7.5 - Work

The major concepts:

(i) $W = F \times D$, that is, work = force $\times$ distance. (This is for constant force.)

(ii) In any case, constant or not, the work to move an object from a point a to a point b is given by: $W = \int_a^b f(x)\,dx$, where $f(x)$ is the force exerted on the object at position x. In other words, $W = \int$ force.

A note on units: grams and kilograms are units of mass. To change them to their equivalent forces (weight) you must multiply by gravity (9.8 m/s^2 or 32 ft/s^2 depending on whether you are measuring length in meters (m) or feet (ft) at the time.) Pounds (lbs) is a unit of force (weight) so you need NOT multiply this by anything to change it into a force when it shows up. It *is* force.

If grams (g) occurs in a problem, change it to kilograms (kg) and all lengths to meters. With these units (kg for mass and m for length) your force will have units of Newtons (N) and your work will have the unit Joule (J), which is a Newton-meter (Nm or N-m). Though, in general we won't care too much about units here (leave that to the physicists), it makes it look like you know what you're talking about when you use units, and you can always double check your answers by seeing if the units work out to what they're supposed to. When force is given in pounds and length is measured in feet, the unit of work is ft-lb ("foot pounds").

Now on to the first kind of work we will be measuring, that for springs:

HOOKE'S LAW

This law states that the force $f(x)$ needed to stretch or compress a spring by a length x beyond its natural length is given by

$$f(x) = kx$$

where k is called the spring constant. Hence, for springs, the work needed to stretch or compress them from a length a beyond their natural length to a length b beyond their natural length is given by

$$W = \int_a^b kx\,dx$$

An example,

Problem 1: A spring has a length of 15 inches. A force of 750 lbs is needed to compress the spring to 12 inches. Find the work needed to compress it an additional 3 inches. Ans: 3375 inch-lbs.

LIFTING/PULLING PROBLEMS

The approach: When lifting an object piece by piece (for example, pulling up a cable to the top of a roof), (1) find the work needed to lift a single piece of the object (of infinitesimal length or thickness), (2) set up the "work = force × distance" expression, and then (3) integrate to find the work need to move all such pieces, and hence, the whole object. Begin by drawing a diagram, and choosing your "origin". With this set up, you can figure out your lengths and distances and hence set up the right integral.

If you are simply lifting an object all at once, use the good ol' "integral of force" formulation for work. The following examples illustrate:

Problem 2: A 100 ft cable weighs 300 lbs and hangs off the top of a tall building. Find the work needed to lift the cable to the top of the building.

Problem 3: What if a bucket (of negligible weight) with 500 lbs of water in it was attached to the end of the cable. Now what is the work in lifting the cable (and the bucket) to the top of the building?

Problem 4: Now what if water is leaking out the bucket? If the cable is pulled up at a constant rate of 2 ft/sec, and ALL the water leaks out of the bucket by the time it gets to the top of the building. What was the work done in lifting the cable with the leaking bucket?

Another variation: In problem 4, suppose the bucket weighs 50 pounds. What is the work needed now?

PUMPING PROBLEMS

The main formula: $W = \int_a^b \text{volume} \cdot \text{density} \cdot \text{distance}\ dx$

Note that the units of density must be force/(unit volume) in order to be consistent with our force times distance integral. (So you would measure density as lbs/ft^3 instead of kg/ft^3 for example).

The approach: again, start with the force times distance set up for a single "slice" of what you want to pump, usually some mysterious liquid. After finding the work to move that, integrate to get the work to move all such "slices". * It is very important to realize, that the limits on your integral in this set up, do NOT deal with position, rather, they state the "length" of the liquid you want to move. We illustrate how to do this with examples:

Problem 5: An inverted conical tank is filled with liquid of density 200 lb/ft^3. The tank has a height of 12 ft and the radius at the top is 6 ft. Find the work needed to pump the liquid out of the top of the tank.

Problem 6: A rectangular tank of length 10 m, width 5 m and height 5 m is filled with water (density = 1000kg/m^3). Find the work needed to pump the water out the top of the tank. (FYI: the density of water in pounds per cubic foot is 62.4lb/ft^3.)

Problem 7: A spherical tank of radius 10 ft is filled with liquid of density 20 lb/ft^3. The tank has a 5 m spout at its top. Find the work needed to pump the liquid out the tank, through the spout.

Problem 8: A right cylindrical tank has a radius of 3 m and a height of 6 m. It is filled with liquid weighing 10 lb/m^3. Find the work needed to pump the top half of the liquid out the top of the tank.

Problem 9: (From Spring 2008, problem 7 – I will draw the diagram on the board) A 10 ft long tank with height 4 ft has a vertical cross-section given by $y = x^2$. It is filled to a height of 3 ft with liquid having density 30 lb/ft^3. Set up an integral to compute the work necessary to pump the liquid in the tank over the top. Use the integral to compute the work.