THE CITY COLLEGE of NEW YORK
Department of Mathematics

MATH 201 Fall 2021– Final Exam (135 minutes)
Fall 2021

Course Supervisor: Prof. Wolf

12/20/2021

Student's Last Name, First Name: ____________________________________

Instructour's Name: ____

Instructions: This exam contains **12 pages** (including this cover page) and **10 questions**. Students must solve all **10 questions**. Total of **100 points**. No **calculators** or any other electronic devices are allowed during the examination. Turn off cell phones, alarms, and anything else that makes noises. You must show all your work to receive credit. Good luck!

Distribution of Marks

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (10 points) Compute $\frac{dy}{dx}$ for each of the functions below. **You do not need to simplify your answer**.

   (a) (3 points) $y = \dfrac{x^3 - 5x^4 + x^8}{e^{3x}}$.

   (b) (3 points) $y = x^{x+2}$.

   (c) (4 points) $xy^3 - x^2 = e^y - 2$.

   Solution:

   (a) Using the quotient rule we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{(3x^2 - 20x^3 + 8x^7)e^{3x} - 3e^{3x}(x^3 - 5x^4 + x^8)}{(e^{3x})^2} \\ &= \frac{[3x^2 - 20x^3 + 8x^7 - 3(x^3 - 5x^4 + x^8)]e^{3x}}{e^{6x}} \\ &= \frac{-3x^8 + 8x^7 + 15x^4 - 23x^3 + 3x^2}{e^{3x}}.\end{aligned}$$

   (b) Using the logarithmic differentiation we have

$$\begin{aligned}\ln(y) &= \ln(x)^{x+2} = (x+2)\ln(x) \\ \frac{d}{dx}[\ln(y)] &= \frac{d}{dx}[(x+2)\ln(x)] \\ \frac{y'}{y} &= \left(1 \cdot \ln(x) + (x+2) \cdot \frac{1}{x}\right) \\ y' &= \left(\ln(x) + \frac{x+2}{x}\right) \cdot y \\ y' &= \left(\ln(x) + \frac{x+2}{x}\right) \cdot x^{x+2}.\end{aligned}$$

   (c) Using the implicit differentiation we have

$$\begin{aligned}\frac{d}{dx}[xy^3 - x^2] &= \frac{d}{dx}\left(e^y - 2\right) \\ 1 \cdot y^3 + x \cdot 3y^2y' - 2x &= y' \cdot e^y \\ y^3 + 3xy^2y' - 2x &= y' \cdot e^y \\ 3xy^2y' - y'e^y &= 2x - y^3 \\ (3xy^2 - e^y)y' &= 2x - y^3 \\ \frac{dy}{dx} &= \frac{2x - y^3}{3xy^2 - e^y}.\end{aligned}$$

2. (12 points) Evaluate each integral and **simplify your answer**.

   (a) (3 points) $\int x^5 \sin\left(x^6\right)\ dx.$

   (b) (3 points) $\int \frac{2^{1/x^2}}{x^3}\, dx.$

   (c) (4 points) $\int_1^e \frac{2+\ln(x)}{x}\, dx.$

   Solution: Using the substitution we have

   (a) Let $u = x^6 \Longrightarrow du = 6x^5 dx \Longrightarrow dx = \frac{1}{6x^5}\, du$. So the integral becomes

$$\int x^5 \sin\left(x^6\right)\ dx = \int x^5 \sin(u)\ \frac{1}{6x^5}\, du = \frac{1}{6}\int \sin(u)\ du = -\frac{1}{6}\cos(u) = -\frac{1}{6}\cos(x^6)+C.$$

   (b) Let $u = \frac{1}{x^2} \Longrightarrow du = -\frac{2x}{x^4}\, dx = -\frac{2}{x^3}\, dx \Longrightarrow dx = -\frac{1}{2}\, x^3 du$ so that

$$\int \frac{2^{1/x^2}}{x^3}\, dx = -\int \frac{2^u}{x^3}\,\frac{1}{2}\, x^3 du = -\frac{1}{2}\int 2^u\ du = -\frac{1}{2}\cdot\frac{2^u}{\ln 2} = -\frac{2^u}{2\ln 2}+C = -\frac{2^{1/x^2}}{2\ln 2}+C.$$

   We use the formula: $\int a^x dx = \frac{a^x}{\ln a} + C.$

   (c) Let $u = 2+\ln(x) \Longrightarrow du = \frac{1}{x}\, dx \Longrightarrow dx = x du.$

$$\int_1^e \frac{2+\ln(x)}{x}\, dx = \int_1^e \frac{u}{x}\, x du = \int_1^e u\ du = \frac{u^2}{2} = \left.\frac{(2+\ln(x))^2}{2}\right|_1^e = \frac{(2+\ln(e))^2}{2} - \frac{(2+\ln(1))^2}{2}$$
$$= \frac{(3)^2}{2} - \frac{(2)^2}{2} = \frac{5}{2}.$$

3. (10 points) Find the limit or state that the limit does not exist. **You must justify your answer**.

   (a) (3 points) $\lim_{x\to-\infty} \frac{x+3}{\sqrt{4x^2+5x-8}}.$

   (b) (3 points) $\lim_{x\to\infty} \ln\left(x^{1/x}\right).$

   (c) (4 points) $\lim_{x\to 0}\left[\frac{1}{\ln(x+1)} - \frac{1}{x}\right].$

   Solution:

(a) You can solve it the long way as we demonstrate in class or simply use the short/systematic method as

$$\lim_{x\to-\infty}\frac{x+3}{\sqrt{4x^2+5x-8}}=\lim_{x\to-\infty}\frac{x}{\sqrt{4x^2}}=\lim_{x\to-\infty}\frac{x}{2\sqrt{x^2}}=\lim_{x\to-\infty}\frac{x}{2(-x)}=-\frac{1}{2}.$$

(b) Use law of logarithm then L'Hopital's rule as

$$\begin{aligned}\lim_{x\to\infty}\ln x^{1/x}&=\lim_{x\to\infty}\frac{\ln x}{x}=\frac{\infty}{\infty}\\&\overset{H}{=}\lim_{x\to\infty}\frac{1/x}{1}=\lim_{x\to\infty}1/x=0.\end{aligned}$$

(c) Direct substitution gives an indeterminate form $\infty-\infty$.

$$\begin{aligned}\lim_{x\to0}\left[\frac{1}{\ln(x+1)}-\frac{1}{x}\right]&=\lim_{x\to0}\frac{x-\ln(x+1)}{x\ln(x+1)}\overset{H}{=}\lim_{x\to0}\frac{1-\dfrac{1}{x+1}}{x\cdot\dfrac{1}{x+1}+1\cdot\ln(x+1)}\cdot\frac{x+1}{x+1}\\&=\lim_{x\to0}\frac{x}{x+(x+1)\ln(x+1)}=0/0\\&\overset{H}{=}\lim_{x\to0}\frac{1}{1+(x+1)\cdot\dfrac{1}{x+1}+1\cdot\ln(x+1)}=\lim_{x\to0}\frac{1}{1+1+\ln(x+1)}=\frac{1}{2}.\end{aligned}$$

4. (10 points) This question has **part** (a) through (b)

(a) (5 points) State the Fundamental Theorem of Calculus, Part 1, including all hypotheses.

(b) (5 points) Let $F(x)=\displaystyle\int_0^x\frac{t^2}{t^2+t+2}\,dt$ for all real number x. Find $F''(x)$ and determine the concavity of F.

Solution:

(a) Definion:. Suppose f is continuous on closed interval $[a,b]$. Let $F(x)=\displaystyle\int_a^{g(x)}f(t)\,dt$ where $a\le g(x)\le b$; then

$$F'(x)=g'(x)\cdot f\big(g(x)\big). \tag{1}$$

(b) Given $F(x)=\displaystyle\int_0^x\frac{t^2}{t^2+t+2}\,dt$ with $f(x)=\dfrac{t^2}{t^2+t+2}$, $g(x)=x\implies g'(x)=1$ and $a=0$. Using (1) of part(a) we have

$$F'(x)=\frac{x^2}{x^2+x+2}. \tag{2}$$

Use quotient rule to find $F''(x)$; that is

$$F''(x) = \frac{2x(x^2+x+2) - x^2(2x+1)}{(x^2+x+2)^2} = \frac{x^2+4x}{(x^2+x+2)^2} = \frac{x(x+4)}{(x^2+x+2)^2}. \qquad (3)$$

$$F''(x) = 0 \Longrightarrow x^2+4x = 0 \Longrightarrow x(x+4) = 0 \Longrightarrow x = 0, \quad x = -4$$

since the denominator is always positive. Note that the domain $D_F = (-\infty, \infty)$.

- For $x < -4$ and $x > 0 \Longrightarrow F''(x) > 0$, hence F is concave up on $(-\infty, -4) \cup (0, \infty)$.
- For $-4 < x < 0 \Longrightarrow F''(x) < 0$, and hence F is concave down on $(-4, 0)$.

5. (10 points) Water is leaking out of an inverted canonical tank at a rate of 100 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 0.3 m, find the rate at which water is being pumped into the tank.

   Note that the volume of the cone is $v = \dfrac{\pi}{3} r^2 h$.

   Solution: Step 1: Draw a possible picture. See Figure 1 below.

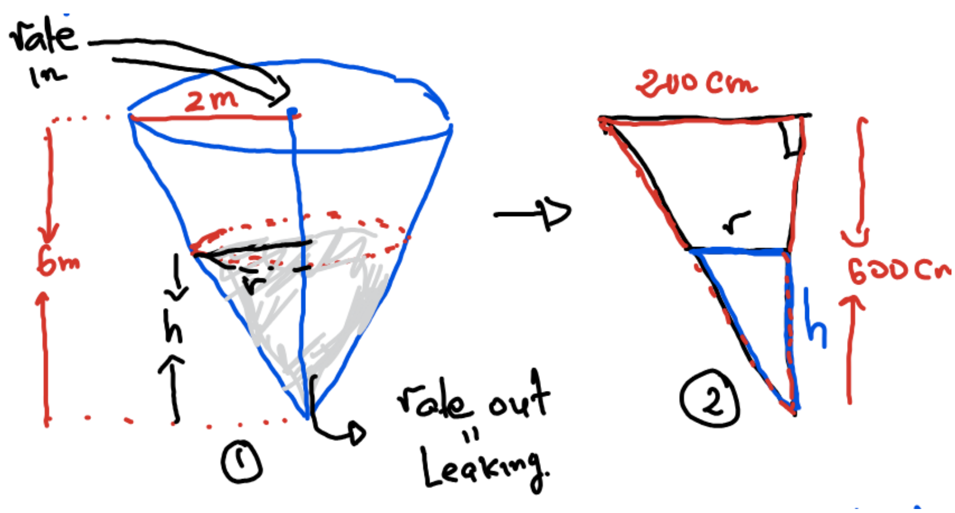


Figure 1: Inverted cone

Step 2: Identify your given (the known) and unknown

- Rate of change pumped in = C (Constant) cm^3/min.
- Rate of change leaked out = 100 cm^3/min.
- $h = h(t)$ = height of the water in the tank at time t. When $h = 0.3\text{m} =. 30$ cm, $\dfrac{dh}{dt} = 20$ cm/min.
- $v = v(t)$ = is the volume of the water at t.
- Unknown: Is to find the constant C.

Rate of change in the tank = Rate of change pumped in − Rate of change leaked out

$$\frac{dv}{dt} = C - 100 \tag{4}$$

Using the similar triangle from triangle 2 of Figure 1, we have

$$\frac{r}{200} = \frac{h}{600} \Longrightarrow r = \frac{200h}{600} \Longrightarrow \qquad r = \frac{h}{3} \tag{5}$$

Substituting $r = \frac{h}{3}$ into the the volume formula we have

$$v = \frac{\pi}{3}\left(\frac{h}{3}\right)^2 h = \frac{\pi}{27}\,h^3 \quad \Longrightarrow \quad v(t) = \frac{\pi}{27}\,[h(t)]^3 \tag{6}$$

Differentiating (7) with respect to t, we have

$$\frac{dv}{dt} = \frac{d}{dt}\Big[\frac{\pi}{27}\,[h(t)]^3\Big] = 3 \cdot \frac{\pi}{27}\,[h(t)]^2 \cdot \frac{dh}{dt} = \frac{\pi}{9}\,(30)^2 \cdot 20 = 2000\pi \text{ cm/min} \tag{7}$$

substitute (8) into (4) to get

$$2000\pi = C - 100 \Longrightarrow C = 2000\pi + 100 \text{ cm}^3/\text{min} \tag{8}$$

6. (10 points) Find all points (x, y) on the graph of $y = f(x) = x^2 + 2$ with tangent lines through the points (x, y) and (3,10).
Solution:

Derivative of f at x = Slope of the line through the points (x, y) and $(3, 10)$

$$2x = \frac{y - 10}{x - 3} = \frac{x^2 + 2 - 10}{x - 3} = \frac{x^2 - 8}{x - 3}$$

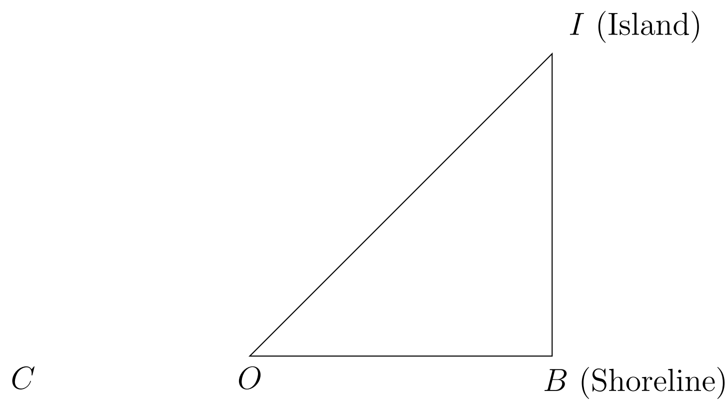
$2x(x-3) = x^2 - 8 \Longrightarrow 2x^2 - 6x = x^2 - 8 \Longrightarrow x^2 - 6x + 8 = 0 \Longrightarrow (x-2)(x-4) = 0 \Longrightarrow$
$x = 2, x = 4.$

So the points (x, y) are

- $x = 2 \Longrightarrow y = (2)^2 + 2 = 6$; the point is (2,6).
- $x = 4 \Longrightarrow y = (4)^2 + 2 = 18$; the point is (4,18).

7. (10 points) An island is $\sqrt{3}$ mi due north of its closest point along a straight shoreline. A guard is staying at a cabin on the shore that is 6 mi west of that point. The guard is planning to go from the cabin to the island. If the guard runs at a rate of 4 mi/h and swims at a rate of 2 mi/h, how far should the guard run before swimming to minimize the time it takes to reach the island?

Solution:



- The. distance from C (Cabin) to B (Shoreline) is $CB = 6$ mi
- Let $R_r = 4$ mi/h be running rate, and t_r be its running time. Let also $x = OC$ be the running distance; then the distance $OB = 6 - x$. So, Distance = rate · time

$$x = R_r \cdot t_r = 4 \cdot t_r \Longrightarrow \qquad t_r = \frac{x}{4} \tag{9}$$

- Let $R_s = 2$ mi/h be the swimming rate, t_s its swimming time and $OI =$ the swimming distance. Using Pythagorean, we have $OI^2 = OB^2 + BI^2$; that is

$$OI^2 = (6-x)^2 + (\sqrt{3})^2 = 36 - 12x + x^2 + 3 = x^2 - 12x + 39 \Longrightarrow \tag{10}$$

$$OI = \sqrt{x^2 - 12x + 39} \Longrightarrow t_s = \frac{OI}{R_s} \Longrightarrow \tag{11}$$

$$t_s = \frac{\sqrt{x^2 - 12x + 39}}{2} \tag{12}$$

The total time T spent is

$$T(x) = t_r + t_s = \frac{x}{4} + \frac{\sqrt{x^2 - 12x + 39}}{2}$$

$$T'(x) = \frac{1}{4} + \frac{1}{2}\,\frac{2x - 12}{2\sqrt{x^2 - 12x + 39}} = \frac{1}{4} \cdot \frac{\sqrt{x^2 - 12x + 39}}{\sqrt{x^2 - 12x + 39}} + \frac{2x - 12}{4\sqrt{x^2 - 12x + 39}} = \frac{\sqrt{x^2 - 12x + 39} + 2x - 12}{4\sqrt{x^2 - 12x + 39}}$$

$T'(x) = 0 \Longrightarrow \sqrt{x^2 - 12x + 39} + 2x - 12 = 0 \Longrightarrow 12 - 2x = \sqrt{x^2 - 12x + 39}$.
Squaring both sides we have

$$(12 - 2x)^2 = (\sqrt{x^2 - 12x + 39})^2 \Longrightarrow 144 - 48x + 4x^2 = x^2 - 12x + 39 = 3x^2 - 36x + 105 = 0 \Longrightarrow$$
$$3(x^2 - 12x + 35) = 0 \Longrightarrow 3(x-5)(x-7) = 0 \Longrightarrow x = 5 \text{ or } x = 7$$

but $x = 7$ is not possible since $x = OC < CB = 6$. Therefore $x = 5$ is minimum, which mean that the guard should run 5 mi before swimming.

8. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $y = f(x) = \dfrac{1}{2}\,x^4 + \dfrac{8}{3}\,x^3 - \dfrac{2}{3}$.
   (a) (2 points) On which intervals is f increasing or decreasing? Explain your reasoning.
   (b) (2 points) At what values of x does f have local maximum or minimum? Explain.
   (c) (2 points) On what interval is f concave up or concave down? Explain your reasoning.
   (d) (2 points) Does f has inflection points? if so, find its x-coordinates.
   (e) (2 points) Sketch the graph of f by using all the information obtained above.

   Solution:

   Since f is a polynomial, domain $D_f = (-\infty, \infty)$.
   (a) $f'(x) = 2x^3 + 8x^2 = 2x^2(x+4) = 0 \Longrightarrow x = 0$ or $x = -4$ are critical numbers.
   When $x < -4$, then $f'(x) < 0$, which implies that f is decreasing on $(-\infty, -4)$.
   When $-4 < x \leq 0$ or $x \geq 0$, then $f'(x) > 0$, which implies that f is increasing on $(-4, \infty)$.
   Therefore, the point $(-4, f(-4))$ is a local minimum and no local maximum.
   (b) at $x = -4$, f has a minimum.
   (c) For the concavity, investigate the solution of $f''(x) = 0$; that is
   $f'(x) = 2x^3 + 8x^2 \Longrightarrow f''(x) = 6x^2 + 16x = 2x(3x+8) = 0 \Longrightarrow 2x = 0$ or $3x + 8 = 0 \Longrightarrow$ $x = 0$ or $x = -8/3$. $f''(x) > 0$ when $x < -8/3$, and $f''(x) < 0$ when $(-4/3, 0)$ and $f''(x) > 0$ when $x > 0$. Hence f concave up on $(-\infty, -8/3) \cup (0, \infty)$ and concave down on $(-8/3, 0)$.
   (d) Since f change signs around $-8/3$ and 0, then f has inflection points with $x-$ coordinates $x = -8/3$ and $x = 0$.
   (e) The graph of f, see Figure 2 on page 9 below.

9. (10 points) This Problems has 3 **parts** labeled (a) through (c).
   (a) (4 points) Use the limit definition of derivative to find $f'(x)$ if $f(x) = \sqrt{x+1}$. **No credit will be given for any other method**.
   (b) (3 points) Find an equation of the tangent line to the curve $y = f(x)$ at $x = 3$.
   (c) (3 points) Use differentials to estimate $f(3.01)$. **You do not need to simplify your answer**.

   Solution:

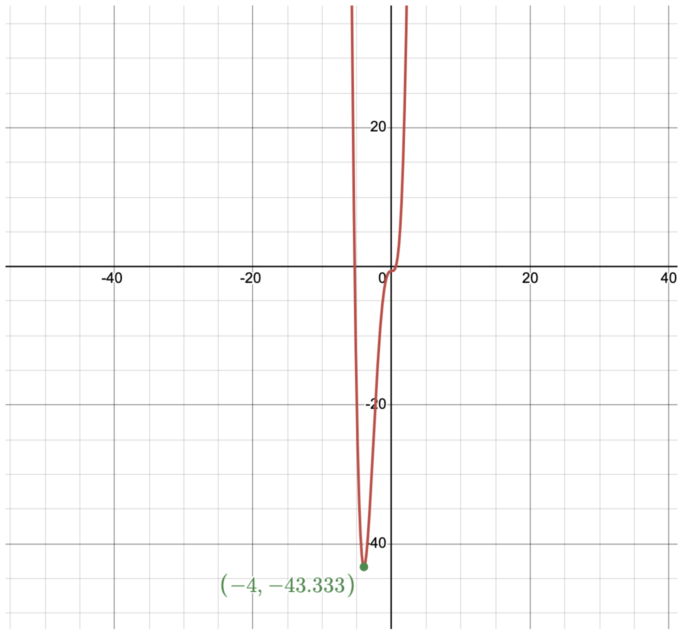


Figure 2: Graph of $f(x) = \frac{1}{2}x^4 + \frac{8}{3}x^3 - \frac{2}{3}$.

(a)

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} && (13)\\
&= \lim_{h\to 0}\frac{\sqrt{x+h+1}-\sqrt{x+1}}{h}\cdot\frac{\sqrt{x+h+1}+\sqrt{x+1}}{\sqrt{x+h+1}+\sqrt{x+1}} && (14)\\
&= \lim_{h\to 0}\frac{(\sqrt{x+h+1})^2-(\sqrt{x+1})^2}{h[\sqrt{x+h+1}+\sqrt{x+1}]} = \lim_{h\to 0}\frac{x+h+1-(x+1)}{h[\sqrt{x+h+1}+\sqrt{x+1}]} && (15)\\
&= \lim_{h\to 0}\frac{h}{h[\sqrt{x+h+1}+\sqrt{x+1}]} = \lim_{h\to 0}\frac{1}{\sqrt{x+h+1}+\sqrt{x+1}} && (16)\\
&= \lim_{h\to 0}\frac{1}{\sqrt{x+1}+\sqrt{x+1}} && (17)\\
&= \frac{1}{2\sqrt{x+1}} && (18)\\
& && (19)
\end{aligned}
$$

(b) The equation of the tangent line to the curve $f(x) = \sqrt{x+1}$ is of the form

$$y = f(a) + f'(a)(x-a) \text{ where } a = 3.$$

We have $f(3) = \sqrt{3+1} = 2$ and $f'(3) = \frac{1}{2\sqrt{3+1}} = \frac{1}{4}$. So the equation is

$y - 2 = \frac{1}{4}(x-3) \Longrightarrow y = \frac{1}{4}x + \frac{5}{4}$.

(c) Using the differentials formulus: $f(a+dx) \approx f(a) + f'(a)dx$ we have

$$f(3.01) \approx f(3) + f'(3)(0.01) = 2 + \frac{1}{4} \cdot (0.01) = 2 + \frac{1}{4} \cdot \frac{1}{100} = \frac{801}{400}.$$

10. (10 points) This Problems has 3 **parts** labeled (a) through (c).

(a) (3 points) Let $f : [a, b] \to \mathbb{R}$ be continuous. Give the definition for $\int_a^b f(x)\,dx$ in terms of Riemann sums.

(b) (3 points) Use part (a) with 4 subintervals and right endpoints to estimate the integral $\int_1^9 \sqrt{\ln x}\,dx$. **You do not need to simplify your answer**.

(c) (4 points) Let $s(t) = t^3 - 10t^2 + 25t$ be the position function of a moving object. Find the object's acceleration each time the speed is zero.

Solution:

(a) Definition: If f is defined for $[a, b]$, we divide the interval $[a, b]$ into n subintervals $[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]$ of equal width $\Delta x = \frac{b-a}{n}$. Let $c_1, c_2, \ldots c_n$ be any sample points in these subintervals, c_i lies in the ith subinterval $[x_{i-1}, x_i]$. Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \Big[f(c_1) + f(c_2) + f(c_3) + f(c_4) + \ldots + f(c_{n-1}) + f(c_n)\Big]\Delta x$$

(b) Using part(a) with $f(x) = \sqrt{\ln x}$, $a = 1$, $b = 9$ and $n = 4$ subintervals. We have

$$\Delta x = \frac{b-a}{n} = \frac{9-1}{4} = 2.$$

$$\begin{aligned} x_0 &= a = 1. \\ x_1 &= x_0 + \Delta x = 1 + 2 = 3. \\ x_2 &= x_1 + \Delta x = 3 + 2 = 5. \\ x_3 &= x_2 + \Delta x = 5 + 2 = 7. \\ x_4 &= x_3 + \Delta x = 7 + 2 = 9. \end{aligned}$$

The sample points in this case are the right endpoints; that is $c_1 = x_1, c_2 = x_2, c_3 = x_3$ and $c_4 = x_4$. So

$$\begin{aligned} \int_1^9 \ln(x)\,dx &= \Big[\sqrt{\ln(c_1)} + \sqrt{\ln(c_2)} + \sqrt{\ln(c_3)} + \sqrt{\ln(c_4)}\Big](2) \\ &= 2\Big[\sqrt{\ln(3)} + \sqrt{\ln(5)} + \sqrt{\ln(7)} + \sqrt{\ln(9)}\Big]. \end{aligned}$$

(c) velocity $v(t) = \dfrac{ds}{dt} = 3t^2 - 20t + 25 = (3t-5)(t-5) = 0 \Longrightarrow t = 5$ or $t = 5/3$

The acceleration $a(t) = \dfrac{dv}{dt} = 6t - 20$.

at $t = 5$, we get $a(5) = \dfrac{dv}{dt} = 6(5) - 20 = 10$.

at $t = 5/3$, we get $a(5/3) = \dfrac{dv}{dt} = 6(5/3) - 20 = -10$.

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