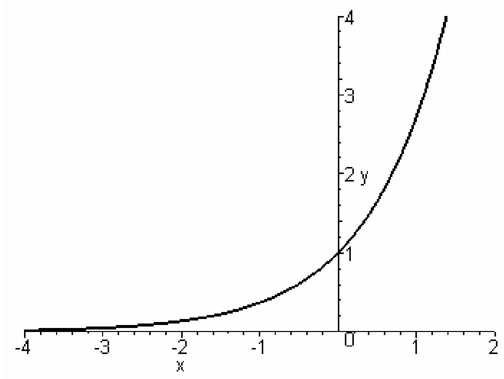
Properties of the Natural Exponential Function: The exponential function $f(x)=e^x$ is an increasing continuous function with domain $(-\infty,\infty)$ and range $(0,\infty)$. Thus $e^x>0$ for all *x*. Also

$$\lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to\infty} e^x = \infty$$



Graph of the function $f(x)=e^x$ is to the right.

When using the rule for derivative of exponential function, it is wise to always use the chain rule:

$$\frac{d}{dx}(e^x)=e^x(1)$$

Laws of Exponents If *A* and *B* are positive numbers and *c* is a rational number, then

$$e^{A+B}=e^Ae^B \qquad e^{A-B}=\frac{e^A}{e^B} \qquad (e^A)^c=e^{cA}$$

$$e^p=N \quad \Leftrightarrow \quad p=\ln N \qquad\qquad e^{\ln x}=x \quad x>0 \qquad\qquad \ln(e^x)=x \quad \text{for all } x$$

$$\int e^x\,dx=e^x+C$$

Additional examples:

2) a) $e^{\ln 15}=15$ b) $\ln\left(\frac{1}{e}\right)=\ln(e^{-1})=-1$

4) a) $\ln e^{\sin x}=\sin x$ b) $e^{x+\ln x}=(e^x)(e^{\ln x})=(e^x)(x)=xe^x$

8) a) $\ln(\ln x)=1 \;\Rightarrow\; \ln x=e^1 \;\Rightarrow\; x=e^e$ b) $e^{e^x}=10 \;\Rightarrow\; e^x=\ln 10 \;\Rightarrow\; x=\ln(\ln 10)=\ln\ln 10$

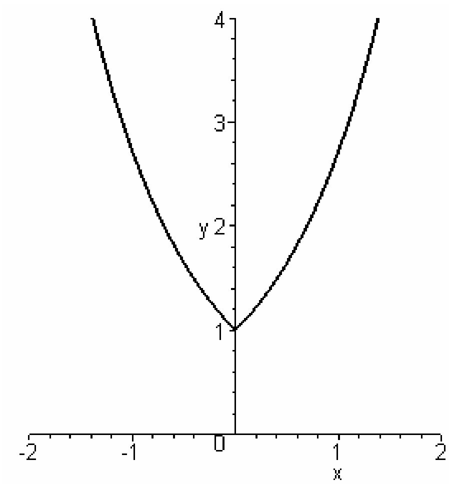
10) a)

$$\begin{aligned} 1&<e^{3x-1}<2\\ \ln 1&<3x-1<\ln 2\\ 0&<3x-1<\ln 2\\ 1&<3x<1+\ln 2\\ \frac{1}{3}&<x<\frac{1+\ln 2}{3} \end{aligned} \quad\Rightarrow\quad \left(\frac{1}{3},\frac{1+\ln 2}{3}\right)$$

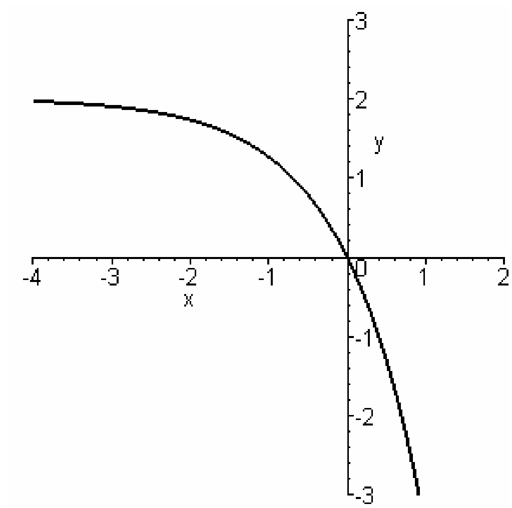
b)

$$\begin{aligned} 1-2\ln x&<3\\ 1-3&<2\ln x\\ -2&<2\ln x\\ -1&<\ln x\\ e^{-1}&<x\\ \frac{1}{e}&<x \end{aligned} \quad\Rightarrow\quad \left(\frac{1}{e},\infty\right)$$

12) $y = e^{|x|}$



14) $y = 2(1 - e^{x})$



16) $f(x) = \ln(2 + \ln x)$

$2 + \ln x > 0$

a) $\ln x > -2 \quad \Rightarrow \quad x > \dfrac{1}{e^2}$ Domain: $\left(\frac{1}{e^2}, \infty\right)$

$x > e^{-2}$

b) $y = \ln(2 + \ln x) \quad \Rightarrow \quad \begin{aligned} e^{y} &= 2 + \ln x \\ e^{y} - 2 &= \ln x \end{aligned} \quad \Rightarrow \quad e^{(e^{y}-2)} = x$

$f^{-1}(x) = e^{(e^{x}-2)}$

Domain : $(-\infty, \infty)$

24) $y = \dfrac{e^{x}}{1 - e^{x}}$

$$\frac{dy}{dx} = \frac{[e^{x}(1)](1-e^{x}) - (e^{x})[-e^{x}(1)]}{(1-e^{x})^2} = \frac{e^{x}\{[1](1-e^{x}) - (1)[-e^{x}]\}}{(1-e^{x})^2} = \frac{e^{x}\{1 - e^{x} + e^{x}\}}{(1-e^{x})^2} = \frac{e^{x}\{1\}}{(1-e^{x})^2} = \frac{e^{x}}{(1-e^{x})^2}$$

28) $y = x^2 e^{-1/x}$

$$\frac{dy}{dx} = [2x](e^{-1/x}) + (x^2)\left[e^{-1/x}\left(\frac{1}{x^2}\right)\right] = e^{-1/x}\left\{[2x](1) + (x^2)\left[\frac{1}{x^2}\right]\right\} = e^{-1/x}\{2x + 1\}$$

32) $y = \dfrac{e^{u} - e^{-u}}{e^{u} + e^{-u}}$

$$\frac{dy}{du} = \frac{[e^{u}(1) - e^{-u}(-1)](e^{u} + e^{-u}) - (e^{u} - e^{-u})[e^{u}(1) + e^{-u}(-1)]}{(e^{u} + e^{-u})^2}$$

$$= \frac{[e^{u} + e^{-u}](e^{u} + e^{-u}) - (e^{u} - e^{-u})[e^{u} - e^{-u}]}{(e^{u} + e^{-u})^2}$$

$$= \frac{(e^{u} + e^{-u})^2 - (e^{u} - e^{-u})^2}{(e^{u} + e^{-u})^2}$$

$$= \frac{(e^{2u} + 2 + e^{-2u}) - (e^{2u} - 2 + e^{-2u})}{(e^{u} + e^{-u})^2} = \frac{4}{(e^{u} + e^{-u})^2}$$

34) $y=\sqrt{1+xe^{-2x}}=(1+xe^{-2x})^{1/2}$

$$\frac{dy}{dx}=\frac{1}{2}(1+xe^{-2x})^{-1/2}([1](e^{-2x})+(x)[e^{-2x}(-2)])=\frac{e^{-2x}-2xe^{-2x}}{2\sqrt{1+xe^{-2x}}}=\frac{e^{-2x}(1-2x)}{2\sqrt{1+xe^{-2x}}}$$

38) $y=\dfrac{e^x}{x}\quad(1,e)$

$$\frac{dy}{dx}=\frac{[e^x(1)](x)-(e^x)[1]}{(x)^2}=\frac{e^x[x-1\}}{x^2}$$

$$m=\left.\frac{dy}{dx}\right|_{x=1}=\frac{e^{(1)}[(1)-1\}}{(1)^2}=0$$

$$y-(e)=0(x-(1))\quad\Leftrightarrow\quad y=e$$

40) $y=Ae^{-x}+Bxe^{-x}\quad y''+2y'+y=0\quad\Leftrightarrow\quad\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+y=0$

$$\frac{dy}{dx}=A[e^{-x}(-1)]+B\{[1](e^{-x})+(x)[e^{-x}(-1)]\}=-Ae^{-x}+Be^{-x}-Bxe^{-x}=(B-A)e^{-x}-Bxe^{-x}$$

$$\frac{d^2y}{dx^2}=(B-A)[e^{-x}(-1)]-B\{[1](e^{-x})+(x)[e^{-x}(-1)]\}=(B-A)[-e^{-x}]-B\{e^{-x}-xe^{-x}\}$$

$$=(A-B)e^{-x}-Be^{-x}+Bxe^{-x}=(A-2B)e^{-x}+Bxe^{-x}$$

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=\{(A-2B)e^{-x}+Bxe^{-x}\}+2\{(B-A)e^{-x}-Bxe^{-x}\}+Ae^{-x}+Bxe^{-x}$$

$$=Ae^{-x}-2Be^{-x}+Bxe^{-x}+2Be^{-x}-2Ae^{-x}-2Bxe^{-x}+Ae^{-x}+Bxe^{-x}=0$$

50) $g(x)=\dfrac{e^x}{x}\quad x>0$

$$\frac{dg}{dx}=\frac{[e^x(1)](x)-(e^x)[1]}{(x)^2}=\frac{e^x(x-1)}{x^2}$$

$$\frac{d^2g}{dx^2}=\frac{\big[[e^x(1)](x-1)+(e^x)[1]\big](x^2)-\big(e^x(x-1)\big)[2x]}{(x^2)^2}=\frac{xe^x\{[(x-1)+1](x)-(x-1)[2]\}}{x^4}=\frac{e^x\{x^2-2x+2\}}{x^3}$$

Now finding point of extrema and using 2nd derivative test:

$$\frac{e^x(x-1)}{x^2}=0$$

$$e^x(x-1)=0$$

$e^x=0\qquad x-1=0$

discard $\qquad x=1$

$$\left.\frac{d^2g}{dx^2}\right|_{x=1}=\frac{e^{(1)}\{(1)^2-2(1)+2\}}{(1)^3}>0$$

Concave Up

$$g(1)=\frac{e^{(1)}}{(1)}=e$$

64) $\int \frac{(1+e^x)^2}{e^x}\,dx$

$$\int \frac{(1+e^x)^2}{e^x}\,dx = \int \frac{1+2e^x+e^{2x}}{e^x}\,dx = \int\left(\frac{e^{2x}}{e^x}+\frac{2e^x}{e^x}+\frac{1}{e^x}\right)dx = \int\left(e^x+2+\frac{1}{e^x}\right)dx = \int (e^x+2+e^{-x})\,dx$$

$$= e^x + 2x - e^{-x} + C$$

68) $\int_0^1 \frac{\sqrt{1+e^{-x}}}{e^x}\,dx$

$$u = 1+e^{-x} \qquad du = e^{-x}(-1)\,dx = \frac{-1}{e^x}\,dx \qquad (-1\,du) = \frac{1}{e^x}\,dx$$

$$\int \frac{\sqrt{1+e^{-x}}}{e^x}\,dx = \int \frac{(1+e^{-x})^{1/2}}{e^x}\,dx = \int u^{1/2}(-1\,du) = -1\left[\frac{u^{3/2}}{\frac{3}{2}}\right]+C = \frac{-2}{3}u^{3/2}+C = \frac{-2}{3}(1+e^{-x})^{3/2}+C$$

$$= \frac{-2}{3}\left(\sqrt{1+e^{-x}}\right)^3 + C$$

$$\int_0^1 \frac{\sqrt{1+e^{-x}}}{e^x}\,dx = \left[\frac{-2}{3}\left(\sqrt{1+e^{-x}}\right)^3 + C\right]_0^1$$

$$= \left[\frac{-2}{3}\left(\sqrt{1+e^{-(1)}}\right)^3 + C\right] - \left[\frac{-2}{3}\left(\sqrt{1+e^{-(0)}}\right)^3 + C\right]$$

$$= \frac{2}{3}\left(\sqrt{2}\right)^3 - \frac{2}{3}\left(\sqrt{1+e^{-1}}\right)^3$$

$$= \frac{4\sqrt{2}}{3} - \frac{2}{3}\left(\sqrt{1+\frac{1}{e}}\right)^3$$