The formula for integration by parts:

$$\int u\,dv = uv - \int v\,du$$

Make sure to determine how you want to choose *u* and *dv*. Remember that your 1st choice might not be the optimized method of solving by this technique. Also, you need to use derivative method to find du from *u* and integration (anti-derivative) method to find *v* from dv.

There are problems that require you to apply integration by parts more than once.

Additional examples:

6) $\int (x-1)\sin \pi x\,dx$

$$\int (x-1)\sin \pi x\,dx = (x-1)\left(\frac{-1}{\pi}\cos \pi x\right) - \int \left(\frac{-1}{\pi}\cos \pi x\right)(dx)$$

$$= \frac{-1}{\pi}(x-1)\cos \pi x + \frac{1}{\pi}\int \cos \pi x\,dx$$

$$= \frac{-1}{\pi}(x-1)\cos \pi x + \frac{1}{\pi}\left(\frac{1}{\pi}\sin \pi x\right) + C$$

$$= \frac{-1}{\pi}(x-1)\cos \pi x + \frac{1}{\pi^2}\sin \pi x + C$$

$u_1 = (x-1)$ $\quad dv_1 = \sin \pi x\,dx$

$du_1 = dx$ $\quad v_1 = \frac{-1}{\pi}\cos \pi x$

16) $\int t^3 e^t\,dt$

$$\int t^3 e^t\,dt = (t^3)(e^t) - \int (e^t)(3t^2\,dx)$$

$$= t^3 e^t - \int 3t^2 e^t\,dx$$

$$= t^3 e^t - \left\{(3t^2)(e^t) - \int (e^t)(6t\,dx)\right\}$$

$$= t^3 e^t - 3t^2 e^t + \int 6te^t\,dx$$

$$= t^3 e^t - 3t^2 e^t + \left\{(6t)(e^t) - \int (e^t)(6\,dx)\right\}$$

$$= t^3 e^t - 3t^2 e^t + 6te^t - \int 6e^t\,dx$$

$$= t^3 e^t - 3t^2 e^t + 6te^t - 6e^t + C$$

$u_1 = t^3$ $\quad dv_1 = e^t\,dt$

$du_1 = 3t^2\,dx$ $\quad v_1 = e^t$

$u_2 = 3t^2$ $\quad dv_2 = e^t\,dt$

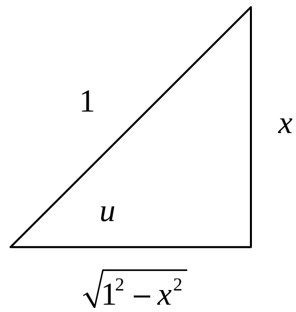
$du_2 = 6t\,dx$ $\quad v_2 = e^t$

$u_3 = 6t$ $\quad dv_3 = e^t\,dt$

$du_3 = 6\,dx$ $\quad v_3 = e^t$

12) $\int \sin^{-1} x \, dx$

$$u = \sin^{-1} x$$
$$\sin u = x$$
$$\cos u \frac{du}{dx} = 1$$
$$\frac{du}{dx} = \frac{1}{\cos u} = \frac{1}{\sqrt{1^2 - x^2}}$$



$$p = 1 - x^2$$
$$dp = -2x \, dx$$
$$\frac{-1}{2} dp = x \, dx$$

$$\int \frac{x}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{p}} \left(\frac{-1}{2} dp \right)$$
$$= \frac{-1}{2} \left(\frac{2}{1} \sqrt{p} \right) + C$$
$$= -\sqrt{1-x^2} + C$$

$$\int \sin^{-1} x \, dx = (\sin^{-1} x)(x) - \int (x) \left(\frac{1}{\sqrt{1-x^2}} dx \right)$$

$$u_1 = \sin^{-1} x \qquad dv_1 = dx$$
$$du_1 = \frac{1}{\sqrt{1-x^2}} dx \qquad v_1 = x$$

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$
$$= x \sin^{-1} x - \left(-\sqrt{1-x^2} \right) + C$$
$$= x \sin^{-1} x + \sqrt{1-x^2} + C$$

18) $\int_0^1 (x^2+1) e^{-x} \, dx$

$$\int (x^2+1)e^{-x} \, dx = (x^2+1)(-e^{-x}) - \int (-e^{-x})(2x \, dx)$$
$$= -(x^2+1)e^{-x} + \int 2x e^{-x} \, dx$$
$$= -(x^2+1)e^{-x} + \left\{ (2x)(-e^{-x}) \int (-e^{-x})(2 \, dx) \right\}$$
$$= -(x^2+1)e^{-x} - 2xe^{-x} + \int 2e^{-x} \, dx$$
$$= -(x^2+1)e^{-x} - 2xe^{-x} - 2e^{-x} + C$$
$$= \frac{-(x^2+1)}{e^x} - \frac{2x}{e^x} - \frac{2}{e^x} + C$$

$$u_1 = (x^2+1) \qquad dv_1 = e^{-x} \, dx$$
$$du_1 = 2x \, dx \qquad v_1 = -e^{-x}$$
$$u_2 = 2x \qquad dv_2 = e^{-x} \, dx$$
$$du_2 = 2 \, dx \qquad v_2 = -e^{-x}$$

$$\int_0^1 (x^2+1)e^{-x} \, dx = \left[\frac{-(x^2+1)}{e^x} - \frac{2x}{e^x} - \frac{2}{e^x} + C \right]_0^1$$
$$= \left[\frac{-((1)^2+1)}{e^{(1)}} - \frac{2(1)}{e^{(1)}} - \frac{2}{e^{(1)}} + C \right] - \left[\frac{-((0)^2+1)}{e^{(0)}} - \frac{2(0)}{e^{(0)}} - \frac{2}{e^{(0)}} + C \right] = \left[\frac{-6}{e} \right] - [-3] = 3 - \frac{6}{e}$$

20) $\int_4^9 \frac{\ln y}{\sqrt{y}}\,dy$

$$\int \frac{\ln y}{\sqrt{y}}\,dy = (\ln y)(2\sqrt{y}) - \int (2\sqrt{y})\left(\frac{1}{y}\,dy\right)$$

$$= 2\sqrt{y}\ln y - \int \frac{2}{\sqrt{y}}\,dy$$

$$= 2\sqrt{y}\ln y - 4\sqrt{y} + C$$

$$u_1 = \ln y \qquad dv_1 = \frac{1}{\sqrt{y}}\,dx$$

$$du_1 = \frac{1}{y}\,dy \qquad v_1 = 2\sqrt{y}$$

$$\int_4^9 \frac{\ln y}{\sqrt{y}}\,dy = \left[2\sqrt{y}\ln y - 4\sqrt{y} + C\right]_4^9$$

$$= \left[2\sqrt{(9)}\ln(9) - 4\sqrt{(9)} + C\right] - \left[2\sqrt{(4)}\ln(4) - 4\sqrt{(4)} + C\right]$$

$$= \left[6\ln 9 - 12\right] - \left[4\ln 4 - 8\right] = 6\ln 9 - 4\ln 4 - 4 = \ln 9^6 - \ln 4^4 - 4 = \ln\frac{9^6}{4^4} - 4$$

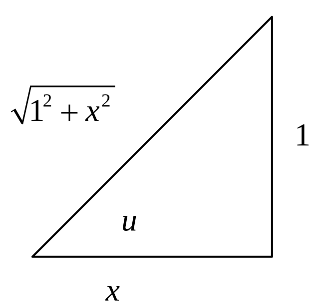
22) $\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right)dx$

$$u = \tan^{-1}\left(\frac{1}{x}\right)$$

$$\tan u = \frac{1}{x}$$

$$\sec^2 u\,\frac{du}{dx} = \frac{-1}{x^2}$$

$$\frac{du}{dx} = \frac{-1}{x^2\sec^2 u} = \frac{-1}{x^2\left(\frac{\sqrt{1^2+x^2}}{x}\right)^2} = \frac{-1}{1+x^2}$$



$$p = 1 + x^2$$

$$dp = 2x\,dx$$

$$\frac{1}{2}dp = x\,dx$$

$$\int \frac{x}{1+x^2}\,dx = \int \frac{1}{p}\left(\frac{1}{2}dp\right)$$

$$= \frac{1}{2}\ln|p| + C$$

$$= \frac{1}{2}\ln\left|1+x^2\right| + C$$

$$\int \tan^{-1}\left(\frac{1}{x}\right)dx = \left(\tan^{-1}\left(\frac{1}{x}\right)\right)(x) - \int (x)\left(\frac{-1}{1+x^2}\,dx\right)$$

$$= x\tan^{-1}\left(\frac{1}{x}\right) + \int \frac{x}{1+x^2}\,dx$$

$$= x\tan^{-1}\left(\frac{1}{x}\right) + \frac{1}{2}\ln\left|1+x^2\right| + C$$

$$u_1 = \tan^{-1}\left(\frac{1}{x}\right) \qquad dv_1 = dx$$

$$du_1 = \frac{-1}{1+x^2}\,dx \qquad v_1 = x$$

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right)dx = \left[x\tan^{-1}\left(\frac{1}{x}\right) + \frac{1}{2}\ln\left|1+x^2\right| + C\right]_1^{\sqrt{3}}$$

$$= \left[(\sqrt{3})\tan^{-1}\left(\frac{1}{(\sqrt{3})}\right) + \frac{1}{2}\ln\left|1+(\sqrt{3})^2\right| + C\right] - \left[(1)\tan^{-1}\left(\frac{1}{(1)}\right) + \frac{1}{2}\ln\left|1+(1)^2\right| + C\right]$$

$$= \left[\sqrt{3}\left(\frac{\pi}{6}\right) + \frac{1}{2}\ln|4|\right] - \left[(1)\left(\frac{\pi}{4}\right) + \frac{1}{2}\ln|2|\right] = \frac{\sqrt{3}\pi}{6} - \frac{\pi}{4} + \frac{1}{2}(\ln 4 - \ln 2) = \frac{\sqrt{3}\pi}{6} - \frac{\pi}{4} + \frac{1}{2}\ln 2$$

24) $\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr$

Method 1: Using integration by parts

$$\int \frac{r^3}{\sqrt{4+r^2}}\,dr = (r^2)\left(\sqrt{4+r^2}\right) - \int \left(\sqrt{4+r^2}\right)(2r\,dr)$$

$$= r^2\sqrt{4+r^2} - 2\int r\sqrt{4+r^2}\,dr$$

$$= r^2\sqrt{4+r^2} - 2\left[\frac{1}{3}\left(\sqrt{4+r^2}\right)^3\right] + C$$

$$= r^2\sqrt{4+r^2} - \frac{2}{3}\left(\sqrt{4+r^2}\right)^3 + C$$

$u_1 = r^2 \qquad dv_1 = \frac{r}{\sqrt{4+r^2}}\,dr$

$du_1 = 2r\,dr \qquad v_1 = \sqrt{4+r^2}$

$$p = 4 + r^2 \quad dp = 2r\,dr \Rightarrow \frac{1}{2}\,dp = r\,dr$$

side computations: $\int \frac{r}{\sqrt{4+r^2}}\,dr = \int \frac{1}{\sqrt{p}}\left(\frac{1}{2}\,dp\right) = \frac{1}{2}\left(\frac{2}{1}\sqrt{p}\right) + C = \sqrt{p} + C = \sqrt{4+r^2} + C$

$$\int \sqrt{4+r^2}\,dr = \int \sqrt{p}\left(\frac{1}{2}\,dp\right) = \frac{1}{2}\left(\frac{2}{3}p^{\frac{3}{2}}\right) + C = \frac{1}{3}\left(\sqrt{4+r^2}\right)^3 + C$$

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr = \left[r^2\sqrt{4+r^2} - \frac{2}{3}\left(\sqrt{4+r^2}\right)^3 + C\right]_0^1$$

$$= \left[(1)^2\sqrt{4+(1)^2} - \frac{2}{3}\left(\sqrt{4+(1)^2}\right)^3 + C\right] - \left[(0)^2\sqrt{4+(0)^2} - \frac{2}{3}\left(\sqrt{4+(0)^2}\right)^3 + C\right]$$

$$= \left[\sqrt{5} - \frac{2}{3}\left(\sqrt{5}\right)^3\right] - \left[0 - \frac{2}{3}(2)^3\right] = \left[\sqrt{5} - \frac{2}{3}\left(5\sqrt{5}\right)\right] - \left[-\frac{16}{3}\right] = \frac{3\sqrt{5}}{3} - \frac{10\sqrt{5}}{3} + \frac{16}{3} = \frac{16 - 7\sqrt{5}}{3}$$

Method 2: using substitution

$$p = 4 + r^2 \Rightarrow p - 4 = r^2 \quad dp = 2r\,dr \Rightarrow \frac{1}{2}\,dp = r\,dr$$

$$\int \frac{r^3}{\sqrt{4+r^2}}\,dr = \int \frac{r^2}{\sqrt{4+r^2}}(r\,dr) = \int \frac{(p-4)}{\sqrt{p}}\left(\frac{1}{2}\,dp\right) = \frac{1}{2}\int\left(\sqrt{p} - \frac{4}{\sqrt{p}}\right)dp = \frac{1}{2}\left[\frac{2}{3}\left(\sqrt{p}\right)^3 - 4\left(\frac{2}{1}\sqrt{p}\right)\right] + C$$

$$= \frac{1}{3}\left(\sqrt{p}\right)^3 - 4\sqrt{p} + C = \frac{1}{3}\left(\sqrt{4+r^2}\right)^3 - 4\sqrt{4+r^2} + C$$

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr = \left[\frac{1}{3}\left(\sqrt{4+r^2}\right)^3 - 4\sqrt{4+r^2} + C\right]_0^1$$

$$= \left[\frac{1}{3}\left(\sqrt{4+(1)^2}\right)^3 - 4\sqrt{4+(1)^2} + C\right] - \left[\frac{1}{3}\left(\sqrt{4+(0)^2}\right)^3 - 4\sqrt{4+(0)^2} + C\right]$$

$$= \left[\frac{1}{3}\left(\sqrt{5}\right)^3 - 4\sqrt{5}\right] - \left[\frac{1}{3}(2)^3 - 4(2)\right] = \left[\frac{5\sqrt{5}}{3} - 4\sqrt{5}\right] - \left[\frac{8}{3} - 8\right] = \left[\frac{5\sqrt{5}}{3} - \frac{12\sqrt{5}}{3}\right] - \left[\frac{8}{3} - \frac{24}{3}\right]$$

$$= \left[\frac{-7\sqrt{5}}{3}\right] - \left[\frac{-16}{3}\right] = \frac{16 - 7\sqrt{5}}{3}$$

26) $\int_0^t e^s \sin(t-s)\, ds$

$$\int e^s \sin(t-s)\, ds = (\sin(t-s))(e^s) - \int (e^s)(-\cos(t-s)\, dx)$$
$$\int e^s \sin(t-s)\, ds = e^s \sin(t-s) + \int e^s \cos(t-s)\, dx$$
$$\int e^s \sin(t-s)\, ds = e^s \sin(t-s) + \left\{(\cos(t-s))(e^s) - \int (e^s)(\sin(t-s)\, dx)\right\}$$
$$\int e^s \sin(t-s)\, ds = e^s \sin(t-s) + e^s \cos(t-s) - \int e^s \sin(t-s)\, ds$$
$$2\int e^s \sin(t-s)\, ds = e^s \sin(t-s) + e^s \cos(t-s)$$
$$\int e^s \sin(t-s)\, ds = \frac{1}{2} e^s \sin(t-s) + \frac{1}{2} e^s \cos(t-s) + C$$

$u_1 = \sin(t-s)$ $\quad dv_1 = e^s\, dx$
$du_1 = -\cos(t-s)\, dx$ $\quad v_1 = e^s$

$u_2 = \cos(t-s)$ $\quad dv_2 = e^s\, dx$
$du_2 = \sin(t-s)\, dx$ $\quad v_2 = e^s$

$$\int_0^t e^s \sin(t-s)\, ds = \left[\frac{1}{2} e^s \sin(t-s) + \frac{1}{2} e^s \cos(t-s) + C\right]_0^t$$
$$= \left[\frac{1}{2} e^{(t)} \sin(t-(t)) + \frac{1}{2} e^{(t)} \cos(t-(t)) + C\right] - \left[\frac{1}{2} e^{(0)} \sin(t-(0)) + \frac{1}{2} e^{(0)} \cos(t-(0)) + C\right]$$
$$= \left[\frac{1}{2} e^t\right] - \left[\frac{1}{2}\sin t + \frac{1}{2}\cos t\right] = \frac{1}{2} e^t - \frac{1}{2}\sin t - \frac{1}{2}\cos t$$

30) $\int_1^4 e^{\sqrt{x}}\, dx$

$$\int e^{\sqrt{x}}\, dx = \int e^p (2p\, dp)$$
$$= \int 2pe^p\, dp$$
$$= (2p)(e^p) - \int (e^p)(2\, dp)$$
$$= 2pe^p - \int 2e^p\, dp$$
$$= 2pe^p - 2e^p + C$$
$$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

$p = \sqrt{x} \quad \Rightarrow \quad p^2 = x$, $\quad 2p\, dp = dx$

$u_1 = 2p$ $\quad dv_1 = e^p\, dp$
$du_1 = 2\, dp$ $\quad v_1 = e^p$

$$\int_1^4 e^{\sqrt{x}}\, dx = \left[2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C\right]_1^4$$
$$= \left[2\sqrt{(4)}e^{\sqrt{(4)}} - 2e^{\sqrt{(4)}} + C\right] - \left[2\sqrt{(1)}e^{\sqrt{(1)}} - 2e^{\sqrt{(1)}} + C\right]$$
$$= \left[4e^2 - 2e^2\right] - \left[2e - 2e\right] = 2e^2$$

36) Prove $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

$$\int x^n e^x\,dx = (x^n)(e^x) - \int (e^x)(nx^{n-1}\,dx)$$

$$= x^n e^x - n\int x^{n-1}e^x\,dx$$

$u_1 = x^n$ $dv_1 = e^x\,dx$

$du_1 = nx^{n-1}\,dx$ $v_1 = e^x$

38) Prove $\int \sec^n x\,dx = \dfrac{\tan x\sec^{n-2}x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2}x\,dx \quad (n \ne 1)$

$u_1 = \sec^{n-2}x$ $dv_1 = \sec^2 x\,dx$

$du_1 = (n-2)\sec^{n-3}x(\sec x\tan x)\,dx$ $v_1 = \tan x$

$$\int \sec^n x\,dx = \int (\sec^{n-2}x)(\sec^2 x\,dx)$$

$$\int \sec^n x\,dx = (\sec^{n-2}x)(\tan x) - \int (\tan x)((n-2)\sec^{n-3}x(\sec x\tan x)\,dx)$$

$$\int \sec^n x\,dx = \tan x\sec^{n-2}x - (n-2)\int \sec^{n-2}x\tan^2 x\,dx$$

$$\int \sec^n x\,dx = \tan x\sec^{n-2}x - (n-2)\int \sec^{n-2}x(\sec^2 x - 1)\,dx$$

$$\int \sec^n x\,dx = \tan x\sec^{n-2}x - (n-2)\left\{\int \sec^n x\,dx - \int \sec^{n-2}x\,dx\right\}$$

$$\int \sec^n x\,dx = \tan x\sec^{n-2}x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2}x\,dx$$

$$(n-1)\int \sec^n x\,dx = \tan x\sec^{n-2}x + (n-2)\int \sec^{n-2}x\,dx$$

$$\int \sec^n x\,dx = \frac{\tan x\sec^{n-2}x + (n-2)\int \sec^{n-2}x\,dx}{(n-1)}$$

$$\int \sec^n x\,dx = \frac{\tan x\sec^{n-2}x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$$

The condition of $(n \ne 1)$ is needed so we don't end up with undefined expression.

40) Using the formula in exercise 36 (applying 4 times)

$$\int x^4 e^x\,dx = \left(x^4e^x - 4\int x^{4-1}e^x\,dx\right)$$

$$= x^4e^x - 4\int x^3e^x\,dx$$

$$= x^4e^x - 4\left(x^3e^x - 3\int x^{3-1}e^x\,dx\right)$$

$$= x^4e^x - 4x^3e^x + 12\int x^2e^x\,dx$$

$$= x^4e^x - 4x^3e^x + 12\left(x^2e^x - 2\int x^{2-1}e^x\,dx\right)$$

$$= x^4e^x - 4x^3e^x + 12x^2e^x - 24\int xe^x\,dx$$

$$= x^4e^x - 4x^3e^x + 12x^2e^x - 24\left(xe^x - 1\int x^{1-1}e^x\,dx\right)$$

$$= x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24\int e^x\,dx$$

$$= x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C$$