Sample Final 2, Fall 2018
Math 201

Instructions: Please read each question carefully, show all work, and check afterwards that you have answered all of each question correctly. **Important: No books, calculators, cell phones, computers, laptops or notes are allowed.** You must show **all** your work to receive credit. Any crossed out work will be disregarded (even if correct). Write **one** clear answer with a coherent derivation for each question. You have 135 minutes to complete this exam. Good luck!

[1] **(16 pts, 4 pts each)** Compute the derivative $f'(x)$ for each of the functions below. You do not need to simplify your answer.

(a) $f(x) = \sin(x^2)\cos^4(\sqrt{x})$

$f'(x) = \frac{d}{dx}[\sin(x^2)]\cos^4(\sqrt{x}) + \frac{d}{dx}[\cos^4(\sqrt{x})]\sin(x^2)$ (product rule)

(chain rule)

$\frac{d}{dx}[\sin(x^2)] = \cos(x^2) \cdot 2x$

$\frac{d}{dx}[\cos^4(\sqrt{x})] = 4\cos^3(\sqrt{x}) \cdot \left(-\sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}\right)$ (chain rule)

$\Rightarrow \boxed{f'(x) = [2x\cos(x^2)] \cdot \cos^4(\sqrt{x}) - \left[2\cos^3(\sqrt{x}) \cdot \sin(\sqrt{x}) \cdot \frac{1}{\sqrt{x}}\right]}$

(b) $f(x) = (\ln x)\left(\sqrt[3]{x} - \frac{1}{x^2}\right)$

$\boxed{f'(x) = \frac{1}{x}\left(\sqrt[3]{x} - \frac{1}{x^2}\right) + \ln x\left(\frac{1}{3}x^{-2/3} + 2x^{-3}\right)}$ (product rule)

(d) $f(x) = \frac{\arcsin(x)}{e^{4x}}$ $= e^{-4x} \cdot \arcsin(x)$ [Here, $|x|<1$].

$$\boxed{f'(x) = -4e^{-4x} \cdot \arcsin x + e^{-4x} \cdot \frac{1}{\sqrt{1-x^2}}}$$

(Product rule again)

Another possibility is to use the quotient rule:

$$f'(x) = \frac{\frac{1}{\sqrt{1-x^2}} \cdot e^{4x} - (4e^{4x} \arcsin(x))}{e^{8x}}$$

, which yields the same answer as above, after simplification.

(e) $f(x) = x^{x+3}$. Let $y = x^{x+3}$. We want to find $\frac{dy}{dx}$.

$\ln y = \ln x^{x+3} = (x+3)\ln x$

Differentiating with respect to x, we get:

$$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln x + (x+3) \cdot \frac{1}{x} = \ln x + \frac{x+3}{x}$$

$$\boxed{\frac{dy}{dx} = y\left(\ln x + \frac{x+3}{x}\right) = x^{x+3}\left[\ln x + \frac{x+3}{x}\right]}$$

[1] (16 pts)

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[2] **(16 pts, 4 pts each)** Find each integral and simplify your answer.

(a) $\int_1^e \frac{\ln(x)}{x}dx$. Let $u = \ln x$. Then, $du = \frac{1}{x}dx$.

Hence, $\int \frac{\ln x}{x}dx = \int u\,du = \frac{u^2}{2}+c = \frac{(\ln x)^2}{2}+c$.

Finally, $\int_1^e \frac{\ln x}{x}dx = \frac{(\ln x)^2}{2}\Big|_1^e = \frac{1}{2}$.

(b) $\int \sin^3(3x)\cos(3x)\,dx$ Let $u = \sin 3x$. Then $du = 3\cos 3x\,dx$.

Hence, $\int \sin^3(3x)\cos 3x\,dx = \frac{1}{3}\int u^3\,du = \frac{1}{3}\left[\frac{u^4}{4}+c\right]$.

$= \boxed{\frac{\sin^4 3x}{12} + \tilde{c}}$

(c) $\int \frac{36dx}{(2x+1)^3}$ Let $u = 2x+1$. Then $du = 2dx$.

$$\Rightarrow \int \frac{36\,dx}{(2x+1)^3} = 18\int \frac{1}{u^3}\,du = 18\left[\frac{-1}{2u^2} + C\right].$$

$$= \boxed{\frac{-9}{(2x+1)^2} + \tilde{C}}$$

(d) $\int x^2 2^{x^3}\,dx$ Let $u = x^3$. Then $du = 3x^2\,dx$.

$$\Rightarrow \int x^2 2^{x^3}\,dx = \frac{1}{3}\int 2^u\,du = \frac{1}{3}\left[\frac{2^u}{\ln 2} + C\right]$$

$$= \frac{2^{x^3}}{3\ln 2} + \tilde{C}.$$

[2] (16 pts)

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[3] **(9 pts, 3 pts each)** Find the limits, or state that the limit does not exist (you must justify your answer):

(a) $\lim_{x\to+\infty}\left(\sqrt{x^2+2x}-\sqrt{x^2-x}\right) = \lim_{x\to+\infty} \frac{(x^2+2x)-(x^2-x)}{\sqrt{x^2+2x}+\sqrt{x^2-x}} = \lim_{x\to+\infty} \frac{3x}{\sqrt{x^2+2x}+\sqrt{x^2-x}} =$

(Dividing all terms by x)

$$= \lim_{x\to+\infty} \frac{3}{\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{1}{x}}} = \frac{3}{1+1} = \frac{3}{2}.$$

(b) $\lim_{x\to0^+}(\ln x - \ln(\sin x))$. Rewrite $\ln x - \ln \sin x = -\ln\left(\frac{\sin x}{x}\right)$.

Since $\lim_{x\to0^+} \frac{\sin x}{x} = 1$ and the function $f(x) = \ln x$ is continuous

at $x=1$, $\lim_{x\to0^+} -\ln\left(\frac{\sin x}{x}\right) = -\ln(1) = 0$

(c) $\lim_{x\to+\infty} x^{1/(\ln x)}$. Let $y = x^{1/\ln x} \Rightarrow \ln y = \frac{1}{\ln x}\cdot \ln x = 1.$

Hence, $y = e$ (a constant), so

$$\lim_{x\to+\infty} y = e$$

[3] (9 pts)

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[4] **(6 pts, 3 pts each)**

(a) Using the limit definition of derivative, find $f'(x)$ for $f(x) = \sqrt{x}$. No credit will be given for any other method.

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{x+h-x}{\sqrt{x+h}+\sqrt{x}}\right] = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{x}}}$$

(b) Find an equation of the tangent line to the graph $y = \sqrt{x}$ at the point where $x = 9$.

At $x=9$, $y=3$, and $m = f'(9) = \frac{1}{6}$.

Eqn of line through (x_0, y_0) with slope m : $y - y_0 = m(x - x_0)$

In this case,

$$\boxed{y - 3 = \frac{1}{6}(x-9)}$$

[4] (6 pts)

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[5] (8 pts, 4 pts each)

(a) Let $F(x) = \int_0^{\tan(x)} \sqrt{1-t^3}\,dt$. Find $F'(x)$.

Fundamental Theorem of Calculus: $\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$.

- Note that $F(x) = G \circ H(x)$, where

$G(x) = \int_0^x \sqrt{1-t^3}\,dt$, and $H(x) = \tan x$.

$\Rightarrow \quad F'(x) = G'(H(x)) \cdot H'(x)$

$= \sqrt{1-\tan^3(x)} \cdot \sec^2 x$.

(b) Find $\frac{dy}{dx}$ for the curve $e^{2x} = \sin(x+3y)$.

$e^{2x} = \sin(x+3y)$ $\quad \frac{d}{dx}$ (equation)

$2e^{2x} = \cos(x+3y) \cdot \frac{d}{dx}(x+3y)$

$2e^{2x} = \cos(x+3y)\left(1+3\frac{dy}{dx}\right)$

Isolating $\frac{dy}{dx}$, we get

$$\boxed{\frac{dy}{dx} = \frac{2e^{2x} - \cos(x+3y)}{3\cos(x+3y)}}$$

[5] (8 pts)

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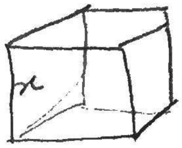
[6] **(8 pts, 4 pts each)**

(a) A cube's surface area increases at the rate of $72\,\text{in}^2/\text{sec}$. At what rate is the cube's volume changing when the edge length is $3\,\text{in}$? Be sure to include units in your answer.

Surface area of a cube of side lenght x:



$S(x) = 6x^2$

We are given; $\frac{dS}{dt} = 72\,\text{in}^2/\text{sec}$. We want: $\frac{dV}{dt}$ when $x = 3\,\text{in}$

$S(x) = 6x^2 \Rightarrow \frac{dS}{dt} = 12x\frac{dx}{dt}$. When $x = 3$, $72 = 12 \cdot 3 \cdot \frac{dx}{dt}$

$\Rightarrow \left.\frac{dx}{dt}\right|_{x=3} = 2\,\text{in/sec}$. Since the volume of the cube of side x is given by

$V(x) = x^3$, $\frac{dV}{dx} = 3x^2\frac{dx}{dt}$. When $x = 3$, $\boxed{\frac{dV}{dx} = 27 \cdot 2 = 54\,\text{in}^3/\text{sec}}$

(b) Find the absolute maximum and minimum values of the function $f(x)$ for x in the interval $[-1, 3]$, where f is given by $f(x) = \begin{cases} x^2 & \text{if } -1 \le x < 2, \\ 8 - 2x & \text{if } 2 \le x \le 3. \end{cases}$

A) Critical points: • $f'(x)$ DNE at $x = 2$

• $f'(x) = 0$: $f'(x) = -2$, if $2 < x < 3$

$f'(x) = 2x$, if $-1 < x < 2$

$\Rightarrow$ Critical points at $x = 0$ and $x = 2$.

Table of candidates to being extreme points: critical points and endpoints.

x	-1	0	2	3
$f(x)$	1	0	4	2

Abs max value: 4, achieved at $x = 2$.

Abs min value: 0, achieved at $x = 0$

[6] (8 pts)

Please leave blank!

[7] **(5 pts)** Use differentials to estimate $\sin(\pi/6 + 0.01)$. You do not need to simplify your answer.

Linear approximation:

$f(x) \simeq L(x) = f(a) + f'(a)(x-a)$.

In this case, $f(x) = \sin(x)$, $a = \frac{\pi}{6}$, $x = \frac{\pi}{6} + 0.01$.

$\Rightarrow \quad f(x) \sim \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right)(0.01) = \frac{1}{2} + \frac{\sqrt{3}}{200}$.

[7] (5 pts)

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[8] **(8 pts)**

(a) (2 pts) State the Intermediate Value Theorem, including the hypotheses.

Let f be a continuous function defined on a closed interval $[a,b]$. Let y_0 be a number between $f(a)$ and $f(b)$. Then, there exists a number c in (a,b) such that $f(c)=y_0$

(b) (2 pts) State the Mean Value Theorem, including the hypotheses.

Let f be a continuous function defined on a closed interval $[a,b]$, and differentiable on (a,b).

Then, there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

(c) (4 pts) Prove that the function $f(x) = 2x^7 - 1$ has **exactly one** real root in the interval $[0,1]$. More precisely, show that there is **at least** one real root, and **at most** one real root.

The function f is continuous and differentiable on $\mathbb{R}$, since it is a polynomial

Note that $f(0)=-1$, and $f(1)=1$. Since $-1<0<1$, there exists one number $c \in (0,1)$ such that $f(c)=0$, due to the Intermediate Value Theorem.

Since $f'(x) = 14x^6$ (which is positive for all $x \neq 0$), the function is increasing for all x, so it can only cross the x-axis at most once.

Therefore, there exists exactly one root of $f(x)$, which lies between 0 and 1.

[8] (8 pts)

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[9] **(8 pts)**

(a) (4 pts) Let $f(x) = \begin{cases} (x-1)^2 & \text{if } x < 1, \\ ax+b & \text{if } 1 \le x \le 4, \\ \sqrt{2x+1} & \text{if } x > 4. \end{cases}$

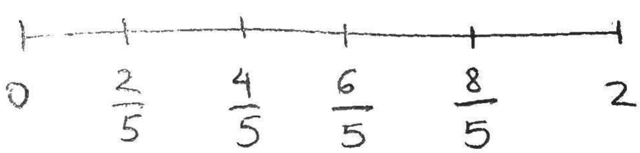
Find a and b so that $f(x)$ is continuous for all x, and justify your answer.

- $\lim\limits_{x\to 1} f(x) = f(1) \Rightarrow 0 = a+b$.
- $\lim\limits_{x\to 4} f(x) = f(4) \Rightarrow 3 = 4a+b$.

solving the system $\begin{cases} a+b=0 \\ 4a+b=3 \end{cases}$, we get $\boxed{a=1,\ b=-1}$

(b) (4 pts) Using Riemann Sums, write down an expression that estimates $\int_0^2 x^2 dx$ using the Left Endpoint Rule with 5 subdivisions. You may leave your answer as a sum of unsimplified fractions.

$\Delta x = \frac{2}{5}$, $f(x) = x^2$



$$\int_0^2 x^2 dx \sim \Delta x \left[f(0) + f\left(\tfrac{2}{5}\right) + f\left(\tfrac{4}{5}\right) + f\left(\tfrac{6}{5}\right) + f\left(\tfrac{8}{5}\right) \right]$$

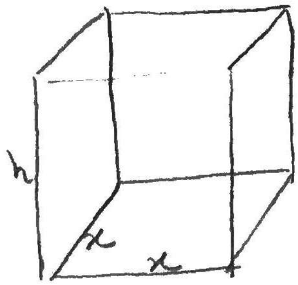
$$= \frac{2}{5}\left[0 + \frac{4}{25} + \frac{16}{25} + \frac{36}{25} + \frac{64}{25} \right]$$

[9] (8 pts)

Please leave blank!

[10] (8 pts) A rectangular box has a square base and an open top. The four sides are made of wood that costs 2 dollars per square foot, while the base is made of aluminum that costs 25 dollars per square foot. If the volume of the box is to be 50 cubic feet, what is its minimum possible cost?

Note: Justify your answer using calculus.



$$C(x,h) = 2\cdot 4xh + 25x^2$$
$$= 8xh + 25x^2$$

Constraint: $V = x^2h = 50 \Rightarrow h = \frac{50}{x^2}$

$\Rightarrow \quad C(x) = \frac{400}{x} + 25x^2$ [It is reasonable to assume $x \neq 0$].

$C'(x) = -\frac{400}{x^2} + 50x = 0 \Rightarrow 50x^3 = 400 \Rightarrow x^3 = 8 \Rightarrow x = 2$.

So $x=2$ is a critical point. We still need to check it is indeed an absolute minimum.

x	$x \in (0,2)$	$x=2$	$x \in (2,+\infty)$
$c'(x)$	negative	$c'(x)=0$	positive

Therefore, the cost is decreasing for x between 0 and 2, and increasing after $x=2$, proving that $x=2$ is an absolute minimum point.

Minimum cost: $C(2) = \frac{400}{2} + 25\cdot 2^2$
$= 200 + 100 = 300$ dollars.

[10] (8 pts)

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[11] **(8 pts)** For the function $f(x) = \dfrac{x^2}{x^2-1}$, you are given (do not compute!) that

$$f'(x) = \frac{-2x}{(x^2-1)^2} \text{ and } f''(x) = \frac{2(3x^2+1)}{(x^2-1)^3}.$$

(a) Find the domain of $f(x)$.

(b) Find the coordinates of all intercepts, and the equations of all asymptotes, of the graph of $y = f(x)$.

(c) In what intervals is the function f increasing? decreasing?

(d) In what intervals is the graph of f concave up? concave down?

(e) Find the coordinates of all local maxima, local minima, and points of inflection of the function f.

(f) Sketch the graph of $y = f(x)$. Label the features you found in items b and e.

a) Domain $(f) = (-\infty,-1) \cup (-1,1) \cup (1,+\infty)$

b) x-intercepts · set $y=0$, get $x=0$ $(0,0)$

y-intercepts set $x=0$, get $y=0$ $(0,0)$

vertical asymptotes: $x=-1$ and $x=1$.

Horizontal asymptotes: $\lim\limits_{x\to+\infty} \dfrac{x^2}{x^2-1} = \lim\limits_{x\to\infty} \dfrac{1}{1-1/x^2} = 1$.

Since the function is even, $y=1$ is the only horizontal asymptote.

c) The denominator in the expression of f' is always positive, so we only need to look at the numerator. So f' is negative for $x>0$, and positive for $x<0$. Hence, f is decreasing on $(0,+\infty)$, and increasing on $(-\infty,0)$.

d) $f''>0$ when $(x^2-1)^3>0 \iff (x^2-1)>0 \Rightarrow x>1$ or $x<-1$

$\Updownarrow$

f concave up

f conc. down $\iff f''<0 \iff (x^2-1)^3<0 \iff (x^2-1)<0 \Rightarrow -1<x<1$

e) crit. points: $f'=0$ or DNE. $f'(x)=0 \iff \boxed{x=0}$.

At 0, $f''<0$, so 0 is a local maximum.

candidate for inflection points where $f''=0$, which does not happen in this example.

f) graph on next page.

[11] (8 pts)

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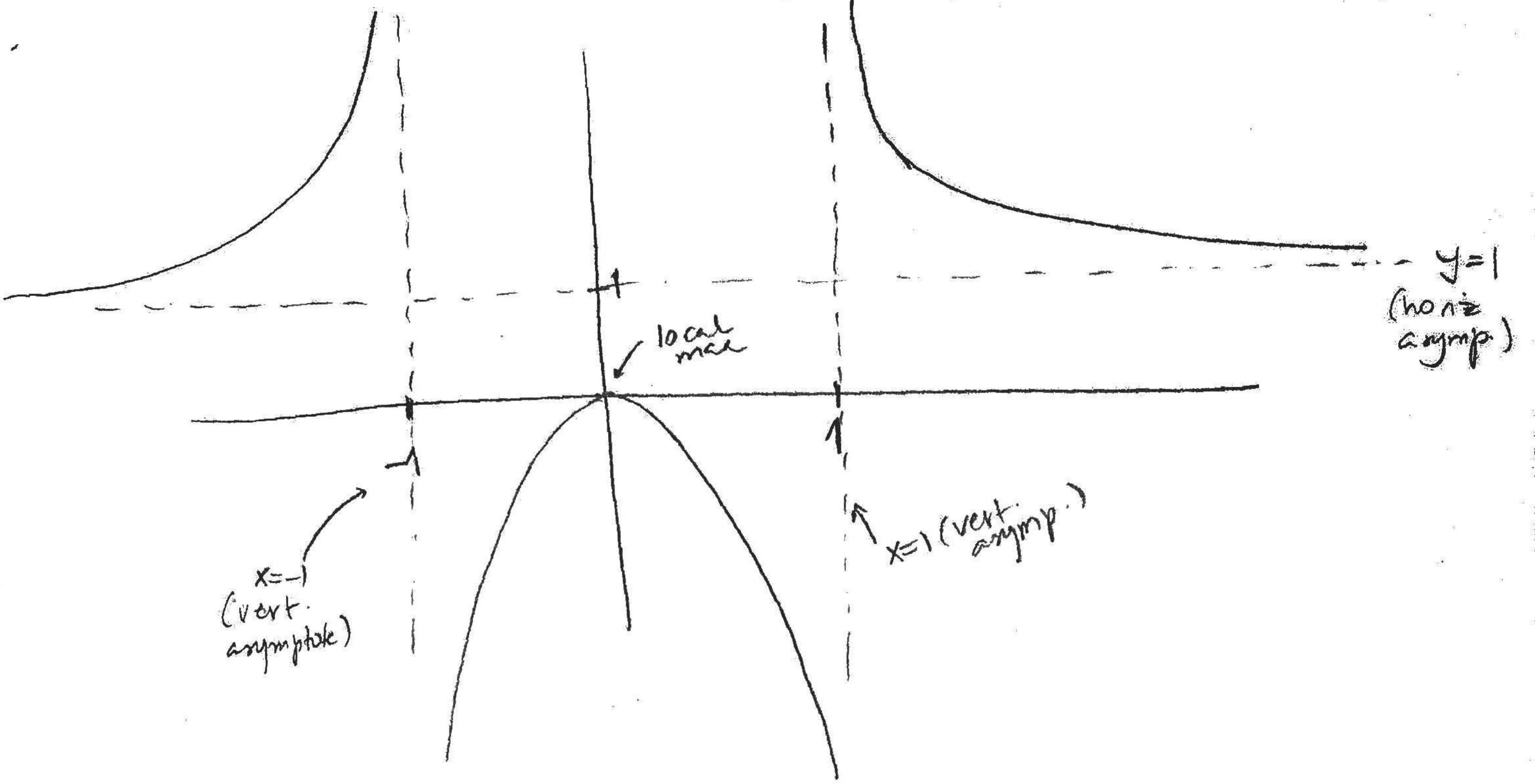

y=1
(horiz
asymp.)
1
local
max
-1
1
x=-1
(vert.
asymptote)
x=1 (vert.
asymp.)