# Trigonometric Integration

Strategy for evaluating $\int \sin^m x \cos^n x\, dx$

a*) If the power of cosine is odd ($n = 2k+1$), then use $\cos^2 x = 1 - \sin^2 x$:
$\int \sin^m x \cos^{2k+1} x\, dx = \int \sin^m x (\cos^2 x)^k \cos x\, dx = \int \sin^m x (1 - \sin^2 x)^k (\cos x\, dx)$ then let $u = \sin x$

b*) If the power of sine is odd ($m = 2k+1$), then use $\sin^2 x = 1 - \cos^2 x$:
$\int \sin^{2k+1} x \cos^n x\, dx = \int (\sin^2 x)^k \sin x \cos^n x\, dx = \int (1 - \cos^2 x)^k \cos^n x (\sin x\, dx)$ then let $u = \cos x$

c) If both powers of sine and cosine are even, then use half angle identities:
$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$.
Sometimes this identity is helpful: $\sin x \cos x = \frac{1}{2} \sin 2x$

* The main trick for these cases you must set aside proper trigonometric function aside from original problem so that you will have your substitution correctly. In other words, you are setting up your $du$ part of the $u$-substitution technique.

Recall the Pythagorean identity in trigonometric form :

$$\cos^2\theta = 1 - \sin^2\theta \quad \sin^2\theta = 1 - \cos^2\theta$$
$$\sec^2\theta = 1 + \tan^2\theta \quad \tan^2\theta = \sec^2\theta - 1$$
$$\csc^2\theta = 1 + \cot^2\theta \quad \cot^2\theta = \csc^2\theta - 1$$

Also, remember that you must have even powers so that you can use the conversions given above.

Strategy for evaluating $\int \tan^m x \sec^n x\, dx$

a*) If the power of secant is even ($n = 2k$), use $\sec^2 x = 1 + \tan^2 x$:
$\int \tan^m x \sec^{2k} x\, dx = \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x\, dx = \int \tan^m x (1 + \tan^2 x)^{k-1} (\sec^2 x\, dx)$ then let $u = \tan x$

b*) If the power of tangent is odd ($m = 2k+1$), use $\tan^2 x = \sec^2 x - 1$:
$\int \tan^{2k+1} x \sec^n x\, dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x\, dx = \int (\sec^2 x - 1)^k \sec^{n-1} x (\sec x \tan x\, dx)$ then let $u = \sec x$

Recall $\int \tan x\, dx = \ln|\sec x| + C \quad \int \sec x\, dx = \ln|\sec x + \tan x| + C$

To evaluate the following:

a) $\int \sin mx \cos nx\, dx$ use $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$

b) $\int \sin mx \sin nx\, dx$ use $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$

c) $\int \cos mx \cos nx\, dx$ use $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

Additional examples:

4) $\int_0^{\frac{\pi}{2}} \sin^5 x\, dx$

$$\int \sin^5 x\, dx = \int (\sin^2 x)^2 (\sin x\, dx) = \int (1-\cos^2 x)^2 (\sin x\, dx)$$

$$= \int (1-2\cos^2 x + \cos^4 x)(\sin x\, dx)$$

$$= \int (1-2p^2+p^4)(-1\, dx)$$

$$= -\left( p - \frac{2}{3}p^3 + \frac{1}{5}p^5 \right) + C$$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$$

$p = \cos x$

$dp = -\sin x\, dx$

$-1\, dp = \sin x\, dx$

$$\int_0^{\frac{\pi}{2}} \sin^5 x\, dx = \left[ -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C \right]_0^{\frac{\pi}{2}}$$

$$= \left[ -\cos\left(\frac{\pi}{2}\right) + \frac{2}{3}\cos^3\left(\frac{\pi}{2}\right) - \frac{1}{5}\cos^5\left(\frac{\pi}{2}\right) + C \right] - \left[ -\cos(0) + \frac{2}{3}\cos^3(0) - \frac{1}{5}\cos^5(0) + C \right]$$

$$= \left[ -(0) + \frac{2}{3}(0)^3 - \frac{1}{5}(0)^5 \right] - \left[ -(1) + \frac{2}{3}(1)^3 - \frac{1}{5}(1)^5 \right]$$

$$= 1 - \frac{2}{3} + \frac{1}{5} = \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$$

6) $\int_0^{2\pi} \sin^2\left(\frac{1}{3}\theta\right) d\theta$

$$\int \sin^2\left(\frac{1}{3}\theta\right) d\theta = \int \frac{1}{2}\left(1-\cos 2\left(\frac{1}{3}\theta\right)\right) d\theta = \int \frac{1}{2}\left(1-\cos\left(\frac{2}{3}\theta\right)\right) d\theta = \frac{1}{2}\left(\theta - \frac{3}{2}\sin\left(\frac{2}{3}\theta\right)\right) + C$$

$$= \frac{1}{2}\theta - \frac{3}{4}\sin\left(\frac{2}{3}\theta\right) + C$$

$$\int_0^{2\pi} \sin^2\left(\frac{1}{3}\theta\right) d\theta = \left[ \frac{1}{2}\theta - \frac{3}{4}\sin\left(\frac{2}{3}\theta\right) + C \right]_0^{2\pi}$$

$$= \left[ \frac{1}{2}(2\pi) - \frac{3}{4}\sin\left(\frac{2}{3}(2\pi)\right) + C \right] - \left[ \frac{1}{2}(0) - \frac{3}{4}\sin\left(\frac{2}{3}(0)\right) + C \right]$$

$$= \left[ \pi - \frac{3}{4}\sin\left(\frac{4\pi}{3}\right) \right] - \left[ (0) - \frac{3}{4}(0) \right] = \pi - \frac{3}{4}(0) = \pi$$

8) $\int_0^{\frac{\pi}{2}} (2-\sin\theta)^2 \, d\theta$

$$\int (2-\sin\theta)^2 \, d\theta = \int (4-4\sin\theta+\sin^2\theta)\, d\theta = \int \left(4-4\sin\theta+\frac{1}{2}(1-\cos(2\theta))\right) d\theta$$

$$= \int \left(\frac{9}{2}-4\sin\theta-\frac{1}{2}\cos(2\theta)\right) d\theta = \frac{9}{2}\theta+4\cos\theta-\frac{1}{4}\sin(2\theta)+C$$

$$\int_0^{\frac{\pi}{2}} (2-\sin\theta)^2 \, d\theta = \left[\frac{9}{2}\theta+4\cos\theta-\frac{1}{4}\sin(2\theta)+C\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{9}{2}\left(\frac{\pi}{2}\right)+4\cos\left(\frac{\pi}{2}\right)-\frac{1}{4}\sin\left(2\left(\frac{\pi}{2}\right)\right)+C\right]-\left[\frac{9}{2}(0)+4\cos(0)-\frac{1}{4}\sin(2(0))+C\right]$$

$$= \left[\frac{9\pi}{4}+4(0)-\frac{1}{4}(0)\right]-\left[0+4(1)-\frac{1}{4}(0)\right] = \frac{9\pi}{4}-4$$

10) $\int_0^{\pi} \cos^6\theta \, d\theta$

$$\int \cos^6\theta \, d\theta = \int (\cos^2\theta)^3 \, d\theta = \int \left(\frac{1}{2}(1+\cos(2\theta))\right)^3 d\theta = \left(\frac{1}{2}\right)^3 \int (1+3\cos(2\theta)+3\cos^2(2\theta)+\cos^3(2\theta))\, d\theta$$

$$= \frac{1}{8}\int \left(1+3\cos(2\theta)+3\left(\frac{1}{2}(1+\cos(4\theta))\right)+\cos^3(2\theta)\right) d\theta$$

$$= \frac{1}{8}\int \left(\frac{5}{2}+3\cos(2\theta)+\frac{3}{2}\cos(4\theta)+\cos^3(2\theta)\right) d\theta$$

$$= \frac{1}{8}\left(\frac{5}{2}\theta+\frac{3}{2}\sin(2\theta)+\frac{3}{8}\sin(4\theta)+\left\{\frac{1}{2}\sin(2\theta)-\frac{1}{6}\sin^3(2\theta)\right\}\right)+C$$

$$= \frac{1}{8}\left(\frac{5}{2}\theta+2\sin(2\theta)+\frac{3}{8}\sin(4\theta)-\frac{1}{6}\sin^3(2\theta)\right)+C$$

$p = \sin(2\theta)$

$dp = 2\cos(2\theta)\, d\theta$

$\frac{1}{2}dp = \cos(2\theta)\, d\theta$

$$\int \cos^3\theta \, d\theta = \int \cos^2\theta(\cos\theta\, d\theta) = \int (1-\sin^2\theta)(\cos\theta\, d\theta)$$

$$= \int (1-p^2)\left(\frac{1}{2}dp\right) = \frac{1}{2}\left(p-\frac{1}{3}p^3\right)+C = \frac{1}{2}\sin(2\theta)-\frac{1}{6}\sin^3(2\theta)+C$$

$$\int_0^{\pi} \cos^6\theta \, d\theta = \left[\frac{1}{8}\left(\frac{5}{2}\theta+2\sin(2\theta)+\frac{3}{8}\sin(4\theta)-\frac{1}{6}\sin^3(2\theta)\right)+C\right]_0^{\pi}$$

$$= \left[\frac{1}{8}\left(\frac{5}{2}(\pi)+2\sin(2(\pi))+\frac{3}{8}\sin(4(\pi))-\frac{1}{6}\sin^3(2(\pi))\right)+C\right]-\left[\frac{1}{8}\left(\frac{5}{2}(0)+2\sin(2(0))+\frac{3}{8}\sin(4(0))-\frac{1}{6}\sin^3(2(0))\right)+C\right]$$

$$= \left[\frac{1}{8}\left(\frac{5\pi}{2}+2(0)+\frac{3}{8}(0)-\frac{1}{6}(0)^3\right)\right]-\left[\frac{1}{8}\left((0)+(0)+(0)-(0)^3\right)\right] = \frac{5\pi}{16}$$

12) $\int x\cos^2 x\,dx$

$$\int x\cos^2 x\,dx = \int x\left(\frac{1}{2}(1+\cos(2x))\right)dx = \int\left(\frac{1}{2}x+\frac{1}{2}x\cos(2x)\right)dx = \int\frac{1}{2}x\,dx+\int\frac{1}{2}x\cos(2x)\,dx$$

$$\int\frac{1}{2}x\cos(2x)\,dx = \left(\frac{1}{2}x\right)\left(\frac{1}{2}\sin(2x)\right)-\int\left(\frac{1}{2}\sin(2x)\right)\left(\frac{1}{2}\,dx\right)$$

$$=\frac{1}{4}x\sin(2x)-\int\frac{1}{4}\sin(2x)\,dx$$

$$u_1=\frac{1}{2}x \qquad dv_1=\cos(2x)\,dx$$

$$du_1=\frac{1}{2}\,dx \qquad v_1=\frac{1}{2}\sin(2x)$$

$$=\frac{1}{4}x\sin(2x)-\frac{1}{4}\left(\frac{-1}{2}\cos(2x)\right)+C$$

$$=\frac{1}{4}x\sin(2x)+\frac{1}{8}\cos(2x)+C$$

$$\int x\cos^2 x\,dx = \int x\left(\frac{1}{2}(1+\cos(2x))\right)dx = \int\left(\frac{1}{2}x+\frac{1}{2}x\cos(2x)\right)dx = \int\frac{1}{2}x\,dx+\int\frac{1}{2}x\cos(2x)\,dx$$

$$=\frac{1}{2}\left(\frac{x^2}{2}\right)+\left\{\frac{1}{4}x\sin(2x)+\frac{1}{8}\cos(2x)+C\right\}=\frac{1}{4}x^2+\frac{1}{4}x\sin(2x)+\frac{1}{8}\cos(2x)+C$$

16) $\int\cos^2 x\sin(2x)\,dx$

$$\int\cos^2 x\sin(2x)\,dx = \int\cos^2 x(2\sin x\cos x)\,dx$$

$$=\int 2\cos^3 x\sin x\,dx$$

$$p=\cos x$$

$$dp=-\sin x\,dx$$

$$-1\,dp=\sin x\,dx$$

$$=\int 2p^3(-1\,dp)$$

$$=-2\left(\frac{p^4}{4}\right)+C=\frac{-1}{2}\cos^4 x+C$$

20) $\int(\tan^2 x+\tan^4 x)\,dx$

$$\int(\tan^2 x+\tan^4 x)\,dx=\int\tan^2 x(1+\tan^2 x)\,dx$$

$$=\int\tan^2 x\sec^2 x\,dx$$

$$=\int\tan^2 x(\sec^2 x\,dx)$$

$$p=\tan x$$

$$dp=\sec^2 x\,dx$$

$$=\int p^2\,(dp)$$

$$=\frac{1}{3}p^3+C$$

$$=\frac{1}{3}\tan^3 x+C$$

22) $\int_0^{\frac{\pi}{4}} \sec^4\theta \tan^4\theta \, d\theta$

$$\int \sec^4\theta \tan^4\theta \, d\theta = \int \sec^2\theta \tan^4\theta (\sec^2\theta \, d\theta) = \int (1+\tan^2\theta)\tan^4\theta(\sec^2\theta \, d\theta)$$
$$= \int (\tan^4\theta + \tan^6\theta)(\sec^2\theta \, d\theta)$$
$$= \int (p^4 + p^6)(dp) \quad \leftarrow \quad p = \tan\theta, \quad dp = \sec^2\theta \, d\theta$$
$$= \frac{1}{5}p^5 + \frac{1}{7}p^7 + C = \frac{1}{5}\tan^5\theta + \frac{1}{7}\tan^7\theta + C$$

$$\int_0^{\frac{\pi}{4}} \sec^4\theta \tan^4\theta \, d\theta = \left[\frac{1}{5}\tan^5\theta + \frac{1}{7}\tan^7\theta + C\right]_0^{\frac{\pi}{4}}$$
$$= \left[\frac{1}{5}\tan^5\left(\frac{\pi}{4}\right) + \frac{1}{7}\tan^7\left(\frac{\pi}{4}\right) + C\right] - \left[\frac{1}{5}\tan^5(0) + \frac{1}{7}\tan^7(0) + C\right]$$
$$= \left[\frac{1}{5}(1)^5 + \frac{1}{7}(1)^7\right] - \left[\frac{1}{5}(0) + \frac{1}{7}(0)\right] = \frac{1}{5} + \frac{1}{7} = \frac{12}{35}$$

26) $\int_0^{\frac{\pi}{4}} \tan^4 t \, dt$

$$\int \tan^4 t \, dt = \int (\tan^2 t)(\tan^2 t) \, dt$$
$$= \int (\tan^2 t)(\sec^2 t - 1) \, dt$$
$$= \int (\sec^2 t \tan^2 t - \tan^2 t) \, dt$$
$$= \int \tan^2 t(\sec^2 t \, dt) - \int \tan^2 t \, dt$$
$$= \int \tan^2 t(\sec^2 t \, dt) - \int (\sec^2 t - 1) \, dt$$
$$= \int \tan^2 t(\sec^2 t \, dt) - \int \sec^2 t \, dt + \int 1 \, dt$$
$$= \frac{1}{3}\tan^3 t - \tan t + t + C$$

$p = \tan t$, $dp = \sec^2 t \, dt$ $\rightarrow$

$$\int \tan^2 t(\sec^2 t \, dt) = \int p^2 \, (dp) = \frac{1}{3}p^3 + C = \frac{1}{3}\tan^3 t + C$$

$$\int_0^{\frac{\pi}{4}} \tan^4 t \, dt = \left[\frac{1}{3}\tan^3 t - \tan t + t + C\right]_0^{\frac{\pi}{4}}$$
$$= \left[\frac{1}{3}\tan^3\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{4}\right) + \left(\frac{\pi}{4}\right) + C\right] - \left[\frac{1}{3}\tan^3(0) - \tan(0) + (0) + C\right]$$
$$= \left[\frac{1}{3}(1)^3 - (1) + \frac{\pi}{4}\right] - \left[\frac{1}{3}(0) - (0) + (0)\right] = \frac{\pi}{4} - \frac{2}{3} = \frac{3\pi - 8}{12}$$

30) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^3 x\,dx$

$$\int \cot^3 x\,dx = \int (\cot^2 x)(\cot x)\,dx$$
$$= \int (\csc^2 x - 1)(\cot x)\,dx$$
$$= \int (\csc^2 x \cot x - \cot x)\,dx$$
$$= \int \cot x(\csc^2 x\,dx) - \int \cot x\,dx$$
$$= \frac{-1}{2}\cot^2 x - \ln|\sin x| + C$$

$p = \cot x$

$dp = -\csc^2 x\,dx$

$-1\,dp = \csc^2 x\,dx$

$$\int \cot x(\csc^2 x\,dx) = \int p(-1\,dp)$$
$$= \frac{-1}{2}p^2 + C$$
$$= \frac{-1}{2}\cot^2 x + C$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^3 x\,dx = \left[\frac{-1}{2}\cot^2 x - \ln|\sin x| + C\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$= \left[\frac{-1}{2}\cot^2\left(\frac{\pi}{2}\right) - \ln\left|\sin\left(\frac{\pi}{2}\right)\right| + C\right] - \left[\frac{-1}{2}\cot^2\left(\frac{\pi}{4}\right) - \ln\left|\sin\left(\frac{\pi}{4}\right)\right| + C\right]$$
$$= \left[\frac{-1}{2}(0)^2 - \ln|(1)|\right] - \left[\frac{-1}{2}(1)^2 - \ln\left|\frac{1}{\sqrt{2}}\right|\right] = \frac{1}{2} + \ln\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \ln 1 - \ln\sqrt{2} = \frac{1}{2} - \frac{1}{2}\ln 2$$

32) $\int \csc^4 x \cot^6 x\,dx$

$$\int \csc^4 x \cot^6 x\,dx = \int \csc^2 x \cot^6 x(\csc^2 x\,dx) = \int (\cot^2 x + 1)\cot^6 x(\csc^2 x\,dx)$$
$$= \int (\cot^8 x + \cot^6 x)(\csc^2 x\,dx)$$
$$= \int (p^8 + p^6)(-1\,dp)$$
$$= -\left(\frac{1}{9}p^9 + \frac{1}{7}p^7\right) + C = \frac{-1}{9}\cot^9 x - \frac{1}{7}\cot^7 x + C$$

$p = \cot x$

$dp = -\csc^2 x\,dx$

$-1\,dp = \csc^2 x\,dx$

36) $\int \frac{dx}{\cos x - 1}$

$$\int \frac{dx}{\cos x - 1} = \int\left(\frac{1}{\cos x - 1}\right)\left(\frac{\cos x + 1}{\cos x + 1}\right)dx = \int \frac{\cos x + 1}{\cos^2 x - 1}\,dx = \int \frac{\cos x + 1}{-(1 - \cos^2 x)}\,dx = \int \frac{\cos x + 1}{-\sin^2 x}\,dx$$
$$= -\int \frac{\cos x}{\sin^2 x}\,dx - \int \frac{1}{\sin^2 x}\,dx = -\int \frac{1}{\sin^2 x}(\cos x\,dx) - \int \csc^2 x\,dx = -\left(\frac{-1}{\sin x}\right) - (-\cot x) + C$$
$$= \csc x + \cot x + C$$

$p = \sin x$

$dp = \cos x\,dx$

$$\int \frac{1}{\sin^2 x}(\cos x\,dx) = \int \frac{1}{p^2}(dp) = \frac{-1}{p} + C = \frac{-1}{\sin x} + C$$

38) a) Using the addition formulas for Sine

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

$$+$$

$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$

$$||$$

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B$$

$$\frac{1}{2}(\sin(A+B)+\sin(A-B))=\sin A\cos B$$

b) Let $A=(3x),\quad B=(x)$

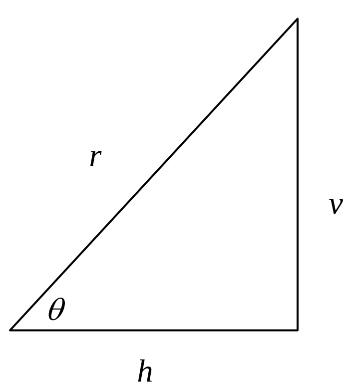
$$\int \sin 3x\cos x\,dx=\int \sin(3x)\cos(x)\,dx=\int \frac{1}{2}(\sin((3x)+(x))+\sin((3x)-(x)))\,dx$$

$$=\frac{1}{2}\int (\sin(4x)+\sin(2x))\,dx=\frac{1}{2}\left(\frac{-1}{4}\cos(4x)-\frac{1}{2}\cos(2x)\right)+C$$

$$=\frac{-1}{8}\cos(4x)-\frac{1}{4}\cos(2x)+C$$

Trigonometric Substitution (triangulation):

For this section, it will be easier to recall the basic trigonometry of a right triangle. Given triangle below:




Recall: SOH CAH TOA

$\sin\theta=\frac{opp}{hyp}=\frac{v}{r}\quad \cos\theta=\frac{adj}{hyp}=\frac{h}{r}\quad \tan\theta=\frac{opp}{adj}=\frac{v}{h}$

By Pythagorean theorem: $r^2=v^2+h^2$

If we solve for hypotenuse and each of the legs, we get:

$r=\sqrt{v^2+h^2}\quad v=\sqrt{r^2-h^2}\quad h=\sqrt{r^2-v^2}$

The trick to this section is to recognize which of the following is present in the problem:

$$\sqrt{v^2+h^2}\quad \sqrt{r^2-h^2}\quad \sqrt{r^2-v^2}$$

$$v^2+h^2\quad r^2-h^2\quad r^2-v^2$$

If this expression ($\sqrt{v^2+h^2}$ or $v^2+h^2$) is present then this part represents the hypotenuse of the triangle; therefore, each part represent the legs of the triangle.

If these expressions ($\sqrt{r^2-h^2}$ or $r^2-h^2$) or ($\sqrt{r^2-v^2}$ or $r^2-v^2$) are present then this part represents the leg of the triangle; therefore, the fist part is the hypotenuse and second the other leg of the triangle.

Note: $v^2+h^2=\left(\sqrt{v^2+h^2}\right)^2$, $r^2-h^2=\left(\sqrt{r^2-h^2}\right)^2$, and $r^2-v^2=\left(\sqrt{r^2-v^2}\right)^2$

Now use this triangle to pick out 2 trigonometric relationships that involve the pairs given below:

$\sin^2\theta+\cos^2\theta=1\quad \tan^2\theta+1=\sec^2\theta\quad 1+\cot^2\theta=\csc^2\theta$

This is the encoding step. Then use techniques of trigonometric integration to solve the problem. After solving, use the triangle we set up for encoding to decode our solution.

Additional examples:

42) $\int_0^1 x^3\sqrt{1-x^2}\,dx$

method 1:

$$\int x^3\sqrt{1-x^2}\,dx = \int x^2\sqrt{1-x^2}\,(x\,dx) = \int (1-p)\sqrt{p}\left(\frac{-1}{2}\,dp\right)$$

$$= \frac{-1}{2}\int\left(\sqrt{p} - p^{\frac{3}{2}}\right)dp$$

$$= \frac{-1}{2}\left(\left(\frac{2}{3}p^{\frac{3}{2}}\right) - \left(\frac{2}{5}p^{\frac{5}{2}}\right)\right) + C$$

$$= \frac{1}{5}\left(1-x^2\right)^{\frac{5}{2}} - \frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}} + C$$

$$= \frac{1}{5}\left(\sqrt{1-x^2}\right)^5 - \frac{1}{3}\left(\sqrt{1-x^2}\right)^3 + C$$

$p = 1-x^2$

$dp = -2x\,dx$

$\frac{-1}{2}\,dp = x\,dx$

$x^2 = (1-p)$

method 2:

$$\int x^3\sqrt{1-x^2}\,dx = \int (\sin\theta)^3(\cos\theta)(\cos\theta\,d\theta)$$

$$= \int \sin^3\theta\cos^2\theta\,d\theta$$

$$= \int \sin^2\theta\cos^2\theta(\sin\theta\,d\theta)$$

$$= \int (1-\cos^2\theta)\cos^2\theta(\sin\theta\,d\theta)$$

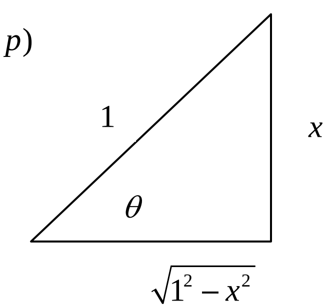
$$= \int (\cos^2\theta - \cos^4\theta)(\sin\theta\,d\theta)$$

$$= \int (p^2 - p^4)(-1\,dp) = -\left(\frac{1}{3}p^3 - \frac{1}{5}p^5\right) + C$$

$$= \frac{1}{5}\cos^5\theta - \frac{1}{3}\cos^3\theta + C$$

$$= \frac{1}{5}\left(\frac{\sqrt{1-x^2}}{1}\right)^5 - \frac{1}{3}\left(\frac{\sqrt{1-x^2}}{1}\right)^3 + C$$

$$= \frac{1}{5}\left(\sqrt{1-x^2}\right)^5 - \frac{1}{3}\left(\sqrt{1-x^2}\right)^3 + C$$




$\frac{x}{1} = \sin\theta$

$x = \sin\theta$

$dx = \cos\theta\,d\theta$

$\frac{\sqrt{1^2-x^2}}{1} = \cos\theta$

$\sqrt{1^2-x^2} = \cos\theta$

$p = \cos\theta$

$dp = -\sin\theta\,d\theta$

$-1\,dp = \sin\theta\,d\theta$

$$\int_0^1 x^3\sqrt{1-x^2}\,dx = \left[\frac{1}{5}\left(\sqrt{1-x^2}\right)^5 - \frac{1}{3}\left(\sqrt{1-x^2}\right)^3 + C\right]_0^1$$

$$= \left[\frac{1}{5}\left(\sqrt{1-(1)^2}\right)^5 - \frac{1}{3}\left(\sqrt{1-(1)^2}\right)^3 + C\right] - \left[\frac{1}{5}\left(\sqrt{1-(0)^2}\right)^5 - \frac{1}{3}\left(\sqrt{1-(0)^2}\right)^3 + C\right]$$

$$= \left[\frac{1}{5}\left(\sqrt{0}\right)^5 - \frac{1}{3}\left(\sqrt{0}\right)^3\right] - \left[\frac{1}{5}\left(\sqrt{1}\right)^5 - \frac{1}{3}\left(\sqrt{1}\right)^3\right] = [0-0] - \left[\frac{1}{5} - \frac{1}{3}\right]$$

$$= -\left[\frac{3}{15} - \frac{5}{15}\right] = \frac{2}{15}$$

44) $\int_0^2 x^3\sqrt{x^2+4}\,dx$

method 1:

$$\int x^3\sqrt{x^2+4}\,dx = \int x^2\sqrt{x^2+4}(x\,dx) = \int (p-4)\sqrt{p}\left(\frac{1}{2}\,dp\right)$$

$$= \int\left(\frac{1}{2}p^{\frac{3}{2}} - 2\sqrt{p}\right)dp$$

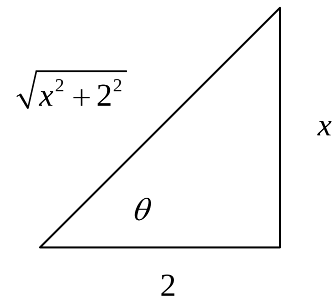
$$= \frac{1}{2}\left(\frac{2}{5}p^{\frac{5}{2}}\right) - 2\left(\frac{2}{3}p^{\frac{3}{2}}\right) + C$$

$$= \frac{1}{5}\left(x^2+4\right)^{\frac{5}{2}} - \frac{4}{3}\left(x^2+4\right)^{\frac{3}{2}} + C$$

$$= \frac{1}{5}\left(\sqrt{x^2+4}\right)^5 - \frac{4}{3}\left(\sqrt{x^2+4}\right)^3 + C$$

$$p = x^2+4$$
$$dp = 2x\,dx$$
$$\frac{1}{2}\,dp = x\,dx$$
$$x^2 = (p-4)$$

method 2:




$$\int x^3\sqrt{x^2+4}\,dx = \int (2\tan\theta)^3(2\sec\theta)(2\sec^2\theta\,d\theta)$$

$$= 2^5\int \sec^3\theta\tan^3\theta\,d\theta$$

$$= 2^5\int \sec^2\theta\tan^2\theta(\sec\theta\tan\theta\,d\theta)$$

$$= 2^5\int \sec^2\theta(\sec^2\theta - 1)(\sec\theta\tan\theta\,d\theta)$$

$$= 2^5\int (\sec^4\theta - \sec^2\theta)(\sec\theta\tan\theta\,d\theta)$$

$$= 2^5\left(\frac{1}{5}\sec^5\theta - \frac{1}{3}\sec^3\theta\right) + C$$

$$= 2^5\left(\frac{1}{5}\left(\frac{\sqrt{x^2+2^2}}{2}\right)^5 - \frac{1}{3}\left(\frac{\sqrt{x^2+2^2}}{2}\right)^3\right) + C$$

$$= \frac{1}{5}\left(\sqrt{x^2+2^2}\right)^5 - \frac{4}{3}\left(\sqrt{x^2+2^2}\right)^3 + C$$

$$\frac{x}{2} = \tan\theta$$
$$x = 2\tan\theta$$
$$dx = 2\sec^2\theta\,d\theta$$
$$\frac{\sqrt{x^2+2^2}}{2} = \sec\theta$$
$$\sqrt{x^2+2^2} = 2\sec\theta$$

$$\int_0^2 x^3\sqrt{x^2+4}\,dx = \left[\frac{1}{5}\left(\sqrt{x^2+4}\right)^5 - \frac{4}{3}\left(\sqrt{x^2+4}\right)^3 + C\right]_0^2$$

$$= \left[\frac{1}{5}\left(\sqrt{(2)^2+4}\right)^5 - \frac{4}{3}\left(\sqrt{(2)^2+4}\right)^3 + C\right] - \left[\frac{1}{5}\left(\sqrt{(0)^2+4}\right)^5 - \frac{4}{3}\left(\sqrt{(0)^2+4}\right)^3 + C\right]$$

$$= \left[\frac{1}{5}\left(2\sqrt{2}\right)^5 - \frac{4}{3}\left(2\sqrt{2}\right)^3\right] - \left[\frac{1}{5}(2)^5 - \frac{4}{3}(2)^3\right] = \left(2\sqrt{2}\right)^3\left[\frac{1}{5}\left(2\sqrt{2}\right)^2 + \frac{4}{3}\right] - (2)^3\left[\frac{1}{5}(2)^2 + \frac{4}{3}\right]$$

$$= 16\sqrt{2}\left[\frac{8}{5} - \frac{4}{3}\right] - 8\left[\frac{4}{5} - \frac{4}{3}\right] = 16\sqrt{2}\left[\frac{24-20}{15}\right] - 8\left[\frac{12-20}{15}\right]$$

$$= 16\sqrt{2}\left[\frac{4}{15}\right] - 8\left[\frac{-4}{15}\right] = \frac{64\sqrt{2}-64}{15}$$

48) $\int \frac{t^5}{\sqrt{t^2+2}}\,dt$

method 1:

$$\int \frac{t^5}{\sqrt{t^2+2}}\,dt = \int \frac{(t^2)^2}{\sqrt{t^2+2}}(t\,dt) = \int \frac{(p-2)^2}{\sqrt{p}}\left(\frac{1}{2}\,dp\right) = \int \frac{(p^2-4p+4)}{2\sqrt{p}}\,dp$$

$$= \int\left(\frac{1}{2}p^{\frac{3}{2}} - 2p^{\frac{1}{2}} + 2p^{\frac{-1}{2}}\right)dp$$

$$= \frac{1}{2}\left(\frac{2}{5}p^{\frac{5}{2}}\right) - 2\left(\frac{2}{3}p^{\frac{3}{2}}\right) + 2\left(\frac{2}{1}p^{\frac{1}{2}}\right) + C$$

$$= \frac{1}{5}\left(t^2+2\right)^{\frac{5}{2}} - \frac{4}{3}\left(t^2+2\right)^{\frac{3}{2}} + 4\left(t^2+2\right)^{\frac{1}{2}} + C$$

$$= \frac{1}{5}\left(\sqrt{t^2+2}\right)^5 - \frac{4}{3}\left(\sqrt{t^2+2}\right)^3 + 4\sqrt{t^2+2} + C$$

$$p = t^2+2$$
$$dp = 2t\,dt$$
$$\frac{1}{2}\,dp = t\,dt$$
$$t^2 = p-2$$

method 2:

$$\frac{t}{\sqrt{2}} = \tan\theta$$
$$t = \sqrt{2}\tan\theta$$
$$dt = \sqrt{2}\sec^2\theta\,d\theta$$

$$\frac{\sqrt{t^2+(\sqrt{2})^2}}{\sqrt{2}} = \sec\theta$$
$$\sqrt{t^2+(\sqrt{2})^2} = \sqrt{2}\sec\theta$$

$$\int \frac{t^5}{\sqrt{t^2+2}}\,dt = \int \frac{t^5}{\sqrt{t^2+(\sqrt{2})^2}}\,dt$$

$$= \int \frac{(\sqrt{2}\tan\theta)^5}{(\sqrt{2}\sec\theta)}(\sqrt{2}\sec\theta\,d\theta)$$

$$= (\sqrt{2})^5\int \sec\theta\tan^5\theta\,d\theta$$

$$= (\sqrt{2})^5\int (\tan^2\theta)^2(\sec\theta\tan\theta\,d\theta)$$

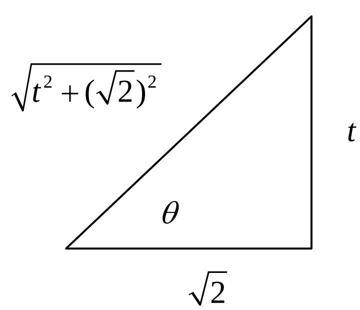
$$= (\sqrt{2})^5\int (\sec^2\theta-1)^2(\sec\theta\tan\theta\,d\theta)$$

$$= (\sqrt{2})^5\int (\sec^4\theta-2\sec^2\theta+1)(\sec\theta\tan\theta\,d\theta)$$

$$= (\sqrt{2})^5\left(\frac{1}{5}\sec^5\theta - \frac{2}{3}\sec^3\theta + \sec\theta\right) + C$$

$$= (\sqrt{2})^5\left(\frac{1}{5}\left(\frac{\sqrt{t^2+(\sqrt{2})^2}}{\sqrt{2}}\right)^5 - \frac{2}{3}\left(\frac{\sqrt{t^2+(\sqrt{2})^2}}{\sqrt{2}}\right)^3 + \left(\frac{\sqrt{t^2+(\sqrt{2})^2}}{\sqrt{2}}\right)\right) + C$$

$$= \frac{1}{5}\left(\sqrt{t^2+2}\right)^5 - \frac{4}{3}\left(\sqrt{t^2+2}\right)^3 + 4\sqrt{t^2+2} + C$$



52) $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$

method 1:

$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \int \frac{1}{\sqrt{1+x^2}}(x\,dx) = \int \frac{1}{\sqrt{p}}\left(\frac{1}{2}\,dp\right)$$

$$= \frac{1}{2}\left(\frac{2}{1}\sqrt{p}\right) + C$$

$$= \sqrt{1+x^2} + C$$

$p = 1 + x^2$

$dp = 2x\,dx$

$\frac{1}{2}\,dp = x\,dx$

method 2:

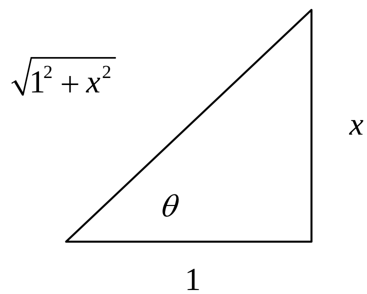
$\frac{x}{1} = \tan\theta$

$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$\frac{\sqrt{1^2+x^2}}{1} = \sec\theta$

$\sqrt{1^2+x^2} = \sec\theta$




$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \int \frac{x}{\sqrt{1^2+x^2}}\,dx = \int \frac{(\tan\theta)}{(\sec\theta)}(\sec^2\theta\,d\theta)$$

$$= \int \sec\theta\tan\theta\,d\theta = \sec\theta + C = \sqrt{1^2+x^2} + C = \sqrt{1+x^2} + C$$

58) $\displaystyle\int_0^1 \frac{dx}{(x^2+1)^2}$

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{1}{\left(\sqrt{x^2+1}\right)^4}\,dx = \int \frac{1}{(\sec\theta)^4}(\sec^2\theta\,d\theta)$$

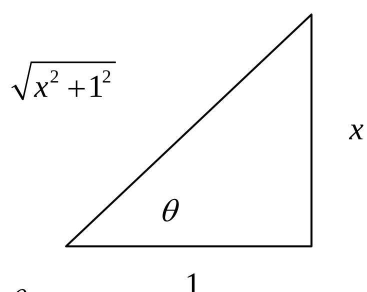
$$= \int \frac{1}{\sec^2\theta}\,d\theta = \int \cos^2\theta\,d\theta = \int \frac{1}{2}(1+\cos(2\theta))\,d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C = \frac{1}{2}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{1}{2}\left((\tan^{-1}x) + \left(\frac{x}{\sqrt{x^2+1^2}}\right)\left(\frac{1}{\sqrt{x^2+1^2}}\right)\right) + C$$

$$= \frac{1}{2}\tan^{-1}x + \frac{x}{2\left(\sqrt{x^2+1^2}\right)^2} + C$$

$$= \frac{1}{2}\tan^{-1}x + \frac{x}{2(x^2+1)} + C$$




$\frac{x}{1} = \tan\theta$

$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$\frac{\sqrt{x^2+1^2}}{1} = \sec\theta$

$\sqrt{x^2+1^2} = \sec\theta$

$\theta = \tan^{-1}x$

$$\int_0^1 \frac{dx}{(x^2+1)^2} = \left[\frac{1}{2}\tan^{-1}x + \frac{x}{2(x^2+1)} + C\right]_0^1$$

$$= \left[\frac{1}{2}\tan^{-1}(1) + \frac{(1)}{2((1)^2+1)} + C\right] - \left[\frac{1}{2}\tan^{-1}(0) + \frac{(0)}{2((0)^2+1)} + C\right] = \frac{\pi}{8} + \frac{1}{4} = \frac{\pi+2}{8}$$

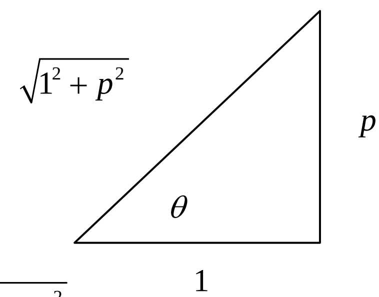
60) $\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt$

$$\int \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt = \int \frac{1}{\sqrt{1+\sin^2 t}}(\cos t\,dt)$$

$$= \int \frac{1}{\sqrt{1^2+p^2}}(dp)$$

$p = \sin t$

$dp = \cos t\,dt$



$$= \int \frac{1}{\sec\theta}(\sec^2\theta\,d\theta)$$

$$= \int \sec\theta\,d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \ln\left|\sqrt{1^2+p^2} + p\right| + C$$

$$= \ln\left|\sqrt{1^2+(\sin t)^2} + \sin t\right| + C$$

$$= \ln\left|\sqrt{1+\sin^2 t} + \sin t\right| + C$$

$\frac{p}{1} = \tan\theta$

$p = \tan\theta$

$dp = \sec^2\theta\,d\theta$

$\frac{\sqrt{1^2+p^2}}{1} = \sec\theta$

$\sqrt{1^2+p^2} = \sec\theta$

$$\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt = \left[\ln\left|\sqrt{1+\sin^2 t} + \sin t\right| + C\right]_0^{\frac{\pi}{2}}$$

$$= \left[\ln\left|\sqrt{1+\sin^2\left(\frac{\pi}{2}\right)} + \sin\left(\frac{\pi}{2}\right)\right| + C\right] - \left[\ln\left|\sqrt{1+\sin^2(0)} + \sin(0)\right| + C\right]$$

$$= \left[\ln\left|\sqrt{1+(1)^2} + (1)\right|\right] - \left[\ln\left|\sqrt{1+(0)^2} + (0)\right|\right]$$

$$= \left[\ln\left|1+\sqrt{2}\right|\right] - \left[\ln|1|\right] = \left[\ln(1+\sqrt{2})\right] - [0] = \ln(1+\sqrt{2})$$

62) $\int \frac{x^2}{\left(3+4x-4x^2\right)^{\frac{3}{2}}}\,dx$

First we must fix the trinomial in the denominator such that it becomes either sum or difference of squares.

$$3+4x-4x^2 = 3+(1-1)+4x-4x^2$$

$$= 3+1-1+4x-4x^2$$

$$= 3+1-(1-4x+4x^2)$$

$$= 4-(4x^2-4x+1)$$

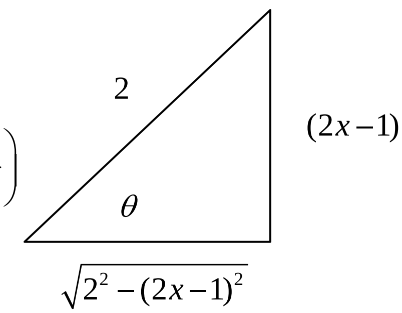
$$= 4-(2x-1)^2$$

$$= 2^2-(2x-1)^2$$

$$\frac{(2x-1)}{2} = \sin\theta$$
$$2x-1 = 2\sin\theta$$
$$2x = 2\sin\theta + 1$$
$$x = \sin\theta + \frac{1}{2}$$
$$dx = \cos\theta\, d\theta$$

$$\frac{\sqrt{2^2-(2x-1)^2}}{2} = \cos\theta$$
$$\sqrt{2^2-(2x-1)^2} = 2\cos\theta$$

$$\theta = \sin^{-1}\left(\frac{2x-1}{2}\right)$$




$$\int \frac{x^2}{\left(3+4x-4x^2\right)^{\frac{3}{2}}}\,dx = \int \frac{x^2}{\left(\sqrt{3+4x-4x^2}\right)^3}\,dx = \int \frac{x^2}{\left(\sqrt{2^2-(2x-1)^2}\right)^3}\,dx = \int \frac{\left(\sin\theta+\frac{1}{2}\right)^2}{(2\cos\theta)^3}(\cos\theta\, d\theta)$$

$$= \int \frac{\left(\sin\theta+\frac{1}{2}\right)^2}{8\cos^2\theta}\,d\theta = \frac{1}{8}\int \frac{\left(\sin^2\theta+\sin\theta+\frac{1}{4}\right)}{\cos^2\theta}\,d\theta$$

$$= \frac{1}{8}\left\{\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta + \int \frac{\sin\theta}{\cos^2\theta}\,d\theta + \frac{1}{4}\int \frac{1}{\cos^2\theta}\,d\theta\right\}$$

$$= \frac{1}{8}\left\{\int \tan^2\theta\, d\theta - \int \frac{(-\sin\theta)}{\cos^2\theta}\,d\theta + \frac{1}{4}\int \sec^2\theta\, d\theta\right\}$$

$$= \frac{1}{8}\left\{\int (\sec^2\theta - 1)\, d\theta - \int \frac{(-\sin\theta)}{\cos^2\theta}\,d\theta + \frac{1}{4}\int \sec^2\theta\, d\theta\right\}$$

$$= \frac{1}{8}\left\{\frac{5}{4}\int \sec^2\theta\, d\theta - \int \frac{1}{\cos^2\theta}(-\sin\theta\, d\theta) - \int 1\, d\theta\right\}$$

$$= \frac{1}{8}\left\{\frac{5}{4}\tan\theta - \frac{1}{\cos\theta} - \theta\right\} + C = \frac{1}{8}\left\{\frac{5}{4}\tan\theta - \sec\theta - \theta\right\} + C$$

$$= \frac{1}{8}\left\{\frac{5}{4}\left(\frac{2x-1}{\sqrt{2^2-(2x-1)^2}}\right) - \left(\frac{2}{\sqrt{2^2-(2x-1)^2}}\right) - \left(\sin^{-1}\left(\frac{2x-1}{2}\right)\right)\right\} + C$$

$$= \frac{1}{8}\left\{\frac{5}{4}\left(\frac{2x-1}{\sqrt{3+4x-4x^2}}\right) - \left(\frac{2}{\sqrt{3+4x-4x^2}}\right) - \sin^{-1}\left(\frac{2x-1}{2}\right)\right\} + C$$

$$= \frac{1}{8}\left\{\frac{10x-5}{4\sqrt{3+4x-4x^2}} - \frac{8}{4\sqrt{3+4x-4x^2}} - \sin^{-1}\left(\frac{2x-1}{2}\right)\right\} + C$$

$$= \frac{1}{8}\left\{\frac{10x-13}{4\sqrt{3+4x-4x^2}} - \sin^{-1}\left(\frac{2x-1}{2}\right)\right\} + C$$

$$= \frac{10x-13}{32\sqrt{3+4x-4x^2}} - \frac{1}{8}\sin^{-1}\left(\frac{2x-1}{2}\right) + C$$