Solution To Final Exam
Fall 2021

HE CI Y COLLEGE of NEW YORK
Department of Mathematics

MA H 201– Final Exam (135 minutes)
Fall 2021

nstructour's Name ______________________

Student's Last Name, First Name ______________________

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nstructions: This exam contains **13 pages** (including this cover page) and **10 questions**. Students must solve all **10 questions**. Total of **100 points**. No **calculators** or any other electronic devices are allowed during the examination. Turn o cell phones, alarms, and anything else that makes noises. You must show all your work to receive credit. Good luck!

Distribution of Marks

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

1. (10 points) Compute $\frac{y}{x}$ for each of the functions below. **You do not need to simplify your answer**.

   (a) (3 points) $y = \frac{x \quad 5x^4 + x^8}{e}$.

use quotient rule $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$$\frac{dy}{dx} = \frac{(3x^2 - 20x^3 + 8x^7)\, e^{3x} - (x^3 - 5x^4 + x^8) \cdot 3e^{3x}}{(e^{3x})^2}$$

$$\boxed{\frac{dy}{dx} = \frac{(3x^2 - 20x^3 + 8x^7)\, e^{3x} - 3(x^3 - 5x^4 + x^8)\, e^{3x}}{e^{6x}}}$$

   (b) (3 points) $y = x^{\ +2}$.

Use logarithmic differentiation:

$\ln y = \ln x^{x+2} \Rightarrow \ln y = (x+2)\ln(x) \Rightarrow \frac{d}{dx}\ln(y) = \frac{d}{dx}\left[(x+2)\ln(x)\right]$

$$\frac{y'}{y} = 1 \cdot \ln(x) + (x+2) \cdot \frac{1}{x}$$

$$y' = \left(\ln(x) + \frac{x+2}{x}\right) \cdot y$$

$$\boxed{y' = \left(\ln(x) + \frac{x+2}{x}\right) \cdot x^{x+2}}$$

   (c) (4 points) $xy \quad x^2 = e^y \quad 2$.

Implicit differentiation:

$$1 \cdot y^3 + 3xy^2 y' - 2x = y' e^y$$

$$3xy^2 y' - y' e^y = 2x - y^3$$

$$(3xy^2 - e^y)\, y' = 2x - y^3$$

$$\boxed{y' = \frac{2x - y^3}{3xy^2 - e^y}}$$

2. (10 points) Evaluate each integral and **simplify your answer**.

(a) (3 points) $x^5 \sin \; x^6) \quad x$.

use substitution method.

Let 1) $u = x^6$

2) $du = 6x^5 dx$

3) $dx = \frac{du}{6x^5}$

$$\int x^5 \sin(x^6)\, dx = \int \cancel{x^5} \sin(u) \cdot \frac{du}{6\cancel{x^5}}$$

$$= \frac{1}{6} \int \sin(u)\, du$$

$$= -\frac{1}{6} \cos(u)$$

$$= \boxed{-\frac{1}{6} \cos(x^6) + C}$$

(b) (3 points) $\frac{2^{1/}}{x} \quad x$

1) $u = \frac{1}{x^2}$

2) $du = -\frac{2}{x^3} dx$

3) $dx = -\frac{x^3}{2} du$

$$= \int \frac{2^u}{\cancel{x^3}} \cdot -\frac{\cancel{x^3}}{2}\, du$$

$$= -\frac{1}{2} \int 2^u\, du$$

$$= \boxed{-\frac{1}{2} \cdot \frac{2^u}{\ln 2} + C}$$

we use formula

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

(c) (4 points) ${}_{1}^{e} \quad \frac{2 + \ln(x)}{x} \quad x$ $= \int_2^3 \frac{u}{\cancel{x}} \cdot \cancel{x}\, du$

1) $u = 2 + \ln(x)$

2) $du = \frac{1}{x}\, dx$

3) $dx = x\, du$

we can change limits.

- $x = 1 \Rightarrow u = 2 + \cancel{\ln(1)}^{0} = \boxed{2}$
- $x = e \Rightarrow u = 2 + \ln(e) = 2 + 1 = \boxed{3}$

$$= \int_2^3 u\, du$$

$$= \frac{1}{2}\left[u^2\right]_2^3$$

$$= \frac{1}{2}\left[3^2 - 2^2\right]$$

$$= \frac{1}{2}\left[9 - 4\right)$$

$$= \boxed{5/2}$$

3. (10 points) Find the limit or state that the limit does not exist. **You must justify your answer**.

(a) (3 points) $\lim_{-\infty} \frac{x+3}{\sqrt{4x^2+5x \quad 8}}$

$$= \lim_{x\to-\infty} \frac{x}{\sqrt{4x^2}} = \lim_{x\to-\infty} \frac{x}{\sqrt{4}\sqrt{x^2}}$$

$$= \lim_{x\to-\infty} \frac{x}{2(-x)} = \lim_{x\to-\infty} \frac{\cancel{x}}{-2\cancel{x}} = \boxed{-\frac{1}{2}}$$

NB: you also can use the Long method and use the fact $\lim_{x\to\infty} \frac{1}{x^n} = 0$ for $n>0$

(b) (3 points) $\lim_{\infty} \ln \; x^{1/} \;)$

$$= \lim_{x\to\infty} \frac{1}{x}\ln(x)$$

$$= \lim_{x\to\infty} \frac{\ln(x)}{x} = \text{“}\frac{\infty}{\infty}\text{”}$$

L'Hopital's rule

$$\overset{H}{=} \lim_{x\to\infty} \frac{1/x}{1}$$

$$= \lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = \boxed{0}$$

(c) (4 points) $\lim_{0}\left[\frac{1}{\ln(x+1)} \quad \frac{1}{x}\right] = \frac{1}{\ln(0+1)} - \frac{1}{0} = \frac{1}{\ln 1} - \frac{1}{0} = \frac{1}{0} - \frac{1}{0} = \text{“}\infty - \infty\text{”}$

$$= \lim_{x\to0} \frac{x - \ln(x+1)}{x\ln(x+1)} \overset{H}{=} \lim_{x\to0} \frac{1 - \frac{1}{x+1}}{1\cdot\ln(x+1) + x\cdot\frac{1}{x+1}}$$

$$= \lim_{x\to0} \frac{x+1-1}{(x+1)\ln(x+1) + x} = \lim_{x\to0} \frac{x}{(x+1)\ln(x+1)+x} = \text{“}\frac{0}{0}\text{”}$$

$$\overset{H}{=} \lim_{x\to0} \frac{1}{1\cdot\ln(x+1) + \cancel{(x+1)}\cdot\frac{1}{\cancel{(x+1)}} + 1} = \lim_{x\to0} \frac{1}{\ln(x+1)+2}$$

$$= \lim_{x\to0} \frac{1}{\ln(0+1)+2} = \frac{1}{0+2} = \boxed{\frac{1}{2}}$$

4. (10 points) This question has **part** (a) and (b).

(a) (5 points) State the Fundamental Theorem of Calculus, Part 1 including all hypotheses.

Suppose f is continuous on $[a,b]$. Let $F(x) = \int_a^{g(x)} f(t)\,dt$ where $a < g(x) < b$; then

$$F'(x) = f(g(x)) \cdot g'(x)$$

$1 - 4(1)(2)$

(b) (5 points) Let $F(x) = \,_{0}\ \frac{t^2}{t^2+t+2}\ t$ for all real number x. Find $F''(x)$ and determine the concavity of F.

Let $g(x) = x$, $a = 0$, $f(t) = \frac{t^2}{t^2+t+2}$. Using Part (a) above

$$F'(x) = f(g(x)) \cdot g'(x) = f(x) \cdot (x)' = f(x) \cdot 1 = \frac{x^2}{x^2+x+2}.$$

$$F''(x) = \frac{d}{dx}\left(\frac{x^2}{x^2+x+2}\right) = \frac{2x(x^2+x+2) - x^2(2x+1)}{(x^2+x+2)^2}$$

$$= \frac{\cancel{2x^3} + 2x^2 + 4x \cancel{- 2x^3} - x^2}{(x^2+x+2)^2} = \boxed{\frac{x^2 - 4x}{(x^2+x+2)^2}}$$

$$F''(x) = 0 \Rightarrow x^2 - 4x = 0 \Rightarrow x(x-4) = 0 \Rightarrow \boxed{x = 0,\ x = 4}$$

Since the denominator $(\underbrace{x^2+x+2}_{>0})^2 > 0$ is always positive.

we have

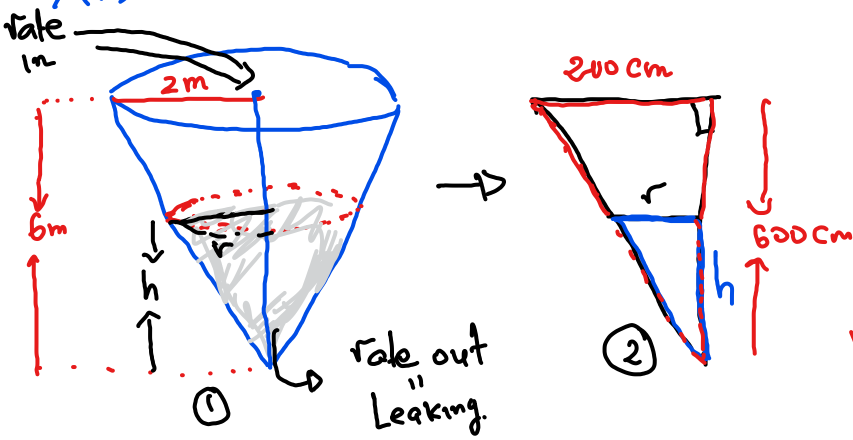
	$-\infty$	0		4	$+\infty$
$F''(x)$	+	0	−	0	+
$F(x)$	Concave up		Concave Down		Concave up

F Concave up on $(-\infty, 0) \cup (4, +\infty)$ and down on $(0, 4)$

5. (10 points) Water is leaking out of an inverted conical tank at a rate of 100 cm /min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 0.3 m, find the rate at which water is being pumped into the tank.

Note that the volume of the cone is $V = \frac{\ }{3} r^2 h$. Also 1m=100cm

Answer: 1) Draw the tank.



Given:

- rate of change out $= 100 \text{ cm}^3/\text{min}$
- rate of chang in = constant $= C \text{ cm}^3/\text{min}$.
- h = height of water.

when $h = 0.3 \text{ m} = 30 \text{ cm}$, then $\frac{dh}{dt} = 20 \text{ cm/min}$.

Question: Find the constant C?

Rate of change of water in the tank = Rate of change in (Pumped) — Rate of change out (Leaking)

V : volume

$$\boxed{\frac{dV}{dt} = C - 100} \quad \ldots (*)$$

Using similar triangle from ②

$$\frac{r}{200} = \frac{h}{600} \Rightarrow r = \frac{200h}{600} \Rightarrow \boxed{r = \frac{h}{3}}$$

$$\Rightarrow V = \frac{\pi r^2 h}{3} = \frac{\pi}{3} \cdot \left(\frac{h}{3}\right)^2 \cdot h \Rightarrow$$

$$\Rightarrow V = \frac{\pi}{3} \frac{h^2}{9} \cdot h = \frac{\pi}{27} h^3$$

$$\Rightarrow \boxed{V(t) = \frac{\pi}{27} (h(t))^3}$$

$$\frac{dV}{dt} = \frac{\pi}{27} \cdot 3 (h(t))^2 \cdot \frac{dh}{dt}$$

$$= \frac{\pi}{27} \cdot 3 (30)^2 \cdot 20$$

$$\boxed{\frac{dV}{dt} = 2000\pi \text{ cm}/\text{min}}$$

Substituting into (*) we have.

$$\frac{dV}{dt} = C - 100$$

$$2000\pi = C - 100$$

$$\Rightarrow \boxed{C = (2000\pi + 100) \text{ cm}^3/\text{min}}$$

6. (10 points) Let $s(t) = t \quad 10t^2 + 25t$ be the position function of a moving object. Find the object's acceleration each time the speed is zero.

Solution: Find we need to find the acceleration $a(t)$, which is the Second derivative of Position function, that is

Velocity $= v(t) = s'(t)$ and acceleration $a(t) = v'(t) = s''(t)$

$v(t) = \frac{d}{dt}\left(t^3 - 10t^2 + 25t\right)$

$v(t) = 3t^2 - 20t + 25$

$= (3t-5)(t-5)$

Speed $= |v(t)| = 0 \Rightarrow$

$|3t-5|\,|t-5| = 0$

$|3t-5| = 0$ or $|t-5| = 0$

$\boxed{t = 5/3 \quad \text{or} \quad t = 5}$

$a(t) = \frac{d}{dt}\left(3t^2 - 20t + 25\right)$

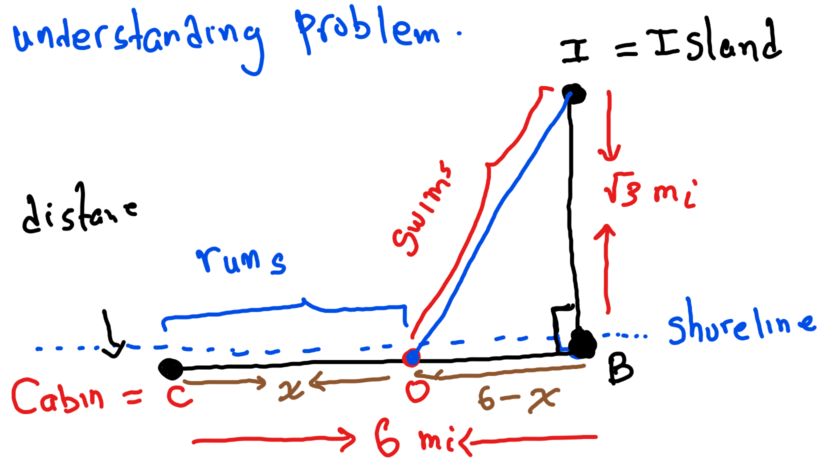
$a(t) = 6t - 20$

- when $t = 5 \Rightarrow a(t) = 6t - 20\big|_{t=5}$
  $= 6(5) - 20$
  $= \boxed{10}$

- when $t = \frac{5}{3}$
  $a(t) = 6\left(\frac{5}{3}\right) - 20$
  $= 10 - 20$
  $\boxed{a(t) = -10}$

7. (10 points) An island is $\sqrt{3}$ mi due north of its closest point along a straight shoreline. A guard is staying at a cabin on the shore that is 6 mi west of that point. The guard is planning to go from the cabin to the island. If the guard runs at a rate of 4 mi/h and swims at a rate of 2 mi/h, how far should the guard run before swimming to minimize the time it takes to reach the island?

Solution: First draw picture after understanding problem.



Given:

- Running: rate = 4 mi/h

Let $x = OC$ be the running distance

then: distance = rate · time

$$\Rightarrow x = 4t_{run}$$

$$\Rightarrow \boxed{t_{run} = \frac{x}{4}}$$

- Swimming: rate = 2 mi/h and distance = OC

Using Pythagorean, we have $OI^2 = OB^2 + BI^2$

$$\Rightarrow OI^2 = (6-x)^2 + (\sqrt{3})^2 = 36 - 12x + x^2 + 3 = 39 - 12x + x^2.$$

$$\Rightarrow \boxed{OI = \sqrt{39 - 12x + x^2}} \Rightarrow t_{swim} = \frac{OI}{rate} = \frac{\sqrt{39 - 12x + x^2}}{2}$$

So, the total time spent is $\boxed{T = t_{run} + t_{swim}}$ $\Rightarrow$

$$T(x) = \frac{x}{4} + \frac{\sqrt{39 - 12x + x^2}}{2}$$ is a function of x.

$$T'(x) = \frac{1}{4} + \frac{1}{2} \cdot \frac{-12 + 2x}{2\sqrt{39 - 12x + x^2}} = 0$$

$$= \frac{1 - 12 + 2x}{4\sqrt{39 - 12x + x^2}} = 0$$

$$\Rightarrow -11 + 2x = 0 \Rightarrow 2x = 11 \Rightarrow \boxed{x = \frac{11}{2} = 5.5}$$

		$11/2$	
$T'(x)$	$-$	0	$+$
$T(x)$	$\searrow$	min	$\nearrow$

So the guard should run $\frac{11}{2}$ mi before swiming.

8. (10 points) Let f $\to$ be defined by $y = f(x) = \frac{1}{2}x^4 + \frac{8}{3}x \quad \frac{2}{3}$.

(a) (2 points) On which intervals is f increasing or decreasing? Explain your reasoning.

$f'(x) = 2x^3 + 8x^2 = 2x^2(x+4) = 0$ To find critical numbers

$\Rightarrow 2x^2 = 0$ or $x + 4 = 0$

$\boxed{x = 0 \quad \text{or} \quad x = -4}$

	$-\infty$		-4		0		$+\infty$
$f'(x)$		$-$	0	$+$	0	$+$	
$f(x)$	$+\infty$	$\searrow$	$f(-4)$	$\nearrow$	$f(0) = -\frac{2}{3}$	$\nearrow$	$+\infty$

f is decreasing on $(-\infty, -4)$ and increasing on $(0, +\infty)$.

(b) (2 points) At what values of x does f have local maximum or minimum? Explain.

f has a Local minimum at $x = -4$ with value $f(-4)$.
and NO Local maximum.

(c) (2 points) On what interval is f concave up or concave down? Explain your reasoning.

We have $f'(x) = 2x^3 + 8x^2$

$\Rightarrow f''(x) = 6x^2 + 16x = 2x(3x+8) = 0$

$\Rightarrow 2x = 0$ or $3x + 8 = 0$

$\Rightarrow \boxed{x = 0}$ $\qquad$ $\boxed{x = -8/3}$

	$-\infty$		$-8/3$		0		$+\infty$
$f''(x)$		$+$	0	$-$	0	$+$	
$f(x)$		Concave up		Concave Down		Concave up	

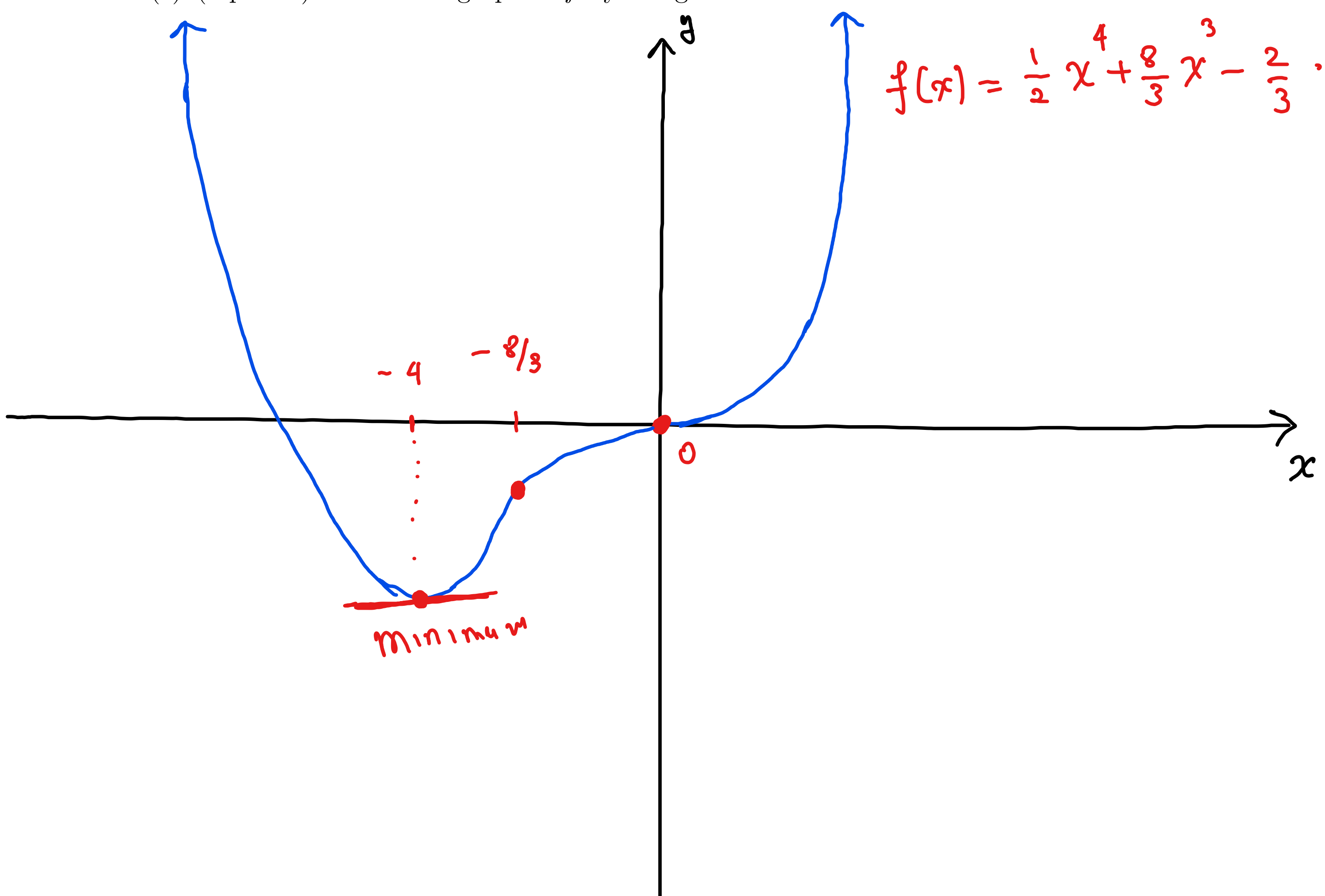
f Concave up on $(-\infty, -8/3) \cup (0, +\infty)$ and down on $(-8/3, 0)$.

(d) (2 points) Does f has inflection points? if so, find its x-coordinates.

Since f changes signs around $x = -\frac{8}{3}$, $x = 0$

$\Rightarrow$ $x = -8/3$ and $x = 0$ are coordinates of inflection points.

(e) (2 points) Sketch the graph of f by using all the information obtained above.



9. (10 points) This Problems has 3 **parts** labeled (a) through (c).

(a) (4 points) Use the limit definition of derivative to find $f'(x)$ if $f(x) = \sqrt{x+1}$. **No credit will be given for any other method.**

Def: $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}}$

$= \lim_{h\to 0} \frac{(\sqrt{x+h+1})^2 - (\sqrt{x+1})^2}{h(\sqrt{x+h+1} + \sqrt{x+1})} = \lim_{h\to 0} \frac{\cancel{x}+h+\cancel{1} - \cancel{x} - \cancel{1}}{h(\sqrt{x+h+1} + \sqrt{x+1})}$

$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h+1} + \sqrt{x+1})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}}$

$= \frac{1}{\sqrt{x+0+1} + \sqrt{x+1}} = \boxed{\frac{1}{2\sqrt{x+1}} = f'(x)}$

(b) (3 points) Find an equation of the tangent line to the curve $y = f(x)$ at $x = 3$.

$x = 3 \Rightarrow y = \sqrt{3+1} = 2$

$(3, 2)$

$x_0 \quad y_0$

Tangent equation is

$y - y_0 = m(x - x_0)$

$y - 2 = \frac{1}{4}(x - 3)$

$\boxed{y = \frac{1}{4}(x-3) + 2}$

$m = f'(3) = \frac{1}{2\sqrt{3+1}}$

$\boxed{m = \frac{1}{4}}$

(c) (3 points) Use di erentials (or linear approximations) to estimate $f(3.01)$. **You do not need to simplify your answer.**

$f(3.01) \approx \frac{1}{4}(3.01 - 3) + 2$

$\approx \frac{1}{4}(0.01) + 2$

$\approx \frac{1}{400} + 2$

$\boxed{f(3.01) \approx \frac{801}{400}}$

10. (10 points) This Problems has 2 **parts** labeled (a) and (b).

(a) (5 points) Let f be a continuous function. Give the definition for $\int_a^b f(x)\, dx$ in terms of Riemann sums.

Solution: The partition of $[a,b]$ is $a_0 = x_0 < x_1 < x_2 < x_3 \cdots < x_{n-1} < x_n = b$

Right end point: $\int_a^b f(x)\,dx \cong R_n = \Delta x\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right]$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

n = # of subinterval

Left end pound: $\int_a^b f(x)\,dx \cong L_n = \Delta x\left[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right]$

$x_i = a + (i-1)\Delta x$

n, Δx are the same as in R_n

(b) (5 points) Use part (a) with 4 subintervals and right endpoints to estimate the integral $\int_1^9 \sqrt{\ln x}\, dx$. **You do not need to simplify your answer.**

Solution: From Part (a): $n = 4$, $[a,b] = [1,9]$

- $\Delta x = \frac{b-a}{n} = \frac{9-1}{4} = \frac{8}{4} = 2$
- $x_0 = a = 1$
- $x_1 = 1 + \Delta x = 1 + 2 = 3$
- $x_2 = 3 + 2 = 5$
- $x_3 = 5 + 2 = 7$
- $x_4 = 7 + 2 = 9$

$R_4 = \Delta x\left[f(x_1) + f(x_2) + f(x_3) + f(x_4)\right]$

$= 2\left[f(3) + f(5) + f(7) + f(9)\right]$

$= \boxed{2\left[\sqrt{\ln 3} + \sqrt{\ln 5} + \sqrt{\ln 7} + \sqrt{\ln 9}\right]}$

$f(x) = \sqrt{\ln(x)}$

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