\right)^3 + 2\right) + \ln\left(\left(\frac{57}{10}\right)^3 + 2\right) + \ln\left(\left(\frac{59}{10}\right)^3 + 2\right)\right]$$

$$T_{10}=\frac{\left(\frac{1}{5}\right)}{2}\left[\begin{aligned}&\ln\left(\left(\frac{20}{5}\right)^3+2\right)+2\left(\ln\left(\left(\frac{21}{5}\right)^3+2\right)\right)+2\left(\ln\left(\left(\frac{22}{5}\right)^3+2\right)\right)+2\left(\ln\left(\left(\frac{23}{5}\right)^3+2\right)\right)\\&+2\left(\ln\left(\left(\frac{24}{5}\right)^3+2\right)\right)+2\left(\ln\left(\left(\frac{25}{5}\right)^3+2\right)\right)+2\left(\ln\left(\left(\frac{26}{5}\right)^3+2\right)\right)+2\left(\ln\left(\left(\frac{27}{5}\right)^3+2\right)\right)\\&+2\left(\ln\left(\left(\frac{28}{5}\right)^3+2\right)\right)+2\left(\ln\left(\left(\frac{29}{5}\right)^3+2\right)\right)+\ln\left(\left(\frac{30}{5}\right)^3+2\right)\end{aligned}\right]$$

$$S_{10}=\frac{\left(\frac{1}{5}\right)}{2}\left[\begin{aligned}&\ln\left(\left(\frac{20}{5}\right)^3+2\right)+4\left(\ln\left(\left(\frac{21}{5}\right)^3+2\right)\right)+2\left(\ln\left(\left(\frac{22}{5}\right)^3+2\right)\right)+4\left(\ln\left(\left(\frac{23}{5}\right)^3+2\right)\right)\\&+2\left(\ln\left(\left(\frac{24}{5}\right)^3+2\right)\right)+4\left(\ln\left(\left(\frac{25}{5}\right)^3+2\right)\right)+2\left(\ln\left(\left(\frac{26}{5}\right)^3+2\right)\right)+4\left(\ln\left(\left(\frac{27}{5}\right)^3+2\right)\right)\\&+2\left(\ln\left(\left(\frac{28}{5}\right)^3+2\right)\right)+4\left(\ln\left(\left(\frac{29}{5}\right)^3+2\right)\right)+\ln\left(\left(\frac{30}{5}\right)^3+2\right)\end{aligned}\right]$$

16) $\int_0^1 \sqrt{z}e^{-z}\,dx=\int_0^1 \frac{\sqrt{z}}{e^z}\,dx \qquad n=10$

$$\Delta x=\frac{(1)-(0)}{(10)}=\frac{1}{10}=\frac{2}{20} \qquad \frac{\Delta x}{2}=\frac{1}{20}$$

$\frac{0}{20}$	$\frac{1}{20}$		$\frac{3}{20}$		$\frac{5}{20}$		$\frac{7}{20}$		$\frac{9}{20}$	
z_0	$\bar{z}_1$	z_1	$\bar{z}_2$	z_2	$\bar{z}_3$	z_3	$\bar{z}_4$	z_4	$\bar{z}_5$	z_5
$0=\frac{0}{10}$		$\frac{1}{10}$		$\frac{2}{10}$		$\frac{3}{10}$		$\frac{4}{10}$		$\frac{5}{10}$

$\frac{11}{20}$		$\frac{13}{20}$		$\frac{15}{20}$		$\frac{17}{20}$		$\frac{19}{20}$	
$\bar{z}_6$	z_6	$\bar{z}_7$	z_7	$\bar{z}_8$	z_8	$\bar{z}_9$	z_9	$\bar{z}_{10}$	z_{10}
	$\frac{6}{10}$		$\frac{7}{10}$		$\frac{8}{10}$		$\frac{9}{10}$		$\frac{10}{10}$

$$M_{10} = \frac{\left(\frac{1}{10}\right)}{1}\left[\frac{\sqrt{\left(\frac{1}{20}\right)}}{e^{\left(\frac{1}{20}\right)}} + \frac{\sqrt{\left(\frac{3}{20}\right)}}{e^{\left(\frac{3}{20}\right)}} + \frac{\sqrt{\left(\frac{5}{20}\right)}}{e^{\left(\frac{5}{20}\right)}} + \frac{\sqrt{\left(\frac{7}{20}\right)}}{e^{\left(\frac{7}{20}\right)}} + \frac{\sqrt{\left(\frac{9}{20}\right)}}{e^{\left(\frac{9}{20}\right)}} + \frac{\sqrt{\left(\frac{11}{20}\right)}}{e^{\left(\frac{11}{20}\right)}} + \frac{\sqrt{\left(\frac{13}{20}\right)}}{e^{\left(\frac{13}{20}\right)}} + \frac{\sqrt{\left(\frac{15}{20}\right)}}{e^{\left(\frac{15}{20}\right)}} + \frac{\sqrt{\left(\frac{17}{20}\right)}}{e^{\left(\frac{17}{20}\right)}} + \frac{\sqrt{\left(\frac{19}{20}\right)}}{e^{\left(\frac{19}{20}\right)}}\right]$$

$$T_{10} = \frac{\left(\frac{1}{10}\right)}{2}\left[\frac{\sqrt{\left(\frac{0}{10}\right)}}{e^{\left(\frac{0}{10}\right)}} + 2\left(\frac{\sqrt{\left(\frac{1}{10}\right)}}{e^{\left(\frac{1}{10}\right)}}\right) + 2\left(\frac{\sqrt{\left(\frac{2}{10}\right)}}{e^{\left(\frac{2}{10}\right)}}\right) + 2\left(\frac{\sqrt{\left(\frac{3}{10}\right)}}{e^{\left(\frac{3}{10}\right)}}\right) + 2\left(\frac{\sqrt{\left(\frac{4}{10}\right)}}{e^{\left(\frac{4}{10}\right)}}\right) + 2\left(\frac{\sqrt{\left(\frac{5}{10}\right)}}{e^{\left(\frac{5}{10}\right)}}\right) + 2\left(\frac{\sqrt{\left(\frac{6}{10}\right)}}{e^{\left(\frac{6}{10}\right)}}\right) + 2\left(\frac{\sqrt{\left(\frac{7}{10}\right)}}{e^{\left(\frac{7}{10}\right)}}\right) + 2\left(\frac{\sqrt{\left(\frac{8}{10}\right)}}{e^{\left(\frac{8}{10}\right)}}\right) + 2\left(\frac{\sqrt{\left(\frac{9}{10}\right)}}{e^{\left(\frac{9}{10}\right)}}\right) + \frac{\sqrt{\left(\frac{10}{10}\right)}}{e^{\left(\frac{10}{10}\right)}}\right]$$

$$S_{10} = \frac{\left(\frac{1}{10}\right)}{2}\left[\frac{\sqrt{\left(\frac{0}{10}\right)}}{e^{\left(\frac{0}{10}\right)}} + 4\left(\frac{\sqrt{\left(\frac{1}{10}\right)}}{e^{\left(\frac{1}{10}\right)}}\right) + 2\left(\frac{\sqrt{\left(\frac{2}{10}\right)}}{e^{\left(\frac{2}{10}\right)}}\right) + 4\left(\frac{\sqrt{\left(\frac{3}{10}\right)}}{e^{\left(\frac{3}{10}\right)}}\right) + 2\left(\frac{\sqrt{\left(\frac{4}{10}\right)}}{e^{\left(\frac{4}{10}\right)}}\right) + 4\left(\frac{\sqrt{\left(\frac{5}{10}\right)}}{e^{\left(\frac{5}{10}\right)}}\right) + 2\left(\frac{\sqrt{\left(\frac{6}{10}\right)}}{e^{\left(\frac{6}{10}\right)}}\right) + 4\left(\frac{\sqrt{\left(\frac{7}{10}\right)}}{e^{\left(\frac{7}{10}\right)}}\right) + 2\left(\frac{\sqrt{\left(\frac{8}{10}\right)}}{e^{\left(\frac{8}{10}\right)}}\right) + 4\left(\frac{\sqrt{\left(\frac{9}{10}\right)}}{e^{\left(\frac{9}{10}\right)}}\right) + \frac{\sqrt{\left(\frac{10}{10}\right)}}{e^{\left(\frac{10}{10}\right)}}\right]$$

20) $\int_0^1 e^{x^2}\,dx \qquad |E_S| \le 0.00001$

$$y = e^{x^2} \qquad f^{(4)}(x) = \frac{d^4y}{dx^4} = e^{x^2}(12 + 48x^2 + 16x^4) \qquad |E_S| \le \frac{K(b-a)^5}{180n^4} \qquad \left|f^{(4)}(x)\right| \le K$$

$$\left|f^{(4)}(x)\right| \le K = \left|e^{(1)^2}(12 + 48(1)^2 + 16(1)^4)\right| = 76e$$

$$0.00001 \le \frac{(76e)((1)-(0))^5}{180n^4} \quad\Rightarrow\quad \frac{1}{100000} = \frac{(76e)(1)^5}{180n^4},\quad n^4 = \frac{(76e)(100000)}{180},\quad n^4 = \frac{380000e}{9} \quad\Rightarrow\quad n = \sqrt[4]{\frac{380000e}{9}},\quad n = 18.40597773,\quad n \approx 20$$