Math 308 - Solutions to the review for test #1

`Answer 1a:` The set $\mathbb{Z}$ is the set of all integers including negative numbers. We can see that if $n = 0, 1, 2, -1, -2$ then $n^2 < 5$ but if n is larger than 3 or less than -3 then n^2 will be larger than 5. Therefore, this set explicitly is $\{-2, -1, 0, -1, 2\}$.

`Answer 1b:` One can simply write out all the elements out and count them. But let us look for patterns. The set $\{0\}$ has cardinality 1. The set $\{0, 2\}$ has cardinality 2. The set $\{0, 2, 4\}$ has cardinality 3. The set $\{0, 2, 4, 6\}$ has cardinality 4. The pattern should be clear, take the largest number, divide it by two, and then add one. Therefore, $\{0, 2, 4, ..., 2014\}$ has cardinality $\frac{2014}{2} + 1 = 1007 + 1 = 1008$.

`Answer 1c:` Let $A = \{\emptyset\}$ and $B = \emptyset$. Then $B \subset A$, as the empty set is a subset of every set, however we can see that $B \in A$ also. This is just one examples, there are are endless list of such examples.

`Answer 1d:`

$$\mathcal{P}(S) = \Big\{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\Big\}$$

`Answer 1e:` This is the set consisting of the elements $\emptyset$, $\{\emptyset\}$, and $\{\emptyset, \{\emptyset\}\}$. Therefore, the cardinality is three. It is true that the elements themselves are sets also, but we do not count their elements.

`Answer 1f:` $\mathcal{P}(A) = \Big\{\emptyset, \{\emptyset\}\Big\} \Longrightarrow \mathcal{P}(\mathcal{P}(A)) = \Big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\Big\}$

`Answer 2a:` π is irrational. The original statement is false.

`Answer 2b:` For some $n \in \mathbb{N}, n + 1 < 2$. The original statement is false because for $n = 0$ we have that $0 + 1 \geq 2$ is a false statement.

`Answer 2c:` There is a $x \in \mathbb{R}$, such that for every $y > 0$, we have that $xy \neq x$. The original statement is true because if we choose $y = 1$ for every chosen x then we clearly have that $xy = x$.

`Answer 2d:` There is a $\varepsilon > 0$, such that for every $\delta > 0$, we have that $\delta \geq \varepsilon$. The original statement is true because for any positive real number we can choose half of it to get a real number which is smaller than it.

`Answer 2e:` There exists an $\varepsilon > 0$ such that for any $N \in \mathbb{N}$ there is an $n > N$ so that $\left|\sqrt[n]{2} - 1\right| \geq \varepsilon$.

`Answer 2f:` There exists an $\varepsilon > 0$ so that for any $\delta > 0$ there is an x with $|x| < \delta$ such that $|\sin x| \geq \varepsilon$.

`Answer 3:` (Left to the reader :p)

`Answer 4:` The statement $P(x) \implies Q(x)$ for all x is true. If $x \neq -2$ then $P(x)$ is false so that $P(x) \implies Q(x)$ is automatically true because a conditional statement with a false hypothesis/premise is automatically true. If however $x = -2$ then $P(x)$ happens to be true, but so is $Q(x)$ true, and so $P(-2) \implies Q(-2)$ is a true statement because both parts of the conditional happen to be true.

The statement $Q(x) \implies P(x)$ for all x is false. To show that it is false we need to come up with one example of x when $Q(x) \implies P(x)$ becomes false. Let us take $x = 2$ then $Q(2)$ happens to be true while $P(2)$ is false. Therefore, $Q(2) \implies P(2)$ is a false conditional statement as the hypothesis/premise is true while the conclusion is false.

The statement $P(x) \iff Q(x)$ for all x is false. The reason for this is that the biconditional statement is defined as both $P(x) \implies Q(x)$ and $Q(x) \implies P(x)$ are true statements. We observed that $P(x) \implies Q(x)$ is true while $Q(x) \implies P(x)$ is false, therefore the biconditional statement is not true.

`Answer 5:` The first equation in the hint can be derived from adding the other two equations in the hint. So we know that $|A \cup B| = |A| + |B| - |A \cap B|$. We now apply this idea to $|A \cup B \cup C|$.

Note that

$$\begin{aligned}
|A \cup B \cup C| &= |(A \cup B) \cup C| \\
&= |A \cup B| + |C| - |(A \cup B) \cap C| \\
&= |A \cup B| + |C| - |(A \cap C) \cup (B \cap C)| \\
&= |A \cup B| + |C| - \Big(|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|\Big) \\
&= |A \cup B| + |C| - \Big(|A \cap C| + |B \cap C| - |A \cap B \cap C|\Big) \\
&= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|
\end{aligned}$$

`Answer 6a:` This is a vacous statement. The hypothesis, $|n-1|+|n+1| \leq 1$ is false for every n. Therefore, the conditional statement is automatically true for any n regardless of what the conclusion of the conditional statement is. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$.

`Answer 6b:` If x is odd then it means we can write $x = 2n+1$ for some integer n. This means that $9x+5 = 9(2n+1)+5 = 18n+9+5 = 18n+14 = 2(9n+7)$. This number is immediately seen to be even as $2(9n+7)$ has an even factor. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$.

`Answer 6c:` If $1-n^2 > 0$ then it means that $n = 0$, because for any other natural number n^2 will exceed 1. Therefore, saying $1-n^2 < 0$ is another way of saying that $n = 0$. But if $n = 0$ then we clearly see that $3n-2 = -2$ which is even. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$.

`Answer 6d:` We will prove this by cases. The first case is when x is assumed to be even and the second case is when x is assumed to be odd.

If x is an even integer then we can write $x = 2n$. Thus, $x^3 - x = (2n)^3 - 2n = 8n^3 - 2n = 2n(4n^2-1)$. This is clearly an even integer as it has a factor of 2.

If x is an odd integer then we can write $x = 2m+1$. Thus, $x^3 - x = (2m+1)^3 - (2m+1) = 8m^3 + 12m^2 + 6m + 1 - 2m - 1 = 4m(2m^2+3m+1)$. This is clearly an even integer as it has a factor of 4 (which is divisible by 2). $\mathbb{Q}.\mathbb{E}.\mathbb{D}$.

(We could have also observed that $x^3 - x = x(x-1)(x+1)$, in which, again by cases, we see that either x or $(x-1)$ is even.)

`Answer 7`: If $a|n$ and $a|m$ then this means that $n = as$ and $m = at$ for some integers s and t (this is the definition of divisibility). We wish to prove that $a|(nx+my)$ for any choice of x and y. What this means is that we need to show that we can find an integer k such that $nx + my = ka$ (this is again the definition of divisibility). As $n = as$ and $m = at$ we can write $nx + my = (as)x + (at)y = (sx + ty)a$. Thus, we have shown that $nx + my = ka$ where $k = sx + ty$. This means, by definition of divisiblity, that $a|(nx + my)$. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$.

`Answer 8`: The meaning of $a \equiv b(\text{mod} n)$ means that $a - b$ is divisible by n. The meaning of $b \equiv c(\text{mod} n)$ means that $b - c$ is divisible by n. Therefore, $a - b = nx$ for some integer x and $b - c = ny$ for some integer y. Now let us add these two equations together $(a-b)+(b-c) = nx + ny$ and we get that $a - c = n(x + y)$. Therefore, by definition of divisibility, we have that n divides $a - c$, and hence, $a \equiv c(\text{mod} n)$. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$.

`Answer 9`: Here is a very simple way of proving this inequality. Recall that in class we proved the arithmetic-geometry mean inequality. It says that if $x, y \geq 0$ then $\frac{x+y}{2} \geq \sqrt{xy}$. Let $x = \frac{a}{b}$ and $y = \frac{b}{a}$ then we have,

$$\frac{\frac{a}{b}+\frac{b}{a}}{2} \geq \sqrt{\frac{a}{b}\cdot\frac{b}{a}} = 1 \implies \frac{a}{b}+\frac{b}{a} \geq 2$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}$.

Note, we could have also done something similar to what we did in class. Note that, for $a, b \in \mathbb{R}^+$, $(a-b)^2 \geq 0 \Rightarrow a^2 - 2ab + b^2 \geq 0 \Rightarrow a^2 + b^2 \geq 2ab$. And since $a, b > 0$ we have that $ab > 0$ and so we can divide by it without affecting the inequality sign to obtain the desired result.

`Answer 10`: We know that $\sin^2 x + \cos^2 x = 1$ by cubing both sides we get,

$$\sin^6 x + 3\sin^4 x\cos^2 x + 3\sin^2 x\cos^4 x + \cos^6 x = 1$$

Let us rewrite this as,

$$\sin^6 x + 3\sin^2 x\cos^2 x\underbrace{(\sin^2 x + \cos^2 x)}_{1} + \cos^6 x = 1$$

Thus, $\sin^6 x + 3\sin^2 x\cos^2 x + \cos^6 x = 1$. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$.

`Answer 11`: (Diagram left to the reader)

Let $x \in A \cup (B \cap C)$. Thus, $x \in A$ or $x \in B \cap C$. Let us say that $x \in A$. Then clearly $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$. Now if $x \in B \cap C$ rather, then $x \in B$ and $x \in C$, so $x \in A \cup B$ and $x \in A \cup C$ which means that $x \in (A \cup B) \cap (A \cup C)$. In either case we see that $x \in (A \cup B) \cap (A \cup C)$. This means that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Now let $x \in (A \cup B) \cap (A \cup C)$. Therefore, $x \in A \cup B$ and $x \in A \cup C$. This means that, $x \in A$ or $x \in B$, and, $x \in A$ or $x \in C$. Let us say that $x \in A$, then clearly $x \in A \cup (B \cap C)$. If $x \notin A$ then it would mean $x \in B$ and $x \in C$, so that, $x \in B \cap C$ which then means that $x \in A \cup (B \cap C)$. Thus, in either case, $x \in A \cup (B \cap C)$. Thus, $A \cup (B \cap C) \subseteq A \cup (B \cap C)$.

As these two sets are subsets of each other it means that they are equal. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

`Answer 12:` We need to prove that if $-b \leq a \leq b$ then $|a| \leq b$. Aftrerwards we need to prove that if $|a| \leq b$ then $-b \leq a \leq b$.

Let us prove the first half, that is if $-b \leq a \leq b$ then $|a| \leq b$. We do it by cases. If $a \geq 0$ then $|a| = a \leq b$ since we are assuming that $a \leq b$. If $a < 0$ then $|a| = -a$. But since $-b \leq a$ it means that $-a \leq b$ (multiplying both sides by a negative reverses the inequality sign). Thus, $|a| = -a \leq b$. In either case we see that $|a| \leq b$.

Now we prove the second half, that is, if $|a| \leq b$ then $-b \leq a \leq b$. First of all b is a non-negative number since $|a| \leq b$, so, $0 \leq |a| \leq b$, which means that $b \geq 0$, and hence b is a non-negative number. Now we proceede by two cases again. If $a \geq 0$ then $|a| = a$, since $|a| \leq b$ it means $a \leq b$. The other part of the inequality that $-b \leq a$ is automatic, since $-b$ is non-positive and a is at least 0, hence, $-b \leq a \leq b$. If $a < 0$ then $|a| = -a$, since $|a| \leq b$ it means $-a \leq b$. Thus, $-b \leq a$, the other part of the inequality that $a \leq b$ is automatic since b is non-negative and a is negative, hence, $-b \leq a \leq b$. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

`Answer 13 (i):` Consider the interval $A = (0, 1] \subset \mathbb{R}$.

It is not open, since if $x = 1$, then $x \in A$ but there is no open interval I such that $x \in I$ and $I \subseteq A$. Also, $(0, 1]$ is not closed, since $(0, 1] = \mathbb{R} \backslash [0, 1)$. And $[0, 1)$ is not open by the same argument that $(0, 1]$ is not open–using $x = 0$ instead of $x = 1$.

`Answer 13 (ii):` We will need a lemma.

Lemma: The set $A \subseteq \mathbb{R}$ is open iff $\forall x \in A$ there exists an open interval I centered at x that is contained in A. That is, I has the form $(x - r, x + r)$ for some real number $r > 0$.

Proof of lemma: $(\Rightarrow)$ Assume $A \subseteq \mathbb{R}$ is open and let $x \in A$. Then, by definition, there exists some $(a, b) \subseteq \mathbb{R}$ such that $x \in (a, b)$ and $(a, b) \subseteq A$. Choose $r = \min\{|x - a|, |x - b|\}$. Then $I = (x - r, x + r)$ works. (We call r the *radius*) of I.

$(\Leftarrow)$ Let $x \in A$ and assume $I \subseteq A$ is an open interval centered at x. Then A is open by definition. $\square$

Now to prove the claim in `(ii)`.

Assume U and V re open subsets of $\mathbb{R}$. Then $\forall x \in U$ and $\forall y \in V$, we have open intervals I_1 and I_2 such that $I_1 \subseteq U$ is centered at x with radius r_1 and $I_2 \subseteq V$ is centered at y with radius r_2. Now let $a \in U \cap V$. Then if we set $r = \min\{r_1, r_2\}$, we have that $I = (a - r, a + r) \subseteq U \cap V$ and $a \in I$. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

`Answer 13 (iii):` Assume that C and D are closed subsets of $\mathbb{R}$. Then $C = \mathbb{R} \backslash U$ and $D = \mathbb{R} \backslash V$ for open sets U and V. Thus we see that $C \cup D = (\mathbb{R} \backslash U) \cup (\mathbb{R} \backslash V) = \mathbb{R} \backslash (U \cap V)$. By `13 (ii)`, $U \cap V$ is open, so that $C \cup D$ is closed by definition.

`Answer 13 (iv):` Assume $U_1, U_2, \ldots$ are open subsets of $\mathbb{R}$ and let $x \in \bigcap_{n=1}^{\infty} U_n$. Then $x \in U_i$ for some $i \in \mathbb{N}$. This means there is an open interval $I \subseteq U_i$ that contains x. Clearly if $I \subseteq U_i$ then $I \subseteq \bigcap_{n=1}^{\infty} U_n$.

`Answer 13 (v):` Assume $C_1, C_2, \ldots$ are closed subsets of $\mathbb{R}$. Then, for each $i \in \mathbb{N}$,$C_i = \mathbb{R} \backslash U_i$, where U_i is some open set. Then $\bigcap_{n=1}^{\infty} C_n = \bigcap_{n=1}^{\infty} \mathbb{R} \backslash U_n = \mathbb{R} \backslash \bigcup_{n=1}^{n}$. This last set is clearly closed since ${}_{n=1}^{n}$ is open by `13 (iv)`.

`Answer 13 (vi):` Define $U_n = (-1/n, 1/n)$ for $n \in \mathbb{N}$. Then $\bigcap_{n=1}^{n=\{0\}}$ which is not open since there is no open inerval I contained in the union.

Let $C_n = \bigcup_{n=1}^{\infty} [1/n, 1 + 1/n]$. Then each C_n is closed, but $\bigcup_{n=1}^{\infty} C_n = (0, 1]$, which is not closed, as shown in `13 (i)`.

`Answer 14:` Suppose that $q > 0$ was the smallest positive rational number. Then $q/2$ is a rational number and $q/2 > 0$. But $q/2 < q$, this contradicts the fact that q was the smallest positive rational number. Thus, it must mean that there is no smallest positive rational number. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

`Answer 15:` Let x be an irrational number and q be a non-zero rational number. We need to prove that qx is irrational. We will prove this by contradiction. Suppose that $qx = r$ where r was rational. Then it would mean that $x = r/q$, note we are using the fact that $q \neq 0$ so we can divide by it! However, r/q is a rational number, as the quotient of two rational numbers is rational, but x is irrational, this a contradiction. Thus, it must mean that qx is irrational. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

`Answer 16:` Let us suppose that $\sqrt{3}$ was a rational number. Then we can write $\sqrt{3} = \dfrac{a}{b}$ where a and b are positive integers in reduced form, i.e. they have no positive common factor (other than 1), basically, we are saying that $\dfrac{a}{b}$ is a fraction written in lowest terms. This would mean that,

$$3 = \left(\frac{a}{b}\right)^2 \implies 3 = \frac{a^2}{b^2} \implies 3b^2 = a^2$$

Note that the left hand side is divisible by 3 therefore the right hand side must be divisible by three. The only way that can happen is for a itself to be divisible by 3 also. Therefore, $a = 3c$ for some integer c. This means that $3b^2 = 9c^2 \implies b^2 = 3c^2$. Now the right side is divisible by 3, therefore the left hand side b^2, needs to be divisible by 3 the only way for this to happen is if b itself is divisible by 3. We have shown that both a and both b are divisible by 3. But this is a contradiction as a and b were assumed to have no positive common factor other than 1! Thus, it must be the case that $\sqrt{3}$ is an irrational number. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

`Answer 17:` Suppose that $\log_2 3$ is rational, so we can write $\log_2 3 = \dfrac{a}{b}$. Thus,

$$2^{a/b} = 3 \implies \left(2^{a/b}\right)^b = 3^b \implies 2^a = 3^b$$

Now the left hand side is even while right hand side is odd, so this is impossible. This is a contradiction, therefore, $\log_2 3$ must be an irrational number. The proof that $\log_3 2$ is irrational is similar.

Now let $x = \log_2 3$ and $y = \log_3 2$. Therefore, $2^x = 3$ and $3^y = 2$. Thus,

$$2^x = 3 \implies (3^y)^x = 3 \implies 3^{xy} = 3 \implies xy = 1 \implies \log_2 3 \cdot \log_3 2 = 1$$

So the product of these two irrational numbers, $\log_2 3$ and $\log_3 2$, is a rational number. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$.

`Answer 18:` We can rewrite this equation as $a^2 = 3^b - 3$. Note that both 3^b and 3 are divisible by 3, therefore, $3^b - 3$ is divisible by three also. This means that the left hand side, a^2, is divisible by 3. Now the only way a^2 can be divisible by 3 is if a itself is divisible by 3, so, we can write $a = 3c$. Putting that into the equation we find that,

$$(3c)^2 = 3^b - 3 \implies 9c^2 = 3^b - 3 \implies 9c^2 - 3^b = -3$$

If $b > 1$ then 3^b is divisible by $3^2 = 9$, also $9c^2$ is divisible by 9, so, in this case, $9c^2 - 3^b$ is divisible by 9, while the right hand side is not divisible by 9, this is a contradiction. Thus, it must mean that b cannot be larger than 1.

So the last remaining case is $b = 1$. In this case we have $9c^2 = 0$, and so $c = 0$ but then $a = 3c = 0$. This is a contradiction because a is a positive integer, hence $a > 0$. In conclusion, we have shown that the equation $a^2 = 3^b - 3$ has no solutions in terms of positive integers. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$.

`Answer 19a:` Clearly $a \neq 0, b \neq 0$. If $|b| > 1$ then $|1| = |a||b| > |a|$, from where it will follow that $a = 0$, this is impossible. By a similar argument $|a| > 1$ is impossible as well. Therefore, $|a| = 1$ and $|b| = 1$. From here the claim follows. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$.

`Answer 19b:` We can factor this equation as $(x - 2y)(x + 2y) = 1$. We know that $x - 2y$ and $x + 2y$ are integers, therefore, it must mean that $x - 2y = 1$ and $x + 2y = 1$, or that $x - 2y = -1$ and $x + 2y = -1$. The solution to the first pair of equations is that $x = 1$ and $y = 0$. The solution to the second pair of equations is that $x = -1$ and $y = 0$. This means that the only solutions to the equation $x^2 - 4y^2 = 1$ is that $(x, y) = (1, 0), (-1, 0)$.

Note: Let n be a positive integer. The equation $x^2 - ny^2 = 1$ is known as *Pell's equation* from number theory. If n turns out to be a perfect square, for example, $n = 4$, as above, then we can factor the equation and by a similar process show that $(-1, 0)$ and $(1, 0)$ are the only solutions to this equation. But what if n is no longer a perfect square? For example, $x^2 - 3y^2 = 1$? We cannot factor it as we did before in terms of intergers. We may wonder whether $(1, 0)$ and $(-1, 0)$ are the only solutions still. It turns out, rather, that there are now infinitely many solutions! Things get a lot more interesting.

`Answer 20a:` The polynomial $x^2 + 1$ has degree two. Suppose that it was possible to factor it into a product of two polynomials of smaller degrees. Then those two smaller degree polynomials must have degree one (because their product must have degree two). Suppose that $(x^2 + 1) = (ax + b)(cx + d)$ is a product of two linear polynomials (where a, b, c, d are integers). Expand the right hand side $x^2+1 = acx^2+(ad+bc)x+bd$. Now compare coefficients, x^2 has coefficient 1 on LHS and so $ac = 1$, x has coefficient 0 on LHS and so $ad + bc = 0$, the constant term on LHS is 1 and so $bd = 1$. Since $ac = 1$ and $bd = 1$ it means $a = c = 1$ or $a = c = -1$ and $b = d = 1$ or $b = d = -1$. But regardless of what choices we pick for a, b, c, d the condition $ad + bc = 0$ is never satisfied. Thus, we have a contradiction, which means that it is impossible to factor $x^2 + 1$ into two polynomials of smaller degree, therefore $x^2 + 1$ is an irreducible polynomial.

`Answer 20b:` Let $f(x)$ be a polynomial of some positive degree n. If $f(x)$ is irreducible then the proof is over as $f(x)$ is already factored in terms of irreducibles. If $f(x)$ is not irreducible, that is, it is reducible, then it means, by definition, that $f(x) = f_1(x)f_2(x)$ where the degree of $f_1(x)$ and $f_2(x)$ are smaller than n. If each $f_1(x)$ and $f_2(x)$ are irreducible polynomials then the proof is complete as $f(x)$ has been factored into irreducible polynomials. Otherwise, if at least one of them is reducible, say $f_1(x)$, we can factor $f_1(x) = f_3(x)f_4(x)$ and so $f(x) = f_3(x)f_4(x)f_2(x)$. Now if each factor is irreducible then the proof is complete, otherwise, keep on factoring each polynomial further that is not irreducible. Each time we factor the degree of each polynomial gets smaller. Since the degree was n of $f(x)$ and the degrees of the factors get smaller this process cannot continue indefinitely and hence terminates at some point. Whatever the remaining factors are they must all be irreducible and so $f(x)$ can be factored in terms of irreducible polynomials.

`Answer 20c:` Let $f(x)$ be some polynomial, from the previous problem we know that $f(x) = p_1(x)p_2(x) \cdot ... \cdot p_r(x)$ where each $p_1(x)$,...,$p_r(x)$ is an irreducible polynomial (not necessarily all distinct factors). Therefore, each one of these factors divides $f(x)$. Pick any one of these factors, does not matter and call it $p(x)$, so we see that $f(x)$ is divisible by $p(x)$. We have proved that given a polynomial $f(x)$ there exists an irreducible polynomial $p(x)$ such that divides $f(x)$. Furthermore, if $f(x)$ is monic then the $p_1(x), ..., p_r(x)$ have to be monic also, otherwise when they be multiplied out the highest degree coefficient on the RHS would not be 1, and so for every monic polynomial $f(x)$ there is a monic irreducible polynomial $p(x)$ so that $p(x)$ divides $f(x)$.

Suppose there were finitely many monic irreducible polynomials, call them $q_1(x), ..., q_n(x)$. Now define the polynomial $g(x) = q_1(x) \cdot q_2(x) \cdot ... \cdot q_n(x) + 1$. Note $g(x)$ is monic also. By the above paragraph $g(x)$ must be divisible by some irreducible monic polynomial $p(x)$, but since we are assuming there are finitely many irreducible polynomials, this $p(x)$ has to be one of the $q_1(x), ..., q_n(x)$. This is a contradiction since $g(x)$ is not divisible by any one of those polynomials. For example, let us check if $q_1(x)$ divides $g(x)$,

$$\frac{g(x)}{q_1(x)} = \frac{q_1(x) \cdot q_2(x) \cdot ... \cdot q_n(x) + 1}{q_1(x)} = q_2(x) \cdot ... \cdot q_n(x) + \frac{1}{q_1(x)}$$

So there is a remainder of 1 left over when $g(x)$ is divided by $q_1(x)$. By a similar argument we can show that none of the other $q_2(x), ..., q_n(x)$ divides $g(x)$. This is a contradiction, hence, there must be infinitely many monic irreducible polynomials. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$.

`Answer 21:` To show that $(A \cap B)^C = A^C \cup B^C$ we need to show that any element of $(A \cap B)^C$ is an element of $A^C \cup B^C$ and that any element of $A^C \cup B^C$ is an element of $(A \cap B)^C$.

Let $x \in (A \cap B)^C$ therefore $x \notin A \cap B$. The definition of $A \cap B$ are elements common to both A and B, so if x is not in $A \cap B$ it means x is either not in A or x is not in B. That is, $x \notin A$ or $x \notin B$. Thus, $x \in A^C$ or $x \in B^C$ as A^C is defined as those elements that do not belong to A, and similarly for B^C. Now as $x \in A^C$ or $x \in B^C$ it means that $x \in A^C \cup B^C$ because the definition of the union of two sets are those elements in one of the two sets. This shows that $(A \cap B)^C \subseteq A^C \cup B^C$.

Let $x \in A^C \cup B^C$. This means that $x \in A^C$ or $x \in B^C$, from here it follows that $x \notin A$ or $x \notin B$. As $A \cap B$ is defined as the set of element common to both A and B, and as x is not in one of the two sets we have that $x \notin A \cap B$. But then it means that $x \in (A \cap B)^C$. This shows that $A^C \cup B^C \subseteq (A \cap B)^C$. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$.

(Venn Diagram is left to the reader.)

`Answer 22:` Set $f(x) = x^{2015} + x^{2013} + \cdots + x^3 + x + 1$. Clearly f is continuous and differentiable everywhere since it is a polynomials. Thus we may apply the Intermediate Value Theorem (IVT) and Rolle's theorem on any interval we choose. Since $f(-1) < 0$ and $f(1) > 0$, the IVT gives that there exists $c \in [-1, 1]$ such that $f(c) = 0$. And so, f has at least one root. We show that f has exactly one. For assume to the contrary that there were more. Suppose $f(c_1) = 0$ for $c_1 \neq c$. Then Rolle's theorem would give us that there must be a point x in the open interval defined by c and c_1 such that $f'(x) = 0$. But $f'(x) = 2015x^{2014} + \cdots + 1$ which is greater than 0 for all x. A contradiction. Therefore, $f(x)$ has exactly one root.

`Answer 23:` Let $P(n)$ be the open sentence that $0^3 + 1^3 + 2^3 + ... + n^3 = \frac{1}{4}n^2(n+1)^2$.

We want to prove that this equation is true for all $n \in \mathbb{Z}_0^+$. Clearly, it is true for $n = 0$ as both the left and right hand side of the equations are equal to 0. Thus, the base case of induction is true.

It remains to prove that $P(k) \implies P(k+1)$. Thus, let us suppose that this statement is true for some $k \geq 0$:

$$0^3 + 1^3 + 2^3 + ... + k^3 = \frac{1}{4}k^2(k+1)^2$$

Let us add $(k+1)^3$ to both sides obtaining:

$$0^3 + 1^3 + ... + k^3 + (k+1)^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

Now by basic manipulation we see that,

$$\frac{1}{4}k^2(k+1)^2 + (k+1)^3 = \frac{1}{4}(k+1)^2\{k^2 + 4k + 4\} = \frac{1}{4}(k+1)^2(k+2)^2$$

This proves that $P(k+1)$ is true also.

Now from mathematical induction it follows that $P(n)$ is true for all $n \in \mathbb{Z}_0^+$. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$.

`Answer 24:` We see that this statement is true for $n = 1$. Now we assume that it is true for k,

$$1 + \frac{1}{\sqrt{2}} + ... \frac{1}{\sqrt{k}} \geq \sqrt{k}$$

Adding $1/\sqrt{k+1}$ to both sides to get,

$$1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k} + \frac{1}{\sqrt{k+1}}$$

If we can show that,

$$\sqrt{k} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k+1}$$

Then the statement would also be true for $k+1$ completing the proof by induction. It reamins to prove this inequality. This inequality would be true provided that,

$$\sqrt{k} \cdot \sqrt{k+1} + 1 \geq k + 1$$

Which will follow if we can show,

$$\sqrt{k} \cdot \sqrt{k+1} \geq k$$

But this is true, because $\sqrt{k+1} \geq \sqrt{k}$ so $\sqrt{k+1} \cdot \sqrt{k} \geq \sqrt{k} \cdot \sqrt{k} = k$. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$.

`Answer 25:` (By minimum counterexmple) Let $S = \{n \in \mathbb{N} : 10 \not| (3^{4n} - 1)\}$. If it is true that $10|(3^{4n} - 1)$ for any natural number n then S is empty. Let us suppose by contradiction that $10 \not| (3^{4n} - 1)$ for some n, then S is non-empty. By the well-ordering principle as S is a non-empty set of natural numbers it has a minimal element, call it m. Note that as $10|(3^{4\cdot1} - 1)$ it means that $m \geq 2$. Thus, in particular, $m - 1$ is a natural number. As $m - 1 < m$, and m is *minimal*, it means that $m - 1 \notin S$, therefore, 10 divides $3^{4(m-1)} - 1$. This means there exists an integer k such that $3^{4(m-1)} - 1 = 10k$. Therefore, $3^4 \cdot 3^{4(m-1)} - 3^4 = 3^4 \cdot 10k$. Thus, $3^{4m} - 81 = 810k$. From here it follows that $3^{4m} - 1 = 810k + 80 = 10(81k + 8)$ which means, by definition, that 10 divides $3^{4m} - 1$. But then $m \notin S$, however m is the minimal element in S, which is a contradiction. Thus, S must be empty which means that 10 divides $3^{4n} - 1$ for any n.

Note: It is possible to prove Problem 25 using ordinary induction.

Let $P(n)$ be the open sentence that $10|(3^{4n} - 1)$. Clearly, $P(1)$ is true. We assume $P(k)$, for some natural number k, and try to prove $P(k + 1)$. That is, we assume $10|(3^{4k} - 1)$ and try to prove that $10|(3^{4(k+1)} - 1)$. Since $10|(3^{4k} - 1)$ we can write $3^{4k} - 1 = 10j$ for some integer j. Thus, $3^4 \cdot 3^{4k} - 3^4 = 3^4 \cdot 10j$ from which it follows that $3^{4(k+1)} - 1 = 810j + 80$, and so $10|(3^{4(k+1)} - 1)$. We have proved $P(k + 1)$, now applying induction we get $P(n)$ is true for all $n \in \mathbb{N}$. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$.

`Answer 26:` Let $P(n)$ be the open sentence that: for any choice of n real numbers, $a_1, ..., a_n$ we have that $|a_1 + ... + a_n| \leq |a_1| + ... + |a_n|$. (Here $n \geq 1$, because the problem starts counting at 1). We can check that $P(1)$ is true since $|a_1| \leq |a_1|$ for any real number a_1. We have proved $P(2)$ in class, namely, given any two real numbers x and y (instead of calling them a_1 and a_2) we have that $|x + y| \leq |x| + |y|$. (The triangle inequality.) Now we assume $P(k)$, namely, that given any collection of k real numbers, call them, $a_1, a_2, ..., a_k$ we have that $|a_1 + a_2 + ... + a_k| \leq |a_1| + |a_2| + ... + |a_k|$. We want to prove $P(k + 1)$. Let $a_1, a_2, ..., a_k, a_{k+1}$ be any collection of $k + 1$ real numbers. Now write,

$$|a_1 + a_2 + ... + a_k + a_{k+1}| = |(a_1 + a_2 + ... + a_k) + a_{k+1}| \leq |a_1 + ... + a_k| + |a_{k+1}|$$

This inequality follows from $|x + y| \leq |x| + |y|$ where $x = a_1 + ... + a_k$ and $y = a_{k+1}$. Now apply the induction assumption $P(k)$ to get,

$$|a_1 + a_2 + ... + a_k + a_{k+1}| \leq |a_1| + |a_2| + ... + |a_k| + |a_{k+1}|$$

Thus, we see that $P(k + 1)$ is true. We can conclude that $P(n)$ is true for any $n \geq 1$. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$

`Answer 27:` We want to show that any two adjancent Fibonnaci numbers have no common positive divisor other than 1. We will prove this by contradiction and assume that there exist two adjacent Fibonnaci numbers that have a common factor. Let us define the set:

$$S = \{n \in \mathbb{N} \mid f_n \text{ and } f_{n+1} \text{ have a divisor greater than } 1\}$$

By assumption this set is non-empty as we are assuming (for the sake of contradiction) that there are two adjacent Fibonnaci numbers that have a common divisor greater than 1. This is a set of natural numbers and so by the well-ordering principle it has a minimal element, call it m. Therefore, f_m and f_{m+1} have a common divisor greater than 1. Note that $m \neq 0$, because $f_0 = 1$ and $f_1 = 1$ so clearly f_0 and f_1 only have 1 as a common divisor. Therefore, whatever m is we know it is greater than 0. Consider $m - 1$, that is also a natural number (as $m > 0$). Since $m - 1 < m$, by the minimal property of m it means that $m - 1 \notin S$, hence, f_{m-1} and f_m have no common divisor greater than 1. By the Fibonnaci sequence we have that $f_{m-1} + f_m = f_{m+1}$, so $f_{m-1} = f_{m+1} - f_m$. Now f_m and f_{m+1} have some common divisor, call it d, such that $d > 1$. As d divides both f_m and d divides f_{m+1} it also means that d divides $f_{m+1} - f_m = f_{m-1}$. Thus, d is also a divisor for f_{m-1}. Thus, we see that $d > 1$ is also a common divisor for f_m and f_{m-1}. But that means that $m - 1 \in S$, however, by the minimal property of m we had that $m - 1 \notin S$. This is a contradiction. Therefore, it must be the case that any two Fibonnaci numbers are relatively prime. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

`Answer 28:` Let $P(n)$ be the statement that: n is a sum of distinct Fibonnaci numbers (with domain $n \geq 1$). Clearly, $P(1)$ is true as 1 is already a Fibonnaci number. We will use strong induction, our assumption will be that $P(1), P(2), ..., P(N-1)$ are all true (where N is a positive integer bigger than 1). We will prove $P(N)$, i.e. that N is a sum of distinct Fibonnaci numbers. We can further assume that N is not a Fibonnaci number. Because otherwise the proof is complete. List the Fibonnaci numbers (ignoring f_0) as the following increasing string of numbers,

$$f_1 < f_2 < f_3 < f_4 < f_5 < f_6 < ...$$

Our number N is going to end up somewhere in this string between two adjacent Fibonnaci numbers. The reason being is that the Fibonnaci sequence is an integer sequence that only gets bigger, so at some point it will surpass N. Furthermore, as we are assuming that N is not a Fibonnaci number itself it is not equal to any of the numbers in this string. Thus, there is an integer k such that,

$$f_1 < f_2 < ... < f_{k-1} < f_k < N < f_{k+1} < f_{k+2} < ...$$

The property of this Fibonnaci number f_k is that, $f_k < N$ but $f_{k+1} > N$. Consider the number $N - f_k$. This is a positive integer (as $N > f_k$) and it is less than N (as we are subtracting something off N). By strong induction assumption, the integer $N - f_k$ is therefore a sum of distinct Fibonnaci numbers. Let us therefore write,

$$N - f_k = a_1 + a_2 + ... + a_r$$

Where $a_1, ..., a_r$ are some distinct Fibonnaci numbers (not necessary in any order, and not necessary that $a_j = f_j$, this is why we call them a_j instead of f_j). Therefore, we get the representation $N = a_1 + ... + a_r + f_k$. We see that N is a sum of Fibonnaci numbers. It remains to check that all of these numbers are distinct. The first r numbers are already distinct as we said. It remains to show that $f_k \neq a_j$ for any $j = 1, 2, .., r$. We will show that $f_k > a_j$. We have, (use that $N < f_{k+1}$),

$$a_j \leq a_1 + ... + a_r = N - f_k < f_{k+1} - f_k = f_{k-1}$$

Where on the last equality we used the Fibonnaci equation $f_{k+1} = f_k + f_{k-1} \implies f_{k+1} - f_k = f_{k-1}$ (the reason why the first inequality has $\leq$ instead of $<$ because it is conceivable that $r = 1$). Now we can conclude that $a_j < f_{k-1} < f_k$. $\mathbb{Q}.\mathbb{E}.\mathbb{D}.$

Note: We can add any two adjacent Fibonnaci numbers (if there are any) together to get a new Fibonnaci number. We do this process until we are left with a Fibonnaci representation of N consisting of no two adjacent Fibonnaci numbers. One can show that this is a sum of distinct Fibonnaci numbers. Furthermore, what is more surprising, is that this representation is unique. There is no other way to express N as a sum of distinct Fibonnaci numbers so that no two are adjacent (other than the other in which we write the sum), this is known as **Zackendorf's Theorem**. This representation of N is referred to as the Zackendorf repesentation and the Fibonnaci numbers which come up in the representation of N are referred to as the

Zackendorf digits. Surprisingly, this Zackendorf representation has applications to the study of plants in biology.

`Answer 29:` Let $S = \{a - nb : n \in \mathbb{N} \text{ and } a - nb \geq 0\}$. This set is non-empty because $a \in S$ as we can let $n = 0$ to get $a - 0 \cdot b = a \geq 0$. By the well-ordering principle this set has a minimal number, call it r. We will prove by contradiction that $r < b$. Suppose otherwise, that $r \geq b$. So far we know that $r = a - qb$ for some $q \in \mathbb{N}$. But as $r \geq b$ it means that $r - b \geq 0$. Subtracting b from both sides of the equation we get $r - b = a - qb - b$, that is, $r - b = a - (q + 1)b \geq 0$. Thus, if $r \geq b$ then $r - b \in S$ with $r - b < r$, contradicting the fact that r was minimal. Thus, it must be that $r < b$. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$.

`Answer 30:` Let $P(n)$ be the statement that $4^n > n^3$. We will prove that $P(n)$ is true for all $n \geq 0$ by strong induction. $P(1), P(2) and P(3)$ is true, since $4^1 = 4 > 1^1 = 1, 4^2 = 16 > 2^3 = 8, 4^3 = 64 > 3^3 = 27$, that takes care of the base cases.

Now we show that if $P(k)$ holds for values up to k, where k is at least 3. Then we have that

$$\begin{aligned} 4^{k+1} &= 4 \cdot 4^k \\ &> 4 \cdot k^3 \\ &= k^3 + k \cdot k^2 + k^2 \cdot k + k^3 \\ &\geq k^3 + 3k^2 + 9k + 27 \text{ since } k \text{ is at least } 3 \\ &> k^3 + 3k^2 + 3k + 1 \\ &= (k+1)^3 \end{aligned}$$

$\Longrightarrow P(k + 1)$. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$

`Answer 31:` To show that $A \cup B$ is well-ordered we need to show that any non-empty subset of $A \cup B$ has a least element. Let $C \subseteq A \cup B$ be a non-empty subset, we will show it has a least element. Consider the sets $C \cap A$ and $C \cap B$. Note that $C \cap A \subseteq A$ and $C \cap B \subseteq B$.

Case I: $C \cap A = \emptyset$. If $C \cap B = \emptyset$ also then C has nothing in common with A nor B, which means that C is the empty set, which cannot happen because C was chosen to be a non-empty subset. Thus, in this case, $C = C \cap B \subseteq B$, as B is well-ordered every non-empty subset has a least element, so C has a least element.

Case II: $C \cap B = \emptyset$. If $C \cap A = \emptyset$ also then C has nothing in common with A nor B, which means that C is the empty set, which cannot happen because C was chosen to be a non-empty subset. Thus, in this case, $C = C \cap A \subseteq A$, as A is well-ordered every non-empty subset has a least element, so C has a least element.

Case III: $C \cap A \neq \emptyset, C \cap B \neq \emptyset$. As A is well-ordered and $C \cap A$ is a non-empty subset of A it means that $C \cap A$ has a least element, call it a. Similarly, as B is well-ordered and $C \cap B$ is a non-empty subset of B it means that $C \cap B$ has a least element, call it b. Now define $m = \min(a, b)$, that is, m is the smaller of the two numbers a and b. Note, that $m \in C$. We claim that m is the least element for C. Let $x \in C$. As $C \subseteq A \cup B$ it means that $x \in A$ or $x \in B$. If $x \in A$ then $x \in C \cap A$ so that $m \leq a \leq x$. If $x \in B$ then $x \in C \cap B$ so that $m \leq b \leq x$. Thus, $m \leq x$ for any $x \in C$. This proves that m is the minimal element. $\mathbb{Q}.\mathbb{E}.\mathbb{D}$.