

Rotation of Axes

At the beginning of Chapter 5 we stated that all equations of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

represented a conic section, which might possibly be degenerate. We saw in Section 5.2 that the graph of the quadratic equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

is a parabola when $A = 0$ or $C = 0$, that is, when $AC = 0$. In Section 5.3 we found that the graph is an ellipse if $AC > 0$, and in Section 5.4 we saw that the graph is a hyperbola when $AC < 0$. So we have classified the situation when $B = 0$. Degenerate situations can occur; for example, the quadratic equation $x^2 + y^2 + 1 = 0$ has no solutions, and the graph of $x^2 - y^2 = 0$ is not a hyperbola, but the pair of lines with equations $y = \pm x$. But excepting this type of situation, we have categorized the graphs of all the quadratic equations with $B = 0$.

When $B \neq 0$, a rotation can be performed to bring the conic into a form that will allow us to compare it with one that is in standard position. To set up the discussion, let us first suppose that a set of xy -coordinate axes has been rotated about the origin by an angle θ , where $0 < \theta < \pi/2$, to form a new set of $\hat{x}\hat{y}$ -axes, as shown in Figure 1(a). We would like to determine the coordinates for a point P in the plane relative to the two coordinate systems.

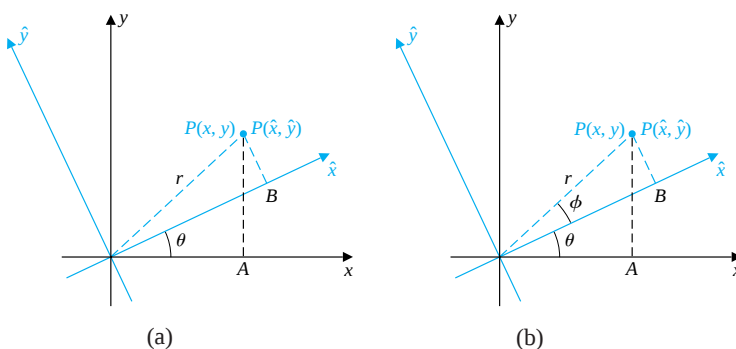


Figure 1

First introduce a new pair of variables r and ϕ to represent, respectively, the distance from P to the origin and the angle formed by the $\hat{x}$ -axis and the line connecting the origin to P , as shown in Figure 1(b).

From the right triangle OBP shown in Figure 2(a) we see that

$$\hat{x} = r \cos \phi \quad \text{and} \quad \hat{y} = r \sin \phi,$$

and from the right triangle OAP shown in Figure 2(b) we see that

$$x = r \cos(\phi + \theta) \quad \text{and} \quad y = r \sin(\phi + \theta).$$

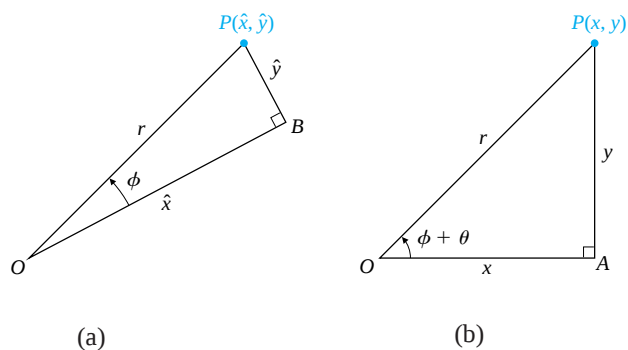


Figure 2

Using the fundamental trigonometric identities for the sum of the sine and the cosine we have

$$x = r \cos(\phi + \theta) = r \cos \phi \cos \theta - r \sin \phi \sin \theta$$

and

$$y = r \sin(\phi + \theta) = r \cos \phi \sin \theta + r \sin \phi \cos \theta.$$

Since $\hat{x} = r \cos \phi$ and $\hat{y} = r \sin \phi$, we have the following result. The second pair of equations is derived by solving for $\hat{x}$ and $\hat{y}$ in the first pair.

Coordinate Rotation Formulas

If a rectangular xy -coordinate system is rotated through an angle θ to form an $\hat{x}\hat{y}$ -coordinate system, then a point $P(x, y)$ will have coordinates $P(\hat{x}, \hat{y})$ in the new system, where (x, y) and $(\hat{x}, \hat{y})$ are related by

$$x = \hat{x} \cos \theta - \hat{y} \sin \theta \quad \text{and} \quad y = \hat{x} \sin \theta + \hat{y} \cos \theta.$$

and

$$\hat{x} = x \cos \theta + y \sin \theta \quad \text{and} \quad \hat{y} = -x \sin \theta + y \cos \theta.$$

EXAMPLE 1 Show that the graph of the equation $xy = 1$ is a hyperbola by rotating the xy -axes through an angle of $\pi/4$.

SOLUTION: Denoting a point in the rotated system by $(\hat{x}, \hat{y})$, we have

$$x = \hat{x} \cos \frac{\pi}{4} - \hat{y} \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}(\hat{x} - \hat{y})$$

and

$$y = \hat{x} \sin \frac{\pi}{4} + \hat{y} \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}(\hat{x} + \hat{y}).$$

Substituting these expressions into the original equation $xy = 1$ produces the equation

$$\frac{\sqrt{2}}{2}(\hat{x} - \hat{y}) \frac{\sqrt{2}}{2}(\hat{x} + \hat{y}) = 1 \quad \text{or} \quad \hat{x}^2 - \hat{y}^2 = 2.$$

In the $\hat{x}\hat{y}$ -coordinate system, then, we have a standard position hyperbola whose asymptotes are $\hat{y} = \pm \hat{x}$. These are the same lines as the x - and y -axes, as seen in Figure 3. ◇

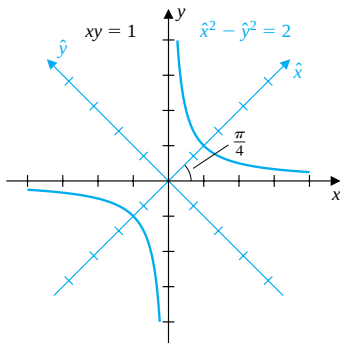


Figure 3

In Example 1 the appropriate angle of rotation was provided to eliminate the $\hat{x}\hat{y}$ -term from the equation. Such an angle θ can always be found so that when the coordinate axes are rotated through this angle, the equation in the new coordinate system will not involve the product $\hat{x}\hat{y}$.

To determine the angle, suppose we have the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad \text{where} \quad B \neq 0.$$

Introduce the variables

$$x = \hat{x} \cos \theta - \hat{y} \sin \theta \quad \text{and} \quad y = \hat{x} \sin \theta + \hat{y} \cos \theta,$$

and substitute for x and y in the original equation. This gives us the new equation in $\hat{x}$ and $\hat{y}$:

$$\begin{aligned} &A(\hat{x} \cos \theta - \hat{y} \sin \theta)^2 + B(\hat{x} \cos \theta - \hat{y} \sin \theta)(\hat{x} \sin \theta + \hat{y} \cos \theta) + C(\hat{x} \sin \theta + \hat{y} \cos \theta)^2 \\ &+ D(\hat{x} \cos \theta - \hat{y} \sin \theta) + E(\hat{x} \sin \theta + \hat{y} \cos \theta) + F = 0. \end{aligned}$$

Performing the multiplication and collecting the similar terms gives

$$\begin{aligned} &\hat{x}^2 (A(\cos \theta)^2 + B(\cos \theta \sin \theta) + C(\sin \theta)^2) \\ &+ \hat{x}\hat{y} [-2A \cos \theta \sin \theta + B((\cos \theta)^2 - (\sin \theta)^2) + 2C \sin \theta \cos \theta] \\ &+ \hat{y}^2 (A(\sin \theta)^2 - B(\sin \theta \cos \theta) + C(\cos \theta)^2) \\ &+ \hat{x}(D \cos \theta + E \sin \theta) + \hat{y}(-D \sin \theta + E \cos \theta) + F = 0. \end{aligned}$$

To eliminate the $\hat{x}\hat{y}$ -term from this equation, choose θ so that the coefficient of this term is zero, that is, so that

$$-2A \cos \theta \sin \theta + B ((\cos \theta)^2 - (\sin \theta)^2) + 2C \sin \theta \cos \theta = 0.$$

Simplifying this equation, we have

$$B ((\cos \theta)^2 - (\sin \theta)^2) = 2(A - C) \cos \theta \sin \theta$$

and

$$\frac{(\cos \theta)^2 - (\sin \theta)^2}{2 \cos \theta \sin \theta} = \frac{A - C}{B}.$$

Using the double angle formulas for sine and cosine gives

$$\frac{\cos 2\theta}{\sin 2\theta} = \frac{A - C}{B}, \quad \text{or} \quad \cot 2\theta = \frac{A - C}{B}.$$

Rotating Quadratic Equations

To eliminate the xy -term in the general quadratic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad \text{where } B \neq 0,$$

rotate the coordinate axes through an angle θ that satisfies

$$\cot 2\theta = \frac{A - C}{B}.$$

EXAMPLE 2 Sketch the graph of $4x^2 - 4xy + 7y^2 - 24 = 0$.

SOLUTION: To eliminate the xy -term we first rotate the coordinate axes through the angle θ where

$$\cot 2\theta = \frac{A - C}{B} = \frac{4 - 7}{-4} = \frac{3}{4}$$

From the triangle in Figure 4, we see that

$$\cos 2\theta = \frac{3}{5}.$$

Using the half angle formulas we have

$$\cos \theta = \sqrt{\frac{1 + \frac{3}{5}}{2}} = \frac{2\sqrt{5}}{5}, \quad \text{and} \quad \sin \theta = \sqrt{\frac{1 - \frac{3}{5}}{2}} = \frac{\sqrt{5}}{5}.$$

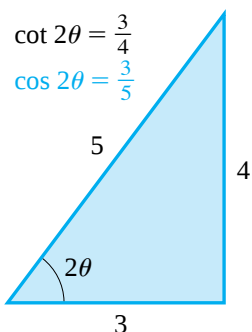


Figure 4

The rotation of axes formulas then give

$$x = \frac{2\sqrt{5}}{5}\hat{x} - \frac{\sqrt{5}}{5}\hat{y} \quad \text{and} \quad y = \frac{\sqrt{5}}{5}\hat{x} + \frac{2\sqrt{5}}{5}\hat{y}.$$

Substituting these into the original equation gives

$$4\left(\frac{2\sqrt{5}}{5}\hat{x} - \frac{\sqrt{5}}{5}\hat{y}\right)^2 - 4\left(\frac{2\sqrt{5}}{5}\hat{x} - \frac{\sqrt{5}}{5}\hat{y}\right)\left(\frac{\sqrt{5}}{5}\hat{x} + \frac{2\sqrt{5}}{5}\hat{y}\right) + 7\left(\frac{\sqrt{5}}{5}\hat{x} + \frac{2\sqrt{5}}{5}\hat{y}\right)^2 - 24 = 0.$$

After simplifying we have

$$\frac{\hat{x}^2}{5} + 8\hat{y}^2 = 24 \quad \text{or} \quad \frac{\hat{x}^2}{120} + \frac{\hat{y}^2}{3} = 1.$$

This is the graph of an ellipse in standard position with respect to the $\hat{x}\hat{y}$ coordinate system. The graph of the equation

$$4x^2 - 4xy + 7y^2 = 24$$

is consequently the graph of an ellipse that has been rotated through an angle $\theta = \arcsin \frac{\sqrt{5}}{5} \approx 26^\circ$, as shown in Figure 5. $\diamond$

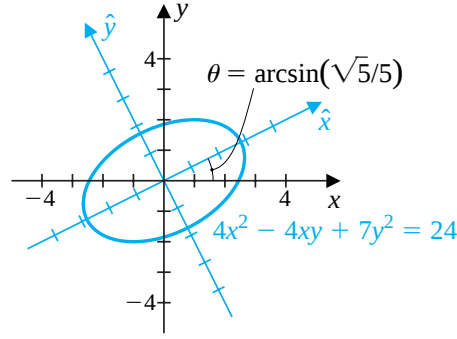


Figure 5

There is an easily applied formula that can be used to determine which conic will be produced once the rotation has been performed. Suppose that a rotation θ of the coordinate axes changes the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

into the equation

$$\hat{A}\hat{x}^2 + \hat{B}\hat{x}\hat{y} + \hat{C}\hat{y}^2 + \hat{D}\hat{x} + \hat{E}\hat{y} + \hat{F} = 0.$$

It is not hard to show, although it is algebraically tedious, that the coefficients of these two equations satisfy

$$B^2 - 4AC = \hat{B}^2 - 4\hat{A}\hat{C}.$$

When the particular angle chosen is such that $\hat{B} = 0$, we have

$$B^2 - 4AC = -4\hat{A}\hat{C}.$$

However, except for the degenerate cases we know that the new equation

$$\hat{A}\hat{x}^2 + \hat{C}\hat{y}^2 + \hat{D}\hat{x} + \hat{E}\hat{y} + \hat{F} = 0$$

will be:

- i) An ellipse if $\hat{A}\hat{C} > 0$;
- ii) A hyperbola if $\hat{A}\hat{C} < 0$;
- iii) A parabola if $\hat{A}\hat{C} = 0$.

Applying this result to the original equation gives the following.

Classification of Conic Curves

Except for degenerate cases, the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

will be:

- i) An ellipse if $B^2 - 4AC = -4\hat{A}\hat{C} < 0$;
- ii) A hyperbola if $B^2 - 4AC = -4\hat{A}\hat{C} > 0$;
- iii) A parabola if $B^2 - 4AC = -4\hat{A}\hat{C} = 0$.

The beauty of this result is that we do not need to know the angle that produces the elimination of the $\hat{x}\hat{y}$ -terms. We only need to know that such an angle exists, and we have already established this fact.

If we apply the result to the examples in this section we see that:

Example 1. The equation $xy = 1$ has

$$B^2 - 4AC = 1^2 - 4(0)(0) = 1 > 0$$

and is therefore a hyperbola.

Example 2. The equation $4x^2 - 4xy + 7y^2 - 24 = 0$ has

$$B^2 - 4AC = (-4)^2 - 4(4)(7) = -56 < 0$$

and is therefore an ellipse.

We strongly recommend that you apply this simple test when rotation is required to graph a general quadratic equation. It provides a final check on a result produced from a considerable amount of algebraic and trigonometric computation. While it will not ensure that you are correct, it can tell you when you are certainly wrong, and this is often more important.

Rotation of Axes Exercises

In Exercises 1–4, determine the $\hat{x}\hat{y}$ -coordinates of the given point if the coordinate axes are rotated through the given angle θ .

1. $(0, 1), \theta = 60^\circ$

2. $(1, 1), \theta = 45^\circ$

3. $(-3, 1), \theta = 30^\circ$

4. $(-2, -2), \theta = 90^\circ$

In Exercises 5–12, **(a)** determine whether the conic section is an ellipse, hyperbola, or parabola, and **(b)** perform a rotation, and if necessary a translation, and sketch the graph.

5. $x^2 - xy + y^2 = 2$

6. $x^2 + 4xy + y^2 = 3$

7. $17x^2 - 6xy + 9y^2 = 0$

8. $x^2 - \sqrt{3}xy = 1$

9. $4x^2 - 4xy + 7y^2 = 24$

10. $5x^2 - 3xy + y^2 = 5$

11. $6x^2 + 4\sqrt{3}xy + 2y^2 - 9x + 9\sqrt{3}y - 63 = 0$

12. $x^2 + 2\sqrt{3}xy + 3y^2 + (4 + 2\sqrt{3})x + (4\sqrt{3} - 2)y + 12 = 0$

13. Find an equation of the parabola with axis $y = x$ passing through the points $(1, 0)$, $(0, 1)$, and $(1, 1)$ **(a)** in $\hat{x}\hat{y}$ -coordinates, and **(b)** in xy -coordinates.