Why do students fail calculus?

Stanley Ocken
Department of Mathematics
The City College of C. U. N. Y.

Abstract: Received doctrine that permeates the literature and practice of K-12 mathematics education, if implemented uncritically, will obstruct the development of skills that are critical for the study of calculus. The following related false dichotomies are addressed in this paper:

- **Blind rote is always bad and leads to error, whereas thinking about what you are doing is always good and leads to success.**
- **Solving problems by using algebra is bad, whereas concretizing problems by using physical and visual models is good.**

Introduction

This paper is an attempt to enlist the assistance of mathematics educators in an effort to reduce the unreasonably high failure rate of students in calculus courses in New York City and nationwide. Any attempt to do so must consider the critical role K-12 educators play in helping students acquire tools for success in college mathematics.

A dichotomy that looms large in discussions of mathematics education is that of "blind rote," typically identified with formal arithmetic or algebraic symbol manipulation, versus "conceptual understanding," as evidenced by the ability to concretize symbolic processes by referring to everyday experience. Many K-12 educators assert that adherence to blind rote is the source of student's weak mathematics performance, which can be ameliorated only by insistence on understanding. Herein that belief is challenged by reference to specific and significant examples that arise in the calculus curriculum. One conclusion is that the standard dogma, if accepted and implemented uncritically by mathematics educators and curriculum developers, poses a clear and present danger to the mathematical development of children who hope to study calculus in college.

The important role of calculus in the college curriculum and the American job market is suggested by two independent statistics. In fall 2000, 1,156,000 June high school graduates began their freshman year at a four year college in the United States. [BLS] At the same time, about 463,000 students in those institutions enrolled in first semester calculus. Comparing the two numbers, one might estimate that forty per cent of freshman took calculus. In fact, the actual percentage was considerably lower, since a significant portion of non-freshmen were re-taking first semester calculus, having failed the course previously one or more times The author is unaware of any reliable estimate of the number of students in this category. There is widespread agreement, however, that far too many students fail calculus and are thereby precluded from pursuing their career of choice.

In the United States and worldwide, one or more years of calculus is required of students who wish to pursue programs in science, mathematics, engineering, business, architecture, medicine, secondary math education, and many other fields. Many of these students will use calculus as an integral part of their professional activities. Nearly all will use the analytic and algebraic skills honed in calculus courses, and it is the frequent lack of precisely these skills that this paper will discuss.

It is borne out by long experience that students with flawless algebra skills usually do well in calculus, while those with even moderate weaknesses in algebra perform poorly. In this respect, mathematics is unique. An essay with a few spelling and grammar mistakes is still intelligible. A couple of missed notes in the performance of a piano sonata don't make a difference. In mathematics, however, there is much less room for error. If a student makes an algebra mistake at the beginning of a problem, the remainder of the solution may be rendered completely irrelevant to the stated problem, and partial credit can be awarded only out of sympathy and without academic justification. The plight of a student who doesn't notice that the algebra has gone astray is analogous to that of a pianist who doesn't notice her hands slipping laterally a key or two. Cacophony ensues.

This essay is an attempt by a mathematician with experience teaching all levels of calculus to engage K-12 educators constructively in an outcome-oriented discussion of current mathematics education reform. The author is no defender of the existing order. For decades, there clearly have been severe deficiencies in the mathematics preparation of entering students in the nation's colleges and universities. However, the present essay argues that present trends in mathematics education will exacerbate rather than alleviate those deficiencies.

The means for improving students' success rate in calculus are already at hand and are embodied in an important and useful principle of mathematics reform: the need for multiple approaches to problem solving. In the author's view, the current wave of curriculum reform has interpreted and implemented this doctrine far too narrowly. The need for multiple approaches applies in a far wider context than most K-12 mathematics educators imagine. Indeed, the examples elucidated herein demonstrate that in a curriculum designed to enable students to succeed in calculus, few patterns of mathematical thought or practice are either all good or all bad. This essay will argue, among other unorthodoxies, that

- Relying on blind rote is often good and is sometimes necessary.
- The attempt to "understand" underlying principles is sometimes irrelevant and counterproductive.
- Manipulatives must be used with discrimination, for reliance thereon can foster habits of mind that damage students' mathematical development.
- Understanding and symbol manipulation skill are closely related.

1. Pedagogy and history

This paper is in part a commentary on visions and perspectives of today's mathematics educators. In large part, new curricula are based on a constructivist approach to childhood education, articulated in the 1989 Standards of the National Council of Teachers of Mathematics. A revised version of the Standards appeared in year 2000, but it is not at all obvious whether there has been a substantive change in viewpoint since the publication of the original document.

A definitive account of the role of constructivism in mathematics education is the foundational treatise *Reconstructing Mathematics Education* [1993], by Deborah Shifter and Cathy Twomey Fosnot. That influential work describes sample topics from a constructivist curriculum and offers detailed accounts of the experiences of K-6 teachers who participated in an NSF sponsored teacher training program.

One teacher comes to

"realize how useless most things are out of context. How nothing makes sense mathematically if there's no concrete picture (even in one's mind) of what it is we're talking about."

Another opines
"I can solve it with algebra, but whenever I go to algebra it is a cop-out. It means that I can't derive what the problem is about at its core, just based on logic and the information given."

In the entire text, there is not a single instance of algebraic notation being used to advantage. Every fraction, every number, and every operation on numbers is reduced to pictures. Every instance of "understanding" is visual (drawing diagrams) or physical (using number sticks and other manipulatives), to the virtual exclusion of mathematical notation. Symbols are viewed as obscurantist. "Understanding" and "constructing meaning" are synonymous with real-world modeling. One teacher, who enters the program knowing how to solve word problems by applying algebraic methods, reaches the following epiphany:

"There seems to be no middle road between understanding and applying formulas."

Insofar as these teachers' statements characterize the pedagogical philosophy of K-12 mathematics educators, it is critical to examine their relevance and validity for students who are relying on those curricula to prepare for the study of calculus.

The historical development of calculus illuminates the present discussion. It is generally understood that Newton and Leibniz, working independently, developed the main geometric and symbolic content of calculus. However, much less well known is that Isaac Barrow, Newton's mentor, has prior claim on many of the foundational concepts of the subject. Nevertheless, Barrow's name is obscured from history books because he failed to develop mathematical notation that would facilitate applying those ideas to the solution of real world problems. In short, Barrow had the right pictures ("concepts"), but without appropriate symbolic notation, he was unable to complete the journey to the top rank of mathematical achievement.

The basic ideas of calculus may be summarized as follows.

- The derivative, obtained by a limit process, describes a rate of change of a quantity with respect to time, as well as the slope of the tangent line to a curve.
- Integration is a summation process that can be used to find areas and volumes.
- Integration and differentiation are inverse processes.

For the benefit of an audience with little formal mathematical background, informal and algebra-free descriptions can convey the importance of these concepts and, in a general way, how they are applicable to scientific inquiry. While lacking in precision, such descriptions are perhaps appropriate for introductory courses in the history of mathematics, or for liberal arts courses once characterized as "math for poets."

Students of poetry are not the object of the current study. Today's calculus students are exposed to and are expected to master the symbol-rich and algebra-based methodology of Newton and Leibniz. Their versions of calculus, not Barrow's, permitted the study and solution of significant problems, ranging from the motion of planets to the study of magnetic and electric fields, planetary and atomic orbits, and the 3-D reconstruction of medical imaging data. All such tasks remain completely intractable without the use of algebraic notation and techniques.

Newton's *Principia Mathematica* opens with the statement: "Herein I explain the system of the World [i.e., the Universe]." That his grandiose claim is not an exaggeration is due in large part to his successful use of notation and formal methods to represent and quantify the problems of mathematical physics.

This brief historical summary suggests that the above quoted teacher's statement "I can solve it with algebra, but whenever I go to algebra it is a cop-out" is one of the most anti-mathematical perspectives in the literature of K-12 mathematics education. The stated view runs completely counter to the needs of high school graduates who are tackling first semester calculus and for whom algebra is the only method that can be used to solve the vast majority of problems in textbooks, examinations, and the real "real-life" problems that they will attack in later courses and during their professional careers.

Anti-algebra bias is even more explicit in the statement "there seems to be no middle road between understanding and applying formulas." The suggested Manichean distinction between understanding and formalism is based on ignorance, for formalism and reference to visual and physical models work hand in hand to permit the resolution of difficult questions. Algebraic techniques are the essential tool for moving from a general and intuitive discussion of a physical problem to a focused and quantitative understanding with predictive power.

All of mathematical physics can be viewed as the use of symbolism to lead a bewildered traveler through a dark wood of difficult concepts. And that is precisely how algebra is used in freshman calculus. Today's and tomorrow's college students need to walk into their first calculus lecture with well honed formal and symbolic skills. Insofar as K-12 curricula and educators de-emphasize algebraic techniques and denigrate algebraic formalism, they will deny high school graduates the opportunity to pursue mathematics-based careers.

2. Examples from the calculus classroom

Here are a few of the algebra errors that appear frequently on calculus exam papers:

- canceling 2 against 4 in the fraction $\frac{x+2}{x+4}$;
- solving $x^2 = 4$ to obtain $x = 2$, thereby omitting the negative solution ;
- rewriting $x^2 + 13^2$ as $(x+13)^2$;
- rewriting $\sqrt{x^2 + 2^2}$ as $x + 2$; or
- dividing both sides of $x^2 = x$ by *x*, concluding that *x* = 1, thereby omitting the solution $x = 0$.

In college mathematics lectures and exams, students need to manipulate fluently dozens, if not hundreds, of symbolic expression, and so they will have many opportunities to make common mistakes similar to the five listed above. Do students make these errors because they adhere to "blind rote?" Would they be less likely to make these errors if they "understood" more? If so, what sort of understanding is required by students if they are to avoid these and similar errors?

A critical observation is that the rational expression $\frac{x+2}{x+4}$, whether it appears in a lecture, text, or exam, is presented to the student and must be confronted as a combination of symbols. In this respect it is unlike a numerical fraction, which in some contexts might be thought of as describing part of a unit, or a ratio, or a property of a physical object. Although the expression $\frac{x+2}{x+4}$ may arise from representation of a real-life model, it is impossible and/or counterproductive for the problem solver to keep track of the relationship between the model and its symbolic representation.

To work successfully with symbolic expressions, students must decide which transformations (usually, but not always, simplifications) of the expression are both legitimate and useful for obtaining the desired answer. The decision process requires both intelligence and judgment. What it does not require, or even permit, is reference to experience. Relying on symbols, to many educators and curriculum developers, implies abandonment of the understanding of the underlying physical model. This is simply not the case in the study of non-trivial mathematical and scientific problems. Instead, the methodology of modern science is to convert an intuitive understanding of a physical or geometric principle into a symbolic representation, in order both to facilitate a deeper understanding of that principle as well as to providing a launching platform for transforming that intuition into a testable prediction.

Student errors on calculus exams can usually be classified as either careless or systematic. An example of a careless error is miscopying the fraction $\frac{2x}{x+2}$ as $\frac{2+x}{x+2}$. That might occur when a student turns the page of his examination booklet.

In contrast, the five errors listed earlier are both frequent and systematic, in the sense that students who make them once make them more or less consistently. For example, it is not unusual for a student to ignore the negative solutions of equations such as $x^2 = 4$ and $y^2 = 7$, or to incorrectly cancel expressions such as $\frac{x+2}{x+4}$, several times throughout the course of a semester's examinations.

In an attempt to expose the reasons for such errors, it is not illuminating to assert that students illegally cancel $\frac{x+2}{x+4}$ because they don't understand mathematical concepts. They certainly need to know the underlying principle: cancellation works only for common factors of numerator and denominator. A precise statement of this principle requires a careful sequence of algebraic definitions, which students should be able to verbalize. In practice, however, students who are both proficient and fluent don't think about the cancellation rule when they are doing mathematics, for they have developed, through extensive practice, the ability to distinguish automatically between those situations in which cancellation is legal and those in which it is not.

Students who hope to succeed in calculus need to develop an automatic facility for distinguishing not only between legitimate and illegitimate symbolic manipulations, but also for deciding which of several possible manipulations is appropriate to the problem of current interest. If they don't, they're going to have trouble getting through a calculus exam or homework assignment.

3. Blind rote

Standard dogma asserts that understanding is the cure for mistakes that would otherwise arise from adherence to blind rote. The clearest counterexample involves substituting an expression for the argument of a function. This requires a bit of explanation.

A function, it is often said, can be thought of as a little black box that acts on a number to produce another number. For example, after carefully observing the path of a falling object, one could in principle express its position, e.g., its height above the ground, as a function of time. Deducing the precise form of the position function is the principal goal of celestial mechanics, ballistics, and all applications that require predicting from *a priori* information the position of a moving object at a given future time.

As an illustrative example, consider the x^2 button on most calculators, a button whose sole purpose in life is to square a number, i.e. to multiply that number by itself. Fulfilling this duty entails three steps:

1. The user types a number, called the **input,** which then appears in the display.
2. The user presses the x^2 button.
3. The calculator display switches to the **output**, or **value**, of the function, which in this example is calculated by multiplying the input by itself.

In the absence of magic boxes such as calculators and computers, a first attempt to communicate to a human the action of the x^2 button might be:

"The value of the squaring function at a number is the number times itself."
This language is cumbersome. A more concise statement (using asterisk to denote multiplication) would be

Square of a number = number*number.

The word "of" is superfluous, and is indicated more clearly by parentheses:
Square (number) = number *number.

In mathematics, brevity is essential, both because it saves paper and because it allows the eye to scan more easily a complex expression. Therefore the function definition is abbreviated to something like: $S(n) = n*n$, with the words "square" and "number" each indicated by their initial letter.

In the algebra sentence $S(n) = n*n$, the first symbol is the name of the function. The name of the input is enclosed in parentheses. The output, the expression to the right of the equal sign, is an algebraic expression that can be figured out by applying arithmetic operations to the input. The entire sentence is the definition of the function S.

In practice, mathematical custom of long standing prefers letters at the end of the alphabet as names of function inputs, and the most common usage in elementary texts is to write the squaring function as $S(x) = x^2$. Reader take note: contrary to certain usages in mathematics and literature, the letter '*x*' does not indicate an unknown, but is rather a variable quantity that can assume any desired numerical value. For example, a student wishing to find the square of 5 might write

Step 1) $S(x) = x*x$, the function definition, and then
Step 2) $S(5) = 5*5$ to indicate that when *x* is 5, the output of the squaring function is 5*5, and finally

Step 3) $S(5)=25$ to express the output as a number.

The alert reader will observe that the equals signs in 1) and 2) mean different things. Step 1) defines a function (a rule, if you will) whereas Step 2) asserts that the result of applying the function to the input 5 is the output 5*5. The distinction between these two usages is a possible source of confusion that should be addressed, in my view, by using alternate notation. In fact, some computer algebra systems employ the notation S[x] := x*x to make it easier for the compiler (the program that translates from algebraic notation to machine language) to understand the programmer's wish to define a function. The remainder of this paper will employ that notation.

Step 2), the principal concern of the present discussion, is a procedure called **argument substitution.** It appears thousands of times in any calculus text and perhaps a dozen times during a typical calculus examination. The above example suggests the following description: replace each symbol '*x*' in the function definition by the symbol for the input. Unfortunately, this first attempt isn't quite correct. To see why, assume that the addition fact 5 + 4 = 9 is temporarily unavailable, and try to evaluate (i.e., find the value of) the squaring function when its input is 5 + 4. According to the proffered description, the value of the squaring function is obtained as follows:

Step 1) $S(x):=x*x$ (the function definition)
Step 2) $S(5+4)=5+4*5+4$ (Erase and replace each '*x*' by '5 + 4.')

Unfortunately, the universal convention that multiplication is performed before addition forces the right hand side of Step 2) to be interpreted as 5 + 20 + 4, which is 29, rather than as 9*9, or 81, the expected function value.

The resolution of this difficulty is both simple and profound. To convey the desired order of operations, add before multiplying, it is necessary to use parentheses:

$S(5+4)=(5+4)*(5+4)$, after which the function value may be simplified as
$S(9)=9*9=81$.

Therefore a complete description of argument substitution is as follows.

Suppose a function is defined by *S*(*x*): = [any algebraic expression]

To find the value of the function at an input, erase each symbol '*x*' to the right of the equals sign and replace it by the input *enclosed in parentheses.*

It is extremely difficult to find this statement in algebra textbooks or in teacher training materials. This is but one instance of how insensitivity to symbolic notation imperils the goal of mathematical precision in student learning.

A critical application of function notation is its use when the input is itself represented by an algebra expression rather than by a number.

Example 2: Given $S[x] := x * x$, find $S(a+b)$ without simplifying.
Solution: $S(a+b) = (a+b) * (a+b)$.

Most students work this problem correctly. However, one occasionally encounters the following error:

Example 3: Given $S[x] := x * x$, find $S(a+2)$
Error: $S(a+2) = a^2 + 2$.

Example 4: Given $F[x] := x + \frac{1}{x}$, find $F(x + h)$.

Error: $F(x+h) = x + \frac{1}{x} + h$ **Correct**: $F(x+h) = x + h + \frac{1}{x+h}$

Example 5: Given $G[x] := 3 - x$, find $G(x+h)$.
Error: $G(x+h) = 3 - x + h$ **Correct:** $G(x+h) = 3 - (x+h) = 3 - x - h$

The listed examples show two different invitations to error.

If a student faithfully follows the argument substitution rule, Example 4 is a giveaway. Therefore, it is surprising that many college fall into the error stated above. But it happens not infrequently, and the error cited often transforms a sophisticated problem into a trivial one that has not been asked. There is no sensible way to assign partial credit. The student has made a systematic error that will result, if not corrected, in a downward spiral of that students' calculus grade and career aspirations.

Why is argument substitution so difficult? Students who are exposed to anti-symbolic bias in their mathematics courses may be troubled by the failure of symbolic process to reflect any model that occurs in nature. For example, the process of evaluating the function $F(x) := x^4 + x^2 - 3$ when $x = 2$ cannot sensibly be illustrated by pictures or manipulatives. Indeed, models are irrelevant, for argument substitution is a purely formal process.

In this writer's opinion and experience, it is pedagogically productive to describe argument substitution as a game played with symbols. It is not a frivolous game by any means. To the contrary, it is one of the important formal tools of mathematics and science. And it is a game that should be played by following the argument substitution rule to the letter.

In the above cited example, students probably are trapped by their attempt to make sense of a purely formal rule. The student writes $F(x + h)$ and notices that h is being added to something. Because the function definition looks like an equation, and because changing an equation requires doing the same thing to both sides, she decides to accord the right side fair treatment by adding *h* to it. Unfortunately for such a student, a function definition isn't an equation. Argument substitution is a pure "erase and replace" operation that has no direct connection with addition, subtraction, fractions, or any mathematical, arithmetical, or algebraic idea. It is simply a procedure, a recipe if you will, for altering the appearance of the original function definition.

My conclusion, shocking though it may sound to some readers, is that argument substitution should be carried out by "blind rote," and that attempting to understand what's going on may lead to errors. Indeed, the following standard dichotomy:

- Rote is bad, and leads to error.
- Thinking about what you are doing is good, and leads to success based on understanding.

is incorrect, or is at best irrelevant, in the current context. To the contrary, a student who feels that he must "understand" every new mathematical idea is likely to fall into the sort of trap illustrated by the above example, whereas one accustomed to plugging in symbols and playing with them by following rules is likely to carry out the argument substitution rule with ease and consistent accuracy.

Many subjects in mathematics offer ample opportunity to think deeply about mathematics and to proceed with eyes wide open. Argument substitution is not one of them.

4. An example using limits

A different sort of example is taken from a calculus lesson. As part of an important discussion of why the absolute value function $f(x) = |x|$ fails to have a derivative at $x = 0$, the instructor asks the class to simplify the expression

$$\lim_{h\to 0}\frac{|0+h|-|0|}{h}$$

The reader who doesn't understand this expression or the previous sentence is advised not to worry. What is important for participating as a student in the following dialogue is to pay careful attention to the instructor's language.

Instructor: Who can help me simplify this expression?
Students :(Silence.)

Instructor: (2^{nd} try): Well, just do one step: anything that will make the expression simpler or shorter:
Students: (Silence.)

Instructor: (desperate, 3rd try): I don't care if you understand what you are doing. Inside that scary-looking formula is something that you see every day, a few symbols that can be replaced by just one symbol. Please!
Students: (many hands raised): Change $0 + h$ to h!

In my view, the students were initially catatonic because they were worried about the "meaning" of the expression, and so succumbed to a syndrome that I call "symbol shock." They were worried about what the limit symbol meant, or perhaps by the use of vertical bars to denote absolute value. The teacher's third attempt encouraged them to overcome their fears and realize that '$0 + h$,' even when embedded in a complicated expression, should (usually) be replaced by 'h.'[1]

At this juncture, the reader of a certain persuasion may well be thinking that the examples just considered are not the essence of mathematics; that that they involve useless playing with symbols; that a computer could do this sort of thing, and so forth. These concerns will be addressed in the following sections.

4. Standard misconceptions

In today's sometimes acrimonious discussions of mathematics education, one encounters assertions that the very nature of mathematics is itself a subject for debate. Some have gone so far as to misinterpret Wittgenstein to buttress a claim that mathematics is socially constructed. For the purposes of this essay, all such philosophical discussion is irrelevant. The only concern is to prepare K-12 students for the specific conception of mathematics defined by the STEM calculus curriculum at nearly all American colleges.

A critical subtext of mathematics education reform in the twenty-first century is that symbol manipulation may once have been important, but is now rendered obsolete by technology. Initially, the availability of calculators suggested to some mathematics educators and curriculum developers that there is no longer a need for K-6 students to memorize, practice, or even to learn the traditional algorithms of arithmetic. Later, when computer algebra systems were implemented on hand calculators, the argument was extended to call into question the need for students to learn algorithms for manipulating algebraic expressions and solving equations.

In this writer's view, the availability of computer technology is a dubious blessing for mathematically challenged college students. The buck stops in the nation's science and engineering classrooms. Whether the subject is chemistry, biology, physics, engineering, economics, or finance, a student's most challenging task is to follow and absorb page after page and blackboard after blackboard of formulas and equations. Science books cannot be shorn of algebra, for the simple reason that all technological innovation depends on an algebraic description of physical phenomena.

---

[1] A more precise statement is that the symbolic expression "0+h" should be replaced by "h" whenever that expression signifies adding 0 to h. Examples in which this is not the case are "30+h" and "2^0+h."

The K-12 traditional curriculum is structured, partly by design and partly by historical accident, to bring students to the level needed to achieve the facility with formal skills needed to speak effectively the language of science. It follows that the availability of computing devices that do either arithmetic or algebra is irrelevant to students' needs to be able to read and to listen to mathematical formalism. Although computer algebra systems have a legitimate place in advanced courses, they pose a clear liability to students who are not already skilled at algebraic manipulation. Students who don't get adequate practice doing (i.e., writing) algebra will encounter the greatest difficulty understanding texts and lectures. Indeed, at my college, instructors of calculus courses that include a calculator or computer algebra system component note that students who are weak in algebra cannot enter formulas correctly into the computing device.

Most disturbing to many university mathematicians is the suggestion, implemented by a number of NCTM Standards-based curricula, that traditional algorithms of arithmetic be de-emphasized or dropped entirely. The oft-stated objection that students should not waste time finding the answer to a problem that could be solved with a calculator entirely misses the core issue, which is the continuum of logic and skills development that runs through the curriculum, from arithmetic to algebra to calculus.

If students don't learn the long division algorithm for whole numbers, they will have difficulty following the long division algorithm for polynomials when they reach that topic in high school algebra. If they don't know long division of polynomials, they won't have any feel for techniques of integration in elementary calculus or for Laplace transforms that are introduced in more advanced courses. Even if hand computation of the answer to a division problem is no longer important, the algebraic patterns and processes that lead to the answer are very important indeed.

In summary, K-12 students headed for calculus need to practice algebra because all advanced science texts speak a universal language written with variables and is punctuated with subscripts, superscripts, summation signs, and other complicated notation. If they're not comfortable with writing algebra, students won't be comfortable reading it. If they haven't devoted significant effort to hands-on manipulation of rational expressions, they will be in a total fog by the end of the first chapter of a typical science or engineering textbook.

Nothing stated in this essay should be taken to suggest that proficiency with formal mathematics is sufficient for success in science. Rather, such proficiency is simply an absolute necessity without which students cannot survive. While there is substantial room for improvement at all levels of the curriculum, there is no place whatsoever for a perspective that calculus students can make do without algebra, or even with less algebra than they currently encounter. Rather, the accelerating mathematization of formerly descriptive sciences such as biology makes it all the more crucial for a growing cadre of students to acquire fluent algebra skills before they begin college.

Jelly beans and limits

In "Reconstructing Mathematics Education," [FS] Fosnot and Schifter discuss at length the classroom experience of Ginny, a teacher who models division by distributing jelly beans among class members. Such a model is the sensible sort of initial learning experience that any capable educator or parent would provide when introducing this subject. From the perspective of the current paper, the critical challenge of early mathematics education is to follow up these initial experiences with an appropriately paced transition to symbolic representations. To do so, it is critical to address the details of that transition process. Schifter and Fosnot purport to do so in the following passage.

> After enough experience distributing objects, the children in Ginny's class will be able to imagine sharing without having to enact it, and reflection on their actions will allow them to generalize from specific cases to more abstract notions of division, although reference to their jelly bean [sharing] problem can be called up as needed. In just a few years, children's understanding of division of whole numbers will form the basis for constructing an understanding of division of fractions, and later still, for algebraic rational expressions, hyperbolic functions, or the limit of $1/x$ as x increases without limit.

It is important to examine the credibility of the multiple transitions enumerated in the last sentence. The first, from whole numbers to fractions, requires a delicate balance of concrete representation and abstract notation. It is critical for children to understand the core notion that unit fractions such as $\frac{1}{5}$ represent the result of equal subdivision of a unit into pieces, but it is also essential for them to encounter and absorb, as early as possible, the purely algebraic statement that $\frac{3}{5}$ is an abbreviation for $3\times\frac{1}{5}$ To see why, it is useful to study a standard example of what is often attacked as blind rote: the rule for multiplying fractions.

A balanced discussion begins with the statement that the product $\frac{3}{5}\times\frac{4}{7}$ is an abbreviation for $3\times\frac{1}{5}\times4\times\frac{1}{7}$, which in turn equals $3\times4\times\frac{1}{5}\times\frac{1}{7}=12\times\frac{1}{5}\times\frac{1}{7}$ after factors are rearranged . A modest reliance on notation has reduced the original problem to a much easier one: finding the product of unit fractions. A variety of methods, including the formal notion of inverse, or cake models, can be invoked to explain why that product is the unit fraction obtained by multiplying denominators of the factors. It follows immediately that the desired product is $12\times\frac{1}{35}=\frac{12}{35}$ Conclusion: to multiply fractions, multiply numerators and multiply denominators.

The argument just presented adheres to the doctrine of using multiple representations by combining symbolic and model-based arguments. In contrast, mathematics educators have developed an uncritical adherence to a purely geometrical model: the cake model for multiplying fractions. The author's observation of teacher training sessions suggests that this purely pictorial model is not easy to absorb, for at least two reasons.

First, the pictorial representation for multiplying improper fractions is not consistent with that of multiplying proper fractions, in that the former produces a "product cake" that completely covers and therefore hides the original unit cake. Second, the cake argument unnecessarily implicates an area interpretation of fractions in a situation where a linear representation, as the length of a stick, is more than sufficient. Indeed, teachers in an observed training session were unable to respond to the trainer's inquiry as to whether the product cake argument is based on counting or on area.

While it is possible to entertain Schifter and Fosnot's suggestion that division of whole number fractions can be modeled adequately by experience of sharing, the remaining quoted transitions from [FS] are implausible and unsupported. Specifically, the move from numerical fractions to rational expressions (quotients of polynomials) requires abandonment of physical models. A student who first encounters the expression $\frac{x+2}{x+4}$ enters a new world of notation in which it is impossible and misleading to suggest that $\frac{x+2}{x+4}$ refers to an activity of sharing. It is irrelevant or counterproductive to describe this expression as standing for the distribution of x + 8 jellybeans among x + 4 children, not least because *x* is seldom a whole number in the surrounding context. The most probable consequence of a sharing-based exposition is that students will block out the variables, focus on the numerals, and cancel illegally. Precisely for this reason, it is critical to impress upon students that the expression $\frac{x+2}{x+4}$ is NOT related to a situation in which 2 is divided by 4. Only a careful explanation of factoring algebraic expressions will permit students to learn when cancellation is, or is not, legitimate.

Next, the inclusion of hyperbolic functions in a list of activities supposedly related to division is puzzling indeed. For example, the role of division in the hyperbolic cosine function $\cosh(x) = \frac{e^x + e^{-x}}{2}$ is minimal. Hyperbolic functions are not about division, nor do they occupy a significant position in the basic calculus curriculum.

Finally, the reference to limits invites close attention as well. In recent years, most college mathematics instructors have been encountering more and more students who are less and less well equipped to understand the rigorous definition of limit. Indeed, that definition has been banished from the standard first-semester calculus course at all but a handful of top-rated institutions, precisely because few students graduate high school with the algebraic skills needed to understand and work with a rigorous formulation of the limit definition.

College students' understanding of this topic has deteriorated to the point that a paper in JRME reports a survey of students' understanding of limits that divides student responses into two categories, neither of which is correct. For example, the many students who suggested:

> The limit as x approaches 2 of f(x) = x*x is 4 because as x gets close to 2, then x*x gets close to 4,

are endorsing not a definition, but a vague substitute, one that is imprecise and contains potential for misunderstanding. Students who offered that response apparently believe that an observer first contemplates the distance from *x* to 2 and afterward verifies that *x*x* is close to 4. Precisely the opposite is the case: a hypothetical challenger offers an arbitrarily stringent criterion for *x*x* being close to 4, and the respondent must counter with a notion of *x* being close to 2 that suffices to meet the challenge.

This hypothetical interaction is summarized elegantly in the traditional ε - δ definition of limit. Unfortunately, an increasing portion of calculus instructors, confronted by students who immediately collapse into symbol shock when they see the variables ε and δ , are forced to revert to a mathematically imprecise quasi-definition.

The only way to restore integrity to the calculus curriculum is to provide students with a rich experience of symbol manipulation before they get to college. To the extent that reform mathematics curricula de-emphasize algebra and notation, they will enlarge rather than reduce the cohort of students who are improperly prepared for calculus. It is the responsibility of curriculum developers to provide a clear map, assuming that one exists, for the transition from jellybean sharing to symbolic mathematics.

What is to be done?

Given the increasing need for many more high school graduates to be completely fluent with algebra, how might one improve the algebra instruction that is currently achieving the desired impact on all too few students in American high schools? The author suggests that students will achieve the appropriate kinds of understanding not by using models and manipulatives, but rather by forming a coherent overview of algebra.

Students should be required to do written work and participate in discussions in which they detail explicitly both the definitions of the building blocks of symbolic algebra (term, factor, coefficient, expression,,,) and answer global procedural questions such as

What's the first thing to look at when you solve a polynomial equation?
When do you use the quadratic formula? completing the square? factoring?

What is the precise symbolic description of a function?
What is the precise definition of the graph of a function?
How do you convert between a function's symbolic definition and its graph?

A dozen years after the alleged revision of perspective in the NCTM Standards, it is simply unacceptable that the opening “Functions” chapter of a major curriculum such as [CPM] fails completely to answer any of the above three questions.

As soon as students learn about a topic, they should be asked to write about common pitfalls and how to avoid them. Consider the following hypothetical precalculus quiz.

$a)\ 3x = 5$

$b)\ 3x + 5 = 0$

$c)\ x^2 = x$

$d)\ x^3 = x$

$e)\ x^2 + 4x = 5$

$f)\ x^2 + 4x = 6$

Every student who wishes to achieve the facility with algebra required on typical calculus exams should be able to solve all of the following equations in less than five minutes. In fact, relatively few students complete this quiz without falling into one or more of a number of systematic errors.

a) is easy: divide both sides by 3 to get x = 5/3.

b) is drummed into students so often that they tend to get it correct as well, by subtracting 5 from both sides and then dividing by 3.

c) trips up quite a few students. By analogy with a), students divide both sides by x and conclude that $x = 1$. That’s clearly wrong, since x could also be 0. Any approach to polynomial solving must constrain students to avoid this error. The error arises because fractions whose denominators are 0 are meaningless, undefined, or illegal, depending on mathematical context as well as linguistic taste.

d) is even worse. not only do students divide by x and miss the solution $x = 0$, but they compound their error by missing the negative solution of the resulting equation $x^2 = 1$. Very few students find all three roots.

The difficulties in the previous two problems would evaporate if students knew that to solve a nonlinear polynomial equation, you move everything to one side, factor that side completely, and then set each factor to zero.

e) Students sometimes factor the left side to obtain $x(x + 4) = 5$. This information is useless. Nevertheless, it is common to see such students proceed further by writing $x = 5$ and $x + 4 = 5$, or perhaps $x = 5$ and $x + 4 = 1$. These students know that factoring helps solve a quadratic equation. Unfortunately, they are generalizing the operative principle:

If a product is zero, one of the factors is zero to an incorrect but linguistically analogous statement about nonzero products. A correct solution is once more obtained by following the general polynomial equation solution procedure stated above.

In f), even students who correctly transform the problem to $x^2 + 4x - 6 = 0$ tend to get stuck because they try too hard to factor this expression. They don't realize that solving a quadratic equation involves a decision process as to which of a number of methods is appropriate in a given situation.

Some students have an intuitive affinity for symbolic notation and solve the listed problems automatically, even with a minimum of formal instruction. The vast majority do not, and therefore require the careful attention of curriculum developers and mathematics educators. They are served poorly by curricula that de-emphasize symbolic manipulation.

Conclusion

A principal goal of this paper has been to expose as both misguided and counter-productive the rhetoric that identifies symbol manipulation with lack of understanding. Indeed, like any other subject, symbol manipulation can be studied and tested with or without understanding, and students are in dire need of a shift from the latter to the former paradigm.

From a pragmatic perspective, a reasonable, albeit imperfect, measure of students' understanding of algebra is their ability to perform symbol manipulation flawlessly. Students who consistently perform algebra without errors are *prima facie* demonstrating an internalized understanding of definitions and decision procedures. Even for these students, it would be beneficial to articulate their understanding as suggested above.

For the far larger cohort of students whose algebra skills are not automatic, curriculum innovation that encourages students to verbalize algebraic procedures and algorithms, avoids a disjointed presentation of algebra, and provides carefully structured intensive practice, might reverse current trends and increase the number of students who are prepared properly for the study of calculus.

References

[BLS] U. S. Bureau of Labor Statistics, "College Enrollment and Work Activity of Year 2000 High School Graduates," April 2001, available at http:stats.bls.gov/news.release/hsgec.nr0.htm.

[CPM] College Preperatory Mathematics, see www.cpm.org

[FS] *Reconstructing Mathematics Education*, by Deborah Shifter and Cathy Twomey Fosnot, Teachers College Press, 1992