The Three Ancient Geometric Problems

The Three Problems

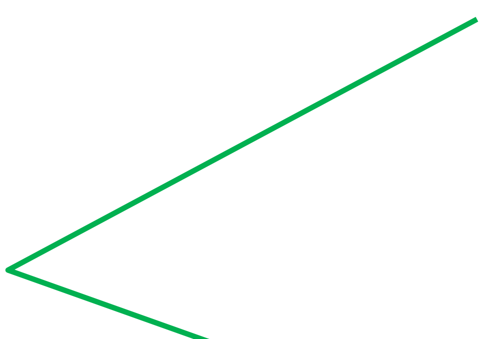
Constructions

- trisect the angle
- double the cube
- square the circle

The Three Problems

trisecting the angle

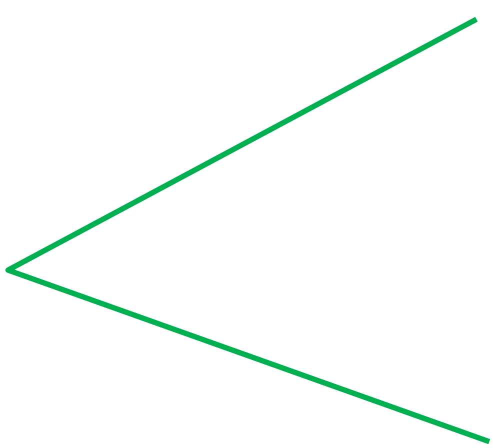
Given an angle,



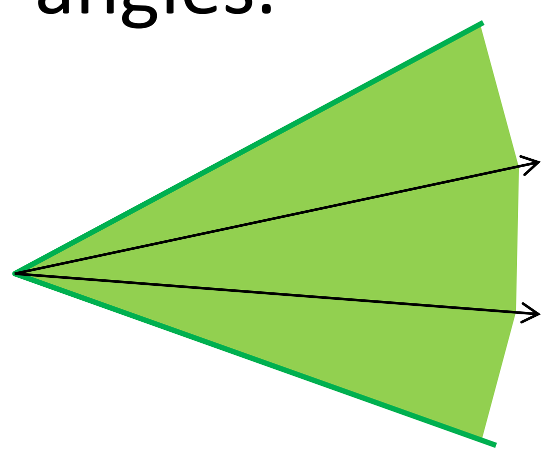
The Three Problems

trisecting the angle

Given an angle,



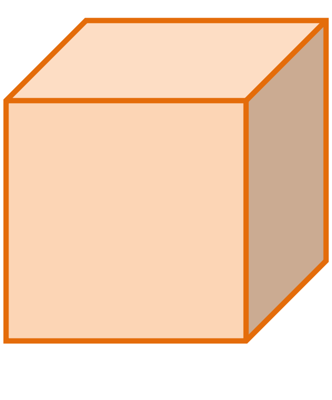
break it up into three equal angles.



The Three Problems

doubling the cube

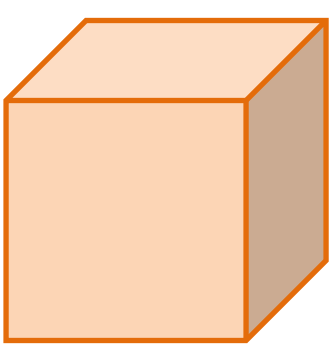
Given a cube,



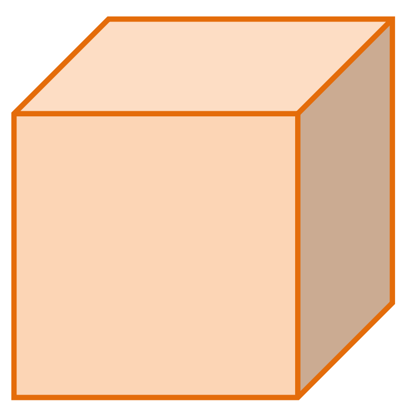
The Three Problems

doubling the cube

Given a cube,



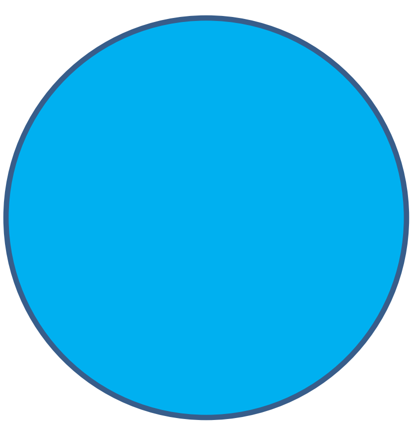
construct one with twice the volume.



The Three Problems

squaring the circle

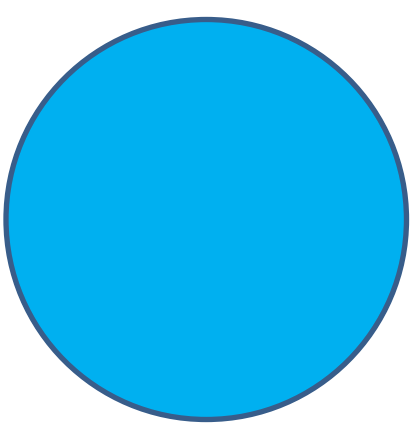
Given a circle,



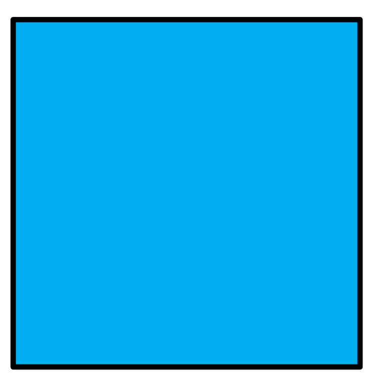
The Three Problems

squaring the circle

Given a circle,



construct a square of the same area.



Constructions

first encountered
in the fifth century BCE

Constructions

first encountered
in the fifth century BCE

the three problems:
Anaxagoras (~ 450)

Constructions

first encountered
in the fifth century BCE

the three problems:
Anaxagoras (~ 450)

straightedge and compass rule
came two centuries later

(Apollonius)

Constructions

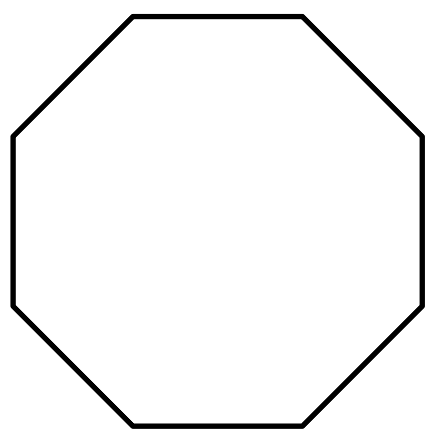
Originally:

Construct a geometric figure with stated properties.

Constructions

Construct a geometric figure with stated properties.

Construct a regular octagon.

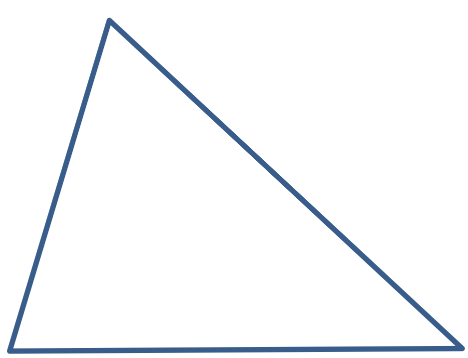


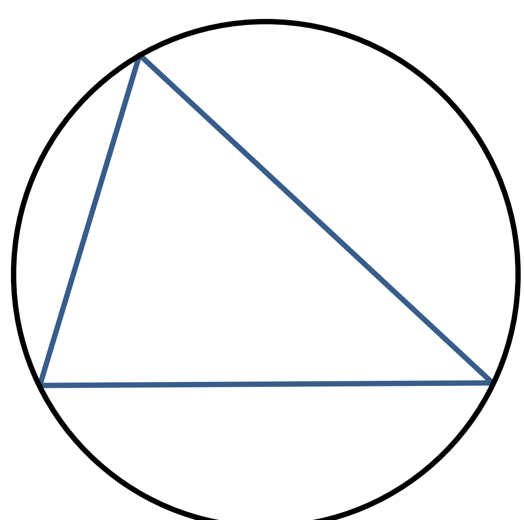
Constructions

Construct a geometric figure with stated properties.

Given a triangle,

construct the circumcircle.





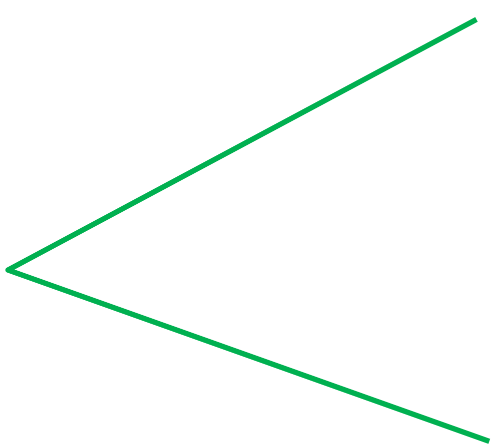
animation

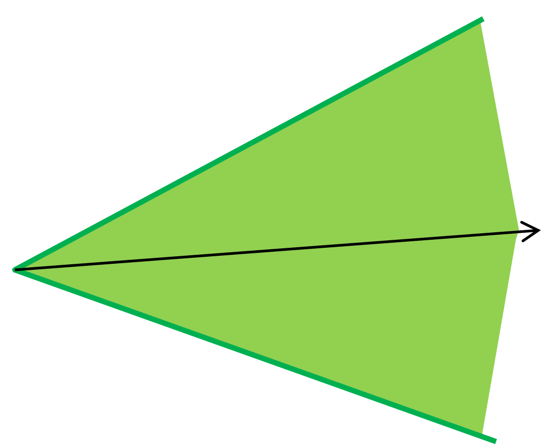
Constructions

Later:

Partition or enlarge.

Given an angle, bisect it.

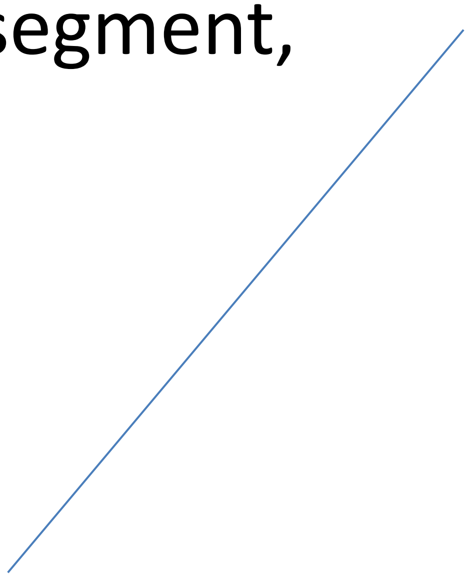




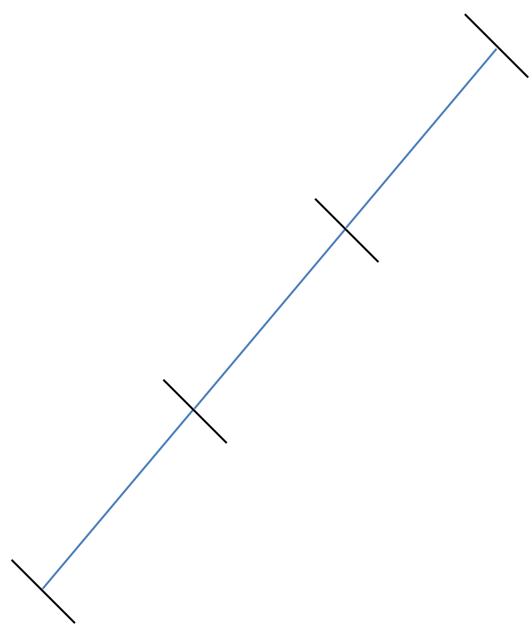
Constructions

Partition.

Given a segment,



trisect it.

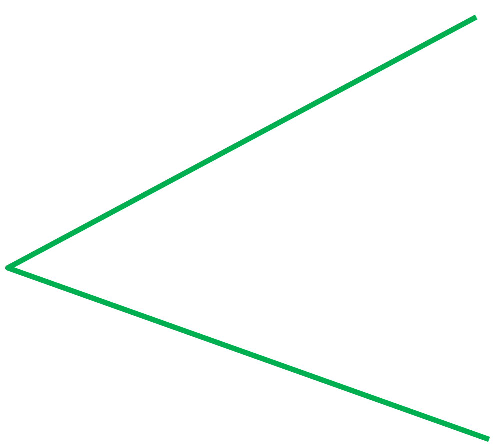


method

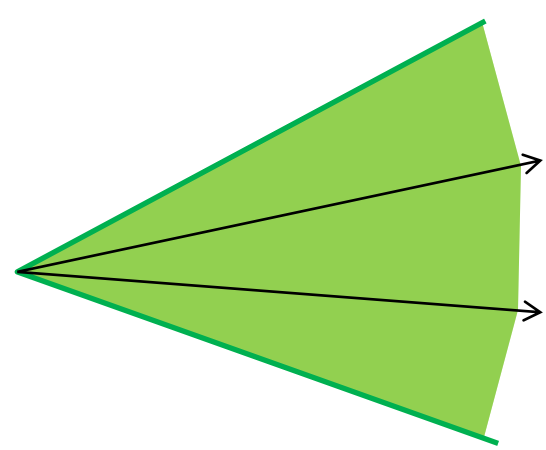
Constructions

Partition.

Given an angle,



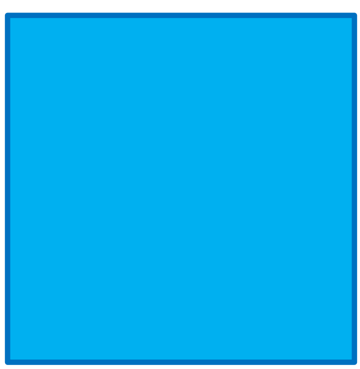
trisect the angle.



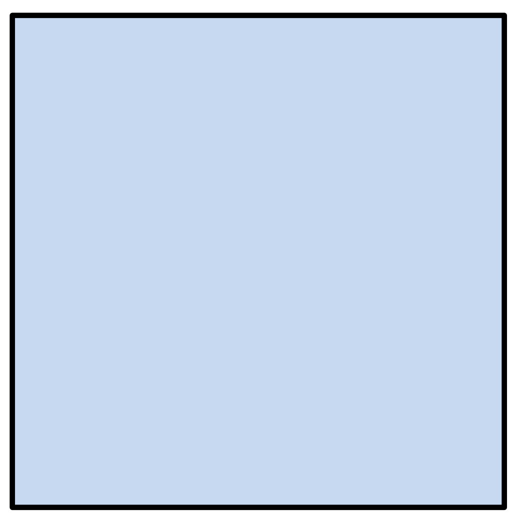
Constructions

Enlarge.

Given a square,



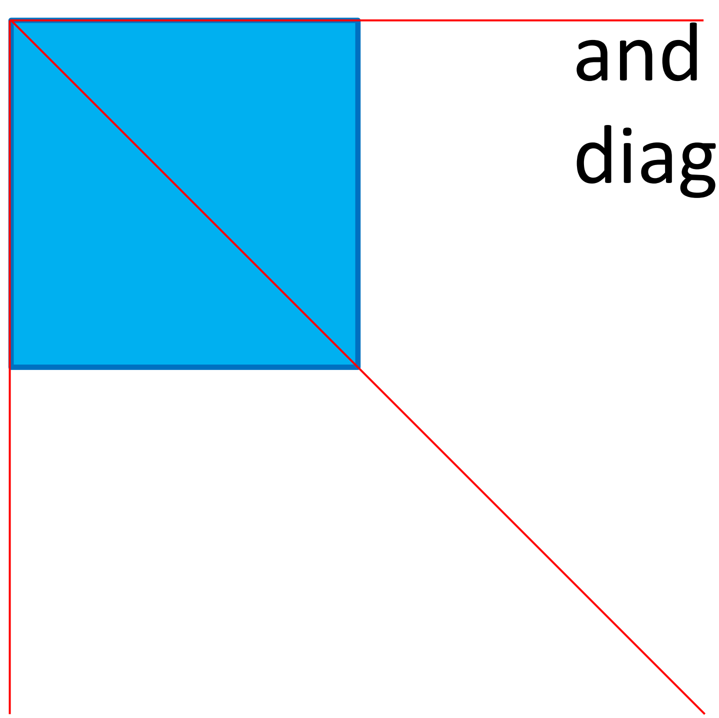
construct one twice as big.



Constructions

Enlarge.

Given a square,

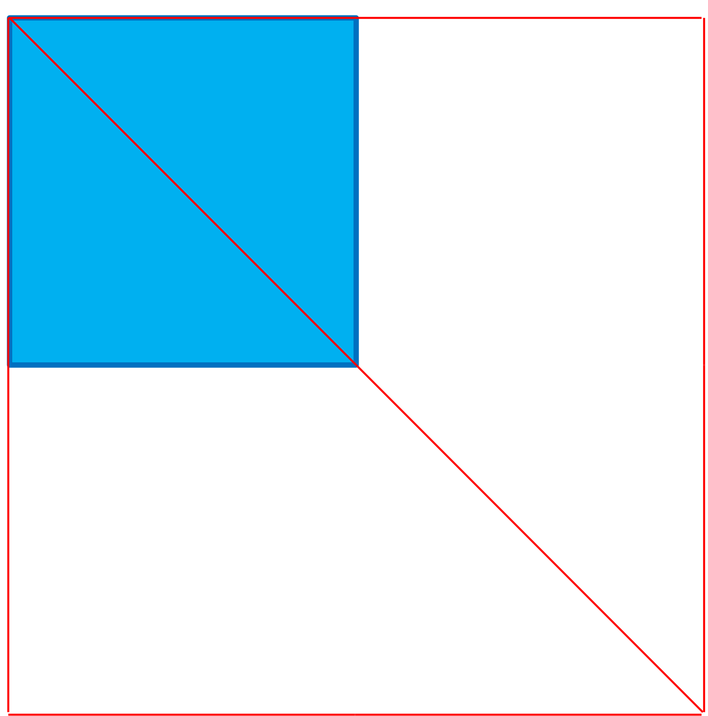


double two adjacent sides and their diagonal.

Constructions

Enlarge.

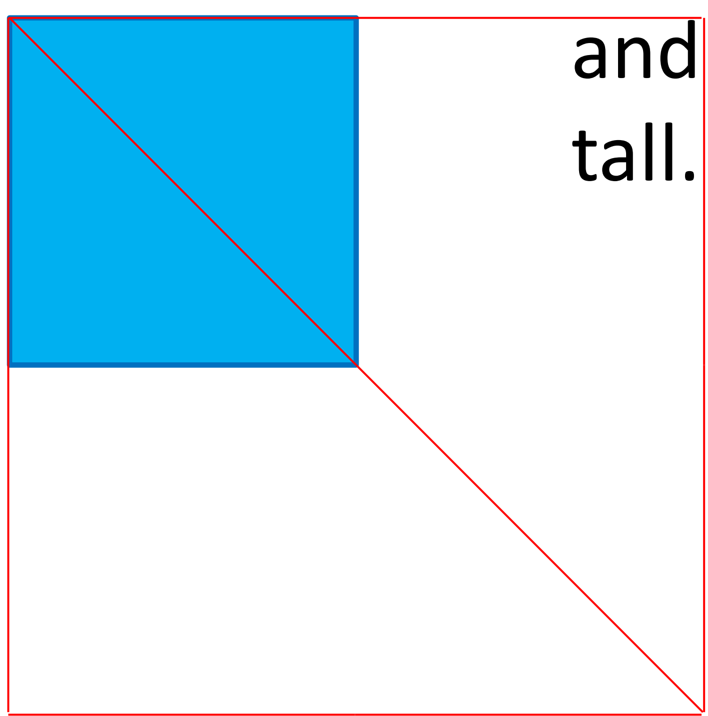
Given a square,   join the ends.



Constructions

Enlarge.

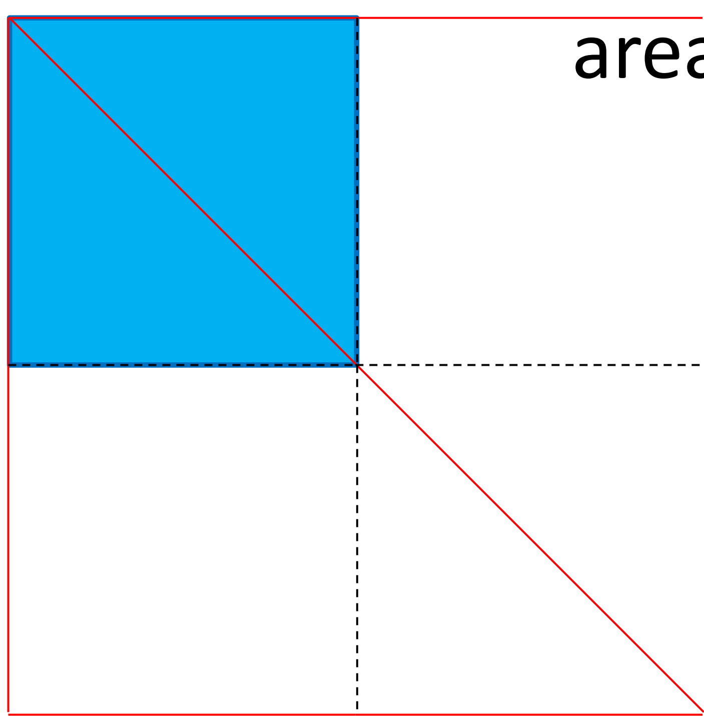
Given a square, the red one is twice as wide and twice as tall.



Constructions

Enlarge.

Given a square, the red one has four times the area.

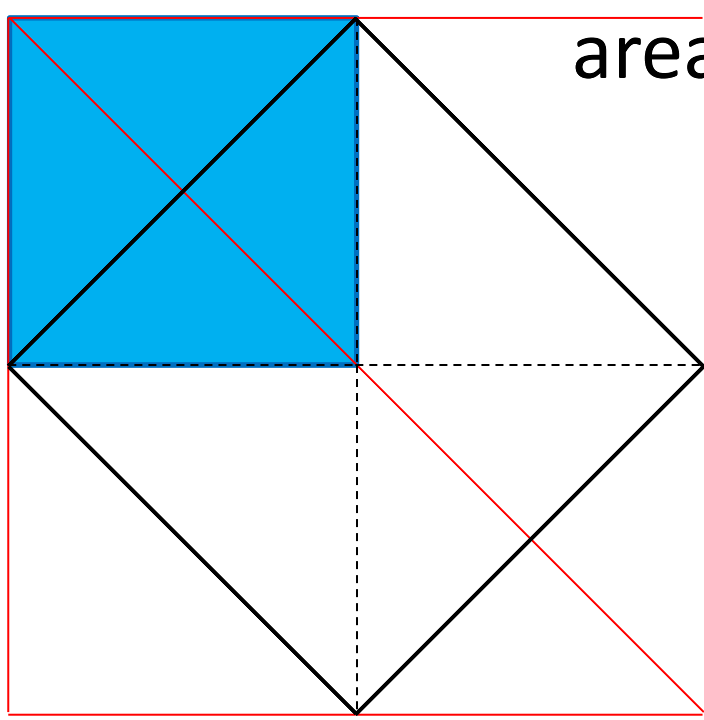


Constructions

Enlarge.

Given a square,

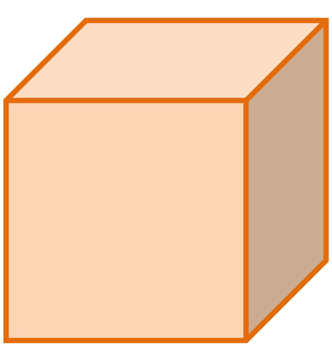
the black one has twice the area.



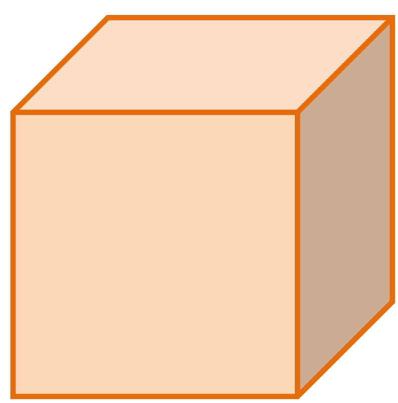
Constructions

Enlarge.

Given a cube,



double the cube.

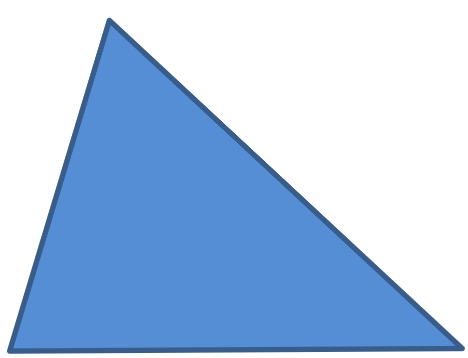


Constructions

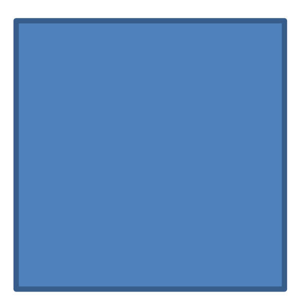
Finally:

Quadratures

Given a triangle,



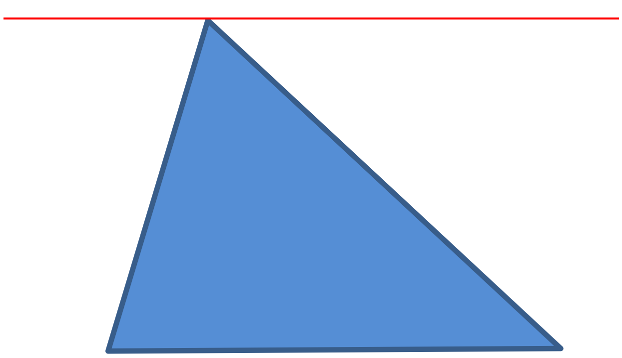
construct a square of the same area.



Constructions

Quadratures

Given a triangle,

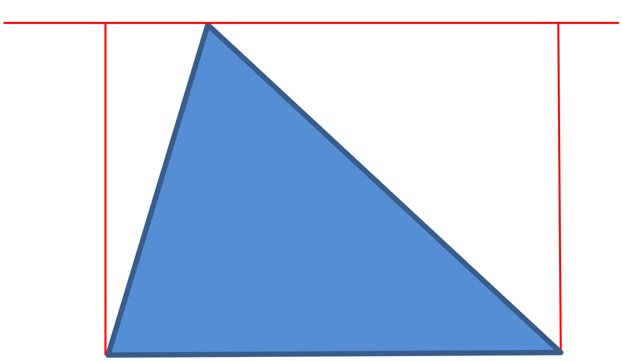


construct the parallel to the base through the top.

Constructions

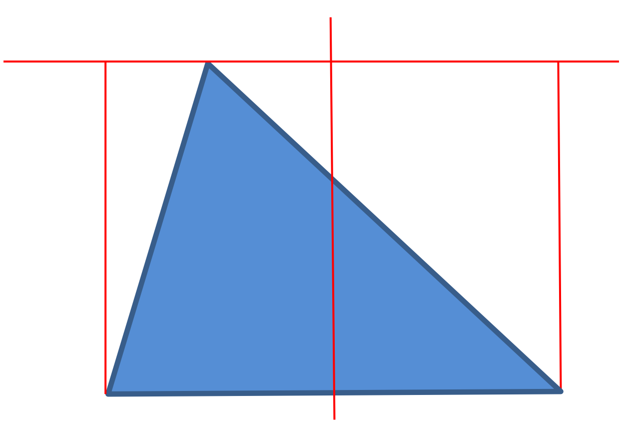
Quadratures

Raise the perpendiculars to the parallel.



Constructions

Quadratures



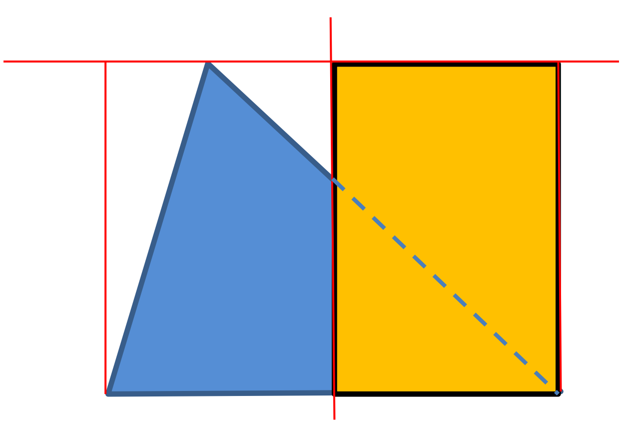
Construct the perpendicular bisector of the base.

Constructions

Quadratures

Given a triangle,

this rectangle has the same area.

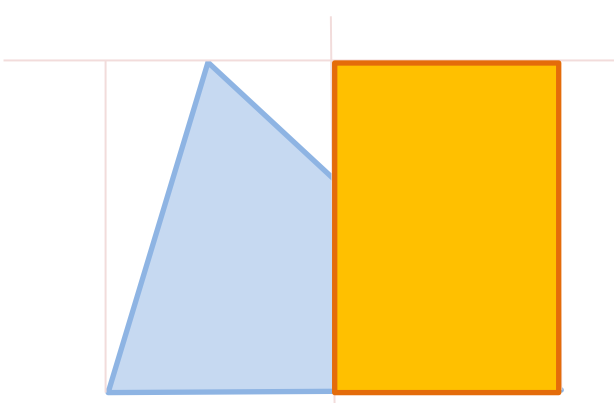


Constructions

Quadratures

Given a rectangle,

construct a square of the same area.



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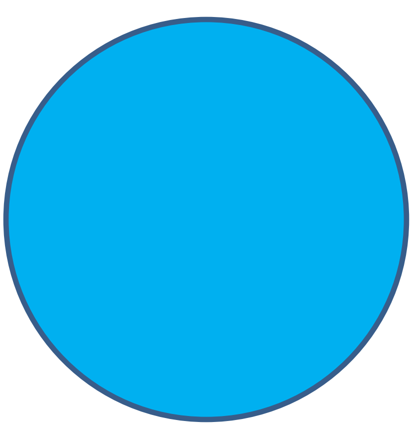
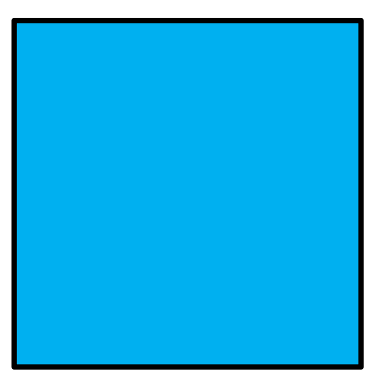
Constructions

Quadratures

Given a circle,



square the circle.

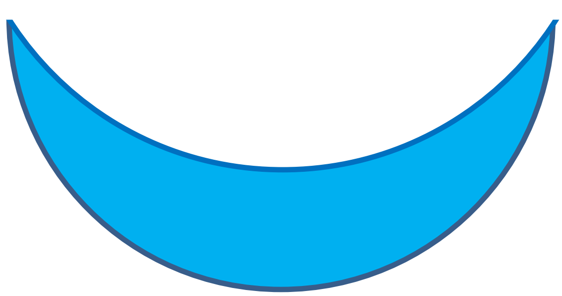


Constructions

Quadratures:

Lunes

region (blue) inside a circle, outside a bigger one

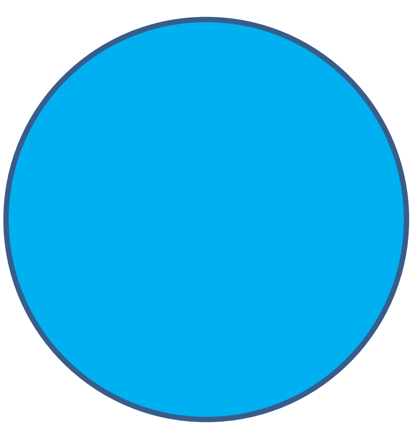


Constructions

Quadratures:

One Lune of Hippocrates

Start with a circle.

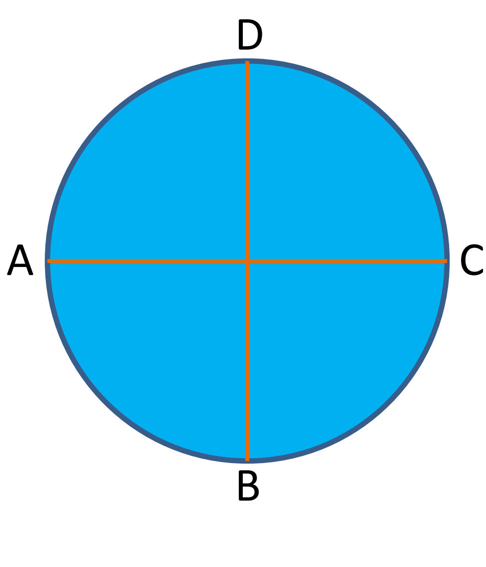


Constructions

Quadratures:

One Lune of Hippocrates

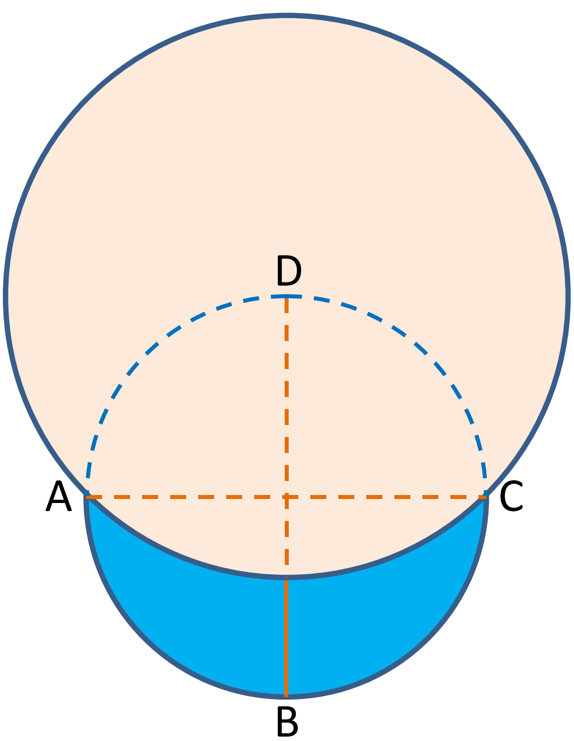
Construct perpendicular diameters.



Constructions

Quadratures:

One Lune of Hippocrates

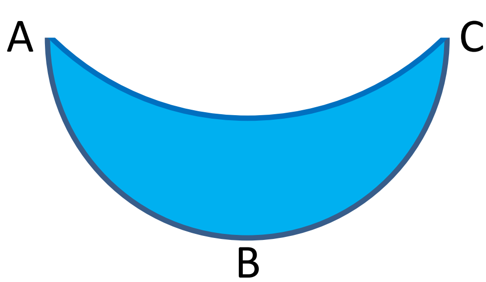


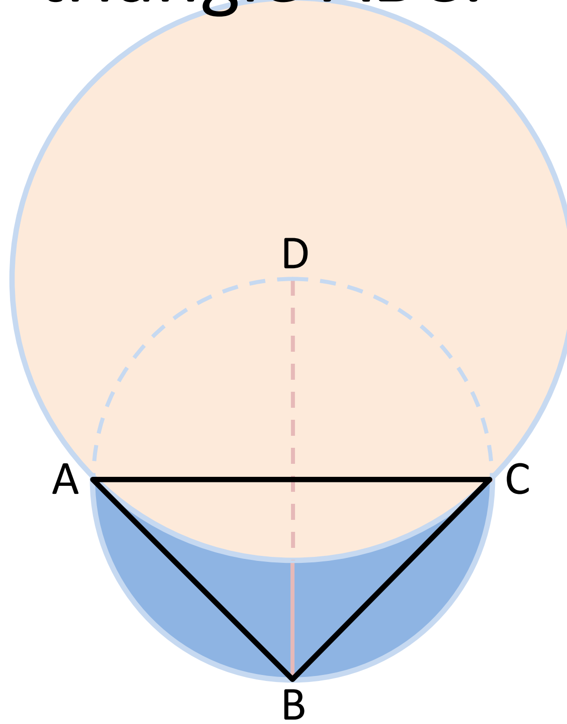
Construct the circle centered at D reaching A and C.

Constructions

Quadratures:
One Lune of Hippocrates

The lune has the same area as triangle ABC.





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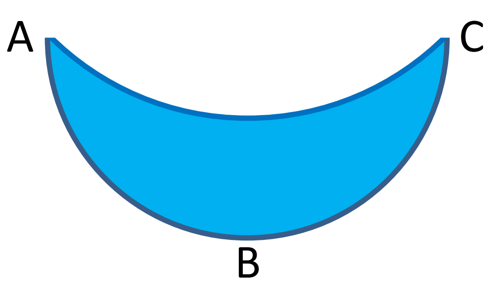
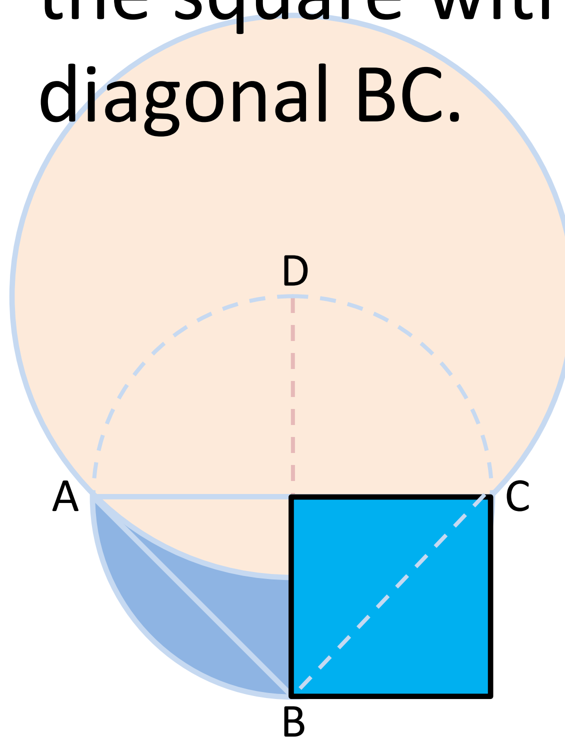
Constructions

Quadratures:
One Lune of Hippocrates

The lune has the same area as the square with diagonal BC.



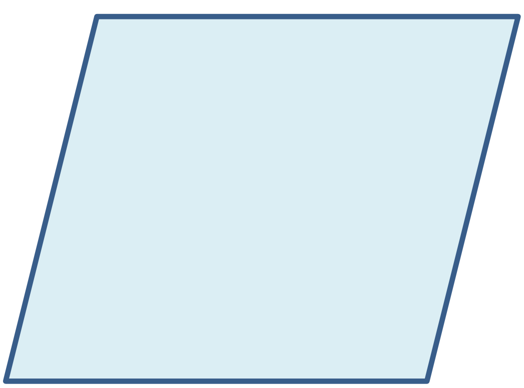


Constructions

Impossibility:

Circumscribing a parallelogram that is not a rectangle

parallelogram



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Impossibility:

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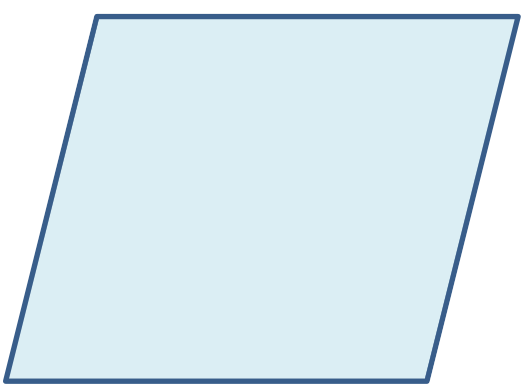
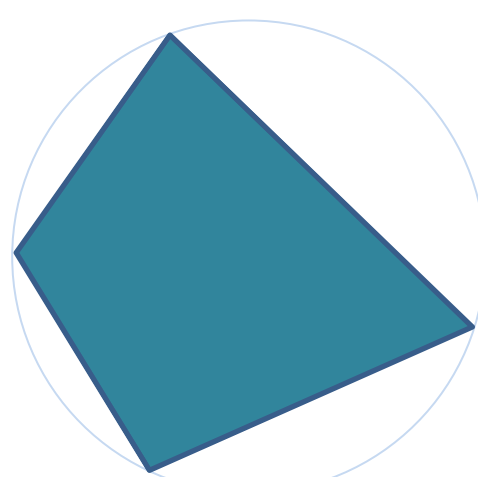


inscribed quadrilateral

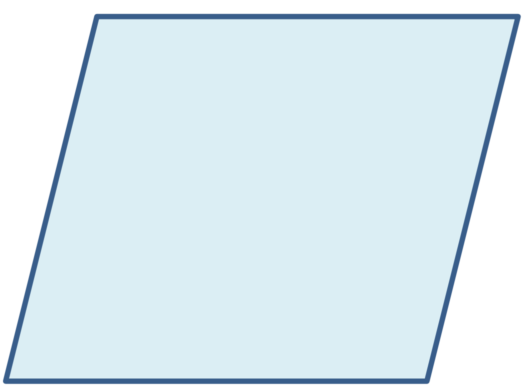


Constructions

Impossibility:

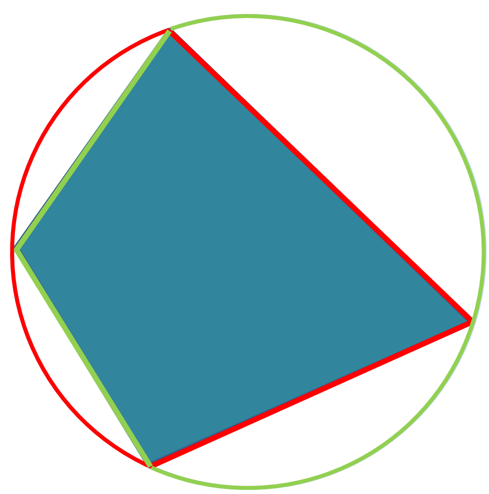
Circumscribing a parallelogram that is not a rectangle

parallelogram



opposite angles are equal

opposites are supplementary



Algebra

Algebra

developed from
Al Khwarismi
(ninth century CE)

wrote the book
Al Jabr…

practical

method-
oriented

example

Algebra

developed from
Al Khwarismi
(ninth century CE)

wrote the book
Al Jabr…

practical

method-
oriented

“algorithmic”

Algebra

through
Leonardo of Pisa
("Fibonacci," ~ 1180-1250)

wrote the book
Liber Abaci

brought algebra
into Latin

methods for
quadratic, even
some cubics

Algebra

through
Geronimo Cardano
(1501-1576)

wrote the book
Ars Magna

solutions of
cubics and
quartics

just solutions;
no thought of
practicality

Algebra

through
Geronimo Cardano
(1501-1576)

wrote the book
Ars Magna

just solutions;
no thought of
practicality

suggested
complex
numbers

Algebra

through
François Viète
(1540-1603)

distinguished unknowns (represented by vowels) from coefficients ("parameters")

not concerned with solution methods

sought relations between roots and coefficients

example

Algebra

to
Évariste Galois
(1811-1832)

completely abstract treatment: “groups”

connected groups with polynomials

proved that no formula solves degree ≥ 5

Mario Livio, *The Equation That Couldn’t Be Solved*

Algebra

to
Évariste Galois
(1811-1832)

completely abstract treatment: “groups”

connected groups with polynomials

connected polynomials with constructions

Galois Theory

Theorem: A number is constructible if and only if it satisfies an integer polynomial and the degree of the minimal such polynomial is a power of 2.

Galois Theory

Theorem: A number is constructible …

Given length 1,

1

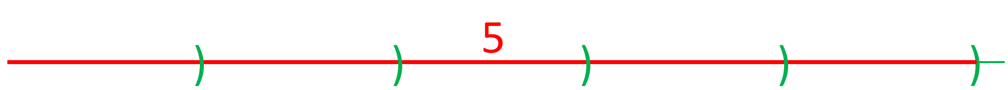
construct any integer.

Galois Theory

Theorem: A number is constructible

…

Given length 1, construct any integer.



Galois Theory

Theorem: A number is constructible …

Given length 1,

1

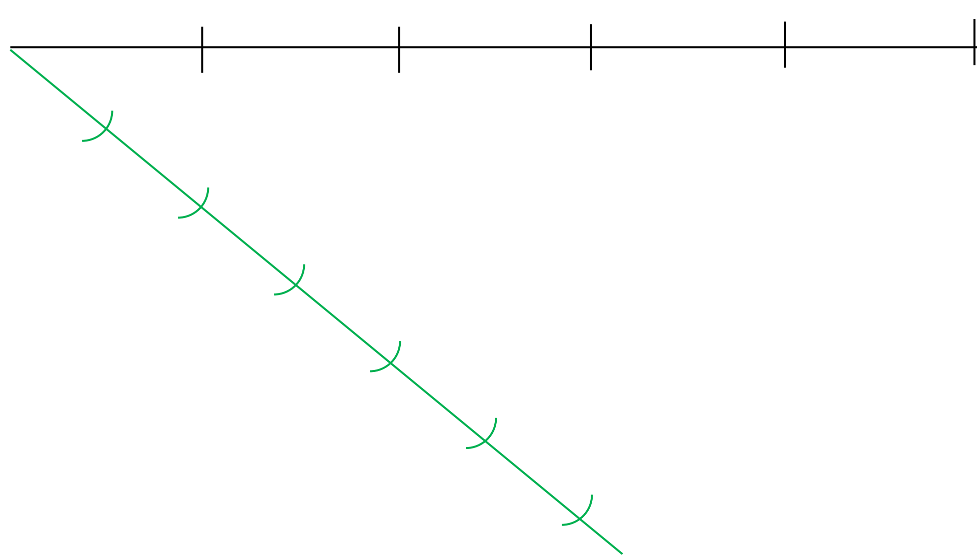
construct any rational number.

Galois Theory

Theorem: A number is constructible

…

Given length 1, construct any rational number.

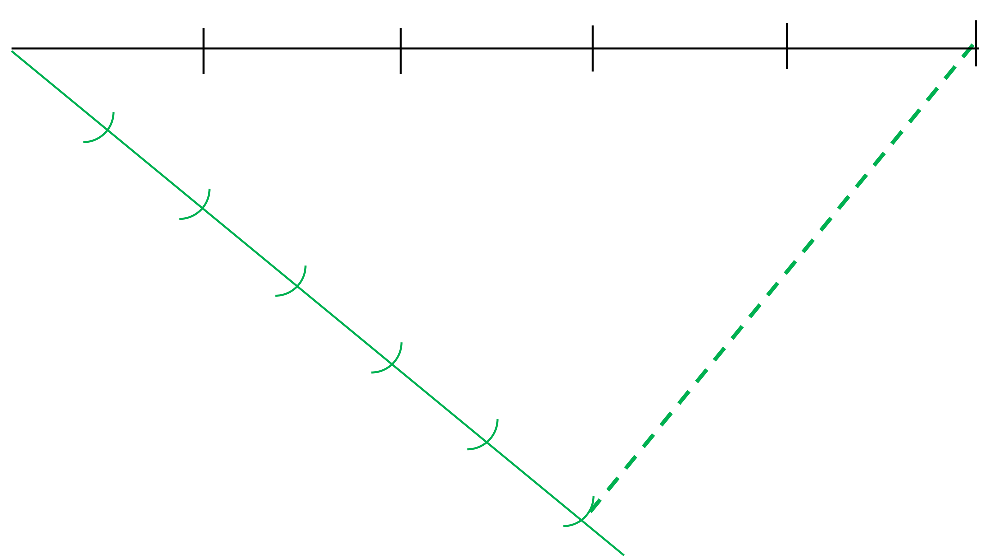


Galois Theory

Theorem: A number is constructible

…

Given length 1, construct any rational number.



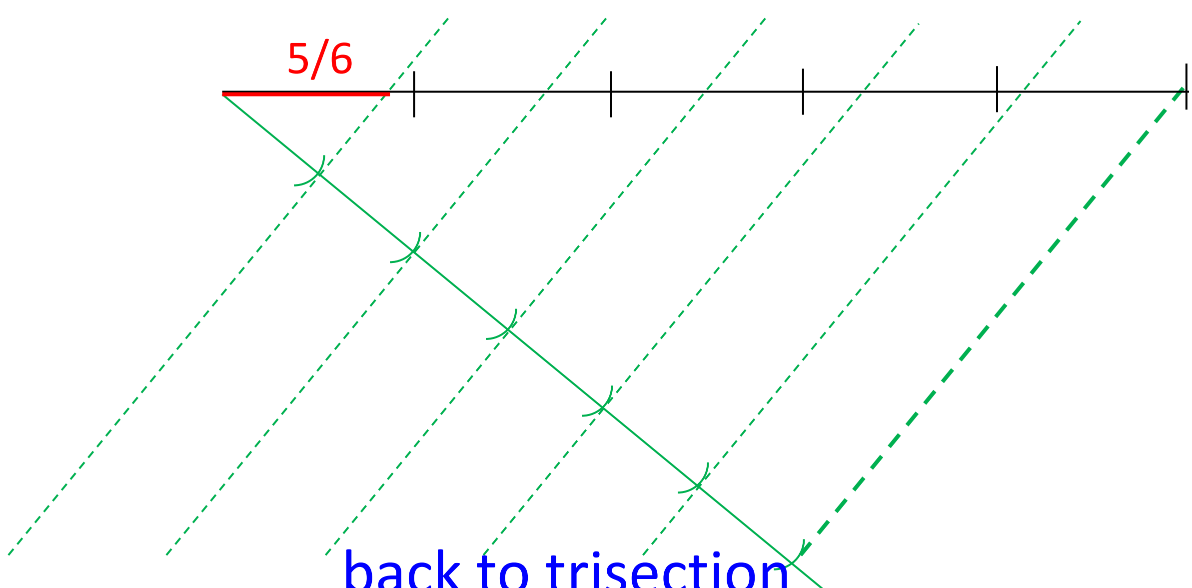
Galois Theory

Theorem: A number is constructible

…

Given length 1, construct any rational number.

1



back to trisection

Galois Theory

Theorem: A number is constructible …

Given length 1,

1

construct an irrational number.

Galois Theory

Theorem: A number is constructible

…

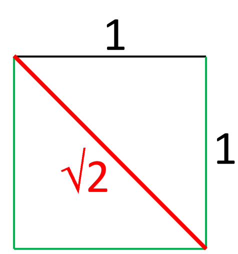
Given length 1, construct $\sqrt{2}$.

Galois Theory

Theorem: A number is constructible

…

Given length 1, construct √2.

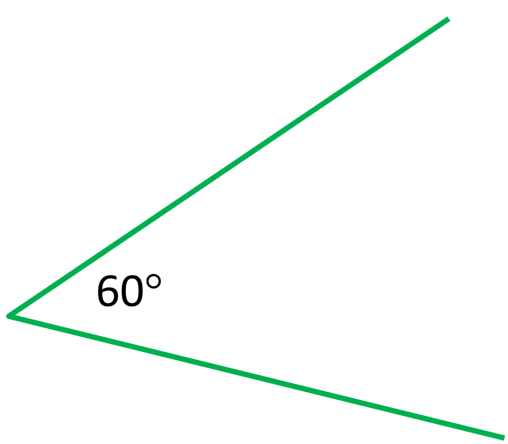


Galois Theory

Theorem: A number is constructible

…

To trisect this angle



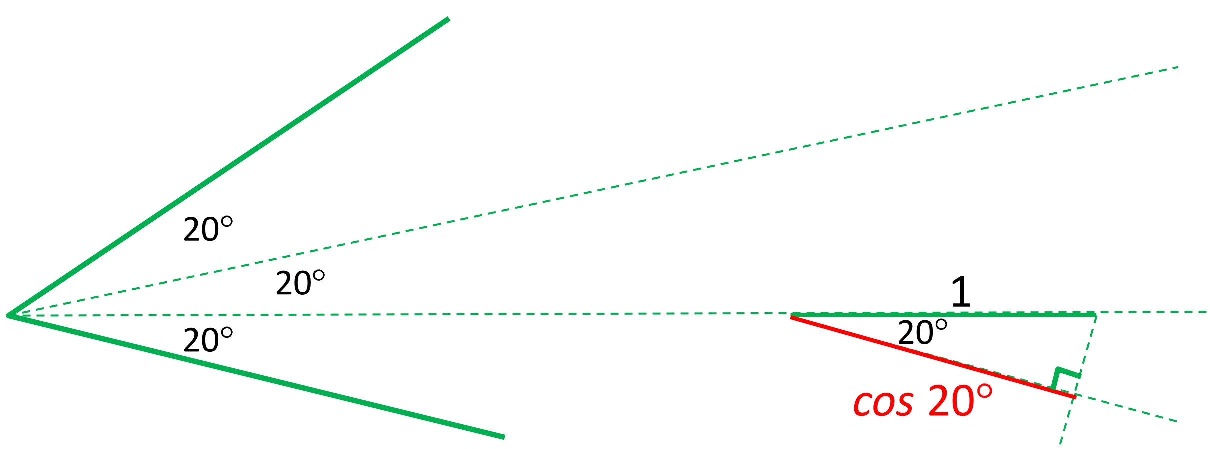
Galois Theory

Theorem: A number is constructible

…

To trisect this angle

is to construct *cos* 20°.

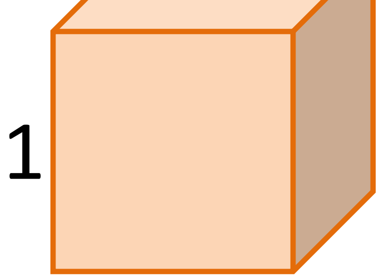


Galois Theory

Theorem: A number is constructible

…

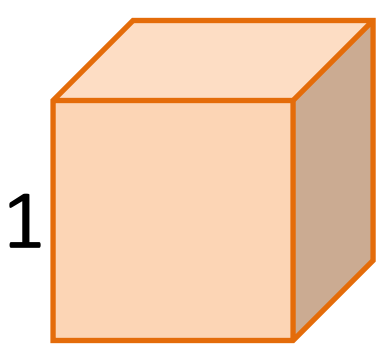
To double this cube



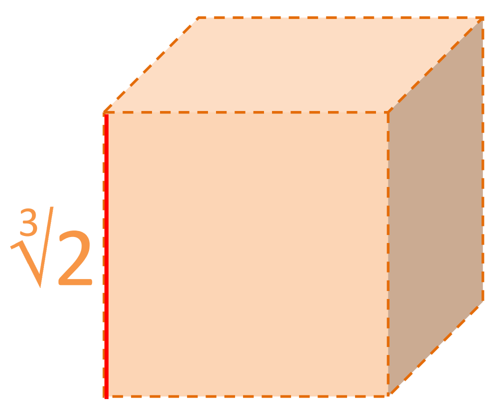
Galois Theory

Theorem: A number is constructible …

To double this cube



is to construct $\sqrt[3]{2}$.

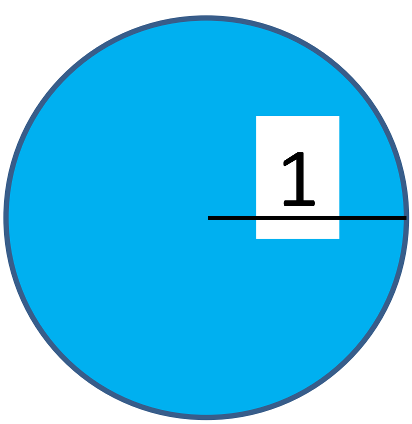


Galois Theory

Theorem: A number is constructible

…

To square this circle

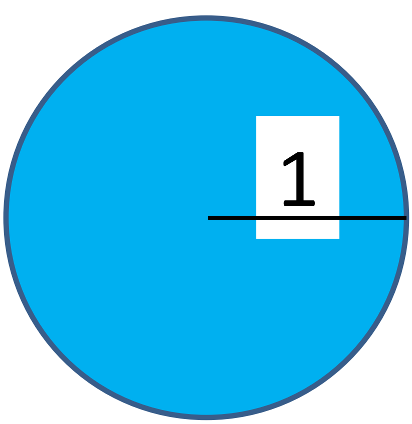


Galois Theory

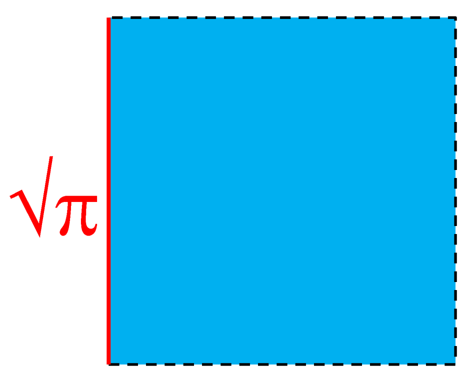
Theorem: A number is constructible

…

To square this circle



is to construct $\sqrt{\pi}$.



Galois Theory

Theorem: A number is constructible
if and only if
…

The following numbers are constructible, and no others.

Galois Theory

Theorem: A number is constructible
if and only if
it satisfies an integer polynomial
…

$\sqrt{2}$ satisfies

$$x^2 - 2 = 0.$$

Galois Theory

Theorem: A number is constructible
if and only if
it satisfies an integer polynomial
…

5/6 satisfies

$$6x - 5 = 0.$$

Galois Theory

Theorem: A number is constructible if and only if it satisfies an integer polynomial …

$\sqrt[3]{2}$ satisfies

$$x^3 - 2 = 0.$$

Galois Theory

Theorem: A number is constructible
if and only if
it satisfies an integer polynomial
and the degree of the minimal
such polynomial …

√2 satisfies

$$x^2 - 2 = 0.$$

Galois Theory

Theorem: A number is constructible
if and only if
it satisfies an integer polynomial
and the degree of the minimal
such polynomial …

$\sqrt{2}$ satisfies

$x^2 - 2 = 0,$

and

$(x^2 - 2)^3 = 0$

Galois Theory

Theorem: A number is constructible
if and only if
it satisfies an integer polynomial
and the degree of the minimal
such polynomial …

$\sqrt{2}$ satisfies

$$x^2 - 2 = 0,$$

and

$$(x^2 - 2)^3 = 0$$

and

$$(x^2 - 2)(x^{15} + 1) = 0.$$

Galois Theory

Theorem: A number is constructible
if and only if
it satisfies an integer polynomial
and the degree of the minimal
such polynomial …

$\sqrt{2}$ satisfies

$$x^2 - 2 = 0.$$

That is minimal.

Galois Theory

Theorem: A number is constructible
if and only if
it satisfies an integer polynomial
and the degree of the minimal
such polynomial is a power of 2.

$\sqrt{2}$ satisfies

$$x^2 - 2 = 0.$$

That is minimal.

Therefore you can construct $\sqrt{2}$.

Galois Theory

Theorem: A number is constructible
if and only if
it satisfies an integer polynomial
and the degree of the minimal
such polynomial is a power of 2.

5/6 satisfies

$$6x - 5 = 0.$$

That is minimal.

Therefore you can construct 5/6.

Galois Theory

Theorem: A number is constructible
if and only if
it satisfies an integer polynomial
and the degree of the minimal
such polynomial is a power of 2.

$\sqrt[3]{2}$ satisfies

$$x^3 - 2 = 0.$$

That is minimal.

evidence

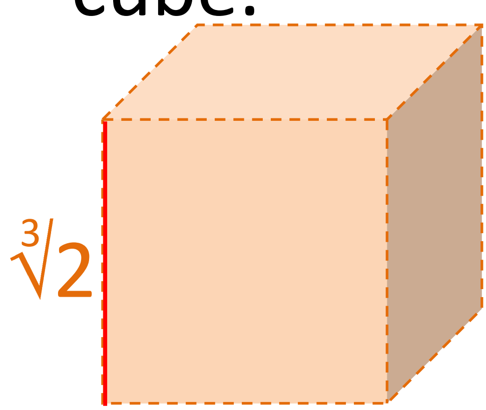
Therefore you CANNOT construct $\sqrt[3]{2}$.

Galois Theory

Theorem: A number is constructible
if and only if
it satisfies an integer polynomial
and the degree of the minimal
such polynomial is a power of 2.

You cannot construct $\sqrt[3]{2}$.

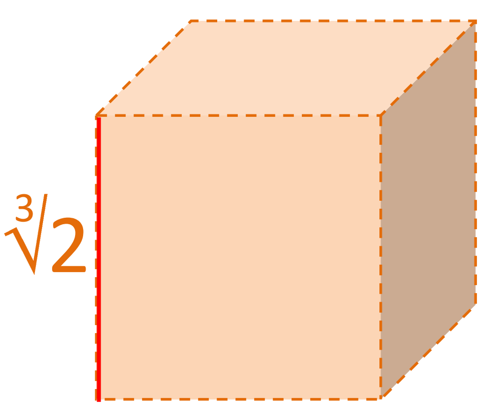
Therefore you cannot construct this cube.



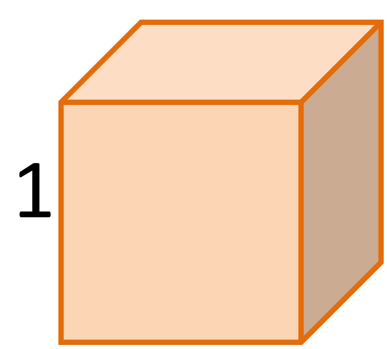
Galois Theory

Theorem: A number is constructible
if and only if
it satisfies an integer polynomial
and the degree of the minimal
such polynomial is a power of 2.

You cannot construct this cube.



Therefore you cannot double this one.



Galois Theory

Theorem: A number is constructible
if and only if
it satisfies an integer polynomial
and the degree of the minimal
such polynomial is a power of 2.

cos 20° satisfies

$8x^3 - 6x - 1 = 0.$

proof

That is minimal.

evidence

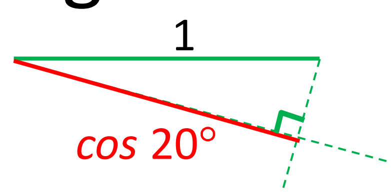
Therefore you CANNOT construct *cos* 20°.

Galois Theory

Theorem: A number is constructible if and only if it satisfies an integer polynomial and the degree of the minimal such polynomial is a power of 2.

You cannot construct *cos* 20°.

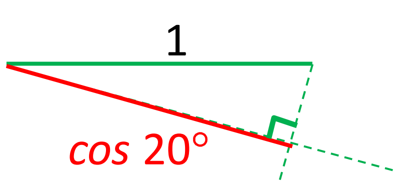
Therefore you cannot construct this triangle:



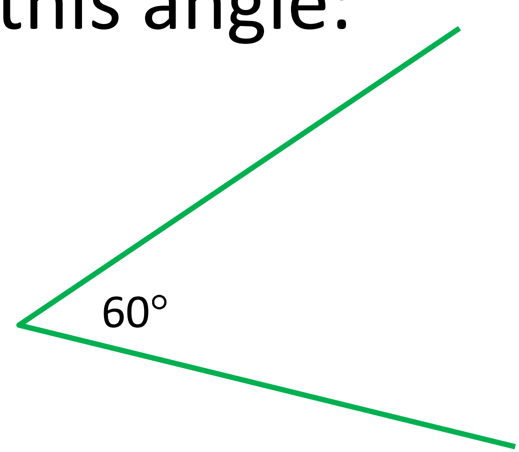
Galois Theory

Theorem: A number is constructible
if and only if
it satisfies an integer polynomial
and the degree of the minimal
such polynomial is a power of 2.

You cannot construct this triangle:



Therefore you cannot trisect this angle:



Galois Theory

Theorem: A number is constructible if and only if it satisfies an integer polynomial and the degree of the minimal such polynomial is a power of 2.

π does not satisfy any polynomial with integer coefficients.

Therefore you cannot construct π.

extremely advanced

Galois Theory

Theorem: A number is constructible
if and only if
it satisfies an integer polynomial
and the degree of the minimal
such polynomial is a power of 2.

You cannot construct π.

Therefore you cannot construct $\sqrt{\pi}$.

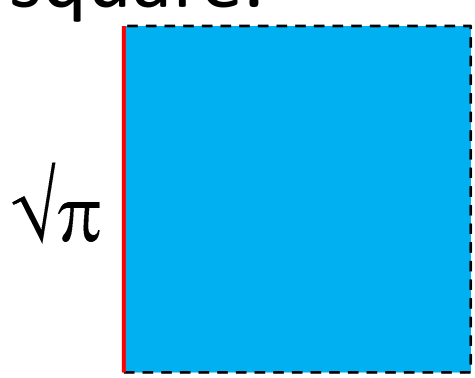
proof

Galois Theory

Theorem: A number is constructible
if and only if
it satisfies an integer polynomial
and the degree of the minimal
such polynomial is a power of 2.

You cannot construct $\sqrt{\pi}$.

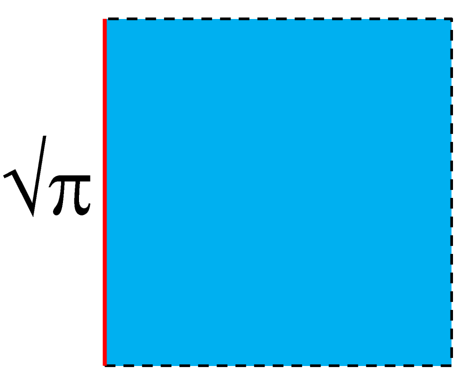
Therefore you cannot construct this square.



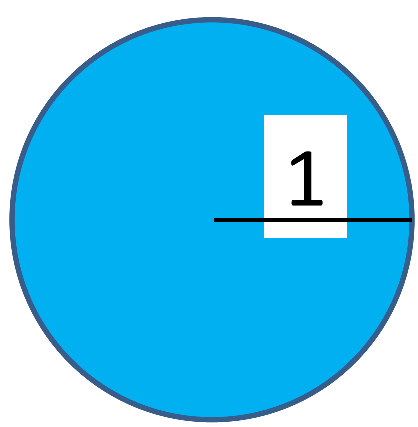
Galois Theory

Theorem: A number is constructible
if and only if
it satisfies an integer polynomial
and the degree of the minimal
such polynomial is a power of 2.

You cannot construct this square.



Therefore you cannot square the unit circle.



Homework

1. Given an angle, construct its bisector.

2. Construct a regular octagon.

3. Construct a regular pentagon. (Extremely hard. Go to pages 15-16 for some needed facts.)

4. Show that you cannot construct a regular nonagon.

$x^3 - 2$ is minimal for $\sqrt[3]{2}$. (Likewise $8x^3 - 6x - 1$ is minimal for $\cos 20°$)

1. If it is not minimal, then the minimal polynomial $m(x)$ has to be a factor of it:
   $x^3 - 2 = m(x)q(x)$.
   Reason: advanced theorem

2. If it factors that way,
   $x^3 - 2 = m(x)q(x)$,
   then either $m(x)$ or $q(x)$ is a linear factor $x - r$.
   Reason: The degrees of $m(x)$ *and* $q(x)$ *have to add up to 3*.

$x^3 - 2$ is minimal for $\sqrt[3]{2}$.
(Likewise $8x^3 - 6x - 1$ is minimal for *cos* 20°)

2. If it factors that way,
   $x^3 - 2 = m(x)q(x)$,
   then either $m(x)$ or $q(x)$ is a linear factor $x - r$.
   Reason: The degrees of $m(x)$ *and* $q(x)$ *have to add up to 3*.
3. If $x - r$ is a factor, then r is a rational root, of $x^3 - 2$.
   (*Reason: Factor Theorem*)
4. That's impossible: $x^3 - 2$ does not have any rational roots.
   (*Reason: Rational Roots Theorem*)

back

$\cos 20^\circ$ is a root of $8x^3 - 6x - 1$

1. This is a "triple-angle formula":

$\cos 3A$

$= \cos (2A + A)$

$= \cos 2A \cos A - \sin 2A \sin A$
(sum formula)

$= (2 \cos^2 A - 1) \cos A - 2 \sin A \cos A \sin A$
(double-angle formulas)

$= (2 \cos^2 A - 1) \cos A - 2 \sin^2 A \cos A$

(equations continue)

$\cos 20^\circ$ is a root of $8x^3 - 6x - 1$

1. This is a "triple-angle formula":

   $\cos 3A$

   $= (2 \cos^2 A - 1) \cos A - 2 \sin^2 A \cos A$

   $= 2 \cos^3 A - \cos A - 2 \sin^2 A \cos A$

   $= 2 \cos^3 A - \cos A - 2 (1 - \cos^2 A) \cos A$

   $= 4 \cos^3 A - 3 \cos A$

$\cos 20^\circ$ is a root of $8x^3 - 6x - 1$

1. This is a "triple-angle formula":

   $\cos 3A$

   $= 4 \cos^3 A - 3 \cos A.$

2. Substitute A = 20°:
   $\cos 60^\circ$

   $= 4 \cos^3 20^\circ - 3 \cos 20^\circ.$

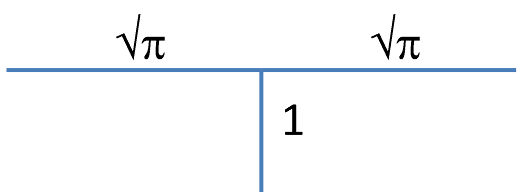
3. $1/2 = 4 \cos^3 20^\circ - 3 \cos 20^\circ$
   rearranges to
   $0 = 8 \cos^3 20^\circ - 6 \cos 20^\circ - 1$.

back

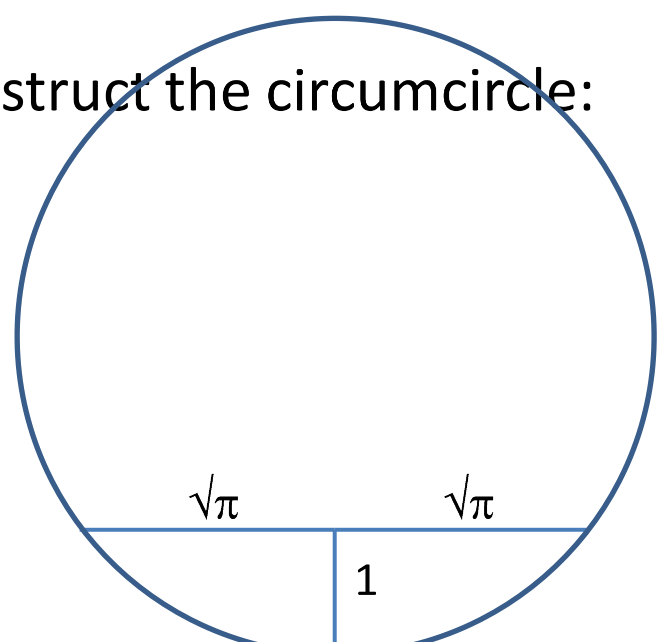
You cannot construct √π because you cannot construct π.

If you could construct √π

1. Double the length and construct the perpendicular bisector to length 1:

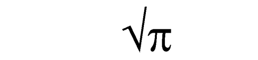


2. Construct the circumcircle:

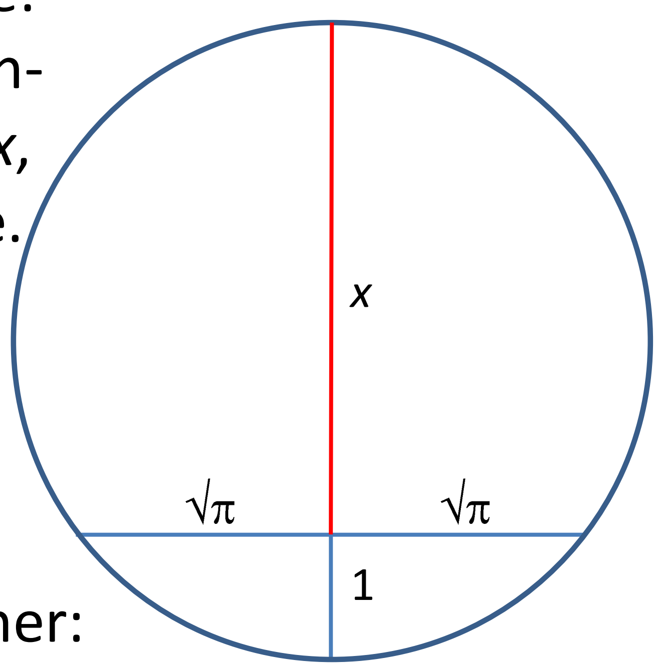


You cannot construct $\sqrt{\pi}$ because you cannot construct π.

If you could construct $\sqrt{\pi}$:



3. In the circumcircle: extend the perpendicular by length *x*, to reach the circle.



4. The product of the pieces of one chord equals the product in the other:

   $x\ 1 = \sqrt{\pi}\ \sqrt{\pi}$

   $x = \pi$;

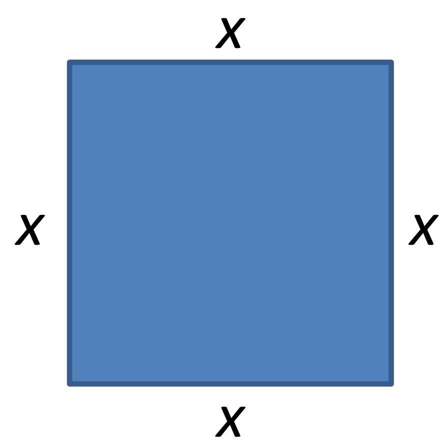
   you would have constructed π.

back

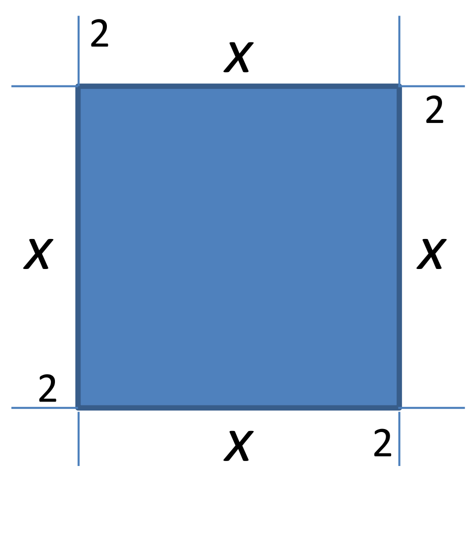
Al Khwarismi's "Completing the Square"

To solve $x^2 + 8x = 65$:

1. Build x^2.



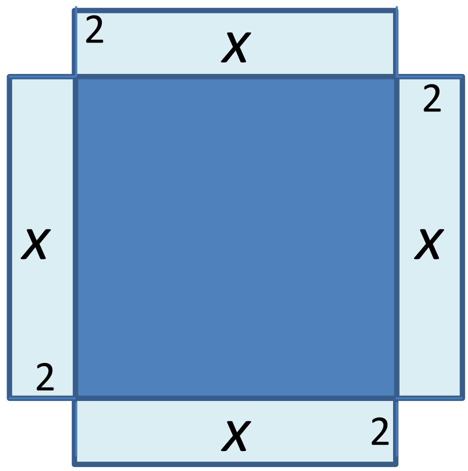
2. Add 1/4 of 8 (coefficient of x) to all the sides.



Al Khwarismi's "Completing the Square"

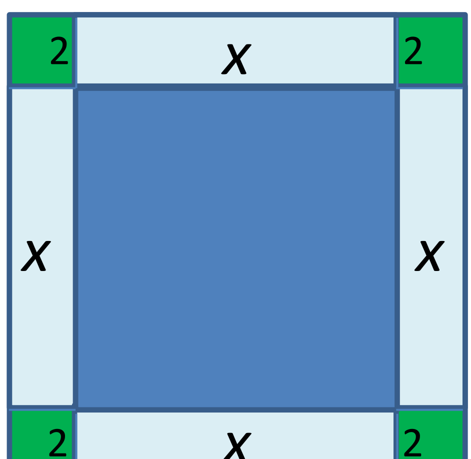
To solve $x^2 + 8x = 65$:

3. The area of this figure is $x^2 + 8x = 65$.



4. "Complete the square":
   $x^2 + 8x + 4(4)$
   $= 65 + 4(4)$

   $(x + 4)^2 = 81$
   $x + 4 = 9$
   $x = 5$



back

Relations Between the Roots and Coefficients in a Quadratic

By the Quadratic Formula, the roots of

$$ax^2 + bx + c = 0$$

are

$$r = (-b + \sqrt{[b^2 - 4ac]})/2a$$

and

$$s = (-b - \sqrt{[b^2 - 4ac]})/2a.$$

1. Add them:

$$r + s = (-b + -b)/2a = -b/a.$$

2. Multiply them:

$$rs = ([-b]^2 - [b^2 - 4ac])/4a^2 = c/a.$$

back