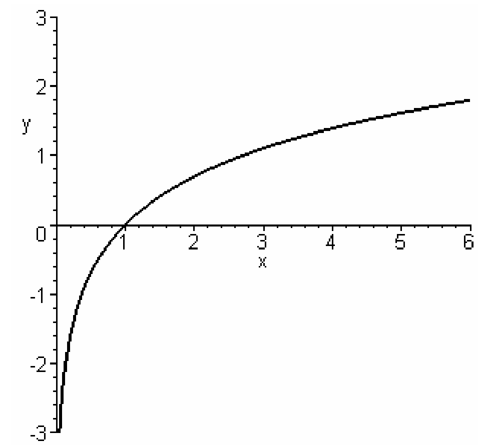
Definition 1: The **natural logarithmic function** is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt \quad x > 0$$

When using the rule for derivative of logarithmic function, it is wise to always use the chain rule:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}(1)$$



Graph of the function $f(x) = \ln x$ is to the right.

$$\lim_{x\to\infty}(\ln x) = \infty \qquad \lim_{x\to 0^+}(\ln x) = -\infty$$

Laws of Logarithms If A and B are positive numbers and c is a rational number, then

$$\ln(AB) = \ln A + \ln B$$

$$\ln\left(\frac{A}{B}\right) = \ln A - \ln B$$

$$\ln\left(A^c\right) = c \ln A$$

$$\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \ln|x| + C$$

Additional examples:

4) $$\ln s^4\sqrt{t\sqrt{u}} = \ln\left(s^4\left(t(u^{1/2})\right)^{1/2}\right) = \ln\left(s^4\left(t^{1/2}(u^{1/2})^{1/2}\right)\right) = \ln\left(s^4 t^{1/2} u^{1/4}\right) = \ln\left(s^4\right) + \ln\left(t^{1/2}\right) + \ln\left(u^{1/4}\right)$$

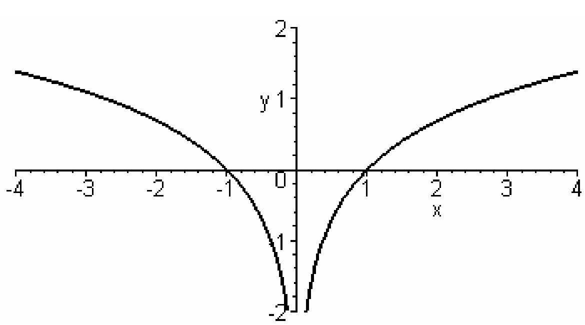
$$= 4\ln s + \frac{1}{2}\ln t + \frac{1}{4}\ln u$$

6) $$\ln 3 + \frac{1}{3}\ln 8 = \ln 3 + \ln 8^{1/3} = \ln 3 + \ln\sqrt[3]{8} = \ln 3 + \ln 2 = \ln\left((3)(2)\right) = \ln 6$$

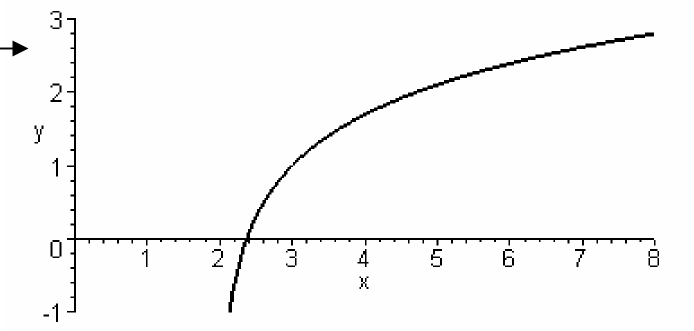
8) $$\ln(a+b) + \ln(a-b) - 2\ln c = \ln(a+b) + \ln(a-b) - \ln c^2 = \ln\left((a+b)(a-b)\right) - \ln c^2$$

$$= \ln\left(\frac{(a+b)(a-b)}{c^2}\right) = \ln\left(\frac{a^2-b^2}{c^2}\right)$$

10) $y = \ln|x|$



12) $y = 1 + \ln(x-2)$ →



16) $f(x) = x\ln x - x$

$$\frac{df}{dx} = \left\{[1](\ln x) + (x)\left[\frac{1}{x}(1)\right]\right\} - 1$$

$$= \{\ln x + 1\} - 1 = \ln x$$

20) $y = \dfrac{1}{\ln x} = (\ln x)^{-1}$

$$\frac{df}{dx} = -1(\ln x)^{-2}\left(\frac{1}{x}(1)\right) = \frac{-1}{x(\ln x)^2} = \frac{-1}{x(\ln^2 x)} = \frac{-1}{x\ln^2 x}$$

24) $f(u) = \dfrac{u}{1+\ln u}$

$$\frac{df}{du} = \frac{[1](1+\ln u) - (u)\left[\frac{1}{u}(1)\right]}{(1+\ln u)^2} = \frac{(1+\ln u)-1}{(1+\ln u)^2} = \frac{\ln u}{(1+\ln u)^2}$$

26) $H(z) = \ln\sqrt{\dfrac{a^2-z^2}{a^2+z^2}} = \dfrac{1}{2}\ln(a^2-z^2) - \dfrac{1}{2}\ln(a^2+z^2)$

$$\frac{dH}{dz} = \frac{1}{2}\left[\frac{1}{a^2-z^2}(-2z)\right] - \frac{1}{2}\left[\frac{1}{a^2+z^2}(2z)\right] = \frac{-z}{a^2-z^2} - \frac{z}{a^2+z^2} = \frac{-z(a^2+z^2)-z(a^2-z^2)}{(a^2-z^2)(a^2+z^2)}$$

$$= \frac{-za^2 - z^3 - za^2 + z^3}{(a^2-z^2)(a^2+z^2)} = \frac{-2za^2}{a^4-z^4} = \frac{2za^2}{z^4-a^4}$$

32) $y = \ln\left|\cos(\ln x)\right|$

$$\frac{dy}{dx} = \frac{1}{\cos(\ln x)}(-\sin(\ln x))\left(\frac{1}{x}(1)\right) = \frac{-\sin(\ln x)}{x\cos(\ln x)} = \frac{-\tan(\ln x)}{x}$$

34) $y = \ln(\sec x + \tan x)$

$$\frac{dy}{dx} = \frac{1}{\sec x + \tan x}(\sec x\tan x + \sec^2 x) = \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x} = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x$$

$$\frac{d^2y}{dx^2} = \sec x\tan x$$

36) $f(x) = \ln\ln\ln x$

$$\frac{df}{dx} = \frac{1}{\ln\ln x}\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}(1)\right) = \frac{1}{x(\ln x)(\ln\ln x)}$$

For domain, we investigate from outer logarithm inwards: $f(x) = \ln(\ln(\ln x))$, 1^{st} $(\ln(\ln x)) > 0$; to make this true, 2^{nd} $(\ln x) > 1$; finally to make this possible, 3^{rd} $x > e$. Therefore, domain is (e, ∞).

38) $f(x)=\dfrac{\ln x}{x} \quad f''(e)=\left.\dfrac{d^2 f}{dx^2}\right|_{x=e}=?$

$$\frac{df}{dx}=\frac{\left[\frac{1}{x}(1)\right](x)-(1-\ln x)[1]}{x^2}=\frac{1-\ln x}{x^2}$$

$$\frac{d^2 f}{dx^2}=\frac{\left[\frac{-1}{x}(1)\right](x^2)-(1-\ln x)[2x]}{(x^2)^2}=\frac{x\left\{\left[\frac{-1}{x}(1)\right](x)-(1-\ln x)[2]\right\}}{x^4}=\frac{-1-2+2\ln x}{x^3}=\frac{2\ln x-3}{x^3}$$

$$\left.\frac{d^2 f}{dx^2}\right|_{x=e}=\frac{2\ln(e)-3}{(e)^3}=\frac{2(1)-3}{e^3}=\frac{-1}{e^3}$$

40) $y=\ln(x^3-7) \quad (2,0)$

$$\frac{dy}{dx}=\frac{1}{x^3-7}(3x^2)=\frac{3x^2}{x^3-7}$$

$$m=\left.\frac{dy}{dx}\right|_{x=2}=\frac{3(2)^2}{(2)^3-7}=\frac{3(4)}{8-7}=12$$

$$y-(0)=12(x-(2)) \quad \Rightarrow \quad y=12x-24$$

54) $y=\dfrac{(x^3+1)^4\sin^2 x}{x^{1/3}}$

$$\ln y=\ln\left(\frac{(x^3+1)^4\sin^2 x}{x^{1/3}}\right)$$

$$\ln y=4\ln(x^3+1)+2\ln(\sin x)-\frac{1}{3}\ln x$$

$$\frac{1}{y}\frac{dy}{dx}=4\left[\frac{1}{x^3+1}(3x^2)\right]+2\left[\frac{1}{\sin x}(\cos x)\right]-\frac{1}{3}\left[\frac{1}{x}(1)\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{12x^2}{x^3+1}+2\cot x-\frac{1}{3x}$$

$$\frac{dy}{dx}=\left\{\frac{12x^2}{x^3+1}+2\cot x-\frac{1}{3x}\right\}y=\left\{\frac{12x^2}{x^3+1}+2\cot x-\frac{1}{3x}\right\}\frac{(x^3+1)^4\sin^2 x}{x^{1/3}}$$

56) $\int_0^3 \frac{dx}{5x+1}$

$$\int \frac{1}{5x+1}\,dx = \int \frac{1}{5(x+\frac{1}{5})}\,dx = \frac{1}{5}\int \frac{1}{x+\frac{1}{5}}\,dx = \frac{1}{5}\ln\left|x+\frac{1}{5}\right| + C$$

$$\int_0^3 \frac{dx}{5x+1} = \left[\frac{1}{5}\ln\left|x+\frac{1}{5}\right| + C\right]_0^3$$

$$= \left[\frac{1}{5}\ln\left|(3)+\frac{1}{5}\right| + C\right] - \left[\frac{1}{5}\ln\left|(0)+\frac{1}{5}\right| + C\right]$$

$$= \frac{1}{5}\left\{\ln\left|\frac{16}{5}\right| - \ln\left|\frac{1}{5}\right|\right\}$$

$$= \frac{1}{5}\left\{\ln\left(\frac{16}{5}\right) - \ln\left(\frac{1}{5}\right)\right\} = \frac{1}{5}\ln\left(\frac{\frac{16}{5}}{\frac{1}{5}}\right) = \frac{1}{5}\ln 16$$

60) $\int_e^6 \frac{dx}{x\ln x}$

$$u = \ln x \quad du = \frac{1}{x}\,dx$$

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln x| + C$$

$$\int_e^6 \frac{dx}{x\ln x} = \left[\ln|\ln x| + C\right]_e^6$$

$$= \left[\ln|\ln(6)| + C\right] - \left[\ln|\ln(e)| + C\right]$$

$$= \ln(\ln 6) - \ln|1|$$

$$= \ln\ln 6$$