Math Review for high school graduates planning to enroll in college STEM courses

The goal of these notes is to help you manage the leap from high school math to college math. In high school, much of the curriculum is devoted to setting up and solving simple real-life problems, but formal algebra instruction is not heavily emphasized.

College is different. Virtually every topic in the STEM curriculum requires mastery of algebra and formulas, which are the key to representing and solving problems in Science, Technology, Engineering, and (of course) Mathematics.

Mathematics is a tool for solving real-life problems, but it is much more. In ancient Greece, it became apparent that the clarity of mind honed by the study of mathematics for its own sake is of fundamental value to society in other ways as well.

Indeed, in Plato's *Republic*, a blueprint for the ideal state, future leaders were required to study mathematics for ten years before going on to more practical affairs of ethics and government. In Book VII, Socrates explains:

In every man there is an eye of the soul which, when by other pursuits lost and dimmed, is by the study of mathematics, purified and re-illuminated.

This eye is far more precious than ten thousand bodily eyes, for by it alone can truth be seen.

These words, written more than two thousand years ago, remain true today. Here is a recent rephrasing by Isaac Asimov, from his foreword to *The History of Mathematics*, by Brooklyn College Professor Carl Boyer:

Mathematics is a unique aspect of human thought... Nothing pertaining to humanity becomes us so well as mathematics. There, and only there, do we touch the human mind at its peak.

You will need to work hard to get to the point where you can really appreciate the beauty and power of the math that you are learning.

To help you reach that goal, this text is written to have a conversation with you. An important innovation is that many procedures are explained, and graphs are drawn, according to your instructions, step by step, so that you can proceed at a comfortable pace.

Enjoy mathematics!

A guide to exploring this text

- Click on Table of contents in the above orange strip. Then click the blue navigation button Chapter 1: Algebra and Plane Geometry to see the material covered in that chapter.
- Click a to go to a section or subsection of the chapter.
- Depending on your browser, you can move forward from frame to frame by rolling the mouse wheel or by using Down/up or Arrow keyboard keys.
- Page and frame numbers are at the right of the bottom orange strip. A page consists of one or more frames that are displayed in sequence, in order to provide step-by-step understanding of complex text or graphics.
- At the bottom right of any page, there are faint Adobe controls. The most important of these is ⥀, which takes you to the subsection that you previously viewed.

After you explore a bit, return to this page and read the next column for details on how to study.

- Begin with Chapter 1, which consists of Sections 1.0 through 1.9. Start by clicking Section 1.0 to list its subsections. Move to the next page for a Preview of its Definitions, Theorems, and Procedures. Clicking any button takes you to the text's complete discussion, which appears as

  Definition/Theorem
  ....
  or
  Procedure
  ....

  To return to the Preview, click the Adobe control ⥀.
- Read Section 1.0 carefully. Quiz 1.0 that follows is a good homework assignment. It lists the Examples worked out in Section 1.0 . Work them out by yourself. If necessary, click the button preceding a Quiz Example to see its solution. Then click ⥀ to get back to the Quiz.
- The Review 1.0 section, following the Quiz, helps you prepare for exams. Each set of 4 questions in the Review includes a Quiz Example as well as 3 similar examples that you should solve to see if you really know that material. Solve each set before you check the answers provided.

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1.0.0 Numbers as stick lengths

Numbers are used for counting and measuring.

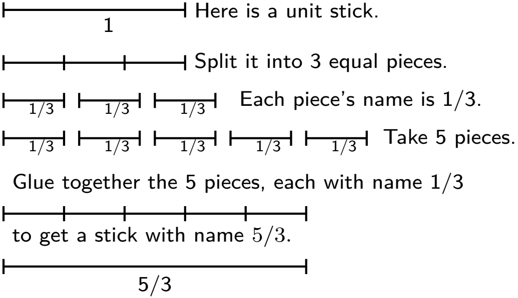
Counting numbers are $0, 1, 2, 3, \ldots$ They tell how many objects are in a collection: 12 people, 7 marbles, 8 sentences. Any such object is destroyed if it is split into pieces.

Counting numbers can also be used to measure things: A stick might be 7 feet long and weigh 17 pounds. But most quantities require in-between measurements.

Measuring numbers describe physical quantities such as distance, mass, and duration ($=$ length of time). These measurements are only occasionally whole numbers.

Take a stick you are friendly with, and call it your *unit stick*. Many other sticks can be constructed from this unit stick, as follows.

If you split the unit stick into 3 equal parts, you get 3 small sticks, each called 1/3 (sometimes written $\frac{1}{3}$) and read as "one third" of a unit stick. When you glue together 5 such small sticks, you get a stick with name $5/3$. Here is the picture.



Each picture is a line segment, whose length (expressed as a number of unit stick lengths) is its name listed above.

Principia numerica: All about numbers

The length of each stick is can be a fraction or a whole number: $1, 2, 3, , ,$ We say that $5/3$ is a *fraction* with *numerator* 5 and *denominator* 3.

All the fractions that can be written as a/b, where a and b are nonzero counting numbers $1, 2, 3, ...$ are called *positive rational numbers.*
The rational number $a/1$ is just the whole number a. Note that b can't be zero: it doesn't make sense to split a stick into zero parts.

In the above paragraph, the letters a and b are just placeholders for numbers. For example the statement that $a/1$ is a is an abbreviation for an infinite list of statements: $1/1 = 1$; $2/2 = 2$, $3/3 = 3$,... The equals sign $=$ is just a math way of writing the English word "is."

What about $a = 0$? Certainly you could think of 0 as the name of a stick consisting of a single point: gluing that stick together with another stick doesn't change the other stick's length. In symbols, for any stick a/b, we have $a/b + 0 = a/b$, a rule called the Identity Law for Addition.

If you glue together sticks with lengths A and B, you get a stick with length $A + B$, the answer obtained when you add A and B. That's why the last stick's length is
$1/3 + 1/3 + 1/3 + 1/3 + 1/3 = 5/3$.

These ideas about stick lengths were presented by Euclid more than two thousand years ago. Algebra came along much later. A crucial idea was to move from the stick picture to the idea of numbers on the number line in order to allow the introduction of negative numbers. Then came the laws of arithmetic. Much later: the laws of arithmetic with letters, which is the subject called algebra as we know it today.

1.0.1 Numbers as coordinates

In ***Geometry,*** we start with points.

Collections (fancy word: sets) of points form

- *lines* (and parts of lines, called *intervals*);
- *planes* (and parts of planes, called *regions*);
- *spaces* (and parts of spaces, called *solids*).

You may have seen that

- a single number x specifies a *point on a line*;
- two numbers (x, y) specify a *point in a plane;*
- three numbers (x, y, z) specify a *point in space.*

You may be less familiar with the idea that four numbers (x, y, z, t) specify a point in space-time, which is the universe we live in. The point (x, y, z, t) specifies an *event* that occurs at point (x, y, z) and at time t.

Not every length is a rational number. In other words, some (in fact, most) numbers are not rational: they are called *irrational numbers.* Two examples:

- If a square's side length is 1 meter, then the length of its diagonal is $D = \sqrt{2}$ meters. It's not hard to show $\sqrt{2}$ can't be written as a fraction, hence D is irrational. However, D satisfies a simple equation: $D^2 = 2$.
- If a circle's diameter is 1 meter, its circumference is π meters. Not only is π irrational, but π (unlike $\sqrt{2}$) doesn't satisfy any polynomial equation whatsoever.

1.0.2 The number line

Think about adding stick lengths. If you want $3+?$ to equal 5, clearly $? = 2$ works: gluing a length 2 stick to a length 3 stick produces a length 5 stick. However, if you want $5+\ ?$ to equal 3, the unknown ? can't be a stick length.

Algebra introduces a number -2, called the *negative* of 2. The basic algebra property of -2 is that $-2+2 = 2+-2 = 0$. It turns out that if you want $5+? = 3$, then $? = -2$ works, since $5+-2 = 3+2+-2 = 3+0 = 3$.

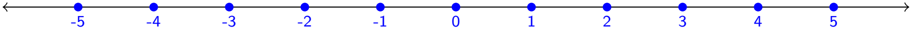
Real numbers consist of measuring numbers and their negatives. They are best pictured as an infinite straight line on which we place reference numbers: 0 and 1, with 1 to the right of 0.

Positive real numbers are the lengths of sticks placed on the number line with their left end at 0. Negative real numbers are the reflections of positive real numbers through the number 0.

Summary: Types of numbers

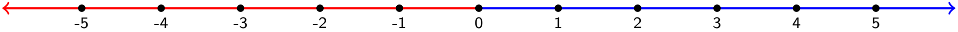
- Whole numbers: 1,2,3,.....
- Counting numbers: 0,1,2,3,....
- Natural numbers: 1,2,3,.....
- Integers: ....,-3,-2,-1,0,1,2,3....
- Measuring numbers: lengths of sticks (including 0)
- Real numbers: Measuring numbers and their negatives.

Here is the picture of the real number line:



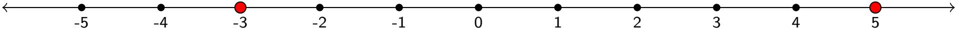
Basic facts about the number line

- Numbers to the left of zero are negative. Numbers to the right of zero are positive:



- The number -5 (read as "negative five" or "minus five") is five units to the left of zero. The number 5 (occasionally written $+5$ and read as 'positive five' or 'five') is five units to the right of zero.

- The *distance* between two numbers on the number line is the larger number (on the right) minus the smaller number (on the left).
  Below, the distance between -3 and 5 is $5-(-3)=5+3=8$.



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- The distance between points is a positive number.
  There is no such thing as negative distance.

- A number is called *non-negative* if it is zero or positive.

- The negative (or opposite) of a given number is its reflection through 0.

  A number and its negative are the same distance from 0, but are on opposite sides of 0.
  For example, the negative of 5 is -5, and the negative of -5 is 5.
  The numbers 5 and -5 are both distance 5 from 0.

Decimal numbers

We often use *decimal numbers,* which are simply rational numbers whose denominator is a power of 10. For example, $\frac{38476}{10^4} = \frac{38476}{10000} = 3.8476$. Numbers of this sort are called terminating decimals: they can be written with a finite number of digits to the right of the decimal point. Every terminating decimal number is a rational number.

Every real number is approximately equal to (written $\approx$) a terminating decimal number. For example, $\sqrt{2} \approx 1.41421356237$ while $\pi \approx 3.14159265359$.
However, lots of rational numbers are not terminating decimal numbers. For example, $1/3$ can be written only as a non-terminating decimal $.333333333....$ This sort of symbol is obviously suspicious: always beware of "...." To be precise about "...", we need the theory of infinite series, usually encountered in third semester calculus and beyond.

Example 1: Rewrite 37.123 as a rational number.
Answer: $37.123 = 37.123 \cdot \frac{1000}{1000} = \frac{37123}{1000}$.

Notation for arithmetic operations

Notation for arithmetic operations

The multiplication sign is $\cdot$, not x or *.
It is required between numbers, but is usually omitted between two letters or after a number followed by a letter.
Examples: $3 \cdot 4 = 12$; $3 \cdot x \cdot y = 3xy$; $3x \cdot 2 = 6x$
The division sign is either $/$ or $\div$.
A fraction $\frac{a}{b}$(same as a/b) equals a divided by b.
Example: $3 \div 5 = 3/5 = \frac{3}{5} = 0.6$

It's best to use vertical fraction notation if numerator or denominator includes an arithmetic operation, since $/$ or $\div$ require parentheses:
$\frac{3+4}{2-5} = (3+4)/(2-5) = (3+4) \div (2-5)$.

1.0.3 Factoring whole numbers

Factoring is used to rewrite a whole number as a product that is easier to understand.

Let a and b be whole numbers. Then b *is a factor of* a if

there is a whole number c with $a = b \cdot c$. This is true if and only if $a \div b$ is a whole number.

Since $12 = 1 \cdot 12 = 3 \cdot 4 = 2 \cdot 6$, factors of 12 are $1, 2, 3, 4, 6$, and 12. There are no other whole number factors of 12.

Two meanings of the word factor

- *Noun:* A *factor* of 12 is a number that appears in some factorization of 12.
- *Verb:* We say that 12 *factors* as $12 = 3 \cdot 4$, or as $12 = 2 \cdot 6$, or as $12 = 2 \cdot 3 \cdot 2$.

The statement $12 = 3 \cdot 4$ breaks down, or factors, 12 as a product of whole numbers. Each such breakdown is called a *factorization* of 12. Notice that 3 is a factor of 12 precisely because $12/3 = 4$ is a whole number.

Factorizations should use powers to abbreviate multiplication.
For example, $2^2 \cdot 3^3$ is a factorization of 108, while 10^6 is a factorization of $1,000,000$.

A *prime number*

is a whole number that has exactly two different factors: 1 and itself.
1 *is not prime, since it has only one factor.*

For example, 7 is prime because its only factors are 1 and 7.

1.0.4 Prime power factorization

The list of primes starts off as
$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \dots$

Let n and p be whole numbers.
n is divisible by p

means: n/p is a whole number.

Therefore p is a factor of n if and only if n is divisible by p.

Fact: n is prime if and only if

it does not have a factor p with $p^2 \leq n$.

To decide if n is prime,

divide n by all primes p with $p^2 \leq n$.
If n/p is a whole number for any such p, then n is not prime. Otherwise, n is prime.

Example 2: Is $n = 437$ prime?
Solution: You need to divide 437 by the primes $p = 2, 3, 5, 7, 11, 13, 17, 19$, since $19^2 = 361 < 437$ and the next prime squared, $23^2 = 529$, is greater than 432. Since $437/19 = 23$, a whole number, 437 is not prime.

Example 3: Is $n = 191$ prime?
Solution: Divide 191 by $2, 3, 5, 7, 11, 13$, since the next prime squared, $17^2 = 289$ is greater than 191. Since none of the quotients $191/2, 191/3, 191/5, 191/7, 191/11, 191/13$ is a whole number, 191 is prime.

A *number is factored completely*
(*or its factorization is complete*)

if it is written as a product of primes.

The *prime power factorization of a number*

factors the number as a product of powers of ***distinct*** prime numbers that increase from left to right.

Factoring whole numbers

There is only one prime power factorization of a number. This would not be true if we allowed 1 to be prime, since then $1 \cdot 2 \cdot 3$ and $1^2 \cdot 2 \cdot 3$ would both be prime power factorizations of 6.

Example 4: Find the (unique) prime power factorization of 240 .

Solution: $240 = 2 \cdot 120 = 2 \cdot 2 \cdot 60 = 2 \cdot 2 \cdot 2 \cdot 30 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 15 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 2^4 \cdot 3 \cdot 5$.

The basic idea was to list primes $2, 3, 5, \ldots$ in order and then pull out the maximum power of each prime that is a factor of 240.

- Factor out 2 four times to get $240 = 2^4 \cdot 15$.
- Factor out 3 one time to get $240 = 2^4 \cdot 3 \cdot 5$.
- We are done since 5 is prime.

This could have been done in other, less systematic, ways. For example:

$240 = 24 \cdot 10 = 6 \cdot 4 \cdot 2 \cdot 5 = 2 \cdot 3 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 2^4 \cdot 3 \cdot 5$.

In some cases, the best method is to first factor out a power of a number. For example,
$2,000,000 = 2 \cdot 10^6 = 2 \cdot (2 \cdot 5)^6 = 2 \cdot 2^6 \cdot 5^6 = 2^7 5^6$.

We now allow fractions of integers to have negative numerator or denominator.

A fraction is $\frac{a}{b}$ (also written a/b)

where nonzero denominator b and numerator a are integers.

The product of fractions $\frac{a}{b}$ and $\frac{c}{d}$ is

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} = (ac)/(bd).$$

1.0.5 Arithmetic with fractions

To reduce a fraction

- Factor the numerator and denominator completely.
- Cancel common factors.
- Reminder: if c is not 0 then $\frac{c}{c} = 1$.

The critical identity used here is $\frac{AC}{BC} = \frac{A}{B} \cdot \frac{C}{C} = \frac{A}{B} \cdot 1 = \frac{A}{B}$. We say that numerator AC and denominator BC have common factor C and write $\frac{A\cancel{C}}{B\cancel{C}} = \frac{A}{B}$.

Here is a shortcut for canceling powers.

$$\frac{7{\cdot}4^5}{3{\cdot}4^3} = \frac{7{\cdot}4{\cdot}4{\cdot}4{\cdot}4{\cdot}4}{3{\cdot}4{\cdot}4{\cdot}4} = \frac{7{\cdot}4{\cdot}4{\cdot}\cancel{4}{\cdot}\cancel{4}{\cdot}\cancel{4}}{3{\cdot}\cancel{4}{\cdot}\cancel{4}{\cdot}\cancel{4}} = \frac{7{\cdot}4{\cdot}4}{3} = \frac{7{\cdot}4^2}{3}$$

Explanation:

The numerator started out with 5 factors of 4. The denominator started out with 3 factors of 4. The 3 factors of 4 in the numerator cancel all 3 factors of 4 in the denominator. Therefore $5 - 3 = 2$ factors of 4 remain in the numerator.

Summary: There's nothing special about the numbers $4, 7, 3$. Let a, b, and c be whole numbers.

The fraction $\frac{ac^m}{bc^n}$ reduces to

- $\frac{ac^{m-n}}{b}$ if $m > n$;
- $\frac{a}{b}$ if $m = n$;
- $\frac{a}{bc^{n-m}}$ if $m < n$

Here are a few more examples:

- $\frac{2^3 y}{2^7 x} = \frac{y}{2^{7-3} x} = \frac{y}{2^4 x}$
- $\frac{2^7 3^8}{2^9 3^4} = \frac{3^{8-4}}{2^{9-7}} = \frac{3^4}{2^2}$
- $\frac{x^7 y^8}{x^9 y^4} = \frac{y^{8-4}}{x^{9-7} x} = \frac{y^4}{x^2}$

Basic fraction rules

Example 5: Reduce $\frac{120}{432}$ by first finding the prime power factorizations of the numerator and denominator.

Solution:

$\frac{120}{432} = \frac{8 \cdot 15}{4 \cdot 108} = \frac{2^3 \cdot 3 \cdot 5}{2^2 \cdot 2^2 \cdot 3^3} = \frac{2^3 \cdot 3^1 \cdot 5}{2^4 \cdot 3^3}$

$= \frac{5}{2^{4-3} \cdot 3^{3-1}} = \frac{5}{2^1 \cdot 3^2} = \boxed{\frac{5}{18}}$

Example 6: Reduce the fraction $\frac{120}{432}$ by using any factorizations you like.

Sample solution:

$\frac{120}{432} = \frac{\not{4} \cdot 30}{\not{4} \cdot 108} = \frac{\not{2} \cdot 15}{\not{2} \cdot 54} = \frac{\not{3} \cdot 5}{\not{3} \cdot 18} = \boxed{\frac{5}{18}}.$

Next we discuss fraction addition.

In precise but clumsy words: The result of adding fractions with the same (= common) denominator is a fraction whose numerator is the sum of the two fractions' numerators and whose denominator is the fractions' common denominator.

A bit less precise, but easier to remember:

To add fractions with the same nonzero denominator

add numerators, keep the common denominator:
$\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C}$

Before you add fractions with different denominators, build them up so that they have the same denominator:

To add fractions with different nonzero denominators

$$\frac{A}{C} + \frac{B}{D} = \frac{A}{C} \cdot \frac{D}{D} + \frac{B}{D} \cdot \frac{C}{C} = \frac{AD}{CD} + \frac{BC}{DC}$$
$$= \frac{AD}{CD} + \frac{BC}{CD} = \frac{AD+BC}{CD}$$

Example 7: $\frac{3}{5} + \frac{2}{3} = \frac{3 \cdot 3 + 5 \cdot 2}{3 \cdot 5} = \frac{19}{15}$.

This method is less useful than it looks, since the denominator BD is often too large to work with comfortably. A better approach: first rewrite the fractions so that they have the same compact denominator.

Adding Fractions by first finding their LCD

As stated above, the basic fraction addition rule is $\frac{A}{B} + \frac{C}{D} = \frac{AD+BC}{BD}$.

However, this formula doesn't work well for examples like $\frac{33}{1000} + \frac{5}{2000}$ (try it).

A better method is to rewrite the first fraction $\frac{33}{1000}$ as follows: multiply numerator and denominator by 2 to get $\frac{33}{1000} = \frac{33}{1000} \cdot \frac{2}{2} = \frac{33 \cdot 2}{1000 \cdot 2} = \frac{66}{2000}$.

Then the fractions can be added easily because their denominators are the same.

$$\frac{33}{1000} + \frac{5}{2000} = \frac{66}{2000} + \frac{5}{2000} = \frac{71}{2000}$$

The best way to add fractions is to rewrite them with a common denominator that is as small as possible. This will be the denominators' least common multiple (LCM).

The *least common multiple* (LCM) of two or more natural numbers

is the smallest natural number that is a multiple of each of them.

To find the LCM of natural numbers

list multiples of the smallest number until you get a number that is a multiple of each of the others.

Example 8: Find the LCM of 36 and 120.
Solution: List the multiples of 36 until you get a multiple of 120, as follows:
$36, 72, 108, 144, 180, 216, 252, 288, 324, 360$, which equals $3 \cdot 120$. Thus the LCD is 360.
This method wouldn't work well, however, if you want to find the LCM of 2 and 37. The following method works faster and generalizes to algebra expressions, as we shall see later on.

Adding Fractions by first finding their LCD

To find the least common multiple (LCM) of two or more natural numbers:

- List the prime power factorizations of those numbers.
- For each prime that appears, write down the *highest power* of that prime that appears anywhere in that list.
- The LCM is the product of those highest powers.

Example 9: Find the LCM of 36 and 120
Solution: $36 = 4 \cdot 9 = 2^2 \cdot 3^2$ and
$120 = 8 \cdot 15 = 2^3 \cdot 3^1 \cdot 5^1$.
The primes that appear are $2, 3,$ and 5.

The highest power of 2 is 2^3, which appears in the factorization of 120.
The highest power of 3 is 3^2, which appears in the factorization of 36.
The highest power of 5 is 5^1, which appears in the factorization of 120.
The product of those highest powers is $2^3 \cdot 3^2 \cdot 5^1$.
Answer: The LCM of 36 and 120 is
$2^3 \cdot 3^2 \cdot 5^1 = 8 \cdot 9 \cdot 5 = 360$.

The *LCD of fractions*

is the LCM of their denominators.

To add fractions by using their LCD:

- Build (rewrite) each fraction with that LCD as its denominator.
- Add the rewritten fractions by adding their numerators and keeping their common denominator.
- Reduce the resulting fraction.

Adding Fractions by first finding their LCD.

Example 10a: Find $\frac{5}{36} + \frac{7}{120}$ by listing multiples of 36.

- $36 = 4 \cdot 9 = 2^2 \cdot 3^2$ and $120 = 8 \cdot 15 = 2^3 \cdot 3^1 \cdot 5^1$. and so the LCD is $2^3 \cdot 3^2 \cdot 5 = 360$
- Since $360 = 36 \cdot 10$, we write $\frac{5}{36} = \frac{5 \cdot 10}{36 \cdot 10} = \frac{50}{360}$

  Since $360 = 120 \cdot 3$, we write $\frac{7}{120} = \frac{7 \cdot 3}{120 \cdot 3} = \frac{21}{360}$
- Add the rewritten fractions: $\frac{50}{360} + \frac{21}{360} = \boxed{\frac{71}{360}}$. This is reduced, since 71 is prime.

Example 10b: Find $\frac{2}{4725} + \frac{8}{2205}$ by building denominators up to the LCD.

- $4725 = 5 \cdot 945 = 5 \cdot\ 5 \cdot 189 = 5 \cdot\ 5 \cdot 3 \cdot 63 = 3^3 \cdot 5^2 \cdot 7$
- $2205 = 5 \cdot 441 = 5 \cdot 21 \cdot 21 = 5 \cdot 3 \cdot 7 \cdot 3 \cdot 7 = 3^2 \cdot 5^1 \cdot 7^2$
- Highest powers of $3, 5, 7$ are $3^3, 5^2, 7^2$ so the LCD is $3^3 \cdot 5^2 \cdot 7^2$
- Build up $\frac{2}{4725} = \frac{2}{3^3 \cdot 5^2 \cdot 7} \cdot \frac{7}{7} = \frac{14}{3^3 \cdot 5^2 \cdot 7^2}$ and $\frac{8}{2205} = \frac{8}{3^2 \cdot 5^1 \cdot 7^2} \frac{3 \cdot 5}{3 \cdot 5} = \frac{120}{3^3 \cdot 5^2 \cdot 7^2}$
- The sum $\frac{2}{4725} + \frac{8}{2205}$ is $\frac{14}{3^3 \cdot 5^2 \cdot 7^2} + \frac{120}{3^3 \cdot 5^2 \cdot 7^2} = \frac{134}{3^3 \cdot 5^2 \cdot 7^2} = \frac{67 \cdot 2}{3^3 \cdot 5^2 \cdot 7^2} = \boxed{\frac{134}{33075}}$

1.0.6 Intervals on the real number line

As we have discussed earlier, the real number line includes:

- natural numbers $1, 2, 3, \ldots$;
- integers $-3, -2, -1, 0, 1, 2, 3, \ldots$;
- decimals such as 3.14567;
- rational numbers (fractions of integers), such as $327/244$;
- irrational numbers such as as $\sqrt{2}$ or π .

The real number line is drawn as a horizontal line on which *the numbers increase from left to right.*

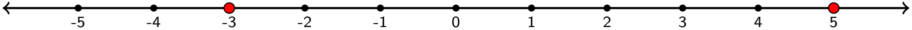
The number line mixes algebra and geometry: algebra is used to measure distance.

Definition: The distance between real numbers a and b on the number line

equals the right number minus the left number.

Intervals on the real number line

Comparing numbers. Suppose you want to compare the positions of the numbers -3 and 5 on the number line.



The following are six different ways of saying the same thing:

- -3 is less than 5;
- 5 is greater than -3;
- $-3 < 5$;
- $5 > -3$;
- -3 is to the left of 5;
- 5 is to the right of -3.

The symbols $\leq$ and $\geq$ are read 'less than or equal to' and 'greater than or equal to.'

- $x \leq 3$ means: $x = 3$ or $x < 3$.
- $x \geq 3$ means: $x = 3$ or $x > 3$.

- $x \leq 3$ means: $x = 3$ or $x < 3$.
- $x \geq 3$ means: $x = 3$ or $x > 3$.

Intervals on the number line

An *interval* on the real number line

is a connected piece of the number line.

The word "connected", in this setting, means: if two points are in the piece, so are all points between those two points.

Example 11: Describe in three different ways the interval consisting of all real numbers to the right of -3 and also to the left of or equal to 5.

Solution:

- inequality notation: $-3 < x \leq 5$;
- interval notation: $(-3, 5]$;
- In the number line below:

  -3 is drawn as a hollow dot, to show that it is not included in (is missing from) the interval.

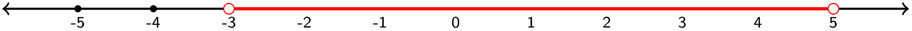
  5 is drawn as a solid dot, to show that it is included in the interval.

There are four types of finite intervals, depending on which endpoint(s) are excluded:

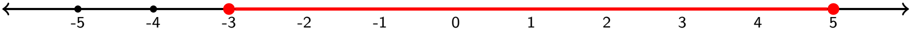
Example 12: Write the four intervals with left endpoint -3 and right endpoint 5 by using interval notation, inequality notation, and a graph.

Solution:

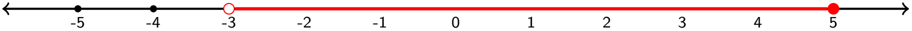
- $(-3, 5)$ is an *open interval*, written in inequality form as $-3 < x < 5$.



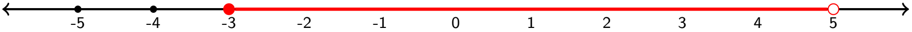
- $[-3, 5]$ is a *closed interval*, written in inequality form as $-3 \leq x \leq 5$.



- $(-3, 5]$ is a *half open interval*, written in inequality form as $-3 < x \leq 5$.



- $[-3, 5)$ is a *half closed interval*, written in inequality form as $-3 \leq x < 5$.



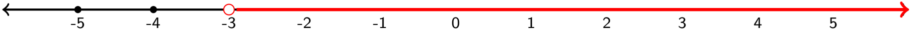
Infinite intervals on the real line

The symbol ∞ or $+\infty$, for (positive) infinity, is used when an interval contains all numbers to the right of a certain point. The entire real number line can be written as $(-\infty, \infty)$.
Note: ∞ and $-\infty$ are not real numbers.

Example 13: Use three kinds of notation to describe all numbers to the right of -3.

Solution:

- inequality notation $-3 < x$;
- interval notation: $(-3, \infty)$;
- a number line graph:

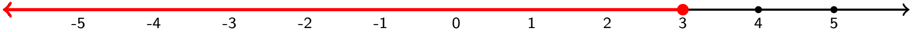


The symbol $-\infty$, for negative infinity, is used when an interval contains all numbers to the left of a certain point.

Example 14: Use three kinds of notation to describe all number to the left of and including 3.

Solution:

- inequality notation: $x \leq 3$;
- interval notation: $(-\infty, 3]$;
- a number line graph:



1.0.7 Absolute value and distance

Absolute value and distance

- *The absolute value of a real number a, written $|a|$,* is the positive number defined by $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a \leq 0$. This is often written as
$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \geq 0 \end{cases} .$$
- For all a: $|-a| = |a|$.
- *The length of the line segment joining points a and b* is the distance between them.
- *The distance between points a and b* is the absolute value of their difference, in either order: $|a-b| = |b-a|$.

Here is a proof that $|a-b| = |b-a|$:
Let $x = b - a$. Use a bit of algebra still to come:
$-x = -(b-a) = -b + a = a - b$ and so
$-x = a - b$.

Since $|x| = |-x|$, $|a-b| = |b-a|$.

For example, $|8-3|$ and $|3-8| = |-5|$ are both equal to 5.

To show that $|a-b|$ is the distance from a to b on the number line: recall that the distance between a and b is the right number minus the left number. There are 3 cases:

- If $a > b$, then a is to the right of b, so the distance from a to b is $a - b$, which equals $|a-b|$ since $a - b > 0$.
- If $a < b$, then b is to the right of a, so the distance from a to b is $b - a$, which equals $|b-a|$ since $b - a > 0$.
- If $a = b$, then $|a-b| = |a-a| = 0$ is the distance from a to b.

Absolute value and distance

Example 15: Use three methods to find the distance between -3 and -7.
Solution:

- Since -3 is to the right of -7, the distance is $-3-(-7)=-3+7=4$.
- The distance is the absolute value of the first number minus the second number: $|-3-(-7)|=|-3+7|=|4|=4$.
- The distance is the absolute value of the second number minus the first number: $|-7-(-3)|=|-7+3|=|-4|=4$.

Absolute value inequalities will be discussed later in this chapter. For the time begin, be aware that

The statement $|a+b|=|a|+|b|$ is FALSE.

This statement is correct if a and b have the same sign but is incorrect if a and b have opposite signs. So it's wrong just half the time! But the laws of algebra must be correct all the time! *Please:*

DON'T INVENT FORMULAS!

Exercises for Section 1.0: Arithmetic

Click on ▸ Wolfram Calculator to find an answer checker.

Click on ▸ Wolfram Algebra Examples to see how to check various types of algebra problems.

1. Find the LCM of a) 300 and 452 b) $.420$ and 300 and 452

2. Find the prime power factorizations of a) 3000, b) 4000, c) 1250, and d) 768 .

3. Find each of the following sums by finding and using the fractions' LCD.
a) $\frac{1}{3000}+\frac{7}{5000}$ b) $\frac{3}{40}-\frac{7}{50}+\frac{11}{60}$ c) $\frac{7}{200}+\frac{7}{300}$ d) $\frac{7}{12}+\frac{5}{16}+\frac{1}{15}$

4. Rewrite each of the following intervals using inequality notation
a) $[-5,17)$ b) $(-9,-4)$ c) $(-\infty,8]$ d) $(80,\infty)$

5. Rewrite each of the following inequalities using interval notation
a) $-5\leq x<17$ b) $x\geq 80$ c) $x<8$ d) $-9<x\leq 42$

Section 1.0 Quiz

Ex. 1.0.1: Rewrite 37.123 as a rational number.

Ex. 1.0.2: Is $n = 437$ prime?

Ex. 1.0.3: Is $n = 191$ prime?

Ex. 1.0.4: Find the prime power factorization of 240.

Ex. 1.0.5: Reduce the fraction $\frac{120}{432}$ by first finding the prime power factorizations of the numerator and denominator.

Ex. 1.0.6: Reduce the fraction $\frac{120}{432}$ by using any factorizations you like.

Ex. 1.0.7: Find $\frac{3}{5} + \frac{2}{3}$.

Ex. 1.0.8: Find the LCM of 36 and 120 by using prime power factorizations.

Ex. 1.0.9: Find the LCM of 36 and 120 by listing multiples.

Ex. 1.0.10: Find a) $\frac{5}{36} + \frac{7}{120}$ and b) $\frac{2}{4725} + \frac{8}{2205}$.

Ex. 1.0.11: Describe in three different ways the interval consisting of all real numbers to the right of -3 and also to the left of (or equal to) 5.

Ex. 1.0.12: There are four intervals with left endpoint -3 and right endpoint 5. Write the four intervals using interval notation, inequality notation, and a graph.

Ex. 1.0.13: Use 3 kinds of notation to describe all numbers to the right of -3.

Ex. 1.0.14: Use 3 kinds of notation to describe all number to the left of and including -3.

Ex. 1.0.15: Use three methods to find the distance between -3 and -7.

Section 1.0 Review: Arithmetic

Ex. 1.0.1: Rewrite as a reduced fraction of integers:

- $37.123 =$
- $.00004 =$
- $-37.1 =$
- $12 =$

Ex. 1.0.1: Rewrite as a reduced fraction of integers:

- $37.123 = = \frac{37123}{1000}$
- $.00004 = \frac{4}{100000}$
- $-37.1 = -\frac{371}{10}$
- $12 = \frac{12}{1}$

Section 1.0 Review: Arithmetic

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Ex. 1.0.2: Is each positive integer prime?

- 191?
- 221?
- 223?
- 1?

Section 1.0 Review: Arithmetic

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Ex. 1.0.3: Is each positive integer prime?

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- 239?
- 399?
- 401?

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Ex. 1.0.4: Find each number's prime power factorization.

- $240 =$
- $1000 =$
- $4000000 =$
- $1 =$

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- $240 = 2^4 \cdot 3 \cdot 5$
- $1000 = 2^3 \cdot 5^3$
- $4000000 = 2^8 \cdot 10^6$
- $1 =$ Does not exist.

Section 1.0 Review: Arithmetic

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Ex. 1.0.5: Reduce each fraction by first finding prime power factorizations of its numerator and denominator.

- $\frac{120}{142} =$
- $\frac{625}{75} =$
- $\frac{147}{194} =$
- $\frac{324}{54} =$

Section 1.0 Review: Arithmetic

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- $\frac{625}{75} = \frac{5^4}{3 \cdot 5 * 2} = \frac{25}{3}$
- $\frac{147}{194} = \frac{3 \cdot 7^2}{2 \cdot 97} = \frac{147}{194}$
- $\frac{324}{54} = \frac{2^2 \cdot 3^4}{2 \cdot 3^3} = \frac{6}{1} = 6$

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Ex. 1.0.6: Reduce each fraction by using a step by step factorization of its numerator and denominator.

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- $\frac{120}{142} = \frac{2 \cdot 60}{71} = \frac{60}{71}$
- $\frac{625}{75} = \frac{25 \cdot 25}{3 \cdot 25} = \frac{25}{3}$
- $\frac{147}{194} = \frac{3 \cdot 7^2}{2 \cdot 97} = \frac{147}{194}$
- $\frac{324}{54} = \frac{18 \cdot 18}{3 \cdot 18} = \frac{6}{1} = 6$

Ex. 1.0.7: Write each sum as a reduced fraction

- $\frac{3}{5} + \frac{2}{3} =$
- $\frac{1}{5} + \frac{2}{7} =$
- $\frac{3}{11} + \frac{5}{17} =$
- $\frac{3}{8} + \frac{4}{5} =$

Section 1.0 Review: Arithmetic

Ex. 1.0.1: Rewrite as a reduced fraction of integers:

- $37.123 = = \frac{37123}{1000}$
- $.00004 = \frac{4}{100000}$
- $-37.1 = -\frac{371}{10}$
- $12 = \frac{12}{1}$

Ex. 1.0.2: Is each positive integer prime?

- 191? Yes
- 221? No
- 223? Yes
- 1? No

Ex. 1.0.3: Is each positive integer prime?

- 437? No
- 239? Yes
- 399? Yes
- 401? No

Ex. 1.0.4: Find each number's prime power factorization.

- $240 = \ 2^4 \cdot 3 \cdot 5$
- $1000 = \ 2^3 \cdot 5^3$
- $4000000 = \ 2^8 \cdot 10^6$
- $1 =$ Does not exist.

Ex. 1.0.5: Reduce each fraction by first finding prime power factorizations of its numerator and denominator.

- $\frac{120}{142} = \frac{2^3 \cdot 3 \cdot 5}{2 \cdot 71} = \frac{60}{71}$
- $\frac{625}{75} = \frac{5^4}{3 \cdot 5 * 2} = \frac{25}{3}$
- $\frac{147}{194} = \frac{3 \cdot 7^2}{2 \cdot 97} = \frac{147}{194}$
- $\frac{324}{54} = \frac{2^2 \cdot 3^4}{2 \cdot 3^3} = \frac{6}{1} = 6$

Ex. 1.0.6: Reduce each fraction by using a step by step factorization of its numerator and denominator.

- $\frac{120}{142} = \frac{2 \cdot 60}{71} = \frac{60}{71}$
- $\frac{625}{75} = \frac{25 \cdot 25}{3 \cdot 25} = \frac{25}{3}$
- $\frac{147}{194} = \frac{3 \cdot 7^2}{2 \cdot 97} = \frac{147}{194}$
- $\frac{324}{54} = \frac{18 \cdot 18}{3 \cdot 18} = \frac{6}{1} = 6$

Ex. 1.0.7: Write each sum as a reduced fraction

- $\frac{3}{5} + \frac{2}{3} = \frac{19}{15}$
- $\frac{1}{5} + \frac{2}{7} = \frac{17}{35}$
- $\frac{3}{11} + \frac{5}{17} = \frac{106}{187}$
- $\frac{3}{8} + \frac{4}{5} = \frac{47}{40}$

Ex. 1.0.8: Find the LCM by using prime power factorizations:

- LCM$(36, 120) =$
- LCM$(30, 85) =$
- LCM$(48, 81) =$
- LCM$(405, 375) =$

Ex. 1.0.8: Find the LCM by using prime power factorizations:

- $\text{LCM}(36, 120) = \text{LCM}(2^2 3^2, 2^3 3 \cdot 5) = 2^3 3^3 5 = 360$
- $\text{LCM}(30, 85) = \text{LCM}(2 \cdot 3 \cdot 5, 5 \cdot 17) = 2 \cdot 3 \cdot 5 \cdot 17 = 510$
- $\text{LCM}(48, 81) = \text{LCM}(2^4 \cdot 3, 3^4) = 2^4 3^4 = 1296$
- $\text{LCM}(405, 375) = \text{LCM}(3^4 \cdot 5, 3 \cdot 5^3) = 3^4 5^3 = 10125$

Ex. 1.0.8: Find the LCM by using prime power factorizations:

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- $\text{LCM}(405, 375) = \text{LCM}(3^4 \cdot 5, 3 \cdot 5^3) = 3^4 5^3 = 10125$

Ex. 1.0.9: Find the LCM by using any convenient factorization

- $\text{LCM}(36, 120)$
- $\text{LCM}(30, 85)$
- $\text{LCM}(48, 81)$
- $\text{LCM}(405, 375)$

Ex. 1.0.8: Find the LCM by using prime power factorizations:

- LCM$(36, 120)$ = LCM$(2^2 3^2, 2^3 3 \cdot 5) = 2^3 3^3 5 = 360$
- LCM$(30, 85)$ = LCM$(2 \cdot 3 \cdot 5, 5 \cdot 17) = 2 \cdot 3 \cdot 5 \cdot 17 = 510$
- LCM$(48, 81)$ = LCM$(2^4 \cdot 3, 3^4) = 2^4 3^4 = 1296$
- LCM$(405, 375)$ = LCM$(3^4 \cdot 5, 3 \cdot 5^3) = 3^4 5^3 = 10125$

Ex. 1.0.9: Find the LCM by using any convenient factorization

- LCM$(36, 120)$ $36 = 4 \cdot 9 = 2^2 3^2$; $120 = 12 \cdot 10 = 2 \cdot 2 \cdot 3 \cdot 2 \cdot 5 = 2^3 \cdot 3 \cdot 5$; LCM $= 2^3 3^3 \cdot 5 = 360$
- LCM$(30, 85)$ $30 = 2 \cdot 3 \cdot 5; 85 = 5 \cdot 17;$LCM $= 2 \cdot 3 \cdot 5 \cdot 17 = 520$
- LCM$(48, 81)$ $48 = 8 \cdot 6 = 2^3 \cdot 2 \cdot 3 = 2^4 3^2$; $81 = 9 \cdot 9 = 3^2 3^2 = 3^4$; LCM $= 2^4 \cdot 3^4 = 1296$
- LCM$(405, 375)$ $405 = 5 \cdot 81 = 3^4 \cdot 5; 375 = 3 \cdot 125 = 3 \cdot 5^3$; LCM $= 3^4 5^3 = 10125$

Section 1.1: Numbers and expressions

- 1.1.1: Real numbers
- 1.1.2: Numerical expressions
- 1.1.3: Algebra, formulas, expressions
- 1.1.4: Polynomials
- 1.1.5: The Distributive Law
- 1.1.6: Simplifying polynomial expressions
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- Procedure 1.1.23 To rewrite a sum of products as a polynomial
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- Procedure 1.1.29 To factor a sum of terms that all have powers of the same letter
- Procedure 1.1.30 To factor $x^2 + bx + c$ if D is a perfect square
- Procedure 1.1.31 To factor $-x^2 + bx + c$
- Procedure 1.1.32 To factor $P = ax^2 + bx + c$ if $D = b^2 - 4ac$ is a perfect square
- Procedure 1.1.33 To rewrite a polynomial expression as a polynomial
- Procedure 1.1.34 To ensure that two degree n polynomials P and Q in x are equal
- Procedure 1.1.35 To solve an equation, find all solutions.
- Procedure 1.1.36 How NOT to solve the degree 2 equation $x^2 = x$
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- Procedure 1.1.38 Let P and Q be polynomial expressions. To solve equation $P = Q$ for x
- Procedure 1.1.39 To solve a degree 1 equation in x
- Procedure 1.1.40 To solve each of the following degree one equations in x

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- Definition 1.1.1 The unit on the real number line consists of all points between and including 0 and 1.
- Definition 1.1.2 The absolute value $|x|$ of a real number x is its distance from 0.
- Definition 1.1.3 Comparing real numbers a and b on the number line
- Definition 1.1.4 Arithmetic operations
- Definition 1.1.5 The reciprocal of any nonzero number K is $\frac{1}{K}$.
- Definition 1.1.6 The product and quotient of two nonzero numbers are
- Definition 1.1.7 Parentheses say "Do me first."
- Definition 1.1.8 E(MD)(AS) describes the precedence of operations.
- Definition 1.1.9 The reason for setting the order of operations to be EMA:
- Definition 1.1.10: The multiply sign $\cdot$ between letters and/or parentheses is usually omitted.
- Definition 1.1.11 Laws of Algebra : for any expressions A, B, C
- Definition 1.1.12 Expressions and formulas
- Definition 1.1.13 Powers, bases and exponents for numbers
- Definition 1.1.14 Power rule examples
- Definition 1.1.15 Power rules for real numbers
- Definition 1.1.16 A factorization of K with factors $A, B, C, \ldots$
- Definition 1.1.17 An integer factorization of K with factors $A, B, C, \ldots$

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- Definition 1.1.18 Powers of expressions
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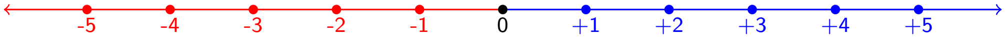
- Definition 1.1.24 The Reverse Distributive Law: $AB + AC = A(B + C)$
- Definition 1.1.25 Sign switch identity: $B - A = -(A - B)$
- Definition 1.1.26 Quadratic polynomials
- Definition 1.1.27 A number is a perfect square if it is the square of some integer.
- Definition 1.1.28 The Greatest common factor (GCF) of integers is
- Definition 1.1.29 Quadratic polynomial $ax^2 + bx + c$ factors as a product $(Ax + B)(Cx + D)$ of two degree one polynomials if
- Definition 1.1.30 A polynomial expression is
- Definition 1.1.31 An equation is a statement $P = Q$ where P and Q are any expressions.
- Definition 1.1.32 The degree of equation $P = Q$ in x is the highest power of x in the polynomial obtained by simplifying $P - (Q)$.

1.1.1 Arithmetic with real numbers



The line above is the real number line.
Each point on the line is a real number.
The integers are shown as dots.
The name of each integer is the signed number below it.

Positive numbers, to the right of 0, have a positive sign $+$.
Negative numbers, to the left of 0, have a negative sign $-$.

We usually don't write or say the sign of a positive number. Write $5+2=10$, not $+5++2=+10$, and say "five plus two equals ten."

Almost all real numbers are the points on the line segments between the dots. Two examples: 2.753 is between 2 and 3, while $-\sqrt{10}$ is between -4 and -3.

The real numbers are a subset (part) of the complex numbers, each of the form $a+bi$, where a and b are real and the imaginary number i has the weird property that $i^2=-1$. More about this later!

Definition of *units*: On the real number line

- The part from 0 to 1 is called a unit.
- -3 (negative 3) is 3 units to the left of 0.
- $+3$ (positive 3) is 3 units to the right of 0.

The *absolute value* $|x|$ of a real number x is its distance from 0.

- 3 is 3 units to the right of 0, so $|3|=3$.
- -3 is 3 units to the left of 0, so $|-3|=3$.
- The answers are the same since distance is always positive.

Comparing real numbers a and b on the number line

- $a=b$ means: a and b are the same number.
- $a\neq b$ means: a and b are different numbers.
- $a<b$ means: a is to the left of b.
- $b>a$ means: b is to the right of a.
- a is *positive* means: $a>0$.
- a is *negative* means: $a<0$.
- $a\leq b$ means: $a<b$ or $a=b$
- $b\geq a$ means: $b>a$ or $b=a$

Arithmetic operations with signed numbers

Definitions for *arithmetic operations*: When you

- *add* numbers, the answer is their *sum*.
- *subtract* numbers, the answer is their *difference*.
- *multiply* numbers, the answer is their *product*.
- *divide* numbers, the answer is their *quotient*.

Adding or subtracting 0 doesn't change a number.

- $0+9=9+0=9-0=9$
- $-9+0=-9-0=0+-9=-9$

The statement $4+5=9$ says: the result of adding 4 and 5 is the number 9.

Read $7+3$ as "7 plus 3" or as "the sum of 7 and 3."

Read $7-3$ as "7 subtract 3" .
However, read the number -12 as "negative 12".
Therefore Read "$-12-4$" as "negative 12 subtract
We will often use the word "minus" instead of
"negative, but never to mean "subtract."

To add two signed numbers with opposite signs

- Find their absolute values.
- Write down the larger absolute value minus the smaller absolute value.
- In front of that difference, write the sign of the number with the larger absolute value.

In all the following sums, the larger absolute value is 9, the smaller is 5. Their difference is $9-5=4$.

- $9+-5=+(9-5)=4$
- $-9+5=-(9-5)=-4$
- $-5+9=+(9-5)=4$
- $5+-9=-(9-5)=-4$

To add two or more signed numbers with the same sign

- Write down the sum of their absolute values.
- In front of their sum, write down that sign.
  - $-5+-9=-(5+9)=-14$
  - $+5++9=+(5+9)=+14$

To use the number line to find $-5 + -9$

- Start at the first number -5.
- Since the second number -9 is *negative*, move 9 units *left* from -5 to arrive at $-5 + -9 = -(5 + 9) = -14$.

To add any two numbers

Start at the first number. Move left if the second number is negative; move right if the second number is positive. The distance you move is the absolute value of the second number.

- $9 + 5 = 14$, the number 5 units right of 9
- $9 + -5 = 4$, the number 5 units left of 9
- $-9 + 5 = -4$, the number 5 units right of -9
- $-9 + -5 = -14$, the number 5 units left of -9

To subtract signed numbers

Add the second number's opposite to the first number. The opposite of 5 is -5 and the opposite of -5 is 5.

- $9 - 5 = 9 + -5 = 4$
- $9 - -5 = 9 + 5 = 14$
- $-9 - -5 = -9 + 5 = -4$
- $-9 - 5 = -9 + -5 = -14$

Warning: You may have heard "two minuses make a plus," or (less likely) "Two negatives make a positive." That's true when you multiply, false when you add!

- Minus times minus = plus: $-9 \cdot -5 = +45$
- Minus + minus = minus: $-9 + -5 = -14$

Two negatives don't have to make a positive!

- Negative plus negative is negative;
- Negative times negative is positive.

To *multiply* two numbers

- The product is 0 if either number is 0. Otherwise:
- multiply their absolute values.
- If the numbers have opposite signs, place a $-$ sign in front of the product.
- The product is zero if either number is zero.

- $0 \cdot 0 = 0 \cdot 4 = 0 \cdot -4 = 0$
- $7 \cdot 4 = |7| \cdot |4| = 7 \cdot 4 = 28$
- $-7 \cdot -4 = |-7| \cdot |-4| = 7 \cdot 4 = 28$
- $-7 \cdot 4 = -|-7| \cdot |4| = -7 \cdot 4 = -28$
- $7 \cdot -4 = -|7| \cdot |-4| = -7 \cdot 4 = -28$

Don't use mixed numbers in algebra/calculus courses.

- For example, leave $\frac{7}{3}$ as a final answer. Do not change it to the "mixed number" $2\frac{1}{3}$
- Reason: The mixed number $2\frac{1}{3}$ means $2+\frac{1}{3}$.

  But in algebra, xy means x times y, so you might think $2\frac{1}{3}$ means 2 times $\frac{1}{3}$.

Write negative fractions with the − sign exactly to the left of the fraction line.

- 10 divided by $-3 = \frac{10}{-3}$.
- It's better to write $\frac{-10}{3}$.
- Best of all, write $-\frac{10}{3}$.

The *reciprocal* of any nonzero number K is $\frac{1}{K}$.

- $\frac{1}{0}$ is *undefined*. 0 does not have a reciprocal.
- $\frac{0}{0}$ is *indeterminate*.
- Any fraction with denominator 0 is the result of an algebra error. Go back and check your work!

To *divide* nonzero numbers:

- Divide their absolute values.
- In front of that quotient, put a − sign if the numbers have opposite signs.
  - $\frac{10}{2} = 5$.
  - $\frac{-10}{-2} = \frac{|-10|}{|-2|} = \frac{10}{2} = 5$
  - $\frac{-10}{2} = -\frac{|-10|}{|-2|} = -\frac{10}{2} = -5$
  - $\frac{10}{-2} = -\frac{|-10|}{|-2|} = -\frac{10}{2} = -5$

To divide by a number, multiply by its reciprocal.

- 10 divided by $2 = 10 \div 2 = 10 \cdot \frac{1}{2} = \frac{10}{2} = 5$
- 10 divided by $3 = 10 \div 3 = 10 \cdot \frac{1}{3} = \frac{10}{3}$
- If $K \neq 0$, then $0 \div K = \frac{0}{K} = 0$

The product and quotient of two nonzero numbers are

- positive if the numbers have the same sign;
- negative if the numbers have opposite signs.

Arithmetic examples with signed numbers

• $-9+5=-(9-5)=-4$ • $5+-9=-(9-5)=-4$ • $9+-5=9-5=4$ • $-5+9=4$	Neg+ Pos Pos + Neg Pos + Neg Pos + Neg	To add numbers with opposite signs, put the sign of the number with larger absolute value in front of(the larger absolute value minus the smaller absolute value).
• $-5+-9=-(5+9)=-14$	Neg + Neg = Neg	Add two negatives: sum their absolute values
• $5+9=14$ • $9+0=0+9=9$ • $0+-9=-9+0=-9$	Pos + Pos = Pos Add 0 Add 0	Ordinary addition Zero + any number = the number.
• $0\cdot 9=9\cdot 0=0=0\cdot 0$	Multiply by 0	Zero times any number = zero.
• $5\cdot 3=15$ • $-5\cdot -3=15$ • $-5\cdot 3=-15$ • $5\cdot -3=-15$	Pos · Pos = Pos Neg · Neg = Pos Neg · Pos = Neg Pos · Neg = Neg	Product of numbers with same sign is the product of their absolute values. Product of numbers with opposite sign is minus the product of their absolute values.
• $10\div 2=\frac{10}{2}=5$ • $-10\div -2=\frac{-10}{-2}=5$ • $10\div -2=\frac{10}{-2}=-5$ • $10\div 3=\frac{10}{3}=\frac{10}{3}$ • $10\div -3=\frac{10}{-3}=-\frac{10}{3}$ • $0\div 10=\frac{0}{10}=0$	Pos ÷ Pos = Pos Neg ÷ Neg = Pos Pos ÷ Neg = Neg Pos ÷ Pos = Pos Pos ÷ Neg = Neg zero ÷ nonzero = zero	Fraction = Numerator ÷ Denominator. Leave $\frac{10}{3}$ as your final answer DO NOT convert $\frac{10}{3}$ to $3\frac{1}{3}$.
• $10\div 0=\frac{10}{0}=$ undefined • $0\div 0=\frac{0}{0}=$ indeterminate	illegal operation illegal operation	

1.1.2: Numerical expressions: numbers and operations

Arithmetic uses *operations* to combine numbers.

- Addition: $5 + 3 = 8$ is the *sum* of 5 and 3.
- Subtraction: $5 - 3 = 2$ is the *difference* of 5 and 3.
- Multiplication: $5 \cdot 3 = 15$ is the *product* of 5 and 3.
- Division: $15/5 = \frac{15}{5} = 3$ is the *quotient* of 15 by 5.
- Exponentiation: $5^3 = 5 \cdot 5 \cdot 5 = 125$ is the 3^{rd} *power* of 5. The power's *base* is 5; its *exponent* is 3.

Parentheses say "Do me first."

$(3 + 4) \cdot 5 = 7 \cdot 5 = 35$ but $3 + (4 \cdot 5) = 3 + 20 = 23$

What if parentheses are missing, as in $3 + 4 \cdot 5$?

E(MD)(AS) describes the *precedence of operations*:

- E comes first. Evaluate E operations from right to left.
- M and D are tied for second place. Evaluate M and/or D operations from left to right.
- Last come A and S, tied for third place. Evaluate A and/or S operations from left to right.

In each of the following, do the red operation first.

- $3 + 4 \cdot 5 = 3 + 20 = 23$ since M precedes A.
- $3 + 20/5 = 3 + 4 = 7$ since D precedes A.
- $3 + 6^2 = 3 + 36 = 39$ since E precedes A.
- $3 \cdot 6^2 = 3 \cdot 36 = 108$ since E precedes M.
- $36/6^2 = 36/36 = 1$ since E precedes D.

If the operations are all A or S, work from left to right.

- $6 - 3 + 2 = 3 + 2 = 5$
- $6 - 3 - 2 = 3 - 2 = 1$
- $16 - 5 - 5 + 4 = 11 - 5 + 4 = 6 + 4 = 10$

If the operations are all M or D, work from left to right.

- $12/3/2 = 4/2 = 2$
- $6/3 \cdot 2 = 2 \cdot 2 = 4$
- $6/3/2 = 2/2 = 1$
- $4 \cdot 5/2/2 \cdot 6 = 20/2/2 \cdot 6 = 10/2 \cdot 6 = 5 \cdot 6 = 30$

However, E operations are evaluated from right to left:

- $2^{2^3} = 2^8 = 256$

The *reason for setting the order of operations to be EMA*

is to ensure that the standard expansion of multidigit decimal numbers can be written without parentheses.

For example, you know that
$378 = 300 + 70 + 8 = 3 \cdot 100 + 7 \cdot 10 + 8 = 3 \cdot 10^2 + 7 \cdot 10 + 8.$
This last expression has no parentheses, but if you evaluate it using EMA you also get $3 \cdot 10^2 + 7 \cdot 10 + 8$
$= 3 \cdot 100 + 7 \cdot 10 + 8 = 300 + 70 + 8 = 370 + 8 = 378.$

The P in PE(MD)(AS) tells you how to deal with parentheses in an expression you need to evaluate:

To evaluate a numerical expression with parentheses:

- Select a pair of parentheses that are innermost: that is, with no other parentheses between them.
- Use E(MD)(AS) to simplify the expression inside the parentheses.
- Repeat the above two steps until you obtain a single number.

Example: $16-(3-(5-7)) = 16-(3-(5-7))$
$= 16-(3-(-2)) = 16-(5) = 11$

Example: $(16-2^2)(2-4(3-5\cdot 3)) =$
$(16-2^2)(2-4(3-15)) = (16-4)(2-4(-12))$
$= (12)(2+48)) = 12(50) = 600$

Here are details about evaluating consecutive operations with the same precedence:

• (Exponents) are evaluated right to left for technical math-history reasons.
$2^{2^3} = 2^{2^3} = 2^8 = 256$, NOT $2^{2^3} = (2^2)^3 = 4^3 = 64$.

• Multiplies and Divides are together because every divide is a multiply: $10 \div 3 = 10/3 = \frac{10}{3} = 10\cdot\frac{1}{3}$.

"Do (MD) from left to right" says:
$10/5\cdot 2 = 10/5\cdot 2 = 2\cdot 2 = 4$, NOT $10/(5\cdot 2) = 1$.

• Adds and Subtracts are grouped together because every subtract is an add : $7-3 = 7+(-3)$
"Do (AS) from left to right" says:
$7-3+5 = 7-3+5 = 4+5 = 9$, NOT $7-(3+5) = -1$

Be careful: (MD) does NOT mean: do multiplies first, then divides.
It does mean: do them together, from left to right.
$12\div 3\cdot 6\div 2 = 12\div 3\cdot 6\div 2 = 4\cdot 6\div 2 = 24\div 2 = 12$.

Be careful: (AS) does NOT mean: do adds first, then subtracts.
It does mean: do them together, from left to right.
$12-3+6-2 = 12-3+6-2 = 9+6-2 = 15-2 = 13$.

A final technical point about minus signs:

To evaluate an expression that begins with a minus sign

- Replace the minus sign by $-1\cdot$ (minus one times)
- Then follow the E(MD)(AS) order of operations.

Here is an important example:
Be careful: $-5^2 = -1\cdot 5^2 = -1\cdot 25 = -25$
NOT $-5^2 = (-5)^2 = 25$

Notation: The *multiply sign* $\cdot$ between letters and/or parentheses is usually omitted

- bh means $b \cdot h = (b)(h)$.
- $7(h+2)$ means $7 \cdot (h+2)$
- $(b+2)(h+2)$ means $(b+2) \cdot (h+2)$
- $(h+2)b$ means $(h+2) \cdot b$, but leave $(h+2) \cdot 3$ as is.

Example 1: Use E(MD)(AS) to evaluate
$2 \cdot 30 \div 5 \cdot 6 - 5 + 6 + 5 \cdot 10 \div 5 \cdot 6$

Solution:
E: There are no exponentiations.

MD: There are 6 multiplies and divides.
Do all 6 from left to right:

$$\begin{aligned} & 2 \cdot 30/5 \cdot 6 - 5 + 6 + 5 \cdot 10/5 \cdot 6 \\ = & 60/5 \cdot 6 - 5 + 6 + 5 \cdot 10/5 \cdot 6 \\ = & 12 \cdot 6 - 5 + 6 + 5 \cdot 10/5 \cdot 6 \\ = & 72 - 5 + 6 + 5 \cdot 10/5 \cdot 6 \\ = & 72 - 5 + 6 + 50/5 \cdot 6 \\ = & 72 - 5 + 6 + 10 \cdot 6 \\ = & 72 - 5 + 6 + 60 \end{aligned}$$

AS: Now do adds and subtracts from left to right.

$$\begin{aligned} = & 72 - 5 + 6 + 60 \\ = & 67 + 6 + 60 = 73 + 60 = \boxed{133} \end{aligned}$$

You can check this answer by typing
$2 \cdot 30 \div 5 \cdot 6 - 5 + 6 + 5 \cdot 10 \div 5 \cdot 6$ into the Google search box or into a calculator.
Warning: not all calculators follow this order of operations.

Example 2: Use PE(MD)(AS) to evaluate
$(3 + (4 - 5 \cdot 2)^2)(4 - (7 + 8))$.
Solution: Innermost pairs of parentheses and their values are indicated in red.

$$\begin{aligned} & (3 + (4 - 5 \cdot 2)^2)(4 - (7 + 8)) \\ = & (3 + ((4 - 10))^2)(4 - (15)) \\ = & (3 + (-6)^2)(4 - 15) \\ = & (3 + 36)(-11) = (39)(-11) = \boxed{-429} \end{aligned}$$

- Be careful: Common mistakes include the following. Fix each error.
  - $10 - 5 + 3 = 10 - 8 = 2$
  - $2 + 3 \cdot 4 = 5 \cdot 4 = 20$
  - $2 \cdot 3^2 = 6^2 = 36$
  - $5 - 3^2 = 2^2 = 4$
  - $2(3x)^2 = 6x^2$
  - $2(3x)^2 = (6x)^2 = 36x^2$

Exercises: rewriting numerical expressions

Find the value of each numerical expression:
First work out anything in parentheses.

Do all questions before you check your answers ⇒.

a) $30 - 5 - 6 - 4$
b) $30 - (5 - 6) - 4$
c) $30 - 5 - (6 - 4)$
d) $30 - (5 - 6 - 4)$
e) $(3 - 5 + 6)(4 - 2 + 7)$
f) $(3 - 5 + 6) - (4 - 2 + 7)$
g) $100/10 \cdot 2$
h) $5 \cdot 3/(5 \cdot 3)$
i) $5 \cdot 3/5 \cdot 3$
j) $24 \div 3 \cdot 4 \div 2$
k) $6 - 3(6 - 3)$
l) $6 - 3(6 - 3(6 - 3))$
m) $6 - 3(6 - 3)(6 - 3)$
n) $-6 - 2(6 - 3)^3$
o) $30 - 2 \cdot 3 + 2 \cdot 3^2$
p) $100 - 3(2 + 3)^2$
q) $30 + 5 \cdot 3 - 4 \cdot 2$
r) $2 \cdot 3 \cdot 2^3$
s) $(2 + 3 \cdot 4)(4 - 2 \cdot 7)$
t) $3 \cdot 10^3 + 2 \cdot 10^2 + 5 \cdot 10 + 5$

Exercises: rewriting numerical expressions

Find the value of each numerical expression:
First work out anything in parentheses.

a) $30 - 5 - 6 - 4$
b) $30 - (5 - 6) - 4$
c) $30 - 5 - (6 - 4)$
d) $30 - (5 - 6 - 4)$
e) $(3 - 5 + 6)(4 - 2 + 7)$
f) $(3 - 5 + 6) - (4 - 2 + 7)$
g) $100/10 \cdot 2$
h) $5 \cdot 3/(5 \cdot 3)$
i) $5 \cdot 3/5 \cdot 3$
j) $24 \div 3 \cdot 4 \div 2$
k) $6 - 3(6 - 3)$
l) $6 - 3(6 - 3(6 - 3))$
m) $6 - 3(6 - 3)(6 - 3)$
n) $-6 - 2(6 - 3)^3$
o) $30 - 2 \cdot 3 + 2 \cdot 3^2$
p) $100 - 3(2 + 3)^2$
q) $30 + 5 \cdot 3 - 4 \cdot 2$
r) $2 \cdot 3 \cdot 2^3$
s) $(2 + 3 \cdot 4)(4 - 2 \cdot 7)$
t) $3 \cdot 10^3 + 2 \cdot 10^2 + 5 \cdot 10 + 5$

Do all questions before you check your answers ⇒.

a) $= 25 - 6 - 4 = 19 - 4 = 15$
b) $= (30 - -1) - 4 = 31 - 4 = 27$
c) $= 30 - 5 - 2 = 25 - 2 = 23$
d) $= 30 - (-1 - 4) = 30 - (-5) = 35$
e) $= (-2 + 6)(2 + 7) = (4)(9) = 36$
f) $= (-2 + 6) - (2 + 7) = 4 - 9 = -5$
g) $= 10 \cdot 2 = 20$
h) $= 15/15 = 1$
i) $= (15/5) \cdot 3 = 3 \cdot 3 = 9$
j) $= (24 \div 3) \cdot 4 \div 2 = 8 \cdot 4 \div 2 = 32 \div 2 = 16$
k) $= 6 - 3(3) = 6 - 9 = -3$
l) $= 6 - 3(6 - 3(3)) = 6 - 3(6 - 9) = 6 - 3(-3) = 6 + 9 = 15$
m) $= 6 - 3(3)(3) = 6 - (9)(3) = 6 - 27 = -21$
n) $-6 - 2(3)^3 = -6 - 2 \cdot 27 = -6 - 54 = -60$
o) $= 30 - 2 \cdot 3 + 2 \cdot 9 = 30 - 6 + 18 = 42$
p) $= 100 - 3(5)^2 = 100 - 3 \cdot 25 = 100 - 75 = 25$
q) $= 30 + 15 - 8 = 45 - 8 = 37$
r) $= 2 \cdot 3 \cdot 8 = 6 \cdot 8 = 48$
s) $= (2 + 12)(4 - 14) = 14 \cdot -10 = -140$
t) $= 3 \cdot 1000 + 2 \cdot 100 + 5 \cdot 10 + 5 = 3000 + 200 + 50 + 5 = 3255$

1.1.3 Algebra, Formulas, Expressions

Algebra is the grammar of mathematics.

You need to master this grammar if you wish to solve math and science problems.

When you write an essay you assemble

- letters to make words;
- words and punctuation to make phrases;
- phrases to make sentences;
- sentences to express ideas.

When you write mathematics you assemble

- numbers, letters, and punctuation to form expressions such as $30x^2(x^3 + \frac{7}{3})$;
- expressions and $=$ to form statements such as $(x+2)(x-2) = x^2 - 4$;
- statements to solve a problem.

The grammar rules for math are at least as strict as those for language.

Numbers are written with digits 0 through 9.

Letters are special symbols.
Sometimes a letter is a variable;
sometimes it is an unknown;
sometimes it is a placeholder for a number;
sometimes it is just a letter!

Expressions are the phrases of mathematics. They are built from letters, numbers, and the following punctuation marks:

- Parentheses: ( )
- Operations: $+$ (add) $-$ (subtract) $\cdot$ (multiply)
- Fraction lines: $\frac{A}{B}$ (sometimes written A/B) is A divided by B.
- Power notation: 5^3 is $5 \cdot 5 \cdot 5$.

The elementary school symbols $\div$ (for division) and $\times$ (for multiplication) are seldom used in college-level math textbooks.

Identities such as $E = F$, where E and F are expressions, are the sentences of mathematics. They assert that each expression can be obtained from the other by applying the

Laws of Algebra: for any expressions A, B, C

Laws for Addition	*Laws for Multiplication*
$A + 0 = A$	$A \cdot 1 = A$
$A + B = B + A$	$A \cdot B = B \cdot A$
$A + (B + C) = (A + B) + C$	$(A \cdot B) \cdot C = A \cdot (B \cdot C)$
$A + -A = 0$	$A \cdot \frac{1}{A} = 1$ if $A \neq 0$.

The connection between Addition and Multiplication is the

Distributive Law: $A \cdot (B + C) = A \cdot B + A \cdot C$

Formulas and expressions

Numerical expressions describe collections that can be counted or things that can be measured:

- Three dozen eggs contain $3 \cdot 12 = 36$ eggs.
- The total weight of 3 apples weighing 2.5 ounces each and 4 oranges weighing 3.4 ounces each is $3 \cdot 2.5 + 4 \cdot 3.4 = 7.5 + 13.6 = 21.1$ ounces.

Algebra works with algebraic expressions that may include letters as well as numbers.

- The letters are not "unknowns."
- They are placeholders for numbers.
- For example, x could stand for any real number.

Definition: *Expressions and formulas*

- An expression combines numbers, letters, math operations, and parentheses as follows:
  - A letter or a number is an expression.
  - If E is an expression, so is (E).
  - If E and F are expressions, so are $-E, E+F, E-F, E \cdot F, E^F$, and $E/F = \frac{E}{F}$.
- A formula states that a letter equals an expression.

Some formulas from high school geometry are

- $A = bh$
- $P = 2w + 2h$
- $V = \frac{1}{3}\pi r^2 h$
- $C = 2\pi r$
- $A = \pi r^2$

Algebra laws (or rules) relate expressions.
They make it possible

- to make choices about the present: If a rectangle has perimeter 27 inches, how long should its sides be to make its area as large as possible?
- and also to predict the future: If a rock is dropped from a cliff that is 100 feet high, after how many seconds will the rock hit the ground?

The simplest use of letters is to abbreviate a general principle. For example:

The area of any rectangle is its base times its height.
Translating to algebra:

Rectangle Area Formula: For any rectangle, *let* b be its base, *let* h be its height, and *let* A be its area.
Then $A = bh$.

The word "Let" is crucial. It means: when you know what numbers are the base and height, substitute those numbers for letters in the formula. For example, if the rectangle has base $b = 6$ and height $h = 5$, then $A = bh$ becomes $A = 65$. Huh?

Substitution in Expressions and Formulas

Our mistake was that we *replaced* b by 6 and h by 5. We should have *substituted* 6 for b and 5 for h.

To *Substitute* a number for a letter

Replace the letter by the number *enclosed in parentheses.*

When you substitute 6 for b and 5 for h, then $A = bh$ becomes $A = (6)(5) = 6 \cdot 5 = 30$, the correct answer. Substitution, not simple replacement, is required to get the right answer.

When you write formulas in science, you usually specify units of measurement. Math formulas omit units, because letters stand for numbers, not measurements.

To evaluate an expression by substituting a number for a letter

- Substitute the number for the letter
- Use the order of operations to rewrite the expression as a single number.

With few exceptions, parentheses are required, even around a one-digit number. For example, substituting 3 for z in $3z^2 - z$ gives $3(3)^2 - 3 = 27 - 3 = 24$.

Leaving out parentheses gives $33^2 - 3 = 1089 - 3 = 1086$, incorrect!

To *Substitute* an expression for a letter

Replace the letter by the expresssion *enclosed in parentheses.*

When you substitute $3x + 2$ for b and $5x - 7$ for h, then $A = bh$ becomes $A = (3x + 2)(6x + 5)$.

To evaluate an expression by substituting an expression for a letter

- Substitute the expression for the letter
- Use the techniques of the following sections to rewrite the expression in the form that will be requested.

Example 3: Find $z = ax^3 + by$ if $a = 2, x = -3, b = 7, y = -5$.

Solution:

$z = ax^3 + by = (2)(-3)^3 + (7)(-5)$ Do E,M,A:

$z = (2)(-27) + (7)(-5) = -54 + (-35)$ $\boxed{z = -89}$

Note: the word "find" just means "evaluate."

Substituting numbers or expressions for a letter

Example 4: Evaluate $z = x - ab - (b - ax^3)$ if $a = 2, x = -3, b = 7, y = -5$.

Solution: First replace $(b - ax^3)$ by $[b - ax^3]$ to make reading easier: $z = x - ab - [b - ax^3]$.

Substitute:	$z = (-3) - (2)(7) - [(7) - (2)(-3)^3]$
E first:	$z = -3 - (2)(7) - [(7) - (2)(-27)]$
Simplify the	$z = -3 - (2)(7) - [(7) - (-54)]$
expression	$z = -3 - (2)(7) - [7 + 54]$
inside [ ]:	$z = -3 - (2)(7) - 61$
Now do M:	$z = -3 - 14 - 61$
Now do AS:	$z = -17 - 61$ so $\boxed{z = -78}$

So far, letters were placeholders for numbers.

Letters can also be placeholders for expressions.

Suppose $z = x^2 + y^2$, $x = a + 2$, and $y = b - 2$. To find the value of z, replace x by $(a + 2)$ and y by $(b - 2)$ to get $z = (a + 2)^2 + (b - 2)^2$.

To substitute an expression or number for a letter, use parentheses!

- To substitute $xy + 3$ for z in expression $3z^2 - z$, replace z by $(xy + 3)$ to obtain $3(xy + 3)^2 - (xy + 3)$.
- If you leave out parentheses, the work from then on will be a waste of time.

The following examples are very important.

- If $E = AB - B^2$, find E if $A = x + 3$ and $B = y - 3$.
  $\boxed{E = (x + 3)(y - 3) - (y - 3)^2}$
- If $E = (x + 2)^3(y - 3)^4 - 3(x + 2)^4(y - 3)^3$, set $U = x + 2$ and $V = y - 3$. $\boxed{E = U^3V^4 - 3U^4V^3}$
- If $A = x + 3$ and $B = y - 2$, state the commutative law $AB = BA$ $\boxed{(x + 3)(y - 2) = (y - 2)(x + 3)}$
- If $y = x^2 - 3x$ and $x = z + 2$, express y in terms of z.
  $\boxed{y = (z + 2)^2 - 3(z + 2)}$
- Rewrite $x^2 - x(x + 1) - (x + 3)^2$ as a polynomial.

Solution:

To simplify	$x^2 - x(x + 1) - (x + 3)^2$
Parenthesize results of ops M and E:	$= x^2 - (x^2 + x) - (x^2 + 6x + 9)$
Distribute $-$ signs:	$= x^2 - x^2 - x - x^2 - 6x - 9$
Collect like terms:	$= \boxed{-x^2 - 7x - 9}$

1.1.4 Powers, monomials and polynomials

Powers, bases and exponents for numbers

- $2^3 = 2 \cdot 2 \cdot 2 = 8$ is the product of three 2's.
- The base is 2. The exponent is 3.
- 2^3 is the 3^{rd} power of 2.
- If A is a number $A^1 = A$. If $m > 1$ is an integer, $A^m = AA... = A \cdot A...$ is the product of m $A's$. For example, $A^5 = AAAAA$

Power rule examples

- To multiply powers of the same base 2 , keep the base 2 and add exponents: $2^3 \cdot 2^5 = 2^{3+5} = 2^8$.
- To raise a power of 2 to a power: Keep the base 2 and multiply exponents: $(2^3)^5 = 2^{3 \cdot 5} = 2^{15}$.
- To multiply powers of different bases, same exponents, multiply the bases and keep the exponent: $2^3 \cdot 5^3 = (2 \cdot 5)^3 = 6^3$.

Algebra laws (or rules) are expressed using letters, often without reference to the fact that letters stand for numbers. That's OK for simple rules such as the commutative law $A + B = B + A$, which is true for any two numbers. However, other rules need to state carefully which numbers are allowed.

Power rules for real numbers

Let A and B be positive real numbers, let m and n be positive integers. Then

- $A^m A^n = A^{m+n}$
- $(A^m)^n = A^{mn}$
- $A^m B^m = (AB)^m$
- If $A \neq 0$, then $A^0 = 1$. However,
- 0^0 is *undefined* (technically, *indeterminate*). Any calculation yielding 0^0 contains an error.

The first identity above says $A^1 A^0 = A^{1+0} = A^1$, so $AA^0 = A$. If $A \neq 0$, dividing by A gives $A^0 = 1$.

Let $K, A, B, C, ...$ be expressions.

Definition: A *factorization* of K with *factors* $A, B, C..$

is a statement that K is a product: $K = ABC...$

An important special case: If $K, A, B, C, ...$ are integers

An *integer factorization* of K with *factors* $A, B, C, ..$

is a statement that K is a product: $K = ABC...$

Powers of expressions

If A is any expression and $n \geq 2$ is an integer, then

- $A^0 = 1$ if $A \neq 0$ but 0^0 is undefined.
- $A^1 = A$
- $A^n = AA... = A \cdot A...$ is the product of n $A's$.
- A^n is the n^{th} power of A. In this power,
- the base is A and the exponent is n.

Power rules for expressions

Let m and n be non-negative integers $0, 1, 2, 3, ...$

- To multiply powers of the same base, add exponents: $A^m A^n = A^m \cdot A^n = A^{m+n}$
- To multiply the same power of different bases, multiply bases: $A^m B^m = (AB)^m$
- To raise a power to a power, multiply exponents: $(A^m)^n = A^{mn}$

For example, let $A = x + y$ and $B = a + 3$. Then substituting in $(AB)^3 = A^3 B^3$ says

$$\big((x+y)(a+b)\big)^3 = (x+y)^3(a+b)^3.$$

Example 5: Rewrite using the above power rules:

$12^0 = \boxed{1}$ $\quad 0^{12} = \boxed{0}$ $\quad (4^2)^3 = 4^{2\cdot 3} = \boxed{4^6}$

$4^3 \cdot 4^2 = 4^{3+2} = \boxed{4^5}$ $\quad 3^2 \cdot 4^2 = (3 \cdot 4)^2 = \boxed{12^2}$

Definition: *Products, Terms, Monomials*

- A term is a product of numbers and/or letters and/or letter powers.
- A letter part is a letter power, or a product of distinct letter powers in alphabetical order,
- A monomial is
  - a number, or
  - a letter part, or
  - a number followed by a letter part.

Simple terms are $0, -5, x, y^3, aycx$, and $-3zy \cdot 31x^2$.

To rewrite a term as a monomial

- Reorder so that numbers are to the left of letters.
- Multiply the numbers.
- Alphabetize the letter powers.
- Multiply powers of the same letter by adding exponents.

Here are examples of terms rewritten as monomials:

- $3 \cdot 4 = 12$ • $xyxyx = x^3y^2$ • $y^5x = xy^5$
- $3x \cdot 5y \cdot -2x = -30x^2y$ • $zyxz(-4)x = -4x^2yz^2$

Definition: *Properties of monomials*

- The letter part of a monomial is the product of its letter powers: $3xyz^3$.
- Like terms are monomials with identical letter parts: $3xyz^3$ and $-2xyz^3$
- A constant monomial is a number c without a letter part: -34567
- The degree of a constant monomial c is 0 if $c \neq 0$ but is undefined if $c = 0$.
- The degree of a non-constant monomial is the sum of the exponents in its letter part: $3xyz^3 = 3x^1y^1z^3$ has degree $1 + 1 + 3 = 5$.
- x by itself has degree 1 since $x = x^1$.
- The monomial's coefficient is
  - 1 if it contains only letters
  - the number at the left otherwise: $3xyz^3$

To add like terms, use the Distributive Law

- $3xy + 7xy = (7 + 3)xy = 10xy$
- The sum of like terms is the sum of their coefficients times their common letter part.
- The sum of unlike terms can't be simplified.

To multiply monomials

Rewrite their product as a monomial.

Example 6: Rewrite each sum or product as a monomial. Remember to first alphabetize each letter part.

- $7x^2 \cdot 2x^3 = 7 \cdot 2 \cdot x^2 \cdot x^3 = 7 \cdot 2x^{2+3} = 14x^5$
- $7x^3 \cdot 2y^3x^2y^4 = 7 \cdot 2x^3x^2y^3y^4 = 14x^5y^7$
- $x^3y^3x^2 \cdot y^4 = x^3x^2y^3y^4 = x^5y^7$
- $7xyx \cdot yz = 7xxyyz = 7x^2y^2z$
- $-8x^3 \cdot y^3x^{80} = -8x^3x^{80}y^3 = -8x^{3+80}y^3 = -8x^{83}y^3$
- $xy + xy^2$ is a sum of unlike terms. Leave as is.
- $3x^2y + 6x^2y = (3 + 6)x^2y = 9x^2y$ (like terms)
- $5x^2 - 7x^2 = 5x^2 + -7x^2 = (5 + -7)x^2 = -2x^2$
- Faster: $5x^2 - 7x^2 = (5 - 7)x^2 = -2x^2$
- $7xyx + 8yx^2 + 9x^2y = 7x^2y + 8x^2y + 9x^2y$
  $= (7 + 8 + 9)x^2y = 24x^2y$

Exercise: Rewrite each sum or product as a monomial:

a) $x^3x^8x^6$ b) $3x^3 \cdot x^8$ c) $3x^3 \cdot -5x^8$
d) $(-2x^3)x(-5x^7)$ e) $3t^3 \cdot 3t^8$ f) $y^3x^3y^5x^7$
g) $3x^2 - 15x^2 + x^2$ h) $3x^2y - 7xyx - 8xxy + 30x^2y$

Please answer all the above questions before you check your answers ⇒

Definition: *Properties of monomials*

- The letter part of a monomial is the product of its letter powers: $3xyz^3$.
- Like terms are monomials with identical letter parts: $3xyz^3$ and $-2xyz^3$
- A constant monomial is a number c without a letter part: -34567
- The degree of a constant monomial c is 0 if $c \neq 0$ but is undefined if $c = 0$.
- The degree of a non-constant monomial is the sum of the exponents in its letter part: $3xyz^3 = 3x^1y^1z^3$ has degree $1+1+3=5$.
- x by itself has degree 1 since $x = x^1$.
- The monomial's coefficient is
  - 1 if it contains only letters
  - the number at the left otherwise: $3xyz^3$

To add like terms, use the Distributive Law

- $3xy + 7xy = (7+3)xy = 10xy$
- The sum of like terms is the sum of their coefficients times their common letter part.
- The sum of unlike terms can't be simplified.

To multiply monomials

Rewrite their product as a monomial.

Example 6: Rewrite each sum or product as a monomial. Remember to first alphabetize each letter part.

- $7x^2 \cdot 2x^3 = 7 \cdot 2 \cdot x^2 \cdot x^3 = 7 \cdot 2x^{2+3} = 14x^5$
- $7x^3 \cdot 2y^3x^2y^4 = 7 \cdot 2x^3x^2y^3y^4 = 14x^5y^7$
- $x^3y^3x^2 \cdot y^4 = x^3x^2y^3y^4 = x^5y^7$
- $7xyx \cdot yz = 7xxyyz = 7x^2y^2z$
- $-8x^3 \cdot y^3x^{80} = -8x^3x^{80}y^3 = -8x^{3+80}y^3 = -8x^{83}y^3$
- $xy + xy^2$ is a sum of unlike terms. Leave as is.
- $3x^2y + 6x^2y = (3+6)x^2y = 9x^2y$ (like terms)
- $5x^2 - 7x^2 = 5x^2 + -7x^2 = (5 + -7)x^2 = -2x^2$
- Faster: $5x^2 - 7x^2 = (5-7)x^2 = -2x^2$
- $7xyx + 8yx^2 + 9x^2y = 7x^2y + 8x^2y + 9x^2y$
  $= (7+8+9)x^2y = 24x^2y$

Exercise: Rewrite each sum or product as a monomial:

a) $x^3x^8x^6$ b) $3x^3 \cdot x^8$ c) $3x^3 \cdot -5x^8$
d) $(-2x^3)x(-5x^7)$ e) $3t^3 \cdot 3t^8$ f) $y^3x^3y^5x^7$
g) $3x^2 - 15x^2 + x^2$ h) $3x^2y - 7xyx - 8xxy + 30x^2y$

Please answer all the above questions before you check your answers ⇒ a) $x^3x^8x^6 = x^{17}$ b) $3x^3 \cdot x^8 = 3x^{11}$
c) $3x^3 \cdot -5x^8 = -15x^{11}$ d) $-2x^3 \cdot x(-5x^7) = 10x^{11}$
e) $3t^3 \cdot 3t^8 = 9t^{11}$ f) $y^3x^3y^5x^7 = x^{10}y^8$
g) $-11x^2$ h) $18x^2y$

1.1.5 The Distributive Law and polynomial operations

To remove or insert parentheses, use a Distributive Law:

For any expressions A, B, C
$A(B+C) = AB + AC$ or $(B+C)A = BA + CA$

Here are some examples:

$A = 2;\ B = 3;\ C = 4$	$2(3+4) = 2(3) + 2(4) = 6 + 8 = 14$
$A = 2;\ B = -3;\ C = 4$	$2(-3+4) = 2(-3) + 2(4) = -6 + 8 = 2$
$A = 2;\ B = x;\ C = -2$	$2(x + -2) = 2(x) + 2(-2) = 2x - 4$
You might prefer:	$2(x-2) = 2(x) - 2(2) = 2x - 4$

Make sure you know what A, B, C are in the following.

- $3(x+2) = 3(x) + 3(2) = 3x + 6$
- $x(x+2) = x(x) + x(2) = x^2 + 2x$
- $3(x+y) = 3x + 3y$
- $3(x-2y) = 3x + 3(-2y) = 3x - 6y$ OR
  $3(x-2y) = 3x - 3(2y) = 3x - 6y$
- $-3(2x+3y) = -3(2x) - 3(3y) = -6x - 9y$
- $3(10A + 6B) = 3(10A) + 3(6B) = 30A + 18B$
- $2x(3x-7) = 2x(3x) + 2x(-7) = 6x^2 - 14x$ OR
- $2x(3x-7) = 2x(3x) + -2x(-) = 6x^2 - 14x$
- $3(x-2y) = 3x - 3(2y) = 3x - 6y$
- $(x+3)(x+2) = x(x+2) + 3(x+2)$
  $= x(x) + x(2) + 3(x) + 3(2)$
  $= x^2 + 2x + 3x + 6 = x^2 + 5x + 6.$

In the following examples, monomial degrees decrease or remain the same as you read from left to right.

- $x^5 + 3x^3 + 17$, with monomial degrees $5, 3, 0$
- $x^4 - x^2y + xy^2 - x = x^4 + -x^2y + xy^2 + -x$
  a sum of four terms with degrees $4, 3, 3, 1$

Definitions: *Polynomial* and *Polynomial Degree*

- A *polynomial* is a sum of monomials with no like terms, whose degrees decrease or remain the same from left to right.
- The degree of a polynomial is the degree of its first term, which is the highest degree of its terms.

To rewrite a sum of products as a polynomial

- Rewrite each product as a monomial with letters in alphabetical order.
- Reorder the sum so that like terms are adjacent.
- Use the Distributive Law to add like terms.
- Reorder so that monomial degrees decrease or remain the same from left to right.
- If there is more than one letter, write terms with the same degree in alphabetical order.

Please note:
Some books call $x^5 + x^3 + 3x^5$ a polynomial even though it has like terms.
Some books say that $x^2 + 7x^3$ is a polynomial.
We reserve the word *polynomial* for a simplified (collected and powers ordered downward) sum of terms.

To add polynomials P and Q

Rewrite $P + Q$ as a polynomial.

Example 7: Rewrite each sum as a polynomial:

- $5x^2 - 2xy + 7x^2 + 8xy$
  $= 5x^2 + 7x^2 - 2xy + 8xy = \boxed{12x^2 + 6xy}$
- $3 - 5x^2 + x^8 + 7x^2 - 15 + 4x^8$
  $= 4x^8 + 1x^8 - 5x^2 + 7x^2 + 3 - 15$
  $= (4 + 1)x^8 + (-5 + 7)x^2 - 12$
  $= \boxed{5x^8 + 2x^2 - 12}$
- $xy^3 + x^4 + x^2y^2$
  $= xyyy + xxxx + xxyy$ Alphabetize terms:
  $= xxxx + xxyy + xyyy = \boxed{x^4 + x^2y^2 + xy^3}$
- If $P = x^2 + 7x + 3$ and $Q = x - x^3 + 9$, then

$P + Q = x^2 + 7x + 3 + x - x^3 + 9$
$= -x^3 + x^2 + 7x + x + 3 + 9 = \boxed{-x^3 + x^2 + 8x + 12}$

Definition: Polynomial *Leading term* and *coefficient*

For a nonzero polynomial P in one letter:

- The leading term of P is the term with the highest exponent.
- The leading coefficient of P is the coefficient of the leading term.

Exercise: Rewrite each polynomial with exponents decreasing from left to right:

a) $x - x^3$
b) $3x - 2x^2 - 4x - 8x^3 + 3x^2$
c) $t - 3t - 5 - 6t^2 + t^3$
d) $x^3 + x^2 + 4x^3 + 5x^2 + x^4 - x^6$
e) $x^2y + 3xy^2 + x^4 + y^4 + 7xy^2 - 3x^2y$

Please answer all the above questions before you check your answers ⇒.

Please note:
Some books call $x^5 + x^3 + 3x^5$ a polynomial even though it has like terms.
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- $3 - 5x^2 + x^8 + 7x^2 - 15 + 4x^8$
  $= 4x^8 + 1x^8 - 5x^2 + 7x^2 + 3 - 15$
  $= (4 + 1)x^8 + (-5 + 7)x^2 - 12$
  $= \boxed{5x^8 + 2x^2 - 12}$
- $xy^3 + x^4 + x^2y^2$
  $= xyyy + xxxx + xxyy$ Alphabetize terms:
  $= xxxx + xxyy + xyyy = \boxed{x^4 + x^2y^2 + xy^3}$
- If $P = x^2 + 7x + 3$ and $Q = x - x^3 + 9$, then
  $P + Q = x^2 + 7x + 3 + x - x^3 + 9$
  $= -x^3 + x^2 + 7x + x + 3 + 9 = \boxed{-x^3 + x^2 + 8x + 12}$

Definition: Polynomial *Leading term* and *coefficient*

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a) $x - x^3$
b) $3x - 2x^2 - 4x - 8x^3 + 3x^2$
c) $t - 3t - 5 - 6t^2 + t^3$
d) $x^3 + x^2 + 4x^3 + 5x^2 + x^4 - x^6$
e) $x^2y + 3xy^2 + x^4 + y^4 + 7xy^2 - 3x^2y$

Please answer all the above questions before you check your answers ⇒.

a) $-x^3 + x$
b) $-8x^3 + x^2 - x$
c) $t^3 - 6t^2 - 2t - 5$
d) $-x^6 + x^4 + 5x^3 + 6x^2$
e) $x^4 - 2x^2y + 10xy^2 + y^4$

Subtracting and multiplying polynomials

To distribute a minus sign across a sum of terms in parentheses:

- Remove the parentheses.
- Rewrite the sum with each term multiplied by -1.
- $-(A - B + C + D - E) = -A + B - C - D + E$

Example 8: Rewrite each expression without parentheses:
- $-(4 - x + y - 7z) = -4 + x - y + 7z$
- $x - 3 - (2y - 3z + 3b - 4) = x - 3 - 2y + 3z - 3b + 4.$

How to subtract polynomials

Let P and Q be polynomials. To find $P - (Q)$:

- Distribute the minus sign across (Q).
- Collect like terms.

Example 9:
Rewrite $(x^2 + 2x) - (2x^2 - 3x)$ as a polynomial.

Solution: $= x^2 + 2x - 2x^2 + 3x = \boxed{-x^2 + 5x}$

Be careful:
- $(x^2 + 2x) - (2x^2 - 3x)$ is not $x^2 + 2x - 2x^2 - 3x$. Make sure to distribute the minus sign!
- $-(a + 2)^2$ is not equal to $(-a - 2)^2$.

Reason: The $-$ sign in the first expression means multiply by -1, so EMDAS says to compute the power first. Check that the two expressions are unequal.

To multiply a monomial by a polynomial

- Multiply the monomial by each term of the polynomial.
- Add the resulting products.

This follows from the general distributive law $A(B + C + D + ...) = AB + AC + AD + ...$ In this case, A is the monomial, which may be just a constant.

Example 10: Rewrite each product as a polynomial.
- $3(5x - 3x^2 + 1)$
  $= 3 \cdot 5x + 3 \cdot (-3x^2) + 3 \cdot 1$
  $= 15x - 9x^2 + 3 = \boxed{-9x^2 + 15x + 3}$
- $(2x^3)(x^2 - 5x + 4)$
  $= 2x^3(x^2) - 2x^3(5x) + 2x^3(4) = \boxed{2x^5 - 10x^4 + 8x^3}$
- $(-2x^2y)(x^2y - 5x + 4y^2)$
  $= (-2x^2y)(x^2y) + (-2x^2y)(-5x) + (-2x^2y)(4y^2)$
  $= \boxed{-2x^4y^2 + 10x^3y - 8x^2y^3}.$

To multiply polynomials P and Q

- Multiply each term of P by each term of Q.
- Add the resulting products.
- Collect like terms to get a polynomial.

Example 11: Rewrite each blue expression as a polynomial.

- $x^2(x+x^3) = x^2(x) + x^2(x^3) = \boxed{x^3 + x^5}$
- $(3x+2)(2x+5)$
  $= 3x(2x+5) + 2(2x+5)$
  $= 3x \cdot 2x + 3x \cdot 5 + 2 \cdot 2x + 2 \cdot 5$
  $= 6x^2 + 15x + 4x + 10 = \boxed{6x^2 + 19x + 10}$
- $(3x^4+10)(3x^4-10)$
  $= (3x^4)(3x^4-10) + 10(3x^4-10)$
  $= 9x^8 - 30x^4 + 30x^4 - 100 = \boxed{9x^8 - 100}$

The last two solutions above, the high school *FOIL method, works only for products of polynomials with two terms each.*

Another method: The identity $(A+B)(A-B) = A^2 - B^2$, to be discussed later, lets us rewrite $(3x^4+10)(3x^4-10)$ as $(A+B)(A-B)$ by setting $A = 3x^4$ and $B = 10$.

Then $A^2 - B^2 = (3x^4)^2 - (10)^2 = \boxed{9x^8 - 100}$

- $(2x+3)(x^2-5x+4) = ?$ Use any of the methods below.

Method 1: Distributive law: $(2x+3)(x^2-5x+4) =$

$2x \cdot (x^2-5x+4)$: $2x^3 - 10x^2 + 8x$
$+3 \cdot (x^2-5x+4)$: $3x^2 - 15x + 12$
Add the results: $= 2x^3 - 10x^2 + 8x + 3x^2 - 15x + 12$
Reorder: $= 2x^3 - 10x^2 + 3x^2 + 8x - 15x + 12$
Add like terms: $= \boxed{2x^3 - 7x^2 - 7x + 12}$

Method 2: Use the chart below. Each term is the product of the terms at the top of its column and the left of its row.

	x^2	$-5x$	$+4$
$2x$	$2x^3$	$-10x^2$	$+8x$
$+3$	$3x^2$	$-15x$	$+12$

Add and collect the products: $\boxed{2x^3 - 7x^2 - 7x + 12}$

Method 3: Imitate the decimal multiplication procedure for finding $154 \cdot 32$.

			x^2	$-5x$	$+4$
			times	$2x$	$+3$
$x^2 - 5x + 4$ times 3 =			$3x^2$	$-15x$	$+12$
$x^2 - 5x + 4$ times $2x$ =	$2x^3$	$-10x^2$	$+8x$		
Add to get the answer:	$2x^3$	$-7x^2$	$-7x$	$+12$	

Exercises: multiplying polynomials

Multiply out and rewrite each product as a polynomial. Monomial degrees should decrease from left to right. Please make sure that you have answered all the questions before you check your answers below at the right.

a) $x(2+x)$

b) $x^6(x^2-9)$

c) $3(x+2)(x+1)$

d) $(3x-2)(x^2+x^3)$

e) $(2-x)(2-x^2)(3x)$

f) $(x-1)(x^2+x+1)$

g) $(3x+4)(x^2-2x+1)$

h) $(2-x^2)(x^3+3x)$

i) $(x^2+x+1)(x^2+x+1)$

j) $(x-3y)(x+3y)$

k) $(x-y)(x-2y)(x-3y)$

l) $(x-y)^2$

m) $(x-y)(x+y)^2$

n) $(x+3)^3$

o) $(2x-3)^3$

p) $(x+y+2)^2$

q) $(x^2+2x+3)^2$

Exercises: multiplying polynomials

Multiply out and rewrite each product as a polynomial. Monomial degrees should decrease from left to right. Please make sure that you have answered all the questions before you check your answers below at the right.

a) $x(2+x)$
b) $x^6(x^2-9)$
c) $3(x+2)(x+1)$
d) $(3x-2)(x^2+x^3)$
e) $(2-x)(2-x^2)(3x)$
f) $(x-1)(x^2+x+1)$
g) $(3x+4)(x^2-2x+1)$
h) $(2-x^2)(x^3+3x)$
i) $(x^2+x+1)(x^2+x+1)$
j) $(x-3y)(x+3y)$
k) $(x-y)(x-2y)(x-3y)$
l) $(x-y)^2$
m) $(x-y)(x+y)^2$
n) $(x+3)^3$
o) $(2x-3)^3$
p) $(x+y+2)^2$
q) $(x^2+2x+3)^2$

a) x^2+2x
b) x^8-9x^6
c) $3x^2+9x+6$
d) $3x^4+x^3-2x^2$
e) $3x^4-6x^3-6x^2+12x$
f) x^3-1
g) $3x^3-2x^2-5x+4$
h) $-x^5-x^3+6x$
i) $x^4+2x^3+3x^2+2x+1$
j) x^2-9y^2
k) $x^3-6x^2y+11xy^2-6y^3$
l) $x^2-2xy+y^2$
m) $x^3+x^2y-xy^2-y^3$
n) $x^3+9x^2+27x+27$
o) $8x^3-36x^2+54x-27$
p) $x^2+y^2+2xy+4x+4y+4$
q) $x^4+4x^3+10x^2+12x+9$

The Reverse Distributive Law

The *Reverse Distributive Law*: $AB + AC = A(B + C)$

- is used to factor a sum of terms and
- to cancel fractions: Wait for Section 1.5.

Example 12: Factor $ax^2 + ay^3$
Solution: Let $A = a$, $B = x^2$, $C = y^3$. Then

$$ax^2 + by^3 = AB + AC = A(B + C) = \boxed{a(x^2 + y^3)}$$

If terms in a sum have powers of the same letter

factor out the lowest power of that letter.

Example 13: Factor each blue sum of terms:

- $ax^8y + bx^6z = x^6(ax^{8-6}y + bz) \ = \boxed{x^6(ax^2y + bz)}$
- $x^5 + 3x^7 = x^5(1 + 3^{7-5}) = \boxed{x^5(1 + 3x^2)}$
- $a^5b^8 + a^7b^5 = a^5(b^8 + a^{7-5}b^5) = a^5(b^8 + a^2b^5)$ $= \boxed{a^5b^5(b^3 + a^2)}$
- $a^5b^6 + a^6b^5 = a^5b^5(a^{6-5}b^{6-5}) = \boxed{a^5b^5(a + b)}$

- $(x+1)(x+4) + (x+1)(x+5)$: First factor out $(x + 1)$:
$(x+1)(x+4) + (x+1)(x+5) = (x+1)((x+4) + (x+5))$
$= \boxed{(x+1)(2x+9)}$

Don't be tempted to multiply out
$(x+1)(x+4) + (x+1)(x+5)$
$= x^2 + 5x + 4 + x^2 + 6x + 5 = 2x^2 + 11x + 9$
since this requires difficult factoring: $= \boxed{(x+1)(2x+9)}$

- $(x+1)^5(x+4)^6 + (x+1)^6(x+4)^5$

Substitute a for $(x + 1)$ and b for $(x + 4)$ to get
$a^5b^6 + a^6b^5 = a^5b^5(a + b)$ as shown above. Now substitute in reverse: Let $a = x + 1$ and $b = x + 4$: Then
$a^5b^5(a + b) = (x+1)^5(x+4)^5((x+1) + (x+4)) =$
$= \boxed{(x+1)^5(x+4)^5(2x+5)}$

***Sign switch identity*:** $B - A = -(A - B)$

Proof: Factor out -1
$-(A - B) = -A + B = B + -A = B - A$

Example 14: Rewrite each expression as a polynomial.

- $-(x - 4) = 4 - x = -x + 4$
- $-(3x - 2y) = 2y - 3x = -3x + 2y$

Factoring quadratic polynomials

Definition: *quadratic polynomials*

- A quadratic polynomial in x is a polynomial $ax^2 + bx + c$ where a, b, c are integers and $a \neq 0$.
- The discriminant of $ax^2 + bx + c$ is $D = b^2 - 4ac$.
- The roots are the solutions of $ax^2 + bx + c = 0$.

Definition: A number is a *perfect square* if it is the square of some integer.

The first 20 perfect squares are $0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361$.

Definition:
The *Greatest common factor (GCF) of integers*

- is the largest integer that is a factor of all of them.
- Special case: we say that integers have no common factor if their GCF is 1.

Example 15: Find the GCF of $40, 60, 320$.
Solution: Write the numbers as a triple $(40, 60, 320)$
Find a number that goes into all 3. Clearly 10 works.
Therefore $(40, 60, 320) = 10(4, 6, 32)$.
Keep going. 2 goes into $4, 6$, and 32, so $(40, 60, 320) = 10(4, 6, 32) = 10 \cdot 2(2, 3, 16) = 20(2, 3, 16)$. But $2, 3, 16$ don't have a common factor, so the requested GCF is 20.

Quadratic polynomial $ax^2 + bx + c$ factors as a product $(Ax + B)(Cx + D)$ of two degree one polynomials

if and only if $b^2 - 4ac$ is a perfect square.

Example 16:
- $6x^2 + 13x + 6$ factors because $D = 13^2 - 4 \cdot 6 \cdot 6 = 25 = 5^2$ is a perfect square.
- $x^2 + 4x + 2$ does not factor because $D = 4^2 - 4 \cdot 1 \cdot 2 = 8$ is not a perfect square.

Note that $2x^2 + 8x + 4$, with $D = 32$ not a perfect square, does factor as $2(x^2 + 4x + 2)$

To factor $x^2 + bx + c$ if D is a perfect square:

- Find a pair of integers r and s with $r^2 \leq |c|$ and $c = rs$.
- For at least one such pair, $r + s = b$ and so $x^2 + bx + c = x^2 + (r + s)x + rs = (x + r)(x + s)$

Example 17: Factor $x^2 + 13x + 36$.
Solution: $D = 13^2 - 4 \cdot 1 \cdot 36 = 169 - 144 = 25 = 5^2$ is a perfect square. Since $rs = 36$ and $r + s = 13$ are positive, so are r and s. Since you know your multiplication tables well, you should know that $r = 4$ and $s = 9$ add to 13 and multiply to 36.

If not, make a chart: fill in the columns from left to right. Since $r^2 \leq 36$ and $r > 0$, we need to check only $r = 1, 2, 3, 4, 6$. Since $rs = 36$, $s = 36/r$.

$r =$	1	2	3	4	6
$s = 36/r =$	36	18	12	9	
$r + s =$	37	20	15	$r + s = 13$ Yes!	

When $r = 4$, $r + s = b = 13$ and so

$$x^2 + 13x + 36 = (x + 4)(x + 9)$$

To factor $-x^2 + bx + c$

- Rewrite $-x^2 + bx + c$ as $-1(x^2 - bx - c)$
- Factor $x^2 - bx - c$.

Example 18: Factor $-x^2 + x + 12$
Solution: Rewrite as $-(x^2 - x - 12)$
Then $x^2 - x - 12 = (x + r)(x + s)$ if $r + s = -1$ and $rs = -12$. This works if $r = -4$ and $s = 3$. Then
$-x^2 + x + 12 = -(x^2 - x - 12) = \boxed{-(x - 4)(x + 3)}$

We haven't discussed how to factor $P = ax^2 + bx + c$ with $a \neq \pm 1$. The following "AC method" for factoring such P is useful only if ac isn't too big. Otherwise, it can get very messy. A completely different and much easier method for factoring degree 2 polynomials, which does not even require a, b, c to be integers, will be explained in Section 1.5.

To factor $P = ax^2 + bx + c$ if $D = b^2 - 4ac$ is a perfect square

- Find integers r and s with $rs = ac$ and $r + s = b$
- Rewrite $P = (ax^2 + rx) + (sx + c)$
- Factor each expression in parentheses.
- Apply the Distributive Law.

Example 19: Use the AC method to factor $P = ax^2 + bx + c = 6x^2 + 13x + 6$.
Solution: First solve $rs = ac = 36$ and $r + s = b = 13$ to get $r = 9$ and $s = 4$. Then

$$6x^2 + 13x + 6 = (6x^2 + 9x) + (4x + 6)$$
$$= 3x(2x + 3) + 2(2x + 3) = \boxed{(3x + 2)(2x + 3)}$$

This was OK. However, try using this method to factor $12x^2 + 44x + 35$. That's quite a bit harder.

1.1.6 Simplifying polynomial expressions

So far we have multiplied or added polynomials by using the distributive law and/or collecting terms. Sometimes we start with more complicated expressions.

Definition: A *polynomial expression*

combines polynomials by using PEMAS: Parentheses, operations Multiply, Add, Subtract, and powers with positive integer Exponents.

Examples:

- $\big((3x+7)^2+2x+9(2-x)\big)^3$
- $\big((3x+y)^2+2x+a(2-b)\big)^2+xyz(x+y)$

In the following, recall that a polynomial is a (collected) sum of terms, all of which are unlike.

To rewrite a polynomial expression as a polynomial

- Collect terms inside each pair of innermost parentheses.
- Remove parentheses using order of operations EMDAS.
- As you proceed, place parentheses around the *result* of each operation.
- Repeat until all parentheses have been removed.

Example 20:
Solution:
Rewrite $x^2-(x+2)(x+3)$ as a polynomial.

$x^2-(x+2)(x+3)$	is given. Multiply before you
$=x^2-(x^2+5x+6)$	subtract. Insert parentheses!
$=x^2-x^2-5x-6$	Distribute the minus sign.
$=\boxed{-5x-6}.$	

If you omit the parentheses, as in the following, you get the wrong answer:

$x^2-(x+2)(x+3)$	is given. Multiply before you
$=x^2-x^2+5x+6$	subtract. Oops, no parentheses!
$=\boxed{5x+6}.$	Wrong!

Example 21: Rewrite $(x+2)(x+3)-x^2$ as a polynomial.
Solution:

$(x+2)(x+3)-x^2$	is given.
$=(x^2+5x+6)-x^2$	Multiply before you subtract.
$=x^2+5x+6-x^2$	$=\boxed{5x+6}$

Here we put parentheses around x^2+5x+6, but in this unusual case they weren't needed.

Be careful: Leaving out parentheses *usually* yields an incorrect (no partial credit) answer.

General fact: if an expression begins with a parenthesized expression followed by operation A or S, you may omit the parentheses, as in the third step below.

Example 22: Rewrite $x^2(x+4)-(x+2)^2$ as a polynomial.

E first; Put results in parentheses	$= x^2(x+4) - (x^2+4x+4)$
M before S: Put results in parentheses.	$= (x^3+4x^2) - (x^2+4x+4)$
Omit the first set of parentheses:	$= x^3+4x^2 - (x^2+4x+4)$
Distribute − sign to remove 2nd ( ):	$= x^3+4x^2-x^2-4x-4$
Collect terms:	$= \boxed{x^3+3x^2-4x-4}$

You can perform a *partial* check by substituting a number for x. For example, substitute 1 for x:

$$x^2(x+4)-(x+2)^2 \;=?\; x^3+3x^2-4x-4$$

Substitute 1 for x: $(1)^2(1+4)-(1+2)^2 \;=?\; (1)^3+3\cdot(1)^2-4(1)-4$

$$1(5)-3^2 \;=?\; 1+3-4-4$$
$$5-9 \;=?\; 4-4-4$$
$$-4 \;=?\; 0-4$$
$$-4 \;=?\; -4 \text{ YES!}$$

Therefore the answer and the original agree at $x=1$. However, that's not enough of a check. The rule is

To ensure that two degree n polynomials P and Q in x are equal

Check that they agree at $n+1$ distinct values of x.

In our example, $n=3$ because both $P=x^2(x+4)-(x+2)^2$ and $Q=x^3+3x^2-4x-4$ can be rewritten as degree 3 polynomials. To verify the above solution, check that $P=Q$ at $n+1=4$ convenient values of x, as at the right.

If $x=$	then $P=$	and $Q=$
0	-4	-4
1	-4	-4
-1	2	2
-2	8	8

Conclusion: the boxed answer x^3+3x^2-4x-4 is correct!

Exercises: rewriting polynomial expressions

Example 23: Rewrite each expression as a polynomial.

a) $3 - 4(x+2)$
b) $x^2 - x(x+2)$
c) $(x)(x+4) - (x+7)^2$
d) $(x+3)(x+4) - 3(x+7)^2$

Solution:

a) $$\begin{cases} 3 - 4(x+2) \\ = 3 - (4x+8) \\ = 3 - 4x - 8 \\ = -4x - 5 \end{cases}$$

b) $$\begin{cases} x^2 - x(x+2) \\ = x^2 - (x^2 + 2x) \\ = x^2 - x^2 - 2x \\ = 0 - 2x \\ = -2x \end{cases}$$

c) $$\begin{cases} (x)(x+4) - (x+7)^2 \\ = (x^2 + 4x) - (x^2 + 14x + 49) \\ = x^2 + 4x - x^2 - 14x - 49 \\ = -10x - 49 \end{cases}$$

d) $$\begin{cases} (x+3)(x+4) - 3(x+7)^2 \\ = (x^2 + 7x + 12) - 3(x^2 + 14x + 49) \\ = x^2 + 7x + 12 - 3x^2 - 42x - 147 \\ = x^2 - 3x^2 + 7x - 42x + 12 - 147 \\ = -2x^2 - 35x - 135 \end{cases}$$

Rewrite each expression as a polynomial.

a) $5 - x^2(x+1)^2$
b) $x^3 + (x+1)(x-2)^2$
c) $x^3 - (x+1)(x-2)^2$
d) $(t+1)^2 - 3t(t-1)^2$
e) $(y+x)(x+y) + (x+2)(y+2)$
f) $(2a+b)^3 - (b+2a)^3$
g) $(2-x)^2 - (x-1)^2$
h) $3x^2y^2 + 4xyx + 5y^2x^2 + 5yx^2y + 5xyxy$
i) $x(y+1)^2 - 2y(x+1)^2$

Please answer all the questions before you check your answers ⇒.

Exercises: rewriting polynomial expressions

Example 23: Rewrite each expression as a polynomial.

a) $3 - 4(x+2)$
b) $x^2 - x(x+2)$
c) $(x)(x+4) - (x+7)^2$
d) $(x+3)(x+4) - 3(x+7)^2$

Solution:

a) $$\begin{cases} 3 - 4(x+2) \\ = 3 - (4x+8) \\ = 3 - 4x - 8 \\ = -4x - 5 \end{cases}$$

b) $$\begin{cases} x^2 - x(x+2) \\ = x^2 - (x^2+2x) \\ = x^2 - x^2 - 2x \\ = 0 - 2x \\ = -2x \end{cases}$$

c) $$\begin{cases} (x)(x+4) - (x+7)^2 \\ = (x^2+4x) - (x^2+14x+49) \\ = x^2 + 4x - x^2 - 14x - 49 \\ = -10x - 49 \end{cases}$$

d) $$\begin{cases} (x+3)(x+4) - 3(x+7)^2 \\ = (x^2+7x+12) - 3(x^2+14x+49) \\ = x^2 + 7x + 12 - 3x^2 - 42x - 147 \\ = x^2 - 3x^2 + 7x - 42x + 12 - 147 \\ = -2x^2 - 35x - 135 \end{cases}$$

Rewrite each expression as a polynomial.

a) $5 - x^2(x+1)^2$
b) $x^3 + (x+1)(x-2)^2$
c) $x^3 - (x+1)(x-2)^2$
d) $(t+1)^2 - 3t(t-1)^2$
e) $(y+x)(x+y) + (x+2)(y+2)$
f) $(2a+b)^3 - (b+2a)^3$
g) $(2-x)^2 - (x-1)^2$
h) $3x^2y^2 + 4xyx + 5y^2x^2 + 5yx^2y + 5xyxy$
i) $x(y+1)^2 - 2y(x+1)^2$

Please answer all the questions before you check your answers ⇒.

a) $-x^4 - 2x^3 - x^2 + 5$
b) $2x^3 - 3x^2 + 4$
c) $3x^2 - 4$
d) $-3t^3 + 7t^2 - t + 1$
e) $x^2 + 3xy + y^2 + 2x + 2y + 4$
f) 0
g) $-2x + 3$
h) $18x^2y^2 + 4x^2y$
i) $-2x^2y + xy^2 - 2xy + x - 2y$

Answer each question below by using two different methods. The answers should match.

Method 1: Substitute the given number directly into the formula, and find the numerical value of the result.

Method 2: Rewrite the polynomial expression as a polynomial, and then substitute the given number into the result.

1. Find $y = (x+1)^2 - (x+4)(x+5)$ if $x = -5$
2. Find $y = (x+1)^2 - (x+4)(x+5)$ if $x = 5$
3. Find $y = (x+1)^2 - (x+4)^2(x+5)$if $x = -3$
4. Find $y = (x+1)^2 - (x+4)^2(x+5)$ if $x = -5$
5. Find $z = (x+1)^3 - (x+4)(x+5)$ if $x = -2$
6. Find $z = 3(x+1)^3 - 2(x+4)(x+5)$ if $x = -2$
7. Find $z = (x+1)^2 - (x^2+x+4)(x+5)$ if $x = -4$
8. Find $A = B - (B+1)^2 - (B+4)(5-B)$ if $B = -5$
9. Find $A = 3 - B(B^2 - (7-2B))$ if $B = 3$
10. Find $T = S - (3(S+1) - (S - S^2(S+1)))$ if $S = -5$

Sample solution for Exercise 2

Method 1:

$y = (x+1)^2 - (x+4)(x+5)$

Substitute 5 for x.

$y = (5+1)^2 - (5+4)(5+5)$

$y = (6)^2 - (9)(10) = 36 - 90 = \boxed{-54}$

Method 2:

$y = (x+1)^2 - (x+4)(x+5)$

First rewrite as a polynomial.

$$\begin{aligned} y &= (x^2+2x+1) - (x^2+9x+20) \\ &= x^2+2x+1-x^2-9x-20 \\ &= x^2-x^2+2x-9x+1-20 \\ &= -7x-19 \end{aligned}$$

Substitute 5 for x

$y = -7(5) - 19 = -35 - 19 = \boxed{-54}$.

1.1.7 Solving equations

Definition: An *equation* is a statement $P = Q$

where P and Q are any expressions.

To solve an equation, find all solutions

Let P and Q be polynomial expressions. Then

- $x = S$ is a solution for x of equation $P = Q$ if
  - substituting S for every letter x in P and Q and then simplifying yields an identity *and*
  - the letter x does not appear in S
- To solve $P = Q$ for x, find *all* solutions for x.

- Is $x = 2$ is a solution of $x + 7 =? \ 9$

Solution: In equation $x + 7 \quad =? \ 9$
Substitute 2 for x. $2 + 7 \quad =? \ 9$
Does $9 \quad =? \ 9$ YES

- Are $x = 2$ and $x = -2$ solutions of $x^2 = 4$?

Solution: In equation $x^2 \quad = 4$
Substitute -2 for x $(2)^2 \quad =? \ 4$
$4 \quad =? \ 4$ YES
Substitute -2 for x $(-2)^2 \quad =? \ 4$
$4 \quad =? \ 4$ YES

- Is $x = \frac{x}{2}$ a solution of $2x = x$?

NO, because the alleged solution $\frac{x}{2}$contains the letter x.

It's easy to check whether or not an expression is a solution to an equation. However, the following receives zero partial credit because it doesn't explain why $x = 0$ and $x = 1$ are the only solutions of $x^2 = x$.

How NOT to solve the degree 2 equation $x^2 = x$.

- $x = 0$ is a solution because $(0)^2 = 0$.
- $x = 1$ is a solution because $(1)^2 = 1$.
- Answer: $x = 0$ or $x = 1$.

Another way NOT to solve $x^2 = x$.

$x = \sqrt{x}$ is *not* a solution because x appear s in $\sqrt{x}$.

The general approach to solving equations is as follows. More details are in Section 1.5.

Let P and Q be polynomial expressions.
To solve equation $P = Q$ for x

Do the same things to both sides of $P = Q$ to get an easier equation with the same solutions. Specifically, the solutions of $P = Q$ are the same as the solutions of

- $P + E = Q + E$ and $P - E = Q - E$ for any expression E
- $PE = QE$ and $\frac{P}{E} = \frac{Q}{E}$ if $E \neq 0$.
- When you finish these moves, rewrite the easier equation as $R = 0$.
  - If R is a nonzero number, there is no solution.
  - If R does not contain x, the solution is x is arbitrary.
  - Otherwise, you can rewrite $R = 0$ as the set of solutions in the form $x = \ldots$ or $x = \ldots$ or , , ,

Here are some basic examples:

- Solve $x(x + 2) = (x + 1)^2$ for x

Solution: $x^2 + 2x = x^2 + 2x + 1$
Subtract: $R = (x^2 + 2x) - (x^2 + 2x + 1) = -1$ is a nonzero number.
Answer: No solution.

- Solve $x(x + a) = x^2 + ax$ for x.

Solution: $x^2 + ax = x^2 + xa$
Subtract: $R = x^2 + ax - (x^2 + xa) = x^2 + ax - x^2 - ax = 0$
Answer: x is arbitrary.

- Solve $g(g + 1) - g = gc$ for g.

Solution: $g^2 + g - g = gc$ so $g^2 = gc$
Subtract: $R = g^2 - gc = g(g - c) = 0$ if $g = 0$ or $g = c$:
Answer: $g = 0$ or $g = c$

- Solve $x(x + 1) - x = x$ for x.

Solution: $x^2 + x - x = x$ so $x^2 = x$
Subtract: $R = x^2 - x = x(x - 1) = 0$ if $x = 0$ or $x = 1$
Answer: $x = 0$ or $x = 1$

- Solve $x^2 = c^2$.

The problem doesn't say what to solve for. Bad problem!

- Solve $x^2 = x$.

Again, a bad problem. But since there is only one letter, solve for x and then complain:)

Solving numerical equations for x is easy. For any x

- $0 = 0$ *is true:* $\boxed{x \text{ is arbitrary}}$
- $2 + 2 = 5$ *is false:* $\boxed{\text{No solution}}$

- Solve $x - 4 = 10$. Assume x is a number, then $x - 4$ and 10 are the same number. Add 4 to both numbers $x - 4 + 4 = 10 + 4$, so $x + 0 = 14$. Answer $\boxed{x = 14}$

This seems simple enough. The important thing is that the first step can be reversed. To check the answer, start with $x = 14$. Subtract 4 to get $x - 4 = 14 - 4$ so $x - 4 = 10$, as required by the original problem. This shows that the answer is correct.

The same idea works if you multiply (or divide) both sides of the equation by any *nonzero* number.

- Solve $x + 3 = x + 4$ for x.

Subtracting x from both sides gives $3 = 4$. For any x, this is a false statement, and so for any x, the original equation $x + 3 = x + 4$ is false.

Answer: $\boxed{\text{There is no solution for } x.}$

- Solve $2x = x + 1$ for x

Correct: subtract x from both sides. Answer: $\boxed{x = 1}$.

Incorrect: multiply both sides by zero. Then you get $0 = 0$, which is true no matter what x is.

But this is nonsense. Try $x = 12$, for instance. Clearly $2 \cdot 12 = 12 + 1$ is false!

The error is that you can't work backwards. There is no way to start with $0 = 0$ and end up with $2x = x + 1$!

Next we show how to solve a simple class of equations.

Definition: The *degree of equation* $P = Q$ in x is

the highest power of x in the polynomial obtained by simplifying $P - (Q)$.

Example: $P = Q$ is	$(x + 3)(x + 4) = x^2 + 5x + 3$
Then $P - (Q)$ is	$x^2 + 7x + 12 - (x^2 + 5x + 3)$
Simplify:	$x^2 + 7x + 12 - 5x - 3 = 2x^1 + 9$
Therefore equation	$P = Q$ has degree 1

Here is how the general instructions on the previous page for solving equations are used

To solve a degree 1 equation in x:

- Rewrite the equation with all x terms on the left side and all other terms on the right side.
- Factor the left side to rewrite the equation as $Ax = B$. Examples are on the next page.
  - If $A \neq 0$: there is one solution $x = \frac{B}{A}$.
  - If $A = 0$ and $B = 0$: x is arbitrary.
  - If $A = 0$ and $B \neq 0$: there is no solution.

Here are some easy degree 1 equations in x. Reminder: $\frac{5}{-3}$ and $\frac{-5}{3}$ are equal and should be rewritten as $-\frac{5}{3}$.

Problem	Strategy	Work	Answer
• $x+3=x+4$	Subtract x	$3=4$	No solution.
• $2x=x+x$	Subtract $2x$	$0=0$	x is arbitrary.
• $x-5=8$	Add 5	$x-5+5=8+5$	$x=13$
• $x+4=8$	Subtract 4	$x+4-4=8-4$	$x=4$
• $x-5=-8$	Add 5	$x-5+5=-8+5$	$x=-3$
• $x+5=-8$	Subtract 5	$x+5-5=-8-5$	$x=-13$
• $2x=5$	Divide by 2	$\frac{2x}{2}=\frac{5}{2}$	$x=\frac{5}{2}$
• $-2x=5$	Divide by -2	$\frac{-2x}{-2}=\frac{5}{-2}$	$x=-\frac{5}{2}$
• $2=-5x$	Divide by -5	$\frac{2}{-5}=\frac{-5x}{-5}$	$x=-\frac{2}{5}$
• $-2=-5x$	Divide by -5	$\frac{-2}{-5}=\frac{-5x}{-5}$	$\frac{2}{5}=x$
• $2x-5=4x$	Subtract $2x$ Divide by 2	$2x-5-2x=4x-2x \Rightarrow -5=2x$ $\frac{-5}{2}=\frac{2x}{2}$	$-\frac{5}{2}=x$
• $2x=-4x-5$	Add $4x$ Divide by 6	$2x+4x=-4x-5+4x \Rightarrow 6x=-5$ $\frac{6x}{6}=\frac{-5}{6}$	$x=-\frac{5}{6}$
• $2x-5=4$	Add 5 Divide by 2	$2x=9$	$x=\frac{9}{2}$
• $-2x+4=7$	Subtract 4 Divide by -2	$-2x+4-4=7-4 \Rightarrow -2x=3$ $-\frac{-2x}{-2}=\frac{3}{-2}$	$x=-\frac{3}{2}$

To solve each of the following degree one equations in x

Example 24:	$x - 3 = 6.$
Add 3 to both sides:	$x - 3 + 3 = 6 + 3$
Collect	$\boxed{x = 9}$
Example 25:	$x - ay = 6.$
Add ay to both side:	$\boxed{x = 6 + ay}$
Example 26:	$3x - 45 + 3y = x + 5$
Add $45 - 3y$ to both sides:	$3x = x + 5 + 45 - 3y$
Collect like terms:	$3x = x + 50 - 3y$
Subtract x:	$3x - x = 50 - 3y$
Collect like terms:	$2x = 50 - 3y$
Divide by 2:	$\boxed{x = \dfrac{50 - 3y}{2}}$
Example 27:	$45 + 3y - x = x - 20 + 3x + 15$
Simplify right side:	$45 + 3y - x = 4x - 5$
In this example it's easier	to put all x terms on the right side:
Add $5 + x$ to both sides:	$45 + 3y - x + x + 5 = 4x - 5 + x + 5$
Collect like terms:	$50 + 3y = 5x$
Divide by 5:	$\boxed{x = \dfrac{50 + 3y}{5}}$

Degree 1 equations with more than one letter

In the following examples, each solution $x = S$ will be an expression rather than a real number.
If S is a fraction, the only numerical solutions are those in which the denominator is not zero.

Example 28: Solve $xc + yc + xyu = 2x + u + v$ for x.

Equation to solve:	$xc + yc + xyu = 2x + u + v$
1. Subtract $2x$	$xc + yc + xyu - 2x = u + v$
2. Subtract yc	$xc + xyu - 2x = u + v - yc$
3. Factor out x on left x	$x(c + yu - 2) = u + v - yc$
4. Divide by $c + yu - 2$	$\boxed{x = \dfrac{u + v - yc}{c + yu - 2}}$

In science, quantities of interest are related by a formula, which asserts that two expressions are equal whenever their letters are replaced by real-life numerical values.

Example 29: The formula $PV = kT$ relates Pressure, Volume, and Temperature of an ideal gas. The number k depends on which gas is being studied.

This formula can be solved for any one of these quantities.

To solve for k, divide by T: $\frac{PV}{T} = \frac{k\cancel{T}}{\cancel{T}}$ to obtain $\boxed{k = \frac{PV}{T}}$

To solve for V, divide by P: $\frac{\cancel{P}V}{\cancel{P}} = \frac{kT}{P}$ to obtain $\boxed{V = \frac{kT}{P}}$

To solve for P, divide by V: $\frac{P\cancel{V}}{\cancel{V}} = \frac{kT}{V}$ to obtain $\boxed{P = \frac{kT}{V}}$

To solve for T, divide by k: $\frac{PV}{k} = \frac{\cancel{k}T}{\cancel{k}}$ to get $\boxed{T = \frac{PV}{k}}$

Example 30: If $u = \frac{MV}{M+m}$, solve for M.

Multiply by $M + m$	$u(M + m) = MV$
Remove parentheses	$uM + um = MV$
Subtract uM	$um = MV - uM$
Factor out M	$um = M(V - u)$
Divide by $V - u$	$\boxed{-\dfrac{um}{u - V} = M}$

Example 31:
Solve $x^2y + xy' + y = x^3y' + 3xy$ for y'.
Solution: The symbol y' is used in calculus. Here, just think of it as a fancy letter.

Rewrite the equation	$x^2y + xy' + y = x^3y' + 3xy$
with y' terms on left:	$x^2y + xy' - x^3y' + y = 3xy$
Other terms on right:	$xy' - x^3y' = 3xy - x^2y - y$
Factor out y' on left:	$y'(x - x^3) = 3xy - x^2y - y$
Divide by $x - x^3$:	$y' = \dfrac{3xy - x^2y - y}{x - x^3}$

Later we'll show how to rewrite this as a reduced fraction: $\boxed{y' = \dfrac{y(x^2 - 3x + 1)}{x(x + 1)(x - 1)}}$

Section 1.1 Quiz

Ex. 1.1.1: Use E(MD)(AS) to evaluate $2 \cdot 30 \div 5 \cdot 6 - 5 + 6 + 5 \cdot 10 \div 5 \cdot 6$

Ex. 1.1.2: Find the value of $(3 + (4 - 5 \cdot 2)^2)(4 - (7 + 8))$.

Ex. 1.1.3: Find $z = ax^3 + by$ if $a = 2, x = -3, b = 7, y = -5$.

Ex. 1.1.4: Evaluate $z = x - ab - (b - ax^3)$ if $a = 2, x = -3, b = 7, y = -5$.

Ex. 1.1.5: Rewrite using power rules: • 12^0 ; • 0^{12} ; • $(4^2)^3$; • $4^3 \cdot 4^2$; • $3^2 \cdot 4^2$

Ex. 1.1.6: Rewrite each sum or product as a monomial. Remember to first alphabetize each letter part.
- $7x^2 \cdot 2x^3$ • $7x^3 \cdot 2y^3x^2y^4$ • $x^3y^3x^2 \cdot y^4$ • $7xyx \cdot yz$ • $-8x^3 \cdot y^3x^{80}$
- $xy + 3yx$ • $3x^2y + 6x^2y$ • $5x^2 - 7x^2$ • $7xyx + 8yx^2 + 9x^2y$

Ex. 1.1.7: Rewrite each sum as a polynomial: • $P + Q$ if $P = x^2 + 7x + 3$ and $Q = x - x^3 + 9$
- $5x^2 - 2xy + 7x^2 + 8xy$ • $3 - 5x^2 + x^8 + 7x^2 - 15 + 4x^8$

Ex. 1.1.8: Rewrite each expression without parentheses:
- $-(4 - x + y - 7z)$ • $x - 3 - (2y - 3z + 3b - 4)$

Ex. 1.1.9: Rewrite the difference as a polynomial: $(x^2 + 2x) - (2x^2 - 3x)$

Ex. 1.1.10: Rewrite each product as a polynomial.
- $3(5x - 3x^2 + 1)$ • $(2x^3)(x^2 - 5x + 4)$ • $(-2x^2y)(x^2y - 5x + 4y^2)$

Ex. 1.1.11: Rewrite each expression as a polynomial.
- $x^2(x + x^3)$ • $(3x + 2)(2x + 5)$ • $(3x^4 + 10)(3x^4 - 10)$ • $(2x + 3)(x^2 - 5x + 4)$

Ex. 1.1.12: Factor $ax^2 + ay^3$.

Ex. 1.1.13: Factor each blue sum of terms:
- $ax^8y + bx^6z$ • $x^5 + 3x^7$ • $a^5b^8 + a^7b^5$ • $a^5b^6 + a^6b^5$
- $(x + 1)(x + 4) + (x + 1)(x + 5)$ • $(x + 1)^5(x + 4)^6 + (x + 1)^6(x + 4)^5$

Ex. 1.1.14: Rewrite $x^2(x+4)-(x+2)^2$ as a polynomial.
Apply the sign switch identity to : • $-(x-4)$ • $-(3x-2y)$

Ex. 1.1.15: Find the GCF of $40, 60, 320$.

Ex. 1.1.16: Does each quadratic polynomial factor?

• x^2+4x+2 • $6x^3+13x+6$

Ex. 1.1.17: Factor $x^2+13x+36$.

Ex. 1.1.18: Factor $-x^2+x+12$

Ex. 1.1.19: Use the AC method to factor $P=ax^2+bx+c=6x^2+13x+6$.

Ex. 1.1.20: Rewrite $x^2-(x+2)(x+3)$ as a polynomial.

Ex. 1.1.21: Rewrite $(x+2)(x+3)-x^2$ as a polynomial.

Ex. 1.1.24: Solve $x-3=6$ for x.

Ex. 1.1.25: Solve $x-ay=6$ for x.

Ex. 1.1.26: Solve $3x-45+3y=x+5$ for x.

Ex. 1.1.27: Solve $45+3y-x=x-20+3x+15$ for x.

Ex. 1.1.28: Solve $xc+yc+xyu=2x+u+v$ for x.

Ex. 1.1.29: Solve $PV=kT$ for each of the four letters k, P, T, V.

Ex. 1.1.30: If $u=\frac{MV}{M+m}$, solve for M.

Ex. 1.1.31: Solve $x^2y+xy'+y=x^3y'+3xy$ for y'.

Section 1.1 Review: Numbers and Expressions

Ex. 1.1.1: Use E(MD)(AS) to evaluate

- $2 \cdot 30 \div 5 \cdot 6 - 5 + 6 + 5 \cdot 10 \div 5 \cdot 6$
- $4 \cdot 30 \div 8 \cdot 6 - 5 + 6 + 8 \cdot 10 \div 16 \cdot 6$
- $2 \cdot 8 \div 2 \div 2 - 5 + 6 - 12 \cdot 10 \div 5 \cdot 6$
- $2 \cdot 30 \div 5 \cdot 6 - 5 - 11 - 5 \cdot 6 \div 2 \cdot 3$

Section 1.1 Review: Numbers and Expressions

Ex. 1.1.1: Use E(MD)(AS) to evaluate

- $2 \cdot 30 \div 5 \cdot 6 - 5 + 6 + 5 \cdot 10 \div 5 \cdot 6 \qquad = 133$
- $4 \cdot 30 \div 8 \cdot 6 - 5 + 6 + 8 \cdot 10 \div 16 \cdot 6 \qquad = 121$
- $2 \cdot 8 \div 2 \div 2 - 5 + 6 - 12 \cdot 10 \div 5 \cdot 6 \qquad = -139$
- $2 \cdot 30 \div 5 \cdot 6 - 5 - 11 - 5 \cdot 6 \div 2 \cdot 3 \qquad = 11$

Ex. 1.1.1: Use E(MD)(AS) to evaluate

- $2 \cdot 30 \div 5 \cdot 6 - 5 + 6 + 5 \cdot 10 \div 5 \cdot 6 \quad = 133$
- $4 \cdot 30 \div 8 \cdot 6 - 5 + 6 + 8 \cdot 10 \div 16 \cdot 6 \quad = 121$
- $2 \cdot 8 \div 2 \div 2 - 5 + 6 - 12 \cdot 10 \div 5 \cdot 6 \quad = -139$
- $2 \cdot 30 \div 5 \cdot 6 - 5 - 11 - 5 \cdot 6 \div 2 \cdot 3 \quad = 11$

Ex. 1.1.2: Find the value of

- $(3 + (4 - 5 \cdot 2)^2)(4 - (7 + 8))$
- $(3 + (4 - 5 \cdot 2))(70 - (7 + 8^2))$
- $(71 - (7 + 8^2))(3 + (4 - (5 \cdot 2)^2))^{12}$
- $(80 - (4 - 5 \cdot 2 + 3)^2)$

Ex. 1.1.1: Use E(MD)(AS) to evaluate

- $2 \cdot 30 \div 5 \cdot 6 - 5 + 6 + 5 \cdot 10 \div 5 \cdot 6 \quad = 133$
- $4 \cdot 30 \div 8 \cdot 6 - 5 + 6 + 8 \cdot 10 \div 16 \cdot 6 \quad = 121$
- $2 \cdot 8 \div 2 \div 2 - 5 + 6 - 12 \cdot 10 \div 5 \cdot 6 \quad = -139$
- $2 \cdot 30 \div 5 \cdot 6 - 5 - 11 - 5 \cdot 6 \div 2 \cdot 3 \quad = 11$

Ex. 1.1.2: Find the value of

- $(3 + (4 - 5 \cdot 2)^2)(4 - (7 + 8)) \quad = -429$
- $(3 + (4 - 5 \cdot 2))(70 - (7 + 8^2)) \quad = 3$
- $(71 - (7 + 8^2))(3 + (4 - (5 \cdot 2)^2))^{12} \quad = 0$
- $(80 - (4 - 5 \cdot 2 + 3)^2) \quad = 71$

Ex. 1.1.1: Use E(MD)(AS) to evaluate

- $2 \cdot 30 \div 5 \cdot 6 - 5 + 6 + 5 \cdot 10 \div 5 \cdot 6 \quad = 133$
- $4 \cdot 30 \div 8 \cdot 6 - 5 + 6 + 8 \cdot 10 \div 16 \cdot 6 \quad = 121$
- $2 \cdot 8 \div 2 \div 2 - 5 + 6 - 12 \cdot 10 \div 5 \cdot 6 \quad = -139$
- $2 \cdot 30 \div 5 \cdot 6 - 5 - 11 - 5 \cdot 6 \div 2 \cdot 3 \quad = 11$

Ex. 1.1.2: Find the value of

- $(3 + (4 - 5 \cdot 2)^2)(4 - (7 + 8)) \quad = -429$
- $(3 + (4 - 5 \cdot 2))(70 - (7 + 8^2)) \quad = 3$
- $(71 - (7 + 8^2))(3 + (4 - (5 \cdot 2)^2))^{12} \quad = 0$
- $(80 - (4 - 5 \cdot 2 + 3)^2) \quad = 71$

Ex. 1.1.3: Find

- $z = ax^3 + by$ if $a = 2, x = -3, b = 7, y = -5.$
- $z = ax^3 - by$ if $a = -3, x = -3, b = 7, y = -5.$
- $z = ax^2 + by$ if $a = 4, x = -3, b = 7, y = -5.$
- $z = ax^2 - by$ if $a = -2, x = -3, b = 7, y = -5.$

Section 1.1 Review: Numbers and Expressions

Ex. 1.1.1: Use E(MD)(AS) to evaluate

- $2 \cdot 30 \div 5 \cdot 6 - 5 + 6 + 5 \cdot 10 \div 5 \cdot 6 \quad = 133$
- $4 \cdot 30 \div 8 \cdot 6 - 5 + 6 + 8 \cdot 10 \div 16 \cdot 6 \quad = 121$
- $2 \cdot 8 \div 2 \div 2 - 5 + 6 - 12 \cdot 10 \div 5 \cdot 6 \quad = -139$
- $2 \cdot 30 \div 5 \cdot 6 - 5 - 11 - 5 \cdot 6 \div 2 \cdot 3 \quad = 11$

Ex. 1.1.2: Find the value of

- $(3 + (4 - 5 \cdot 2)^2)(4 - (7 + 8)) \quad = -429$
- $(3 + (4 - 5 \cdot 2))(70 - (7 + 8^2)) \quad = 3$
- $(71 - (7 + 8^2))(3 + (4 - (5 \cdot 2)^2))^{12} \quad = 0$
- $(80 - (4 - 5 \cdot 2 + 3)^2) \quad = 71$

Ex. 1.1.3: Find

- $z = ax^3 + by$ if $a = 2, x = -3, b = 7, y = -5. \quad \Rightarrow z = -89$
- $z = ax^3 - by$ if $a = -3, x = -3, b = 7, y = -5. \quad \Rightarrow z = 116$
- $z = ax^2 + by$ if $a = 4, x = -3, b = 7, y = -5. \quad \Rightarrow z = 1$
- $z = ax^2 - by$ if $a = -2, x = -3, b = 7, y = -5. \quad \Rightarrow z = 17$

Ex. 1.1.1: Use E(MD)(AS) to evaluate

- $2 \cdot 30 \div 5 \cdot 6 - 5 + 6 + 5 \cdot 10 \div 5 \cdot 6 \quad = 133$
- $4 \cdot 30 \div 8 \cdot 6 - 5 + 6 + 8 \cdot 10 \div 16 \cdot 6 \quad = 121$
- $2 \cdot 8 \div 2 \div 2 - 5 + 6 - 12 \cdot 10 \div 5 \cdot 6 \quad = -139$
- $2 \cdot 30 \div 5 \cdot 6 - 5 - 11 - 5 \cdot 6 \div 2 \cdot 3 \quad = 11$

Ex. 1.1.2: Find the value of

- $(3 + (4 - 5 \cdot 2)^2)(4 - (7 + 8)) \quad = -429$
- $(3 + (4 - 5 \cdot 2))(70 - (7 + 8^2)) \quad = 3$
- $(71 - (7 + 8^2))(3 + (4 - (5 \cdot 2)^2))^{12} \quad = 0$
- $(80 - (4 - 5 \cdot 2 + 3)^2) \quad = 71$

Ex. 1.1.3: Find

- $z = ax^3 + by$ if $a = 2, x = -3, b = 7, y = -5. \quad \Rightarrow z = -89$
- $z = ax^3 - by$ if $a = -3, x = -3, b = 7, y = -5. \quad \Rightarrow z = 116$
- $z = ax^2 + by$ if $a = 4, x = -3, b = 7, y = -5. \quad \Rightarrow z = 1$
- $z = ax^2 - by$ if $a = -2, x = -3, b = 7, y = -5. \quad \Rightarrow z = 17$

Ex. 1.1.4: Evaluate

- $z = x - ab - (b - ax^3)$ if $a = 2, x = -3, b = 7, y = -5.$
- $z = y - ab - (b - ax^2)$ if $a = -2, x = -3, b = 7, y = -5.$
- $z = x - ab - 3(b - ax^3)$ if $a = 2, x = -2, b = -3, y = -5.$
- $z = x + ab - 2(b - ax^2)$ if $a = -2, x = -3, b = -1, y = -5.$

Ex. 1.1.1: Use E(MD)(AS) to evaluate

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- $4 \cdot 30 \div 8 \cdot 6 - 5 + 6 + 8 \cdot 10 \div 16 \cdot 6 \quad = 121$
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- $z = ax^2 + by$ if $a = 4, x = -3, b = 7, y = -5. \quad \Rightarrow z = 1$
- $z = ax^2 - by$ if $a = -2, x = -3, b = 7, y = -5. \quad \Rightarrow z = 17$

Ex. 1.1.4: Evaluate

- $z = x - ab - (b - ax^3)$ if $a = 2, x = -3, b = 7, y = -5. \quad \Rightarrow z = -78$
- $z = y - ab - (b - ax^2)$ if $a = -2, x = -3, b = 7, y = -5. \quad \Rightarrow z = -16$
- $z = x - ab - 3(b - ax^3)$ if $a = 2, x = -2, b = -3, y = -5. \quad \Rightarrow z = -35$
- $z = x + ab - 2(b - ax^2)$ if $a = -2, x = -3, b = -1, y = -5. \quad \Rightarrow z = -35$

Ex. 1.1.5: Rewrite using power rules:

- 12^0
- 0^{12}
- $(4^2)^3$
- $4^3 \cdot 4^2$
- $3^2 \cdot 4^2$
- $2^3 \cdot 3^3 \cdot 4^3$
- $2^3 \cdot 2^3 \cdot 2^4$

Ex. 1.1.5: Rewrite using power rules:

- 12^0 $= 1$
- 0^{12} $= 0$
- $(4^2)^3$ $= 4^6$
- $4^3 \cdot 4^2$ $= 4^5$
- $3^2 \cdot 4^2$ $= 12^2$
- $2^3 \cdot 3^3 \cdot 4^3$ $= 24^3$
- $2^3 \cdot 2^3 \cdot 2^4$ $= 2^{10}$

Ex. 1.1.5: Rewrite using power rules:

- 12^0 $= 1$
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- $4^3 \cdot 4^2$ $= 4^5$
- $3^2 \cdot 4^2$ $= 12^2$
- $2^3 \cdot 3^3 \cdot 4^3$ $= 24^3$
- $2^3 \cdot 2^3 \cdot 2^4$ $= 2^{10}$
- $3^3 \cdot (-4)^3$
- $(1^2)^3$
- $(2^3)^2$
- $2^{(3^2)}$
- $(a^5)^3$
- $(-3)^b \cdot (-4)^b$
- $7^2 \cdot (-3)^2$

Ex. 1.1.5: Rewrite using power rules:

- $12^0 = 1$
- $0^{12} = 0$
- $(4^2)^3 = 4^6$
- $4^3 \cdot 4^2 = 4^5$
- $3^2 \cdot 4^2 = 12^2$
- $2^3 \cdot 3^3 \cdot 4^3 = 24^3$
- $2^3 \cdot 2^3 \cdot 2^4 = 2^{10}$
- $3^3 \cdot (-4)^3 = (-12)^3$
- $(1^2)^3 = 1^6$
- $(2^3)^2 = 2^6$
- $2^{(3^2)} = 2^9$
- $(a^5)^3 = a^{15}$
- $(-3)^b \cdot (-4)^b = 12^b$
- $7^2 \cdot (-3)^2 = (-21)^2$

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- $3^2 \cdot 4^2 = 12^2$
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- $2^3 \cdot 2^3 \cdot 2^4 = 2^{10}$
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- $(1^2)^3 = 1^6$
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- $(-3)^b \cdot (-4)^b = 12^b$
- $7^2 \cdot (-3)^2 = (-21)^2$

Ex. 1.1.6: Rewrite each sum or product as a monomial. Remember to first alphabetize each letter part.

- $7x^2 \cdot 2x^3 =$
- $7xyx \cdot yz =$
- $3x^2y + 6x^2y =$
- $-7x^2 + 2x^2 =$
- $7xyxy^3 \cdot y^5z =$
- $3x^2y + 16x^2y =$
- $7s^2 \cdot 2s^3 =$
- $ssts \cdot stz =$
- $3s^2sts^4 + 6s^2st =$
- $7a^2 \cdot 2b^3 =$
- $ssta \cdot sta =$
- $(asa)^4(3sas) =$
- $7x^3 \cdot 2y^3x^2y^4 =$
- $-8x^3 \cdot -3y^3x^6 =$
- $5x^2 - 7x^2 =$
- $5x^3 \cdot 2y^3x^2y^4 =$
- $-8x^3 \cdot y^{30}x^8 =$
- $-5x^2 - 7x^2 =$
- $7s^3 \cdot st^3s^2st^4 =$
- $-8s^3 \cdot st^3s^8 =$
- $5s^4t - 7s^2ts^2 =$
- $7a^3 \cdot at^3s^2st^4a =$
- $-8s^3 \cdot at^3a^8 =$
- $as^6a(3s^6as)^3 =$
- $x^3y^3x^2 \cdot y^4 =$
- $xy + 3yx =$
- $7xyx + 8yx^2 + 9x^2y =$
- $x^3y^6x^2 \cdot y^4 =$
- $x^2y^2 + 3yxyx =$
- $-7xyx + 8yx^2 + 9x^2y =$
- $s^3st^3s^8 \cdot s^7t^4 =$
- $sst + 3sts =$
- $7ssts^3 + 8sts^4 + 9s^4st =$
- $s^3sa^3s^2 \cdot sta^4 =$
- $asa(3sas)^2 =$
- $a^5s^{10}a(3s^4as) =$

Ex. 1.1.5: Rewrite using power rules:

- $12^0 = 1$
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- $(4^2)^3 = 4^6$
- $4^3 \cdot 4^2 = 4^5$
- $3^2 \cdot 4^2 = 12^2$
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- $(1^2)^3 = 1^6$
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Ex. 1.1.6: Rewrite each sum or product as a monomial. Remember to first alphabetize each letter part.

- $7x^2 \cdot 2x^3 = 14x^5$
- $7xyx \cdot yz =$
- $3x^2y + 6x^2y =$
- $-7x^2 + 2x^2 =$
- $7xyxy^3 \cdot y^5z =$
- $3x^2y + 16x^2y =$
- $7s^2 \cdot 2s^3 =$
- $ssts \cdot stz =$
- $3s^2sts^4 + 6s^2st =$
- $7a^2 \cdot 2b^3 =$
- $ssta \cdot sta =$
- $(asa)^4(3sas) =$
- $7x^3 \cdot 2y^3x^2y^4 = 14x^5y^7$
- $-8x^3 \cdot -3y^3x^6 =$
- $5x^2 - 7x^2 =$
- $5x^3 \cdot 2y^3x^2y^4 =$
- $-8x^3 \cdot y^{30}x^8 =$
- $-5x^2 - 7x^2 =$
- $7s^3 \cdot st^3s^2st^4 =$
- $-8s^3 \cdot st^3s^8 =$
- $5s^4t - 7s^2ts^2 =$
- $7a^3 \cdot at^3s^2st^4a =$
- $-8s^3 \cdot at^3a^8 =$
- $as^6a(3s^6as)^3 =$
- $x^3y^3x^2 \cdot y^4 = x^5y^7$
- $xy + 3yx =$
- $7xyx + 8yx^2 + 9x^2y =$
- $x^3y^6x^2 \cdot y^4 =$
- $x^2y^2 + 3yxyx =$
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- $4^3 \cdot 4^2 = 4^5$
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Ex. 1.1.6: Rewrite each sum or product as a monomial. Remember to first alphabetize each letter part.

- $7x^2 \cdot 2x^3 = 14x^5$
- $7xyx \cdot yz = 7x^2y^2z$
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- $7x^3 \cdot 2y^3x^2y^4 = 14x^5y^7$
- $-8x^3 \cdot -3y^3x^6 = 24x^9y^3$
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- $7x^3 \cdot 2y^3x^2y^4 = 14x^5y^7$
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- $5x^3 \cdot 2y^3x^2y^4 =$
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- $12^0 = 1$
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- $s^3sa^3s^2 \cdot sta^4 =$
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Ex. 1.1.5: Rewrite using power rules:

- $12^0 = 1$
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- $7s^2 \cdot 2s^3 =$
- $ssts \cdot stz =$
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- $5x^3 \cdot 2y^3x^2y^4 = 10^5y^7$
- $-8x^3 \cdot y^{30}x^8 = -8x^{11}y^{30}$
- $-5x^2 - 7x^2 =$
- $7s^3 \cdot st^3s^2st^4 =$
- $-8s^3 \cdot st^3s^8 =$
- $5s^4t - 7s^2ts^2 =$
- $7a^3 \cdot at^3s^2st^4a =$
- $-8s^3 \cdot at^3a^8 =$
- $as^6a(3s^6as)^3 =$
- $x^3y^3x^2 \cdot y^4 = x^5y^7$
- $xy + 3yx = 4xy$
- $7xyx + 8yx^2 + 9x^2y = 24x^2y$
- $x^3y^6x^2 \cdot y^4 = x^5y^{10}$
- $x^2y^2 + 3yxyx = 4x^2y^2$
- $-7xyx + 8yx^2 + 9x^2y =$
- $s^3st^3s^8 \cdot s^7t^4 =$
- $sst + 3sts =$
- $7ssts^3 + 8sts^4 + 9s^4st =$
- $s^3sa^3s^2 \cdot sta^4 =$
- $asa(3sas)^2 =$
- $a^5s^{10}a(3s^4as) =$

Ex. 1.1.5: Rewrite using power rules:

- $12^0 = 1$
- $0^{12} = 0$
- $(4^2)^3 = 4^6$
- $4^3 \cdot 4^2 = 4^5$
- $3^2 \cdot 4^2 = 12^2$
- $2^3 \cdot 3^3 \cdot 4^3 = 24^3$
- $2^3 \cdot 2^3 \cdot 2^4 = 2^{10}$
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- $(1^2)^3 = 1^6$
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- $-7x^2 + 2x^2 = -5x^2$
- $7xyxy^3 \cdot y^5z = 7x^2y^9z$
- $3x^2y + 16x^2y = 19x^2y$
- $7s^2 \cdot 2s^3 =$
- $ssts \cdot stz =$
- $3s^2sts^4 + 6s^2st =$
- $7a^2 \cdot 2b^3 =$
- $ssta \cdot sta =$
- $(asa)^4(3sas) =$
- $7x^3 \cdot 2y^3x^2y^4 = 14x^5y^7$
- $-8x^3 \cdot -3y^3x^6 = 24x^9y^3$
- $5x^2 - 7x^2 = -2x^2$
- $5x^3 \cdot 2y^3x^2y^4 = 10^5y^7$
- $-8x^3 \cdot y^{30}x^8 = -8x^{11}y^{30}$
- $-5x^2 - 7x^2 = -12x^2$
- $7s^3 \cdot st^3s^2st^4 =$
- $-8s^3 \cdot st^3s^8 =$
- $5s^4t - 7s^2ts^2 =$
- $7a^3 \cdot at^3s^2st^4a =$
- $-8s^3 \cdot at^3a^8 =$
- $as^6a(3s^6as)^3 =$
- $x^3y^3x^2 \cdot y^4 = x^5y^7$
- $xy + 3yx = 4xy$
- $7xyx + 8yx^2 + 9x^2y = 24x^2y$
- $x^3y^6x^2 \cdot y^4 = x^5y^{10}$
- $x^2y^2 + 3yxyx = 4x^2y^2$
- $-7xyx + 8yx^2 + 9x^2y = 10x^2y$
- $s^3st^3s^8 \cdot s^7t^4 =$
- $sst + 3sts =$
- $7ssts^3 + 8sts^4 + 9s^4st =$
- $s^3sa^3s^2 \cdot sta^4 =$
- $asa(3sas)^2 =$
- $a^5s^{10}a(3s^4as) =$

Ex. 1.1.5: Rewrite using power rules:

- $12^0 = 1$
- $0^{12} = 0$
- $(4^2)^3 = 4^6$
- $4^3 \cdot 4^2 = 4^5$
- $3^2 \cdot 4^2 = 12^2$
- $2^3 \cdot 3^3 \cdot 4^3 = 24^3$
- $2^3 \cdot 2^3 \cdot 2^4 = 2^{10}$
- $3^3 \cdot (-4)^3 = (-12)^3$
- $(1^2)^3 = 1^6$
- $(2^3)^2 = 2^6$
- $2^{(3^2)} = 2^9$
- $(a^5)^3 = a^{15}$
- $(-3)^b \cdot (-4)^b = 12^b$
- $7^2 \cdot (-3)^2 = (-21)^2$

Ex. 1.1.6: Rewrite each sum or product as a monomial. Remember to first alphabetize each letter part.

- $7x^2 \cdot 2x^3 = 14x^5$
- $7xyx \cdot yz = 7x^2y^2z$
- $3x^2y + 6x^2y = 9x^2y$
- $-7x^2 + 2x^2 = -5x^2$
- $7xyxy^3 \cdot y^5z = 7x^2y^9z$
- $3x^2y + 16x^2y = 19x^2y$
- $7s^2 \cdot 2s^3 = 14s^5$
- $ssts \cdot stz =$
- $3s^2sts^4 + 6s^2st =$
- $7a^2 \cdot 2b^3 =$
- $ssta \cdot sta =$
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- $7s^3 \cdot st^3s^2st^4 = 7s^7t^7$
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- $sst + 3sts =$
- $7ssts^3 + 8sts^4 + 9s^4st =$
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- $7xyxy^3 \cdot y^5z = 7x^2y^9z$
- $3x^2y + 16x^2y = 19x^2y$
- $7s^2 \cdot 2s^3 = 14s^5$
- $ssts \cdot stz = s^4t^2z$
- $3s^2sts^4 + 6s^2st =$
- $7a^2 \cdot 2b^3 =$
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- $7a^3 \cdot at^3s^2st^4a =$
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- $s^3st^3s^8 \cdot s^7t^4 = s^{19}t^7$
- $sst + 3sts = 4s^2t$
- $7ssts^3 + 8sts^4 + 9s^4st =$
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- $7xyxy^3 \cdot y^5z = 7x^2y^9z$
- $3x^2y + 16x^2y = 19x^2y$
- $7s^2 \cdot 2s^3 = 14s^5$
- $ssts \cdot stz = s^4t^2z$
- $3s^2sts^4 + 6s^2st = 3s^7t + 6s^3t$
- $7a^2 \cdot 2b^3 =$
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- $7ssts^3 + 8sts^4 + 9s^4st = 24s^5t$
- $s^3sa^3s^2 \cdot sta^4 = a^7s^7t$
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- $(1^2)^3 = 1^6$
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- $xy + 3yx = 4xy$
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- $s^3st^3s^8 \cdot s^7t^4 = s^{19}t^7$
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- $5x^3 \cdot 2y^3x^2y^4 = 10^5y^7$
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- $s^3st^3s^8 \cdot s^7t^4 = s^{19}t^7$
- $sst + 3sts = 4s^2t$
- $7ssts^3 + 8sts^4 + 9s^4st = 24s^5t$
- $s^3sa^3s^2 \cdot sta^4 = a^7s^7t$
- $asa(3sas)^2 = 9a^4s^5$
- $a^5s^{10}a(3s^4as) = 3a^7s^{15}$

Ex. 1.1.7: Rewrite as a polynomial :

- $P + Q$ if $P = x^2 + 7x + 3$ and $Q = x - x^3 + 9$ =
- $P - 2Q$ if $P = x^2 + 7x + 3$ and $Q = x - x^3 + 9$ =
- $P + Q$ if P $= 5x^2 - 2xy + 7x^2 + 8xy$
  and $Q = 3 - 5x^2 + x^8 + 7x^2 - 15 + 4x^8$ =
- $P - 6Q$ if $P = x^2 + 7x + 3$ and $Q = x - x^3 + 9$ =

Ex. 1.1.5: Rewrite using power rules:

- $12^0 = 1$
- $0^{12} = 0$
- $(4^2)^3 = 4^6$
- $4^3 \cdot 4^2 = 4^5$
- $3^2 \cdot 4^2 = 12^2$
- $2^3 \cdot 3^3 \cdot 4^3 = 24^3$
- $2^3 \cdot 2^3 \cdot 2^4 = 2^{10}$
- $3^3 \cdot (-4)^3 = (-12)^3$
- $(1^2)^3 = 1^6$
- $(2^3)^2 = 2^6$
- $2^{(3^2)} = 2^9$
- $(a^5)^3 = a^{15}$
- $(-3)^b \cdot (-4)^b = 12^b$
- $7^2 \cdot (-3)^2 = (-21)^2$

Ex. 1.1.6: Rewrite each sum or product as a monomial. Remember to first alphabetize each letter part.

- $7x^2 \cdot 2x^3 = 14x^5$
- $7xyx \cdot yz = 7x^2y^2z$
- $3x^2y + 6x^2y = 9x^2y$
- $-7x^2 + 2x^2 = -5x^2$
- $7xyxy^3 \cdot y^5z = 7x^2y^9z$
- $3x^2y + 16x^2y = 19x^2y$
- $7s^2 \cdot 2s^3 = 14s^5$
- $ssts \cdot stz = s^4t^2z$
- $3s^2sts^4 + 6s^2st = 3s^7t + 6s^3t$
- $7a^2 \cdot 2b^3 = 14a^2b^3$
- $ssta \cdot sta = a^2s^3t^2$
- $(asa)^4(3sas) = 3a^9s^6$
- $7x^3 \cdot 2y^3x^2y^4 = 14x^5y^7$
- $-8x^3 \cdot -3y^3x^6 = 24x^9y^3$
- $5x^2 - 7x^2 = -2x^2$
- $5x^3 \cdot 2y^3x^2y^4 = 10^5y^7$
- $-8x^3 \cdot y^{30}x^8 = -8x^{11}y^{30}$
- $-5x^2 - 7x^2 = -12x^2$
- $7s^3 \cdot st^3s^2st^4 = 7s^7t^7$
- $-8s^3 \cdot st^3s^8 = -8s^{12}t^3$
- $5s^4t - 7s^2ts^2 = -2s^4t$
- $7a^3 \cdot at^3s^2st^4a = 7a^5s^3t^7$
- $-8s^3 \cdot at^3a^8 = -8a^9s^3t^3$
- $as^6a(3s^6as)^3 = 27a^5s^{27}$
- $x^3y^3x^2 \cdot y^4 = x^5y^7$
- $xy + 3yx = 4xy$
- $7xyx + 8yx^2 + 9x^2y = 24x^2y$
- $x^3y^6x^2 \cdot y^4 = x^5y^{10}$
- $x^2y^2 + 3yxyx = 4x^2y^2$
- $-7xyx + 8yx^2 + 9x^2y = 10x^2y$
- $s^3st^3s^8 \cdot s^7t^4 = s^{19}t^7$
- $sst + 3sts = 4s^2t$
- $7ssts^3 + 8sts^4 + 9s^4st = 24s^5t$
- $s^3sa^3s^2 \cdot sta^4 = a^7s^7t$
- $asa(3sas)^2 = 9a^4s^5$
- $a^5s^{10}a(3s^4as) = 3a^7s^{15}$

Ex. 1.1.7: Rewrite as a polynomial :

- $P + Q$ if $P = x^2 + 7x + 3$ and $Q = x - x^3 + 9$ $= -x^3 + x^2 + 8x + 12$
- $P - 2Q$ if $P = x^2 + 7x + 3$ and $Q = x - x^3 + 9$ $= 2x^3 + x^2 + 5x - 15$
- $P + Q$ if P $= 5x^2 - 2xy + 7x^2 + 8xy$
  and $Q = 3 - 5x^2 + x^8 + 7x^2 - 15 + 4x^8$ $= 5x^8 + 14x^2 + 6xy - 12$
- $P - 6Q$ if $P = x^2 + 7x + 3$ and $Q = x - x^3 + 9$ $= 6x^3 + x^2 + x - 51$

Ex. 1.1.8: Rewrite each expression without parentheses:

- $-(4 - x + y - 7z) =$
- $x - 3 - (2y - 3z + 3b - 4) =$
- $-(4 - 3(x + y - 7)z) =$
- $x - 3 - (2y - 3(z + 3) - b - 4) =$
- $4 - 3(x - y - 7)z =$
- $x - 3 - (-3z + 3(b - 4)) =$
- $-(4 - x - (2y - 7)z) =$
- $x - 3 - (2y - 3z) + (3b - 4) =$

Ex. 1.1.8: Rewrite each expression without parentheses:

- $-(4-x+y-7z)$ $= x-y+7z-4$
- $x-3-(2y-3z+3b-4)$ $= -3b+x-2y+3z+1$
- $-(4-3(x+y-7)z)$ $= 3xz+3yz-21z-4$
- $x-3-(2y-3(z+3)-b-4)$ $= b+x-2y+3z+10$
- $4-3(x-y-7)z$ $= -3xz+3yz+21z+4$
- $x-3-(-3z+3(b-4))$ $= -3b+x+3z+9$
- $-(4-x-(2y-7)z)$ $= x+2yz-7z-4$
- $x-3-(2y-3z)+(3b-4)$ $= 3b+x-2y+3z-7$

Ex. 1.1.8: Rewrite each expression without parentheses:

- $-(4-x+y-7z) \quad = x-y+7z-4$
- $x-3-(2y-3z+3b-4) \quad = -3b+x-2y+3z+1$
- $-(4-3(x+y-7)z) \quad = 3xz+3yz-21z-4$
- $x-3-(2y-3(z+3)-b-4) \quad = b+x-2y+3z+10$
- $4-3(x-y-7)z \quad = -3xz+3yz+21z+4$
- $x-3-(-3z+3(b-4)) \quad = -3b+x+3z+9$
- $-(4-x-(2y-7)z) \quad = x+2yz-7z-4$
- $x-3-(2y-3z)+(3b-4) \quad = 3b+x-2y+3z-7$

Ex. 1.1.9: Rewrite each difference as a polynomial:

- $(x^2+2x)-(2x^2-3x) \quad =$
- $(x^2+2x)-(x^2-3x) \quad =$
- $(x^3x^2+2x)-(2x^2-x^3-3x) \quad =$
- $(x^2+2x^2)-(2x^2-3x^2) \quad =$

Ex. 1.1.8: Rewrite each expression without parentheses:

- $-(4 - x + y - 7z)$ $= x - y + 7z - 4$
- $x - 3 - (2y - 3z + 3b - 4)$ $= -3b + x - 2y + 3z + 1$
- $-(4 - 3(x + y - 7)z)$ $= 3xz + 3yz - 21z - 4$
- $x - 3 - (2y - 3(z + 3) - b - 4)$ $= b + x - 2y + 3z + 10$
- $4 - 3(x - y - 7)z$ $= -3xz + 3yz + 21z + 4$
- $x - 3 - (-3z + 3(b - 4))$ $= -3b + x + 3z + 9$
- $-(4 - x - (2y - 7)z)$ $= x + 2yz - 7z - 4$
- $x - 3 - (2y - 3z) + (3b - 4)$ $= 3b + x - 2y + 3z - 7$

Ex. 1.1.9: Rewrite each difference as a polynomial:

- $(x^2 + 2x) - (2x^2 - 3x)$ $= -x^2 + 5x$
- $(x^2 + 2x) - (x^2 - 3x)$ $= 5x$
- $(x^3x^2 + 2x) - (2x^2 - x^3 - 3x)$ $= x^5 + x^3 - 2x^2 + 5x$
- $(x^2 + 2x^2) - (2x^2 - 3x^2)$ $= 4x^2$

Ex. 1.1.8: Rewrite each expression without parentheses:

- $-(4 - x + y - 7z) = x - y + 7z - 4$
- $x - 3 - (2y - 3z + 3b - 4) = -3b + x - 2y + 3z + 1$
- $-(4 - 3(x + y - 7)z) = 3xz + 3yz - 21z - 4$
- $x - 3 - (2y - 3(z + 3) - b - 4) = b + x - 2y + 3z + 10$
- $4 - 3(x - y - 7)z = -3xz + 3yz + 21z + 4$
- $x - 3 - (-3z + 3(b - 4)) = -3b + x + 3z + 9$
- $-(4 - x - (2y - 7)z) = x + 2yz - 7z - 4$
- $x - 3 - (2y - 3z) + (3b - 4) = 3b + x - 2y + 3z - 7$

Ex. 1.1.9: Rewrite each difference as a polynomial:

- $(x^2 + 2x) - (2x^2 - 3x) = -x^2 + 5x$
- $(x^2 + 2x) - (x^2 - 3x) = 5x$
- $(x^3x^2 + 2x) - (2x^2 - x^3 - 3x) = x^5 + x^3 - 2x^2 + 5x$
- $(x^2 + 2x^2) - (2x^2 - 3x^2) = 4x^2$

Ex. 1.1.10: Rewrite each product as a polynomial:

- $3(5x - 3x^2 + 1) =$
- $(2x^3)(x^2 - 5x + 4) =$
- $(-2x^2y)(x^2y - 5x + 4y^2) =$
- $-2(5x - (3x^2 + 1)) =$
- $(-x^3)(x^2 - 3x + 4) =$
- $(-2x^2y^3)(x^2y - (5x + 4y^2)) =$
- $3x(5x - 3x^2 + 1) =$
- $((2x)^3)(x^2 - 5x + 4) =$
- $(-2x^2y)(x^2y^2 - 5x + 4y^2) =$

Ex. 1.1.8: Rewrite each expression without parentheses:

- $-(4-x+y-7z)$ $= x-y+7z-4$
- $x-3-(2y-3z+3b-4)$ $= -3b+x-2y+3z+1$
- $-(4-3(x+y-7)z)$ $= 3xz+3yz-21z-4$
- $x-3-(2y-3(z+3)-b-4)$ $= b+x-2y+3z+10$
- $4-3(x-y-7)z$ $= -3xz+3yz+21z+4$
- $x-3-(-3z+3(b-4))$ $= -3b+x+3z+9$
- $-(4-x-(2y-7)z)$ $= x+2yz-7z-4$
- $x-3-(2y-3z)+(3b-4)$ $= 3b+x-2y+3z-7$

Ex. 1.1.9: Rewrite each difference as a polynomial:

- $(x^2+2x)-(2x^2-3x)$ $= -x^2+5x$
- $(x^2+2x)-(x^2-3x)$ $= 5x$
- $(x^3x^2+2x)-(2x^2-x^3-3x)$ $= x^5+x^3-2x^2+5x$
- $(x^2+2x^2)-(2x^2-3x^2)$ $= 4x^2$

Ex. 1.1.10: Rewrite each product as a polynomial:

- $3(5x-3x^2+1)$ $= -9x^2+15x+3$
- $(2x^3)(x^2-5x+4)$ $= 2x^5-10x^4+8x^3$
- $(-2x^2y)(x^2y-5x+4y^2)$ $= -2x^4y^2+10x^3y-8x^2y^3$
- $-2(5x-(3x^2+1))$ $= 6x^2-10x+2$
- $(-x^3)(x^2-3x+4)$ $= -x^5+3x^4-4x^3$
- $(-2x^2y^3)(x^2y-(5x+4y^2))$ $= -2x^4y^4+10x^3y^3+8x^2y^5$
- $3x(5x-3x^2+1)$ $= -9x^3+15x^2+3x$
- $((2x)^3)(x^2-5x+4)$ $= 8x^5-40x^4+32x^3$
- $(-2x^2y)(x^2y^2-5x+4y^2)$ $= -2x^4y^3+10x^3y-8x^2y^3$

Ex. 1.1.10 (continued): Rewrite each product as a polynomial.

- $-2x(5x - 3x^2 + 1) \quad =$
- $x^4(2x^5)(5x - (x^2 + 4)) \quad =$
- $(-2x^2y)(x^2y - 5xyx + 4yx^2) \quad =$

Ex. 1.1.10 (continued): Rewrite each product as a polynomial.

- $-2x(5x - 3x^2 + 1) \quad = \quad 6x^3 - 10x^2 - 2x$
- $x^4(2x^5)(5x - (x^2 + 4)) \quad = \quad -2x^{11} + 10x^{10} - 8x^9$
- $(-2x^2y)(x^2y - 5xyx + 4yx^2) \quad = \quad 0$

Ex. 1.1.10 (continued): Rewrite each product as a polynomial.

- $-2x(5x - 3x^2 + 1) = 6x^3 - 10x^2 - 2x$
- $x^4(2x^5)(5x - (x^2 + 4)) = -2x^{11} + 10x^{10} - 8x^9$
- $(-2x^2y)(x^2y - 5xyx + 4yx^2) = 0$

Ex. 1.1.11: Rewrite each expression as a polynomial.

- $x^2(x + x^3) =$
- $(3x + 2)(2x + 5) =$
- $(3x^4 + 10)(3x^4 - 10) =$
- $(2x + 3)(x^2 - 5x + 4) =$

Ex. 1.1.10 (continued): Rewrite each product as a polynomial.

- $-2x(5x - 3x^2 + 1) \quad = \quad 6x^3 - 10x^2 - 2x$
- $x^4(2x^5)(5x - (x^2 + 4)) \quad = \quad -2x^{11} + 10x^{10} - 8x^9$
- $(-2x^2y)(x^2y - 5xyx + 4yx^2) \quad = \quad 0$

Ex. 1.1.11: Rewrite each expression as a polynomial.

- $x^2(x + x^3) \quad = \quad x^3 + x^5$
- $(3x + 2)(2x + 5) \quad = \quad 6x^2 + 19x + 10$
- $(3x^4 + 10)(3x^4 - 10) \quad = \quad 9x^8 - 100$
- $(2x + 3)(x^2 - 5x + 4) \quad = \quad 2x^3 - 7x^2 - 7x + 12$

Ex. 1.1.10 (continued): Rewrite each product as a polynomial.

- $-2x(5x-3x^2+1) = 6x^3-10x^2-2x$
- $x^4(2x^5)(5x-(x^2+4)) = -2x^{11}+10x^{10}-8x^9$
- $(-2x^2y)(x^2y-5xyx+4yx^2) = 0$

Ex. 1.1.11: Rewrite each expression as a polynomial.

- $x^2(x+x^3) = x^3+x^5$
- $(3x+2)(2x+5) = 6x^2+19x+10$
- $(3x^4+10)(3x^4-10) = 9x^8-100$
- $(2x+3)(x^2-5x+4) = 2x^3-7x^2-7x+12$
- $a^2(a+a^3) =$
- $(3a+2)(2a+5) =$
- $(3a^4+10)(3a^4-10) =$
- $(2a+3)(a^2-5a+4) =$

Ex. 1.1.10 (continued): Rewrite each product as a polynomial.

- $-2x(5x - 3x^2 + 1) = 6x^3 - 10x^2 - 2x$
- $x^4(2x^5)(5x - (x^2 + 4)) = -2x^{11} + 10x^{10} - 8x^9$
- $(-2x^2y)(x^2y - 5xyx + 4yx^2) = 0$

Ex. 1.1.11: Rewrite each expression as a polynomial.

- $x^2(x + x^3) = x^3 + x^5$
- $(3x + 2)(2x + 5) = 6x^2 + 19x + 10$
- $(3x^4 + 10)(3x^4 - 10) = 9x^8 - 100$
- $(2x + 3)(x^2 - 5x + 4) = 2x^3 - 7x^2 - 7x + 12$
- $a^2(a + a^3) = a^5 + a^3$
- $(3a + 2)(2a + 5) = 6a^2 + 19a + 10$
- $(3a^4 + 10)(3a^4 - 10) = 9a^8 - 100$
- $(2a + 3)(a^2 - 5a + 4) = 2a^3 - 7a^2 - 7a + 12$

Ex. 1.1.10 (continued): Rewrite each product as a polynomial.

- $-2x(5x-3x^2+1) = 6x^3-10x^2-2x$
- $x^4(2x^5)(5x-(x^2+4)) = -2x^{11}+10x^{10}-8x^9$
- $(-2x^2y)(x^2y-5xyx+4yx^2) = 0$

Ex. 1.1.11: Rewrite each expression as a polynomial.

- $x^2(x+x^3) = x^3+x^5$
- $(3x+2)(2x+5) = 6x^2+19x+10$
- $(3x^4+10)(3x^4-10) = 9x^8-100$
- $(2x+3)(x^2-5x+4) = 2x^3-7x^2-7x+12$
- $a^2(a+a^3) = a^5+a^3$
- $(3a+2)(2a+5) = 6a^2+19a+10$
- $(3a^4+10)(3a^4-10) = 9a^8-100$
- $(2a+3)(a^2-5a+4) = 2a^3-7a^2-7a+12$
- $2x^2(7x-x^3) =$
- $(3x-5)(12x+5) =$
- $(3x^7+2)(3x^7-2) =$
- $(3x^7+3)(3x^7-2) =$

Ex. 1.1.10 (continued): Rewrite each product as a polynomial.

- $-2x(5x - 3x^2 + 1) = 6x^3 - 10x^2 - 2x$
- $x^4(2x^5)(5x - (x^2 + 4)) = -2x^{11} + 10x^{10} - 8x^9$
- $(-2x^2y)(x^2y - 5xyx + 4yx^2) = 0$

Ex. 1.1.11: Rewrite each expression as a polynomial.

- $x^2(x + x^3) = x^3 + x^5$
- $(3x + 2)(2x + 5) = 6x^2 + 19x + 10$
- $(3x^4 + 10)(3x^4 - 10) = 9x^8 - 100$
- $(2x + 3)(x^2 - 5x + 4) = 2x^3 - 7x^2 - 7x + 12$
- $a^2(a + a^3) = a^5 + a^3$
- $(3a + 2)(2a + 5) = 6a^2 + 19a + 10$
- $(3a^4 + 10)(3a^4 - 10) = 9a^8 - 100$
- $(2a + 3)(a^2 - 5a + 4) = 2a^3 - 7a^2 - 7a + 12$
- $2x^2(7x - x^3) = 14x^3 - 2x^5$
- $(3x - 5)(12x + 5) = 36x^2 - 45x - 25$
- $(3x^7 + 2)(3x^7 - 2) = 9x^{14} - 4$
- $(3x^7 + 3)(3x^7 - 2) = 9x^{14} + 3x^7 - 6$

Ex. 1.1.10 (continued): Rewrite each product as a polynomial.

- $-2x(5x - 3x^2 + 1) = 6x^3 - 10x^2 - 2x$
- $x^4(2x^5)(5x - (x^2 + 4)) = -2x^{11} + 10x^{10} - 8x^9$
- $(-2x^2y)(x^2y - 5xyx + 4yx^2) = 0$

Ex. 1.1.11: Rewrite each expression as a polynomial.

- $x^2(x + x^3) = x^3 + x^5$
- $(3x + 2)(2x + 5) = 6x^2 + 19x + 10$
- $(3x^4 + 10)(3x^4 - 10) = 9x^8 - 100$
- $(2x + 3)(x^2 - 5x + 4) = 2x^3 - 7x^2 - 7x + 12$
- $a^2(a + a^3) = a^5 + a^3$
- $(3a + 2)(2a + 5) = 6a^2 + 19a + 10$
- $(3a^4 + 10)(3a^4 - 10) = 9a^8 - 100$
- $(2a + 3)(a^2 - 5a + 4) = 2a^3 - 7a^2 - 7a + 12$
- $2x^2(7x - x^3) = 14x^3 - 2x^5$
- $(3x - 5)(12x + 5) = 36x^2 - 45x - 25$
- $(3x^7 + 2)(3x^7 - 2) = 9x^{14} - 4$
- $(3x^7 + 3)(3x^7 - 2) = 9x^{14} + 3x^7 - 6$
- $x^2(x + x^3 - x(x^2 + 1)) =$
- $x(3x - 5)(10x + 5) =$
- $(3x^4 + 7x^4)(3x^3 + 10x^3) =$
- $(10b + 3)(b^{10} - 5b + 4) =$

Ex. 1.1.10 (continued): Rewrite each product as a polynomial.

- $-2x(5x-3x^2+1) = 6x^3-10x^2-2x$
- $x^4(2x^5)(5x-(x^2+4)) = -2x^{11}+10x^{10}-8x^9$
- $(-2x^2y)(x^2y-5xyx+4yx^2) = 0$

Ex. 1.1.11: Rewrite each expression as a polynomial.

- $x^2(x+x^3) = x^3+x^5$
- $(3x+2)(2x+5) = 6x^2+19x+10$
- $(3x^4+10)(3x^4-10) = 9x^8-100$
- $(2x+3)(x^2-5x+4) = 2x^3-7x^2-7x+12$
- $a^2(a+a^3) = a^5+a^3$
- $(3a+2)(2a+5) = 6a^2+19a+10$
- $(3a^4+10)(3a^4-10) = 9a^8-100$
- $(2a+3)(a^2-5a+4) = 2a^3-7a^2-7a+12$
- $2x^2(7x-x^3) = 14x^3-2x^5$
- $(3x-5)(12x+5) = 36x^2-45x-25$
- $(3x^7+2)(3x^7-2) = 9x^{14}-4$
- $(3x^7+3)(3x^7-2) = 9x^{14}+3x^7-6$
- $x^2(x+x^3-x(x^2+1)) = 0$
- $x(3x-5)(10x+5) = 30x^3-35x^2-25x$
- $(3x^4+7x^4)(3x^3+10x^3) = 130x^7$
- $(10b+3)(b^{10}-5b+4) = 10b^{11}+3b^{10}-50b^2+25b+12$

Ex. 1.1.10 (continued): Rewrite each product as a polynomial.

- $-2x(5x-3x^2+1) = 6x^3-10x^2-2x$
- $x^4(2x^5)(5x-(x^2+4)) = -2x^{11}+10x^{10}-8x^9$
- $(-2x^2y)(x^2y-5xyx+4yx^2) = 0$

Ex. 1.1.11: Rewrite each expression as a polynomial.

- $x^2(x+x^3) = x^3+x^5$
- $(3x+2)(2x+5) = 6x^2+19x+10$
- $(3x^4+10)(3x^4-10) = 9x^8-100$
- $(2x+3)(x^2-5x+4) = 2x^3-7x^2-7x+12$
- $a^2(a+a^3) = a^5+a^3$
- $(3a+2)(2a+5) = 6a^2+19a+10$
- $(3a^4+10)(3a^4-10) = 9a^8-100$
- $(2a+3)(a^2-5a+4) = 2a^3-7a^2-7a+12$
- $2x^2(7x-x^3) = 14x^3-2x^5$
- $(3x-5)(12x+5) = 36x^2-45x-25$
- $(3x^7+2)(3x^7-2) = 9x^{14}-4$
- $(3x^7+3)(3x^7-2) = 9x^{14}+3x^7-6$
- $x^2(x+x^3-x(x^2+1)) = 0$
- $x(3x-5)(10x+5) = 30x^3-35x^2-25x$
- $(3x^4+7x^4)(3x^3+10x^3) = 130x^7$
- $(10b+3)(b^{10}-5b+4) = 10b^{11}+3b^{10}-50b^2+25b+12$

Ex. 1.1.12: Factor

- $ax^2+ay^3 =$
- $bx^2+axy^3 =$
- $ax^2+ax^3 =$
- $ax^2-ax^7 =$

Ex. 1.1.10 (continued): Rewrite each product as a polynomial.

- $-2x(5x - 3x^2 + 1) = 6x^3 - 10x^2 - 2x$
- $x^4(2x^5)(5x - (x^2 + 4)) = -2x^{11} + 10x^{10} - 8x^9$
- $(-2x^2 y)(x^2 y - 5xyx + 4yx^2) = 0$

Ex. 1.1.11: Rewrite each expression as a polynomial.

- $x^2(x + x^3) = x^3 + x^5$
- $(3x + 2)(2x + 5) = 6x^2 + 19x + 10$
- $(3x^4 + 10)(3x^4 - 10) = 9x^8 - 100$
- $(2x + 3)(x^2 - 5x + 4) = 2x^3 - 7x^2 - 7x + 12$
- $a^2(a + a^3) = a^5 + a^3$
- $(3a + 2)(2a + 5) = 6a^2 + 19a + 10$
- $(3a^4 + 10)(3a^4 - 10) = 9a^8 - 100$
- $(2a + 3)(a^2 - 5a + 4) = 2a^3 - 7a^2 - 7a + 12$
- $2x^2(7x - x^3) = 14x^3 - 2x^5$
- $(3x - 5)(12x + 5) = 36x^2 - 45x - 25$
- $(3x^7 + 2)(3x^7 - 2) = 9x^{14} - 4$
- $(3x^7 + 3)(3x^7 - 2) = 9x^{14} + 3x^7 - 6$
- $x^2(x + x^3 - x(x^2 + 1)) = 0$
- $x(3x - 5)(10x + 5) = 30x^3 - 35x^2 - 25x$
- $(3x^4 + 7x^4)(3x^3 + 10x^3) = 130x^7$
- $(10b + 3)(b^{10} - 5b + 4) = 10b^{11} + 3b^{10} - 50b^2 + 25b + 12$

Ex. 1.1.12: Factor

- $ax^2 + ay^3 = a(x^2 + y^3)$
- $bx^2 + axy^3 = x(bx + ay^3)$
- $ax^2 + ax^3 = ax^2(x + 1)$
- $ax^2 - ax^7 = ax^2(1 - x^5)$

Ex. 1.1.13: Factor each blue sum of terms:

- $ax^8y + bx^6z$ $=$
- $x^5 + 3x^7$ $=$
- $a^5b^8 + a^7b^5$ $=$
- $a^5b^6 + a^6b^5$ $=$
- $(x+1)(x+4) + (x+1)(x+5)$ $=$
- $(x+1)^5(x+4)^6 + (x+1)^6(x+4)^5$ $=$

Ex. 1.1.13: Factor each blue sum of terms:

- $ax^8y + bx^6z = x^6(ax^2y + bz)$
- $x^5 + 3x^7 = x^5(1 + 3x^2)$
- $a^5b^8 + a^7b^5 = a^5b^5(b^3 + a^2)$
- $a^5b^6 + a^6b^5 = a^5b^5(a + b)$
- $(x+1)(x+4) + (x+1)(x+5) = (x+1)(2x+9)$
- $(x+1)^5(x+4)^6 + (x+1)^6(x+4)^5 = (x+1)^5(x+4)^5(2x+5)$

Ex. 1.1.13: Factor each blue sum of terms:

- $ax^8y + bx^6z = x^6(ax^2y + bz)$
- $x^5 + 3x^7 = x^5(1 + 3x^2)$
- $a^5b^8 + a^7b^5 = a^5b^5(b^3 + a^2)$
- $a^5b^6 + a^6b^5 = a^5b^5(a + b)$
- $(x+1)(x+4) + (x+1)(x+5) = (x+1)(2x+9)$
- $(x+1)^5(x+4)^6 + (x+1)^6(x+4)^5 = (x+1)^5(x+4)^5(2x+5)$
- $bx^4y + bx^6z =$
- $3x^7 + x^5 - 7x =$
- $a^5c^8 + a^7c^{15} + c^{10} =$
- $3x^5b^6 + a^6b^5x^{10} =$
- $(x+1)(x+4) + 7(x+1)(x+5) =$
- $(x+1)^5(x+4)^5 + 3(x+1)^6(x+4)^5 =$

Ex. 1.1.13: Factor each blue sum of terms:

- $ax^8y + bx^6z = x^6(ax^2y + bz)$
- $x^5 + 3x^7 = x^5(1 + 3x^2)$
- $a^5b^8 + a^7b^5 = a^5b^5(b^3 + a^2)$
- $a^5b^6 + a^6b^5 = a^5b^5(a + b)$
- $(x+1)(x+4) + (x+1)(x+5) = (x+1)(2x+9)$
- $(x+1)^5(x+4)^6 + (x+1)^6(x+4)^5 = (x+1)^5(x+4)^5(2x+5)$
- $bx^4y + bx^6z = bx^4(x^2z + y)$
- $3x^7 + x^5 - 7x = x(3x^6 + x^4 - 7)$
- $a^5c^8 + a^7c^{15} + c^{10} = c^8(a^7c^7 + a^5 + c^2)$
- $3x^5b^6 + a^6b^5x^{10} = b^5x^5(a^6x^5 + 3b)$
- $(x+1)(x+4) + 7(x+1)(x+5) = (x+1)(8x+39)$
- $(x+1)^5(x+4)^5 + 3(x+1)^6(x+4)^5 = (x+1)^5(x+4)^5(3x+4)$

Ex. 1.1.13: Factor each blue sum of terms:

- $ax^8y + bx^6z \quad = x^6(ax^2y + bz)$
- $x^5 + 3x^7 \quad = x^5(1 + 3x^2)$
- $a^5b^8 + a^7b^5 \quad = a^5b^5(b^3 + a^2)$
- $a^5b^6 + a^6b^5 \quad = a^5b^5(a + b)$
- $(x+1)(x+4) + (x+1)(x+5) \quad = (x+1)(2x+9)$
- $(x+1)^5(x+4)^6 + (x+1)^6(x+4)^5 \quad = (x+1)^5(x+4)^5(2x+5)$
- $bx^4y + bx^6z \quad = bx^4(x^2z + y)$
- $3x^7 + x^5 - 7x \quad = x(3x^6 + x^4 - 7)$
- $a^5c^8 + a^7c^{15} + c^{10} \quad = c^8(a^7c^7 + a^5 + c^2)$
- $3x^5b^6 + a^6b^5x^{10} \quad = b^5x^5(a^6x^5 + 3b)$
- $(x+1)(x+4) + 7(x+1)(x+5) \quad = (x+1)(8x+39)$
- $(x+1)^5(x+4)^5 + 3(x+1)^6(x+4)^5 \quad = (x+1)^5(x+4)^5(3x+4)$
- $ax^8y^8 + bx^6z \quad =$
- $-3x^5 + 4x^7 \quad =$
- $-2x^4b^8 + a^7b^5 \quad =$
- $3a^5b^2 + 6a^6b^5 \quad =$
- $(x+3)(x+4) + (x+1)(x+4) \quad =$
- $(x+2)^5(x+4)^6 + (x+4)^6(x+2)^5 \quad =$

Ex. 1.1.13: Factor each blue sum of terms:

- $ax^8y + bx^6z = x^6(ax^2y + bz)$
- $x^5 + 3x^7 = x^5(1 + 3x^2)$
- $a^5b^8 + a^7b^5 = a^5b^5(b^3 + a^2)$
- $a^5b^6 + a^6b^5 = a^5b^5(a + b)$
- $(x+1)(x+4) + (x+1)(x+5) = (x+1)(2x+9)$
- $(x+1)^5(x+4)^6 + (x+1)^6(x+4)^5 = (x+1)^5(x+4)^5(2x+5)$
- $bx^4y + bx^6z = bx^4(x^2z + y)$
- $3x^7 + x^5 - 7x = x(3x^6 + x^4 - 7)$
- $a^5c^8 + a^7c^{15} + c^{10} = c^8(a^7c^7 + a^5 + c^2)$
- $3x^5b^6 + a^6b^5x^{10} = b^5x^5(a^6x^5 + 3b)$
- $(x+1)(x+4) + 7(x+1)(x+5) = (x+1)(8x+39)$
- $(x+1)^5(x+4)^5 + 3(x+1)^6(x+4)^5 = (x+1)^5(x+4)^5(3x+4)$
- $ax^8y^8 + bx^6z = x^6(ax^2y^8 + bz)$
- $-3x^5 + 4x^7 = x^5(4x^2 - 3)$
- $-2x^4b^8 + a^7b^5 = b^5(a^7 - 2b^3x^4)$
- $3a^5b^2 + 6a^6b^5 = 3a^5b^2(2ab^3 + 1)$
- $(x+3)(x+4) + (x+1)(x+4) = 2(x+4)(x+2)$
- $(x+2)^5(x+4)^6 + (x+4)^6(x+2)^5 = 2(x+2)^5(x+4)^6$

Ex. 1.1.13: Factor each blue sum of terms:

- $ax^8y + bx^6z = x^6(ax^2y + bz)$
- $x^5 + 3x^7 = x^5(1 + 3x^2)$
- $a^5b^8 + a^7b^5 = a^5b^5(b^3 + a^2)$
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- $(x + 1)(x + 4) + (x + 1)(x + 5) = (x + 1)(2x + 9)$
- $(x + 1)^5(x + 4)^6 + (x + 1)^6(x + 4)^5 = (x + 1)^5(x + 4)^5(2x + 5)$
- $bx^4y + bx^6z = bx^4(x^2z + y)$
- $3x^7 + x^5 - 7x = x(3x^6 + x^4 - 7)$
- $a^5c^8 + a^7c^{15} + c^{10} = c^8(a^7c^7 + a^5 + c^2)$
- $3x^5b^6 + a^6b^5x^{10} = b^5x^5(a^6x^5 + 3b)$
- $(x + 1)(x + 4) + 7(x + 1)(x + 5) = (x + 1)(8x + 39)$
- $(x + 1)^5(x + 4)^5 + 3(x + 1)^6(x + 4)^5 = (x + 1)^5(x + 4)^5(3x + 4)$
- $ax^8y^8 + bx^6z = x^6(ax^2y^8 + bz)$
- $-3x^5 + 4x^7 = x^5(4x^2 - 3)$
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- $3a^5b^2 + 6a^6b^5 = 3a^5b^2(2ab^3 + 1)$
- $(x + 3)(x + 4) + (x + 1)(x + 4) = 2(x + 4)(x + 2)$
- $(x + 2)^5(x + 4)^6 + (x + 4)^6(x + 2)^5 = 2(x + 2)^5(x + 4)^6$
- $6ax^8y + bx^6z =$
- $-3x^5 + 3x^6 =$
- $a^5x^8 - 2x^7b^5 =$
- $a^2b^4 + 5a^3b^3 =$
- $(y + 1)(x + 4) + 2(y + 1)(x + 5) =$
- $(x + 1)^5(z + 4)^6 + (z + 4)^6(x + 1)^5 =$

Ex. 1.1.13: Factor each blue sum of terms:

- $ax^8y + bx^6z = x^6(ax^2y + bz)$
- $x^5 + 3x^7 = x^5(1 + 3x^2)$
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- $bx^4y + bx^6z = bx^4(x^2z + y)$
- $3x^7 + x^5 - 7x = x(3x^6 + x^4 - 7)$
- $a^5c^8 + a^7c^{15} + c^{10} = c^8(a^7c^7 + a^5 + c^2)$
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- $(x+1)(x+4) + 7(x+1)(x+5) = (x+1)(8x+39)$
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- $3a^5b^2 + 6a^6b^5 = 3a^5b^2(2ab^3 + 1)$
- $(x+3)(x+4) + (x+1)(x+4) = 2(x+4)(x+2)$
- $(x+2)^5(x+4)^6 + (x+4)^6(x+2)^5 = 2(x+2)^5(x+4)^6$
- $6ax^8y + bx^6z = x^6(6ax^2y + bz)$
- $-3x^5 + 3x^6 = 3x^5(x - 1)$
- $a^5x^8 - 2x^7b^5 = x^7(a^5x - 2b^5)$
- $a^2b^4 + 5a^3b^3 = a^2b^3(5a + b)$
- $(y+1)(x+4) + 2(y+1)(x+5) = (3x+14)(y+1)$
- $(x+1)^5(z+4)^6 + (z+4)^6(x+1)^5 = 2(x+1)^5(z+4)^6$

Ex. 1.1.14: Apply the sign switch identity. Then rewrite your answer as a polynomial.

- $-(x-4) =$
- $-(3x-2y) =$
- $-(4-2x) =$
- $-(3a-2b) =$

Ex. 1.1.14: Apply the sign switch identity. Then rewrite your answer as a polynomial.

- $-(x-4) = 4-x$
- $-(3x-2y) = -3x+2y$
- $-(4-2x) = 2x-4$
- $-(3a-2b) = -3a+2b$

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- $-(b^2-4) =$
- $-(3xy-2yz) =$
- $-(x-y-z) =$
- $-(ab-cd) =$

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- $-(3xy-2yz) = 2yz-3xy$
- $-(x-y-z) = z-(x-y) = -x+y+z$
- $-(ab-cd) = cd-ab$

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- $-(3xy-2yz) = 2yz-3xy$
- $-(x-y-z) = z-(x-y) = -x+y+z$
- $-(ab-cd) = cd-ab$

Ex. 1.1.15: Find the GCF of

- $40, 60, 320 =$
- $30, 60, 320 =$
- $45, 60, 300 =$
- $16, 24, 320 =$

Ex. 1.1.14: Apply the sign switch identity. Then rewrite your answer as a polynomial.

- $-(x-4) = 4-x$
- $-(3x-2y) = -3x+2y$
- $-(4-2x) = 2x-4$
- $-(3a-2b) = -3a+2b$
- $-(b^2-4) = 4-b^2$
- $-(3xy-2yz) = 2yz-3xy$
- $-(x-y-z) = z-(x-y) = -x+y+z$
- $-(ab-cd) = cd-ab$

Ex. 1.1.15: Find the GCF of • $40, 60, 320 = 20$ • $30, 60, 320 = 10$ • $45, 60, 300 = 15$ • $16, 24, 320 = 8$

Ex. 1.1.14: Apply the sign switch identity. Then rewrite your answer as a polynomial.

- $-(x-4) = \ 4-x$
- $-(3x-2y) = \ -3x+2y$
- $-(4-2x) = \ 2x-4$
- $-(3a-2b) = \ -3a+2b$
- $-(b^2-4) = \ 4-b^2$
- $-(3xy-2yz) = \ 2yz-3xy$
- $-(x-y-z) = \ z-(x-y) = -x+y+z$
- $-(ab-cd) = \ cd-ab$

Ex. 1.1.15: Find the GCF of • $40, 60, 320 = \ 20$ • $30, 60, 320 = \ 10$ • $45, 60, 300 = \ 15$ • $16, 24, 320 = \ 8$

Ex. 1.1.16: Does each quadratic polynomial factor?

- $x^2+4x+2 =$
- $6x^2+13x+6 =$
- $3x^2+4x+2 =$
- $30x^2+93x+72 =$

Ex. 1.1.14: Apply the sign switch identity. Then rewrite your answer as a polynomial.

- $-(x-4) = 4-x$
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- $-(b^2-4) = 4-b^2$
- $-(3xy-2yz) = 2yz-3xy$
- $-(x-y-z) = z-(x-y) = -x+y+z$
- $-(ab-cd) = cd-ab$

Ex. 1.1.15: Find the GCF of • $40, 60, 320 = 20$ • $30, 60, 320 = 10$ • $45, 60, 300 = 15$ • $16, 24, 320 = 8$

Ex. 1.1.16: Does each quadratic polynomial factor?

- $x^2+4x+2 =$ No, $D=8$
- $6x^2+13x+6 =$ Yes, $D=25$
- $3x^2+4x+2 =$ No, $D=-8$
- $30x^2+93x+72 =$ Yes, $D=9$

Ex. 1.1.14: Apply the sign switch identity. Then rewrite your answer as a polynomial.

- $-(x-4) = \ 4-x$
- $-(3x-2y) = \ -3x+2y$
- $-(4-2x) = \ 2x-4$
- $-(3a-2b) = \ -3a+2b$
- $-(b^2-4) = \ 4-b^2$
- $-(3xy-2yz) = \ 2yz-3xy$
- $-(x-y-z) = \ z-(x-y) = -x+y+z$
- $-(ab-cd) = \ cd-ab$

Ex. 1.1.15: Find the GCF of • $40, 60, 320 = \ 20$ • $30, 60, 320 = \ 10$ • $45, 60, 300 = \ 15$ • $16, 24, 320 = \ 8$

Ex. 1.1.16: Does each quadratic polynomial factor?

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- $6x^2+13x+6 = $ Yes, $D = 25$
- $3x^2+4x+2 = $ No, $D = -8$
- $30x^2+93x+72 = $ Yes, $D = 9$
- $x^2+30x+200 =$
- $5x^3+13x+5 =$
- $20x^2+13x+2 =$
- $8x^2+10x-3 =$

Ex. 1.1.14: Apply the sign switch identity. Then rewrite your answer as a polynomial.

- $-(x-4) = \ 4-x$
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- $-(4-2x) = \ 2x-4$
- $-(3a-2b) = \ -3a+2b$
- $-(b^2-4) = \ 4-b^2$
- $-(3xy-2yz) = \ 2yz-3xy$
- $-(x-y-z) = \ z-(x-y) = -x+y+z$
- $-(ab-cd) = \ cd-ab$

Ex. 1.1.15: Find the GCF of

- $40, 60, 320 = \ 20$
- $30, 60, 320 = \ 10$
- $45, 60, 300 = \ 15$
- $16, 24, 320 = \ 8$

Ex. 1.1.16: Does each quadratic polynomial factor?

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- $3x^2+4x+2 =$ No, $D=-8$
- $30x^2+93x+72 =$ Yes, $D=9$
- $x^2+30x+200 =$ Yes, $D=100$
- $5x^3+13x+5 =$ No, $D=69$
- $20x^2+13x+2 =$ Yes, $D=9$
- $8x^2+10x-3 =$ Yes, $D=196$

Ex. 1.1.14: Apply the sign switch identity. Then rewrite your answer as a polynomial.

- $-(x-4) = 4-x$
- $-(3x-2y) = -3x+2y$
- $-(4-2x) = 2x-4$
- $-(3a-2b) = -3a+2b$
- $-(b^2-4) = 4-b^2$
- $-(3xy-2yz) = 2yz-3xy$
- $-(x-y-z) = z-(x-y) = -x+y+z$
- $-(ab-cd) = cd-ab$

Ex. 1.1.15: Find the GCF of • $40, 60, 320 = 20$ • $30, 60, 320 = 10$ • $45, 60, 300 = 15$ • $16, 24, 320 = 8$

Ex. 1.1.16: Does each quadratic polynomial factor?

- $x^2+4x+2 =$ No, $D=8$
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- $5x^3+13x+5 =$ No, $D=69$
- $20x^2+13x+2 =$ Yes, $D=9$
- $8x^2+10x-3 =$ Yes, $D=196$

Ex. 1.1.17: Factor

- $x^2+13x+36 =$
- $x^2-18x+81 =$
- $x^2-12x+32 =$
- $x^2-18x+56 =$

Ex. 1.1.14: Apply the sign switch identity. Then rewrite your answer as a polynomial.

- $-(x-4) = 4-x$
- $-(3x-2y) = -3x+2y$
- $-(4-2x) = 2x-4$
- $-(3a-2b) = -3a+2b$
- $-(b^2-4) = 4-b^2$
- $-(3xy-2yz) = 2yz-3xy$
- $-(x-y-z) = z-(x-y) = -x+y+z$
- $-(ab-cd) = cd-ab$

Ex. 1.1.15: Find the GCF of • $40, 60, 320 = 20$ • $30, 60, 320 = 10$ • $45, 60, 300 = 15$ • $16, 24, 320 = 8$

Ex. 1.1.16: Does each quadratic polynomial factor?

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- $30x^2+93x+72 =$ Yes, $D=9$
- $x^2+30x+200 =$ Yes, $D=100$
- $5x^3+13x+5 =$ No, $D=69$
- $20x^2+13x+2 =$ Yes, $D=9$
- $8x^2+10x-3 =$ Yes, $D=196$

Ex. 1.1.17: Factor

- $x^2+13x+36 = (x+9)(x+4)$
- $x^2-18x+81 = (x-9)^2$
- $x^2-12x+32 = (x-8)(x-4)$
- $x^2-18x+56 = (x-4)(x-14)$

Ex. 1.1.14: Apply the sign switch identity. Then rewrite your answer as a polynomial.
- $-(x-4) = \ 4-x$
- $-(3x-2y) = \ -3x+2y$
- $-(4-2x) = \ 2x-4$
- $-(3a-2b) = \ -3a+2b$
- $-(b^2-4) = \ 4-b^2$
- $-(3xy-2yz) = \ 2yz-3xy$
- $-(x-y-z) = \ z-(x-y) = -x+y+z$
- $-(ab-cd) = \ cd-ab$

Ex. 1.1.15: Find the GCF of
- $40, 60, 320 = \ 20$
- $30, 60, 320 = \ 10$
- $45, 60, 300 = \ 15$
- $16, 24, 320 = \ 8$

Ex. 1.1.16: Does each quadratic polynomial factor?
- $x^2+4x+2 =$ No, $D=8$
- $6x^2+13x+6 =$ Yes, $D=25$
- $3x^2+4x+2 =$ No, $D=-8$
- $30x^2+93x+72 =$ Yes, $D=9$
- $x^2+30x+200 =$ Yes, $D=100$
- $5x^3+13x+5 =$ No, $D=69$
- $20x^2+13x+2 =$ Yes, $D=9$
- $8x^2+10x-3 =$ Yes, $D=196$

Ex. 1.1.17: Factor
- $x^2+13x+36 = (x+9)(x+4)$
- $x^2-18x+81 = (x-9)^2$
- $x^2-12x+32 = (x-8)(x-4)$
- $x^2-18x+56 = (x-4)(x-14)$

Ex. 1.1.18: Factor
- $-x^2+x+12 =$
- $-x^2-2x+48 =$
- $-x^2+64 =$
- $-x^2-4x+21 =$

Ex. 1.1.14: Apply the sign switch identity. Then rewrite your answer as a polynomial.

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- $-(b^2-4) = 4-b^2$
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Ex. 1.1.18: Factor

- $-x^2+x+12 = -(x-4)(x+3)$
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Ex. 1.1.19: Use the AC method to factor Pax^2+bx+c

- $6x^2+13x+6 =$
- $12x^2+25x+12 =$
- $15x^2+32x+4 =$
- $6x^2-17x+10 =$

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Ex. 1.1.20: Rewrite as a polynomial:
- $x^2-(x+2)(x+3) =$
- $2x^2-(x+7)(x+3) =$
- $-3x^2-(x+2)(3-x) =$
- $x^3-x(x+2)(x+3) =$

Ex. 1.1.14: Apply the sign switch identity. Then rewrite your answer as a polynomial.
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- $-(3x-2y) = -3x+2y$
- $-(4-2x) = 2x-4$
- $-(3a-2b) = -3a+2b$
- $-(b^2-4) = 4-b^2$
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- $6x^2-17x+10 = (6x-5)(x-2)$

Ex. 1.1.20: Rewrite as a polynomial:
- $x^2-(x+2)(x+3) = -5x-6$
- $2x^2-(x+7)(x+3) = x^2-10x-21$
- $-3x^2-(x+2)(3-x) = -2x^2-x-6$
- $x^3-x(x+2)(x+3) = -5x^2-6x$

Ex. 1.1.21: Rewrite as a polynomial:

- $(x+2)(x+3)-x^2 =$
- $x^2-5(x^2+2)(x+3) =$
- $-3x^2-(x+2)(x^2+3) =$
- $x^2-8(x+2)(x+3) =$

Ex. 1.1.21: Rewrite as a polynomial:

- $(x+2)(x+3)-x^2 = 5x+6$
- $-3x^2-(x+2)(x^2+3) = -x^3-5x^2-3x-6$
- $x^2-5(x^2+2)(x+3) = -5x^3-14x^2-10x-30$
- $x^2-8(x+2)(x+3) = -7x^2-40x-48$

Ex. 1.1.21: Rewrite as a polynomial:

- $(x+2)(x+3)-x^2 = 5x+6$
- $-3x^2-(x+2)(x^2+3) = -x^3-5x^2-3x-6$
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- $x^2-8(x+2)(x+3) = -7x^2-40x-48$

Ex. 1.1.22: Rewrite as a polynomial:

- $x^2(x+4)-(x+2)^2 =$
- $x^2(x+5)-(x-3)^2 =$
- $x^2(x-5)-x(x-1)^2 =$
- $x^2(5-x)+x(4x-1)^2 =$

Ex. 1.1.21: Rewrite as a polynomial:

- $(x+2)(x+3)-x^2 = 5x+6$
- $-3x^2-(x+2)(x^2+3) = -x^3-5x^2-3x-6$
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- $x^2-8(x+2)(x+3) = -7x^2-40x-48$

Ex. 1.1.22: Rewrite as a polynomial:

- $x^2(x+4)-(x+2)^2 = x^3+3x^2-4x-4$
- $x^2(x+5)-(x-3)^2 = x^3+4x^2+6x-9$
- $x^2(x-5)-x(x-1)^2 = -3x^2-x$
- $x^2(5-x)+x(4x-1)^2 = 15x^3-3x^2+x$

Ex. 1.1.21: Rewrite as a polynomial:

- $(x+2)(x+3)-x^2 = 5x+6$
- $-3x^2-(x+2)(x^2+3) = -x^3-5x^2-3x-6$
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- $x^2-8(x+2)(x+3) = -7x^2-40x-48$

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- $x^2(5-x)+x(4x-1)^2 = 15x^3-3x^2+x$

Ex. 1.1.23: Rewrite each expression as a polynomial.

- $3-4(x+2) =$
- $x^2-x(x+2) =$
- $(x)(x+4)-(x+7)^2 =$
- $(x+3)(x+4)-3(x+7)^2 =$

Ex. 1.1.21: Rewrite as a polynomial:

- $(x+2)(x+3) - x^2 = 5x + 6$
- $-3x^2 - (x+2)(x^2+3) = -x^3 - 5x^2 - 3x - 6$
- $x^2 - 5(x^2+2)(x+3) = -5x^3 - 14x^2 - 10x - 30$
- $x^2 - 8(x+2)(x+3) = -7x^2 - 40x - 48$

Ex. 1.1.22: Rewrite as a polynomial:

- $x^2(x+4) - (x+2)^2 = x^3 + 3x^2 - 4x - 4$
- $x^2(x+5) - (x-3)^2 = x^3 + 4x^2 + 6x - 9$
- $x^2(x-5) - x(x-1)^2 = -3x^2 - x$
- $x^2(5-x) + x(4x-1)^2 = 15x^3 - 3x^2 + x$

Ex. 1.1.23: Rewrite each expression as a polynomial.

- $3 - 4(x+2) = -4x - 5$
- $x^2 - x(x+2) = -2x$
- $(x)(x+4) - (x+7)^2 = -10x - 49$
- $(x+3)(x+4) - 3(x+7)^2 = -2x^2 - 35x - 135$

Ex. 1.1.21: Rewrite as a polynomial:

- $(x+2)(x+3)-x^2 = 5x+6$
- $-3x^2-(x+2)(x^2+3) = -x^3-5x^2-3x-6$
- $x^2-5(x^2+2)(x+3) = -5x^3-14x^2-10x-30$
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- $x^2-x(x+2) = -2x$
- $(x)(x+4)-(x+7)^2 = -10x-49$
- $(x+3)(x+4)-3(x+7)^2 = -2x^2-35x-135$
- $3-4(7-x) =$
- $7x^2-x(2x-3) =$
- $(x+1)(x+4)-(x+7)^2 =$
- $(s+3)(s+4)+3(x+7)^2 =$

Ex. 1.1.21: Rewrite as a polynomial:

- $(x+2)(x+3)-x^2 = 5x+6$
- $-3x^2-(x+2)(x^2+3) = -x^3-5x^2-3x-6$
- $x^2-5(x^2+2)(x+3) = -5x^3-14x^2-10x-30$
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Ex. 1.1.22: Rewrite as a polynomial:

- $x^2(x+4)-(x+2)^2 = x^3+3x^2-4x-4$
- $x^2(x+5)-(x-3)^2 = x^3+4x^2+6x-9$
- $x^2(x-5)-x(x-1)^2 = -3x^2-x$
- $x^2(5-x)+x(4x-1)^2 = 15x^3-3x^2+x$

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- $3-4(x+2) = -4x-5$
- $x^2-x(x+2) = -2x$
- $(x)(x+4)-(x+7)^2 = -10x-49$
- $(x+3)(x+4)-3(x+7)^2 = -2x^2-35x-135$
- $3-4(7-x) = 4x-25$
- $7x^2-x(2x-3) = 5x^2+3x$
- $(x+1)(x+4)-(x+7)^2 = -9x-45$
- $(s+3)(s+4)+3(x+7)^2 = s^2+7s+3x^2+42x+159$

Ex. 1.1.21: Rewrite as a polynomial:

- $(x+2)(x+3) - x^2 = 5x + 6$
- $-3x^2 - (x+2)(x^2+3) = -x^3 - 5x^2 - 3x - 6$
- $x^2 - 5(x^2+2)(x+3) = -5x^3 - 14x^2 - 10x - 30$
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- $(x+3)(x+4) - 3(x+7)^2 = -2x^2 - 35x - 135$
- $3 - 4(7-x) = 4x - 25$
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- $(x+1)(x+4) - (x+7)^2 = -9x - 45$
- $(s+3)(s+4) + 3(x+7)^2 = s^2 + 7s + 3x^2 + 42x + 159$
- $12 - 3(x-9) =$
- $x^2 - (x+1)x =$
- $(a+1)(a+4) - (2a+7)^2 =$
- $(b+3)(b+4) - 3(8-b)^2 =$

Ex. 1.1.21: Rewrite as a polynomial:

- $(x+2)(x+3)-x^2 = 5x+6$
- $-3x^2-(x+2)(x^2+3) = -x^3-5x^2-3x-6$
- $x^2-5(x^2+2)(x+3) = -5x^3-14x^2-10x-30$
- $x^2-8(x+2)(x+3) = -7x^2-40x-48$

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- $7x^2-x(2x-3) = 5x^2+3x$
- $(x+1)(x+4)-(x+7)^2 = -9x-45$
- $(s+3)(s+4)+3(x+7)^2 = s^2+7s+3x^2+42x+159$
- $12-3(x-9) = -3x+39$
- $x^2-(x+1)x = -x$
- $(a+1)(a+4)-(2a+7)^2 = -3a^2-23a-45$
- $(b+3)(b+4)-3(8-b)^2 = -2b^2+55b-180$

Ex. 1.1.21: Rewrite as a polynomial:

- $(x+2)(x+3)-x^2 = 5x+6$
- $-3x^2-(x+2)(x^2+3) = -x^3-5x^2-3x-6$
- $x^2-5(x^2+2)(x+3) = -5x^3-14x^2-10x-30$
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Ex. 1.1.22: Rewrite as a polynomial:

- $x^2(x+4)-(x+2)^2 = x^3+3x^2-4x-4$
- $x^2(x+5)-(x-3)^2 = x^3+4x^2+6x-9$
- $x^2(x-5)-x(x-1)^2 = -3x^2-x$
- $x^2(5-x)+x(4x-1)^2 = 15x^3-3x^2+x$

Ex. 1.1.23: Rewrite each expression as a polynomial.

- $3-4(x+2) = -4x-5$
- $x^2-x(x+2) = -2x$
- $(x)(x+4)-(x+7)^2 = -10x-49$
- $(x+3)(x+4)-3(x+7)^2 = -2x^2-35x-135$
- $3-4(7-x) = 4x-25$
- $7x^2-x(2x-3) = 5x^2+3x$
- $(x+1)(x+4)-(x+7)^2 = -9x-45$
- $(s+3)(s+4)+3(x+7)^2 = s^2+7s+3x^2+42x+159$
- $12-3(x-9) = -3x+39$
- $x^2-(x+1)x = -x$
- $(a+1)(a+4)-(2a+7)^2 = -3a^2-23a-45$
- $(b+3)(b+4)-3(8-b)^2 = -2b^2+55b-180$
- $3-4(5p+2) =$
- $p^2-(p+2)(p-3) =$
- $(p+1)^2-2(p+2)(p-3) =$
- $(p+3)(p+4)-3(p+4)(p-2) =$

Ex. 1.1.21: Rewrite as a polynomial:
- $(x+2)(x+3)-x^2 = 5x+6$
- $-3x^2-(x+2)(x^2+3) = -x^3-5x^2-3x-6$
- $x^2-5(x^2+2)(x+3) = -5x^3-14x^2-10x-30$
- $x^2-8(x+2)(x+3) = -7x^2-40x-48$

Ex. 1.1.22: Rewrite as a polynomial:
- $x^2(x+4)-(x+2)^2 = x^3+3x^2-4x-4$
- $x^2(x+5)-(x-3)^2 = x^3+4x^2+6x-9$
- $x^2(x-5)-x(x-1)^2 = -3x^2-x$
- $x^2(5-x)+x(4x-1)^2 = 15x^3-3x^2+x$

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- $3-4(x+2) = -4x-5$
- $x^2-x(x+2) = -2x$
- $(x)(x+4)-(x+7)^2 = -10x-49$
- $(x+3)(x+4)-3(x+7)^2 = -2x^2-35x-135$
- $3-4(7-x) = 4x-25$
- $7x^2-x(2x-3) = 5x^2+3x$
- $(x+1)(x+4)-(x+7)^2 = -9x-45$
- $(s+3)(s+4)+3(x+7)^2 = s^2+7s+3x^2+42x+159$
- $12-3(x-9) = -3x+39$
- $x^2-(x+1)x = -x$
- $(a+1)(a+4)-(2a+7)^2 = -3a^2-23a-45$
- $(b+3)(b+4)-3(8-b)^2 = -2b^2+55b-180$
- $3-4(5p+2) = -20p-5$
- $p^2-(p+2)(p-3) = p+6$
- $(p+1)^2-2(p+2)(p-3) = -p^2+4p+13$
- $(p+3)(p+4)-3(p+4)(p-2) = -2p^2+p+36$

Ex. 1.1.24: Solve for x:

- $x - 3 = 6 \Rightarrow$
- $a + 7 = 6 \Rightarrow$
- $b - 87 = 6 \Rightarrow$
- $c - 32 = 63 \Rightarrow$

Ex. 1.1.24: Solve for x:

- $x - 3 = 6 \Rightarrow x = 9$
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- $x - ay = 6 \Rightarrow$
- $x + 7ab = 6 \Rightarrow$
- $ay - x + 3ay = 6 \Rightarrow$
- $x - by = 3b \Rightarrow$

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- $x - ay = 6 \Rightarrow x = ay + 6$
- $x + 7ab = 6 \Rightarrow x = 6 - 7ab$
- $ay - x + 3ay = 6 \Rightarrow x = 4ay - 6$
- $x - by = 3b \Rightarrow x = by + 3b$

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Ex. 1.1.26: Solve for x:

- $3x - 45 + 3y = x + 5 \Rightarrow$
- $7x - 5 = 3y - x + 5 \Rightarrow$
- $3x - 45 + x = 2y + 5 \Rightarrow$
- $3s - 45 + 3x = 2s + 5 \Rightarrow$

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- $3x-45+3y=x+5 \Rightarrow x=25-\frac{3y}{2}$
- $7x-5=3y-x+5 \Rightarrow x=\frac{3y}{8}+\frac{5}{4}$
- $3x-45+x=2y+5 \Rightarrow x=\frac{y+25}{2}$
- $3s-45+3x=2s+5 \Rightarrow x=\frac{50-s}{3}$

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Ex. 1.1.27: Solve for x:

- $45 + 3y - x = 4x - 5 \quad \Rightarrow$
- $2x + x - 45 + 3y = x + 3a + 5x \quad \Rightarrow$
- $2x - 45 + 3y - 8x = 7x + 3 + 5x \quad \Rightarrow$
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- $45 + 3y - x = 4x - 5 \Rightarrow x = \frac{50+3y}{5}$
- $2x + x - 45 + 3y = x + 3a + 5x \Rightarrow x = -a + y - 15$
- $2x - 45 + 3y - 8x = 7x + 3 + 5x \Rightarrow x = \frac{y}{6} - \frac{8}{3}$
- $5x - 3y = 7x + 3 - 11 + 5x \Rightarrow x = \frac{8}{7} - \frac{3y}{7}$

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- $5x-3y=7x+3-11+5x \quad \Rightarrow x=\frac{8}{7}-\frac{3y}{7}$

Ex. 1.1.28: Solve the equation
- for x : $xc+yc+xyu=2x+u+v \Rightarrow$
- for y : $xc+yc+xy+u=2x+u+v \Rightarrow$
- for u : $xc+yc+x+yu=2x+u+v \Rightarrow$
- for c: $xc+yc+xyu=2x+u+v \Rightarrow$

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Ex. 1.1.27: Solve for x:

- $45 + 3y - x = 4x - 5 \quad \Rightarrow x = \frac{50+3y}{5}$
- $2x + x - 45 + 3y = x + 3a + 5x \quad \Rightarrow x = -a + y - 15$
- $2x - 45 + 3y - 8x = 7x + 3 + 5x \quad \Rightarrow x = \frac{y}{6} - \frac{8}{3}$
- $5x - 3y = 7x + 3 - 11 + 5x \quad \Rightarrow x = \frac{8}{7} - \frac{3y}{7}$

Ex. 1.1.28: Solve the equation

- for x : $xc + yc + xyu = 2x + u + v \Rightarrow \quad x = \frac{-cy+u+v}{c+uy-2}$
- for y : $xc + yc + xy + u = 2x + u + v \Rightarrow \quad y = \frac{-cx+v+2x}{c+x}$
- for u : $xc + yc + x + yu = 2x + u + v \Rightarrow \quad u = \frac{-cx-cy+v+x}{y-1}$
- for c: $xc + yc + xyu = 2x + u + v \Rightarrow \quad c = \frac{-uxy+u+v+2x}{x+y}$

Ex. 1.1.29: Solve each equation for each of the letters that appear in it:

- $PV = kT \Rightarrow$
- $ab = a + b \Rightarrow$
- $xy = qx + qy \Rightarrow$
- $GMm = Fr^2 \Rightarrow$

Ex. 1.1.29: Solve each equation for each of the letters that appear in it:

- $PV = kT \Rightarrow$ $P = \frac{kT}{V}$; $V = \frac{kT}{P}$; $k = \frac{PV}{T}$; $T = \frac{PV}{K}$
- $ab = a + b \Rightarrow$ $a = \frac{b}{b-1}$; $b = \frac{a}{a-1}$
- $xy = qx + qy \Rightarrow$ $x = -\frac{qy}{q-y}$; $y = -\frac{qx}{q-x}$; $q = \frac{xy}{x+y}$
- $GMm = Fr^2 \Rightarrow$ $G = \frac{Fr^2}{Mm}$; $M = \frac{Fr^2}{Gm}$; $m = \frac{Fr^2}{GM}$; $F = \frac{GMm}{r^2}$; $r = \pm\sqrt{\frac{GMm}{F}}$

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Ex. 1.1.30: If $u = \frac{MV}{M+m}$, solve

- for M
- for $m \Rightarrow$
- for $V \Rightarrow$

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Ex. 1.1.30: If $u = \frac{MV}{M+m}$, solve

- for $M = -\frac{mu}{u-V}$
- for $m \Rightarrow= M(\frac{V}{u} - 1)$
- for $V \Rightarrow= \frac{u(m+M)}{M}$

Ex. 1.1.29: Solve each equation for each of the letters that appear in it:

- $PV = kT \Rightarrow$ $P = \frac{kT}{V}$; $V = \frac{kT}{P}$; $k = \frac{PV}{T}$; $T = \frac{PV}{K}$
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Ex. 1.1.31: Solve for y' :

- $x^2y + xy' + y = x^3y' + 3xy \Rightarrow$
- $x^2y + 2x^3y' + y' = x^3y' + 3xy \Rightarrow$
- $x^2y - xy' + y = x^3 + 3xyy' + y' \Rightarrow$
- $2x^2y^3 + xy' + y = x^3y' + 3axy \Rightarrow$

Ex. 1.1.29: Solve each equation for each of the letters that appear in it:

- $PV = kT \Rightarrow \quad P = \frac{kT}{V}; V = \frac{kT}{P}; k = \frac{PV}{T}; T = \frac{PV}{K}$
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- $xy = qx + qy \Rightarrow \quad x = -\frac{qy}{q-y}; y = -\frac{qx}{q-x}; q = \frac{xy}{x+y}$
- $GMm = Fr^2 \Rightarrow \quad G = \frac{Fr^2}{Mm}; M = \frac{Fr^2}{Gm}; m = \frac{Fr^2}{GM}; F = \frac{GMm}{r^2}; r = \pm\sqrt{\frac{GMm}{F}}$

Ex. 1.1.30: If $u = \frac{MV}{M+m}$, solve

- for $M = -\frac{mu}{u-V}$
- for $m \Rightarrow = M(\frac{V}{u} - 1)$
- for $V \Rightarrow = \frac{u(m+M)}{M}$

Ex. 1.1.31: Solve for y' :

- $x^2y + xy' + y = x^3y' + 3xy \Rightarrow \quad y' = \frac{(x^2-3x+1)y}{x(x^2-1)}$
- $x^2y + 2x^3y' + y' = x^3y' + 3xy \Rightarrow \quad y' = -\frac{(x-3)xy}{x^3+1}$
- $x^2y - xy' + y = x^3 + 3xyy' + y' \Rightarrow \quad y' = \frac{-x^3+x^2y+y}{3xy+x+1}$
- $2x^2y^3 + xy' + y = x^3y' + 3axy \Rightarrow \quad y' = \frac{-3axy+2x^2y^3+y}{x^3-x}$

Section 1.2: Fractions of numbers and polynomials

- 1.2.1: Fractions of numbers and polynomials
- 1.2.2: Adding and reducing fractions
- 1.2.3: Multiplying and dividing fractions
- 1.2.4: Negating fractions
- 1.2.5: Adding complicated fractions
- 1.2.6: Adding polynomial fractions
- 1.2.7: Polynomial long division
- 1.2.8: Complex fractions
- Section 1.2 Review

Section 1.2 Preview: Procedures

- Procedure 1.2.1: Multiplying a fraction by $\frac{C}{C} =$ doesn't change it.
- Procedure 1.2.2: Use $\frac{C}{C}$ to build $\frac{A}{B}$ to $\frac{AC}{BC}$.
- Procedure 1.2.3: Cancel C to reduce $\frac{AC}{BC}$ to $\frac{A}{B}$.
- Procedure 1.2.4: To add fractions with the same denominator: $\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C}$.
- Procedure 1.2.5: To add fractions with different denominators: $\frac{A}{C} + \frac{B}{D} = \frac{AD+BC}{CD}$.
- Procedure 1.2.6: To reduce completely a fraction $\frac{P}{Q}$ to lowest terms
- Procedure 1.2.7: To reduce fractions of powers of the same base,
- Procedure 1.2.8: If terms in a sum have powers of the same letter, factor out the highest power.
- Procedure 1.2.9: How to decide if fractions should be reduced.
- Procedure 1.2.10: Fraction multiplication: $\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$.
- Procedure 1.2.11: To divide E by a nonzero fraction: $E \div \frac{A}{B} = \frac{EB}{A}$.
- Procedure 1.2.12: If you have a fraction $\frac{N}{D}$ you may multiply or divide both numerator and denominator by the same nonzero expression C.
- Procedure 1.2.13: $\frac{A}{B}$ is not equal to $\frac{A \pm K}{B \pm K}$ (unless $K = 0$) or to $\frac{A^n}{B^n}$ (unless $n = 1$)...
- Procedure 1.2.14: To find the LCM of expressions A, B, C...
- Procedure 1.2.15: To efficiently add fractions with different denominators
- Procedure 1.2.16: To rewrite a complex fraction

Section 1.2 Preview: Definitions

- Definition 1.2.1: The reciprocal of whole number n is $\frac{1}{n}$.
- Definition 1.2.2: The reciprocal of nonzero expression A is $\frac{1}{A}$.
- Definition 1.2.3: The Reverse Distributive Law is $AB + AC = A(B + C)$.
- Definition 1.2.4: The identity $B - A = -(A - B)$ helps reduce fractions: $\frac{A-B}{B-A} = -1$.
- Definition 1.2.5: The Least Common Multiple (LCM) of a set of positive integers is the smallest integer that is a multiple of all of them.
- Definition 1.2.6: A Least Common Multiple (LCM) of a set of polynomials is a common multiple that is a factor of every other common multiple.
- Definition 1.2.7: The Least Common Denominator (LCD) of a set of fractions is the LCM of their denominators.
- Definition 1.2.8: A fraction of polynomials $\frac{P}{Q}$ is proper if the degree of P is less than the degree of Q.
- Definition 1.2.9: A fraction $\frac{N}{D}$ is called complex if N or D contains a fraction.

1.2.1 Fractions of numbers and polynomials

Reciprocals and fractions of numbers

- The reciprocal $\frac{1}{8}$ of 8 is the number obtained by splitting the unit 1 into 8 equal parts. Each part is called "one eighth."
- $1 = 8 \cdot \frac{1}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$
- The fraction $\frac{3}{8} = 3 \div 8 = 3 \cdot \frac{1}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$, obtained by adding together 3 eighths, is obtained by splitting 3 into 8 equal parts.
- 0 does not have a reciprocal.
- Any nonzero number times its reciprocal equals 1.

Reciprocals and fractions of *expressions* A and B

- If $B = 0$, $\frac{A}{B}$ is an illegal (undefined) expression.
- 0 does not have a reciprocal.
- If $B \neq 0$ then the reciprocal of B is $\frac{1}{B}$.
  - $B \cdot \frac{1}{B} = \frac{1}{B} \cdot B = 1.$
  - $\frac{A}{B} = A \div B = A \cdot \frac{1}{B} = \frac{1}{B} \cdot A.$
  - Special case: $\frac{B}{B} = 1.$

Undefined means: doesn't exist or make sense. Don't write $\frac{3}{0} = \frac{4}{0}$. Things that don't exist can't be equal.

Multiplying a fraction by $\frac{C}{C} = 1$

If $C \neq 0$, then $\dfrac{A}{B} = \dfrac{A}{B} \cdot \dfrac{C}{C} = \dfrac{AC}{BC}$.

This gives two important ways to rewrite fractions:

Use C to build $\frac{A}{B}$

$$\frac{A}{B} = \frac{A}{B} \cdot \frac{C}{C} = \frac{AC}{BC}$$

Cancel C to reduce $\frac{AC}{BC}$

$$\frac{AC}{BC} = \frac{A \cdot \cancel{C}}{B \cdot \cancel{C}} = \frac{A}{B}$$

To add fractions with the same denominator

add numerators and keep the denominator: $\dfrac{A}{B} + \dfrac{C}{B} = \dfrac{A + C}{B}$

Example 1:

- $\dfrac{4}{10} + \dfrac{3}{10} = \dfrac{4+3}{10} = \dfrac{7}{10}$
- $\dfrac{x}{ab} + \dfrac{y}{ba} = \dfrac{x}{ab} + \dfrac{y}{ab} = \dfrac{x+y}{ab}$

1.2.2: Adding and reducing fractions

How to add fractions with different denominators

If $B \neq 0$ and $D \neq 0$, $\quad \frac{A}{B} + \frac{C}{D} = \frac{AD + BC}{BD}$

To get this formula, build both fractions so that they have a common denominator:

$$\frac{A}{B} + \frac{C}{D} = \frac{AD}{BD} + \frac{CB}{DB} = \frac{AD}{BD} + \frac{BC}{BD} = \frac{AD + BC}{BD}$$

In all algebra computations, reducing fractions to lowest terms speeds up the work by producing expressions that are as compact as possible.

To reduce completely a fraction $\frac{P}{Q}$ to lowest terms

- Factor P completely.
- Factor Q completely.
- Remove ("cancel") common factors.

Example 2:

- $\frac{12}{14} = \frac{\cancel{2} \cdot 6}{\cancel{2} \cdot 7} = \frac{6}{7}$
- $\frac{100}{64} = \frac{25 \cdot \cancel{4}}{16 \cdot \cancel{4}} = \frac{25}{16}$
- $\frac{450}{1800} = \frac{45 \cdot \cancel{10}}{180 \cdot \cancel{10}} = \frac{45}{180} = \frac{5 \cdot \cancel{9}}{20 \cdot \cancel{9}} = \frac{5}{20} = \frac{\cancel{5} \cdot 1}{4 \cdot \cancel{5}} = \frac{1}{4}$

Be careful: All of the following are wrong, since the expression being "canceled" is not a factor of BOTH numerator and denominator.

- Error: $\frac{5+4}{5+5} = \frac{\cancel{5}+4}{\cancel{5}+7} = \frac{4}{7}$ Correct: $\frac{9}{10}$
- Error: $\frac{5 \cdot 2}{5+7} = \frac{\cancel{5} \cdot 2}{\cancel{5}+7} = \frac{2}{7}$ Correct: $\frac{10}{12} = \frac{5}{6}$
- Error: $\frac{5+7}{5 \cdot 2} = \frac{\cancel{5}+7}{\cancel{5} \cdot 2} = \frac{7}{2}$ Correct: $\frac{12}{10} = \frac{6}{5}$
- Error: $\frac{x^2+4x}{x^2+5x} = \frac{\cancel{x^2}+4x}{\cancel{x^2}+5x} = \frac{4x}{5x}$

  Correct: $\frac{x^2+4x}{x^2+5x} = \frac{\cancel{x}(x+4)}{\cancel{x}(x+5)} = \frac{x+4}{x+5}$
- Error: $\frac{x}{xy} = \frac{\cancel{x}}{\cancel{x}y} = y$

  Correct: $\frac{x}{xy} = \frac{x \cdot 1}{x \cdot y} = \frac{\cancel{x} \cdot 1}{\cancel{x} \cdot y} = \frac{1}{y}$

.

To reduce fractions, use the Reverse Distributive Law to factor numerator and denominator

To reduce fractions of powers of the same base, subtract the smaller exponent from the larger exponent and cancel the lower power.

- $\frac{x^{10}}{5x^7} = \frac{x^{10-7}\cancel{x^7}}{5\cancel{x^7}} = \frac{x^3}{5}$.
- $\frac{5x^7}{x^{10}} = \frac{5\cancel{x^7}}{\cancel{x^7}x^{10-7}} = \frac{5}{x^3}$.
- $\frac{7x^7y^7}{6x^{10}y^2} = \frac{7y^{7-2}}{6x^{10-7}} = \frac{7y^5}{6x^3}$

  If canceling yields a blank, insert 1.
- $\frac{x^{10}}{x^7} = \frac{x^{10-7}\cancel{x^7}}{1\cdot\cancel{x^7}} = \frac{x^3}{1} = x^3$.
- $\frac{x^7}{x^{10}} = \frac{\cancel{x^7}\cdot 1}{\cancel{x^7}x^{10-7}} = \frac{1}{x^3}$.

The *Reverse Distributive Law:* $AB + AC = A(B + C)$

- Factors a sum of terms: $ax^2 + ay^3 = a(x^2 + y^3)$
- Cancels fractions:
  $\frac{ax+ay}{2a} = \frac{a(x+y)}{2a} = \frac{\cancel{a}(x+y)}{2\cancel{a}} = \boxed{\frac{x+y}{2}}$

If terms in a sum have powers of the same letter

factor out the lowest power of that letter.

Example 3: Factor each blue sum of terms:

- $ax^8y + bx^6z = x^6(ax^{8-6}y + bz) \quad = \boxed{x^6(ax^2y + bz)}$
- $x^5 + 3x^7 = x^5(1 + 3^{7-5}) = \quad \boxed{x^5(1 + 3x^2)}$
- $a^5b^8 + a^7b^5 = a^5(b^8 + a^{7-5}b^5) = a^5(b^8 + a^2b^5)$
  $= \boxed{a^5b^5(b^3 + a^2)}$

Example 4: Factor numerator; then reduce:

- $\frac{a^5b^6 + a^6b^5}{a^6b^6} = \frac{a^5b^5(b + a)}{a^6b^6} = \frac{b + a}{a^{6-5}b^{6-5}} \quad = \boxed{\frac{a + b}{ab}}$

Example 5: Reduce: $\frac{(x + 1)(x + 4) + (x + 1)(x + 5)}{2x + 2}$

Method 1: Factor out $(x + 1)$: $\frac{(x + 1)\big((x + 4) + (x + 5)\big)}{2(x + 1)}$

$= \frac{\cancel{(x + 1)}(2x + 9)}{2\cancel{(x + 1)}} = \boxed{\frac{2x + 9}{2}}$

Method 2: Multiply out: $\frac{(x + 1)(x + 4) + (x + 1)(x + 5)}{2x + 2}$

$= \frac{x^2 + 5x + 4 + x^2 + 6x + 5}{2x + 2} = \frac{2x^2 + 11x + 9}{2x + 2}$

Requires hard factoring: $= \frac{\cancel{(x + 1)}(2x + 9)}{2\cancel{(x + 1)}} = \boxed{\frac{2x + 9}{2}}$

1.2.3 Multiplying and dividing fractions

Method 3: This is Example 4 in disguise: Substitute a for $(x+1)$ and b for $(x+4)$ in $\frac{(x+1)^5(x+4)^6+(x+1)^6(x+4)^5}{(x+1)^6(x+4)^6}$ to get

$$\frac{a^5b^6+a^6b^5}{a^6b^6} = \frac{a+b}{ab} = \frac{(x+1)+(x+4)}{(x+1)(x+4)} = \boxed{\frac{2x+5}{(x+1)(x+4)}}$$

How to decide if fractions should be reduced

- A fraction appearing in the middle of a solution should usually be reduced.
- Any fractional final answer should be reduced completely.

Sign switch identities **help reduce fractions**

- $\dfrac{A-B}{B-A} = -1$
- $B-A = -(A-B)$

To multiply fractions $\frac{A}{B}$ and $\frac{C}{D}$

- Reduce completely their product $\dfrac{AC}{BD}$.

Example 6: Reduce each fraction completely:

- $\dfrac{x^2-x}{x^2-x^3} = \dfrac{x(x-1)}{x^2(1-x)} = \dfrac{-1\cancel{(1-x)}}{x^{2-1}\cancel{(1-x)}} = \boxed{-\dfrac{1}{x}}$
- $\dfrac{x^2+4x}{x^2+5x} = \dfrac{\cancel{x}(x+4)}{\cancel{x}(x+5)} = \boxed{\dfrac{x+4}{x+5}}$
- $\dfrac{x+2}{x^2+5x+6} = \dfrac{\cancel{(x+2)}\cdot 1}{\cancel{(x+2)}(x+3)} = \boxed{\dfrac{1}{x+3}}$
- $\dfrac{x^4-x^2}{x^4+x^3} = \dfrac{x^2(x^2-1)}{x^3(x+1)} = \dfrac{\cancel{x^2}(x-1)\cancel{(x+1)}}{x^{3-2}\cancel{(x+1)}} = \boxed{\dfrac{x-1}{x}}$

Example 7: Find $\dfrac{x^3+x^4}{x^2+7x+12}\cdot\dfrac{x^3+3x^2}{x^2+6x+5}$

Solution: The product is: $= \frac{(x^3+x^4)(x^3+3x^2)}{(x^2+7x+12)(x^2+6x+5)}$

Don't multiply out yet!

First factor A,B,C, and D: $= \frac{(x^3)(x+1)x^2(x+3)}{(x+3)(x+4)(x+1)(x+5)}$

Cancel: $= \frac{(x^5)\cancel{(x+1)}\cancel{(x+3)}}{\cancel{(x+3)}(x+4)\cancel{(x+1)}(x+5)}$

$$= \boxed{\frac{x^5}{(x+4)(x+5)}}$$

Example 8: Find $\dfrac{x+3}{4x^2-9} \cdot \dfrac{2x^2+7x-15}{x^2+7x+12}$

Solution: Multiply: $= \dfrac{(x+3)(2x^2+7x-15)}{(4x^2-9)(x^2+7x+12)}$

Factor: $= \dfrac{(x+3)(2x-3)(x+5)}{(2x+3)(2x-3)(x+3)(x+4)}$

$= \dfrac{\cancel{(x+3)}\cancel{(2x-3)}(x+5)}{(2x+3)\cancel{(2x-3)}\cancel{(x+3)}(x+4)}$

$= \boxed{\dfrac{x+5}{(2x+3)(x+4)}}$

Example 9: Find $\dfrac{100}{81} \cdot \dfrac{144}{65}$

Solution Method 1: Factor completely and cancel:

$100 = 10^2 = (2 \cdot 5)^2 = 2^2 5^2$
$81 = 9 \cdot 9 = 3^2 3^2 = 3^4$
$144 = 12^2 = (4 \cdot 3)^2 = (2^2 \cdot 3)^2 = (2^2)^2 3^2 = 2^4 3^2$
$65 = 5 \cdot 13$

The problem becomes

$$\frac{100 \cdot 144}{81 \cdot 65} = \frac{2^2 5^2 \cdot 2^4 3^2}{3^4 \cdot 5 \cdot 13} = \frac{2^6 3^2 5^2}{3^4 \cdot 5 \cdot 13} = \frac{2^6 \cdot 5^{2-1}}{3^{4-2} \cdot 13}$$

$$= \frac{2^6 \cdot 5}{3^2 \cdot 13} = \boxed{\frac{320}{117}}$$

However, it is much easier to just look for common factors and cancel them:

Method 2:

$$\frac{100 \cdot 144}{81 \cdot 65} = \frac{\cancel{5} \cdot 20 \cdot 12 \cdot 12}{9 \cdot 9 \cdot \cancel{5} \cdot 13} = \frac{20 \cdot \cancel{3} \cdot 4 \cdot \cancel{3} \cdot 4}{9 \cdot \cancel{3} \cdot \cancel{3} \cdot 13} = \boxed{\frac{320}{117}}$$

To divide an expression E by a nonzero fraction

- *Invert* the fraction (exchange numerator and denominator). Then *Multiply* E by the result.
- In symbols: $E \div \dfrac{C}{D} = \dfrac{E}{1} \cdot \dfrac{D}{C} = \dfrac{ED}{C}$
- Special case: Expression $E = \dfrac{A}{B}$ is itself a fraction. Then $\dfrac{A}{B} \div \dfrac{C}{D} = \dfrac{A}{B} \cdot \dfrac{D}{C} = \dfrac{AD}{BC}$
- Reduce the answer fraction completely.

Example 10: Perform the indicated divisions:

a) $\dfrac{x+3}{4x^2-9} \div \dfrac{x^2+7x+12}{2x^2+7x-15}$

Invert: $= \dfrac{x+3}{4x^2-9} \cdot \dfrac{2x^2+7x-15}{x^2+7x+12}$

Multiply: $= \dfrac{(x+3)(2x^2+7x-15)}{(4x^2-9)(x^2+7x+12)}$

Reduce: $= \boxed{\dfrac{(x+5)}{(2x+3)(x+4)}}$ as shown in Example 8.

b) $\dfrac{100}{648} \div \dfrac{65}{144}$

Invert: $= \dfrac{100}{81} \cdot \dfrac{144}{65}$ Multiply: $= \dfrac{100 \cdot 144}{81 \cdot 65}$

Reduce: $= \boxed{\dfrac{320}{117}}$ as shown in Example 9

The minus sign $-$ is used in three related ways

- as part of the name of a negative number;
- as a negation sign: $-x$ means $-1 \cdot x$; and
- as a subtract sign.

A negative fraction can be written in three different ways.

For example, $-0.7 = -\frac{7}{10} = \frac{-7}{10} = \frac{7}{-10}$.
The last format is inelegant and should be replaced by one of the first two.

When you reduce a polynomial fraction, or find the LCM of polynomials, it's best if all polynomial factors have a positive leading coefficient. The basic identity you need to know is
$a - x = -x + a = -(x-a) = -1(x-a)$.
Reminder: negation is just multiplication by -1.

Example 11: Reduce a) $\dfrac{4-x}{x-4}$ and b) $\dfrac{x^2-4x+4}{4-x^2}$

a) $\dfrac{4-x}{x-4} = \dfrac{-1\cancel{(x-4)}}{1 \cdot \cancel{(x-4)}} = -1$

b) $\dfrac{x^2-4x+4}{4-x^2} = \dfrac{(x-2)^2}{-(x^2-4)} = \dfrac{\cancel{(x-2)}(x-2)}{-1\cancel{(x-2)}(x+2)}$

$= \dfrac{-1(x-2)}{(x+2)} = \boxed{-\dfrac{x-2}{x+2}}$

Be careful when you do the same thing to a fraction's numerator and denominator!

When you rewrite a fraction, the only legal way to change both numerator and denominator is to multiply (or divide) each of them by the same nonzero expression.

With rare exceptions, if you do anything else, you will change the fraction!

If you have a fraction $\frac{N}{D}$ you may

multiply or divide both numerator and denominator by the same nonzero expression C, as in

- $\frac{A}{B} = \frac{AC}{BC}$, or
- $\frac{A}{B} = \frac{A/C}{B/C}$.

or you may cancel a common factor from N and D as in

- $\frac{AC}{BC} = \frac{A\cancel{C}}{B\cancel{C}} = \frac{A}{B}$

If you have a fraction $\frac{N}{D}$ you must not

- Add an expression to both N and D: $\frac{A}{B} \neq \frac{A+C}{B+C}$ (unless $C = 0$).
- Subtract an expression from both N and D: $\frac{A}{B} \neq \frac{A+C}{B+C}$ (unless $C = 0$).
- Raise N and D to the same power: $\frac{A}{B} \neq \frac{A^n}{B^n}$ (unless $n = 1$ or $\frac{A}{B} = 0$ or 1).
- Take the same root of both N and D: $\frac{A}{B} \neq \frac{\sqrt[n]{A}}{\sqrt[n]{B}}$ (unless $n = 1$ or $\frac{A}{B} = 0$ or 1).

1.2.5 Adding complicated fractions

The addition rule $\frac{A}{B} + \frac{C}{D} = \frac{AD+BC}{BD}$ is always true, but the answer may be hard to reduce. A better way is to first factor B and D, then find the smallest possible expression that is a multiple of both.

The Least Common Multiple (LCM) of a set of positive integers is the smallest positive integer

that is a multiple of all of them.

A Least Common Multiple (LCM) of a set of polynomials is a common multiple of all of them

that is a factor of every other common multiple.

The LCM of distinct primes is their product:
LCM of $2, 3, 5$ is $2 \cdot 3 \cdot 5 = 30$; LCM of x, y, z is xyz.
LCM of $3, x$ and $x+2$ is $3x(x+2)$.

How to find the LCM of expressions $A, B, C \dots$

- Factor each expression as a *product of powers of primes.*
- For each prime that appears, choose the highest power.
- The LCM is the product of those highest powers.

Example 12:

Find the LCM of 35 and 77.
Solution: $35 = 5 \cdot 7$ and $77 = 7 \cdot 11$. The highest power of 5 is $5^1 = 5$; of 7 is $7^1 = 7$; of 11 is $11^1 = 11$. The product of highest powers is $5 \cdot 7 \cdot 11 = \boxed{385.}$

- Find the LCM of $20, 30$, and 50:
Factor and color highest powers
$20 = 2^2 \cdot 5$; $30 = 2 \cdot 3 \cdot 5$; $50 = 2 \cdot 5^2$. The LCM is product of red highest powers: $2^2 \cdot 3 \cdot 5^2 = \boxed{300}$

- Find the LCM of xy^2 and x^2y:
These don't factor further.
Highest powers: xy^2 and x^2y :
LCM is is the product of red highest powers: $= \boxed{x^2y^2}$

- Find the LCM of $x^2 + x$ and $x^2 + 2x$:
Factor and color highest powers:
$x \cdot (x+1)$ and $x(x+2)$. The LCM is $\boxed{x(x+1)(x+2)}$

- Find the LCM of $x^3 + x^2$; $x^3 + 4x^2 + 4x$; and $x^2 + 3x + 2$. Factor and color highest powers:
$x^2 \cdot (x+1)$; $x(x+2)^2$; $(x+1)(x+2)$
The LCM is $\boxed{x^2(x+1)(x+2)^2}$

Definition of Least Common Denominator (LCD)

The LCD of a set of fractions is the LCM of their denominators.

We know that multiplying numerator and denominator by the same expression doesn't change its value.

Example 13: Rewrite $\frac{3}{5}$ as a fraction with denominator 50.

Solution: To get new denominator 50, multiply 5 by 10. To keep the fraction value the same, also multiply the numerator by 10 to get $\frac{3}{5} = \frac{3}{5} \cdot \frac{10}{10} = \frac{30}{50}$

We sometimes call this process building the fraction. Actually, we are building up (multiplying) the numerator and denominator using the same factor, so that the value of the fraction stays the same.

Example 14:

Rewrite $\frac{a}{xy^2}$ as a fraction with denominator x^3y^6.

Solution: Multiply xy^2 by x^2y^4 to get x^3y^6. Thus

$$\frac{a}{xy^2} = \frac{a}{xy^2}\frac{x^2y^4}{x^2y^4} = \boxed{\frac{ax^2y^4}{x^3y^6}}$$

How to add fractions with different denominators

- Find the LCM of *their* denominators.
  This is the LCD of the fractions.
- For each fraction: what multiple of its denominator equals the LCD?
- Multiply numerator and denominator of each fraction by that multiple.
- Add the built-up fractions by adding their numerators.
- Reduce the sum to lowest terms.

Example 15: Rewrite $\frac{7}{2000} + \frac{11}{3000}$ as a reduced fraction.

Solution:

• In this easy example, you don't need to follow the procedure above, since since it's clear that multiplying 2000 by 3 and 3000 by 2 both give 6000.

• Build numerators and denominators.

$$\frac{7}{2000} = \frac{7 \cdot 3}{2000 \cdot 3} = \frac{21}{6000} \text{ and } \frac{11}{3000} = \frac{11 \cdot 2}{3000 \cdot 2} = \frac{22}{6000}$$

• Add the rewritten fractions: $\frac{21}{6000} + \frac{22}{6000} = \boxed{\frac{43}{6000}}$

Since 43 is prime the answer can't be reduced.

1.2.6 Adding polynomial fractions

Example 16: Rewrite $\frac{2}{35} + \frac{5}{77}$ as a reduced fraction.

Solution:

- First find the LCD: $35 = 5 \cdot 7$ and $77 = 7 \cdot 11$.

The highest power of each prime is red.
Multiply highest powers to get LCD $=5 \cdot 7 \cdot 11$.

- Rewrite the fractions by building denominators up to that LCD and add:

$$\frac{2\cdot 11}{5\cdot 7\cdot 11} + \frac{5\cdot 5}{5\cdot 7\cdot 11} = \frac{22}{5\cdot 7\cdot 11} + \frac{25}{5\cdot 7\cdot 11} = \frac{22+25}{5\cdot 7\cdot 11} = \boxed{\frac{47}{385}}$$

None of the denominator factors $5, 7, 11$ goes exactly into 47, so numerator and denominator have no common factor.
Therefore the answer is completely reduced.

Example 17: Find $\frac{3}{20} + \frac{7}{16} + \frac{7}{24}$

Solution:

- Factor $20 = 2^2 \cdot 5$; $16 = 2^4$; and $24 = 2^3 \cdot 3$.

  LCD is the product of highest powers:

  $2^4 \cdot 3 \cdot 5 = 240$

- Find multiples that produce that LCD:

  $20 \cdot 12 = 240 \qquad 16 \cdot 15 = 240 \qquad 24 \cdot 10 = 240$

- Build the fractions and add them:

$$\frac{3\cdot 12}{20\cdot 12} + \frac{7\cdot 15}{16\cdot 15} + \frac{7\cdot 10}{24\cdot 10} = \frac{36}{240} + \frac{105}{240} + \frac{70}{240}$$
$$= \frac{36+105+70}{240} = \boxed{\frac{211}{240}}$$

Example 18: Is $\frac{211}{240}$ a completely reduced fraction?

Solution: The denominator is easy to factor:
$\frac{211}{240} = \frac{211}{2^4\cdot 3\cdot 5}$. None of the factors $2, 3, 5$ go evenly into 211. Therefore 211 and 240 have no common factor.

$\boxed{\frac{211}{240}}$ is completely reduced.

Example 19: Find the LCM of x^3y^4 and $x^2y^3z^5$.

Solution: Highest powers in red: $x^3 \cdot y^4$ and $x^2y^3z^5$

The LCM is the product of highest powers: $\boxed{x^3y^4z^5}$

Example 20: Add $\frac{a}{x^3} + \frac{b}{x^{10}}$

Solution: The highest power of x is x^{10} = the LCD.

$$\frac{a}{x^3} + \frac{b}{x^{10}} = \frac{a}{x^3}\frac{x^7}{x^7} + \frac{b}{x^{10}} = \frac{ax^7}{x^{10}} + \frac{b}{x^{10}} = \boxed{\frac{ax^7+b}{x^{10}}}$$

Example 21: Find $\frac{a}{xy^2} + \frac{b}{x^2y}$

Solution: The LCM of x^2y and xy^2 is the product of highest powers x^2y^2. Build and add:

$$\frac{a}{xy^2} + \frac{b}{x^2y} = \frac{a}{xy^2}\frac{x}{x} + \frac{b}{x^2y}\frac{y}{y} = \boxed{\frac{ax+by}{x^2y^2}}$$

Example 22: Find $\dfrac{3}{x^2} + \dfrac{2}{x^2+x}$

Solution: Factor the denominators: $\dfrac{3}{x^2} + \dfrac{2}{x(x+1)}$

The LCM is the product of highest powers $x^2(x+1)$.

Build and add numerators: $\dfrac{3(x+1)}{x^2(x+1)} + \dfrac{2x}{x(x+1)x} = \dfrac{3x+3}{x^2(x+1)} + \dfrac{2x}{x^2(x+1)} = \dfrac{3x+3+2x}{x^2(x+1)} = \boxed{\dfrac{5x+3}{x^2(x+1)}}$

Example 23: Find $\dfrac{x}{(x^2+2x+1)(x+2)} + \dfrac{3}{(x^2+4x+4)(x+1)}$ This is harder. Don't get scared!

Solution: Factor denominators, color highest powers red: $\dfrac{x}{(x+1)^2(x+2)} + \dfrac{3}{(x+2)^2(x+1)}$

- The LCD is the product of red powers $(x+1)^2(x+2)^2$. Build each fraction so its denominator is that LCD.

$$\frac{x}{(x+1)^2(x+2)}\,\frac{(x+2)}{(x+2)} = \frac{x(x+2)}{(x+1)^2(x+2)^2} = \frac{x^2+2x}{(x+1)^2(x+2)^2}$$
$$\frac{3}{(x+2)^2(x+1)}\,\frac{(x+1)}{(x+1)} = \frac{3(x+1)}{(x+1)^2(x+2)^2} = \frac{3x+3}{(x+1)^2(x+2)^2}$$

- Add the results by adding numerators: $\dfrac{x^2+2x}{(x+1)^2(x+2)^2} + \dfrac{3x+3}{(x+1)^2(x+2)^2} = \boxed{\dfrac{x^2+5x+3}{(x+1)^2(x+2)^2}}$

- The numerator $x^2+5x+3 = ax^2+bx+c$ doesn't factor, since $D = b^2-4ac = 5^2-4\cdot 1\cdot 3 = 25-12 = 13$ is not a perfect square, Therefore the boxed answer cannot be reduced.

Example 24: Find $\dfrac{x}{x^2+3x+2}+\dfrac{x+3}{x^2+4x+4}$

Solution:

- Factor denominators, color highest powers red: $\dfrac{x}{(x+1)\cdot(x+2)}+\dfrac{x+3}{(x+2)^2}$
- The LCD is the product of highest powers $(x+1)(x+2)^2$
- Build each fraction so its denominator is that LCD:

$$\frac{x}{(x+1)(x+2)}=\frac{x(x+2)}{(x+1)(x+2)(x+2)}=\frac{x^2+2x}{(x+2)^2(x+1)}$$
$$\frac{x+3}{(x+2)^2}=\frac{(x+3)(x+1)}{(x+2)^2(x+1)}=\frac{x^2+4x+3}{(x+2)^2(x+1)}$$

- Add the rewritten fractions by adding numerators:

$$\frac{x^2+2x}{(x+2)^2(x+1)}+\frac{x^2+4x+3}{(x+2)^2(x+1)}=\boxed{\frac{2x^2+6x+3}{(x+2)^2(x+1)}}$$

To see if the fraction reduces, try factoring $2x^2+6x+3=ax^2+bx+c$. Since $D=b^2-4ac=6^2-4\cdot2\cdot3=36-24=12$ is not a perfect square, the numerator does not factor and therefore the boxed answer cannot be reduced.

1.2.7 Polynomial long division

Example 25: Use long division with remainder to divide $x^3 + 3x^2 + 3x + 7$ by $x + 2$.

Solution: The dividend is $x^3 + 3x^2 + 3x + 7$.
The Divisor is $x + 2$.

Click *slowly* through the following demonstration.

$$x + 2 \;/\; \overline{\;x^3 + 3x^2 + 3x + 7\;}$$

1.2.7 Polynomial long division

Example 25: Use long division with remainder to divide $x^3 + 3x^2 + 3x + 7$ by $x + 2$.

Solution: The dividend is $x^3 + 3x^2 + 3x + 7$.
The Divisor is $x + 2$.

- Divide the Divisor's leading term into the dividend's leading term: x into x^3 is $x^{3-1} = x^2$, which is written in the Quotient.

Click *slowly* through the following demonstration.

$$x + 2 \,\big/\, \overline{\;x^3 + 3x^2 + 3x + 7\;} \quad \text{quotient: } x^2$$

x into x^3 is x^2.

1.2.7 Polynomial long division

Example 25: Use long division with remainder to divide $x^3 + 3x^2 + 3x + 7$ by $x + 2$.

Solution: The dividend is $x^3 + 3x^2 + 3x + 7$.
The Divisor is $x + 2$.

- Divide the Divisor's leading term into the dividend's leading term: x into x^3 is $x^{3-1} = x^2$, which is written in the Quotient.
- Multiply x^2 by the Divisor to get $x^2(x + 2) = x^3 + 2x^2$.

Click *slowly* through the following demonstration.

$$\begin{array}{r|l} & x^2 \\ \hline x+2 \; / & x^3 + 3x^2 + 3x + 7 \\ & x^3 + 2x^2 \\ \hline \end{array}$$

x into x^3 is x^2.

Multiply x^2 by $x + 2$

1.2.7 Polynomial long division

Example 25: Use long division with remainder to divide $x^3 + 3x^2 + 3x + 7$ by $x + 2$.

Solution: The dividend is $x^3 + 3x^2 + 3x + 7$.
The Divisor is $x + 2$.

- Divide the Divisor's leading term into the dividend's leading term: x into x^3 is $x^{3-1} = x^2$, which is written in the Quotient.
- Multiply x^2 by the Divisor to get $x^2(x + 2) = x^3 + 2x^2$.
- Subtract $x^3 + 2x^2$ from the dividend to get $x^2 + 3x + 7$, which we call Diff.

Click *slowly* through the following demonstration.

$$\begin{array}{r|l} & x^2 \\ \hline x+2 \;/ & x^3 + 3x^2 + 3x + 7 \\ & \underline{x^3 + 2x^2} \\ & x^2 + 3x + 7 \end{array}$$

x into x^3 is x^2.

Multiply x^2 by $x + 2$

Subtract to get Diff $= x^2 + 3x + 7$.

1.2.7 Polynomial long division

Example 25: Use long division with remainder to divide $x^3 + 3x^2 + 3x + 7$ by $x + 2$.

Solution: The dividend is $x^3 + 3x^2 + 3x + 7$.
The Divisor is $x + 2$.

- Divide the Divisor's leading term into the dividend's leading term: x into x^3 is $x^{3-1} = x^2$, which is written in the Quotient.
- Multiply x^2 by the Divisor to get $x^2(x + 2) = x^3 + 2x^2$.
- Subtract $x^3 + 2x^2$ from the dividend to get $x^2 + 3x + 7$, which we call Diff.
- Repeat the above steps using Diff as the new dividend *unless* the degree of Diff is less than degree of the divisor, in which case you stop.

Click *slowly* through the following demonstration.

$$\begin{array}{r l}
 & x^2 \quad + x \\
x+2 \;/ & \overline{x^3 + 3x^2 + 3x + 7} \\
 & \underline{x^3 + 2x^2} \\
 & \qquad x^2 + 3x + 7 \\
 & \qquad \underline{x^2 + 2x}
\end{array}$$

x into x^3 is x^2.

Multiply x^2 by $x + 2$

Subtract to get Diff $= x^2 + 3x + 7$.

x into x^2 is x. Multiply x by $x + 2$

1.2.7 Polynomial long division

Example 25: Use long division with remainder to divide $x^3 + 3x^2 + 3x + 7$ by $x + 2$.

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- Subtract $x^3 + 2x^2$ from the dividend to get $x^2 + 3x + 7$, which we call Diff.
- Repeat the above steps using Diff as the new dividend *unless* the degree of Diff is less than degree of the divisor, in which case you stop.

Click *slowly* through the following demonstration.

$$
\begin{array}{r|l}
 & \quad\quad\quad\quad x^2 \quad + x \quad +1 \\
x+2 \,/ & x^3 + 3x^2 + 3x + 7 \\
 & \underline{x^3 + 2x^2} \\
 & \quad\quad\quad x^2 + 3x + 7 \\
 & \quad\quad\quad \underline{x^2 + 2x} \\
 & \quad\quad\quad\quad\quad\quad x + 7
\end{array}
$$

x into x^3 is x^2.

Multiply x^2 by $x + 2$

Subtract to get Diff $= x^2 + 3x + 7$.

x into x^2 is x. Multiply x by $x + 2$

Subtract to get Diff $= x + 7$.

1.2.7 Polynomial long division

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- Multiply x^2 by the Divisor to get $x^2(x + 2) = x^3 + 2x^2$.
- Subtract $x^3 + 2x^2$ from the dividend to get $x^2 + 3x + 7$, which we call Diff.
- Repeat the above steps using Diff as the new dividend *unless* the degree of Diff is less than degree of the divisor, in which case you stop.

Click *slowly* through the following demonstration.

$$
\begin{array}{r|l}
 & x^2 \quad + x \quad +1 \\
\hline
x+2 \,/ & x^3 + 3x^2 + 3x + 7 \\
 & \underline{x^3 + 2x^2} \\
 & x^2 + 3x + 7 \\
 & \underline{x^2 + 2x} \\
 & x + 7 \\
 & \underline{x + 2}
\end{array}
$$

x into x^3 is x^2.

Multiply x^2 by $x + 2$

Subtract to get Diff $= x^2 + 3x + 7$.

x into x^2 is x. Multiply x by $x + 2$

Subtract to get Diff $= x + 7$.

x into x is 1. Multiply 1 by $x + 2$

1.2.7 Polynomial long division

Example 25: Use long division with remainder to divide x^3+3x^2+3x+7 by $x+2$.

Solution: The dividend is x^3+3x^2+3x+7 .
The Divisor is $x+2$.

- Divide the Divisor's leading term into the dividend's leading term: x into x^3 is $x^{3-1}=x^2$, which is written in the Quotient.
- Multiply x^2 by the Divisor to get $x^2(x+2)=x^3+2x^2$.
- Subtract x^3+2x^2 from the dividend to get x^2+3x+7, which we call Diff.
- Repeat the above steps using Diff as the new dividend *unless* the degree of Diff is less than degree of the divisor, in which case you stop.

Click *slowly* through the following demonstration.

$$\begin{array}{r|l} & x^2 \quad + x \; +1 \\ \hline x+2 \,/ & x^3+3x^2+3x+7 \\ & \underline{x^3+2x^2} \\ & x^2+3x+7 \\ & \underline{x^2+2x} \\ & x+7 \\ & \underline{x+2} \\ & 5 \end{array}$$

x into x^3 is x^2.

Multiply x^2 by $x+2$

Subtract to get Diff $=x^2+3x+7$.

x into x^2 is x. Multiply x by $x+2$

Subtract to get Diff $=x+7$.

x into x is 1. Multiply 1 by $x+2$

Subtract to get Diff $=5$.

1.2.7 Polynomial long division

Example 25: Use long division with remainder to divide $x^3 + 3x^2 + 3x + 7$ by $x + 2$.

Solution: The dividend is $x^3 + 3x^2 + 3x + 7$.
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- Subtract $x^3 + 2x^2$ from the dividend to get $x^2 + 3x + 7$, which we call Diff.
- Repeat the above steps using Diff as the new dividend *unless* the degree of Diff is less than degree of the divisor, in which case you stop.

Click *slowly* through the following demonstration.

$x^2 \quad + x \ +1$	x into x^3 is x^2.
$x + 2 \ / \ x^3 + 3x^2 + 3x + 7$	
$x^3 + 2x^2$	Multiply x^2 by $x + 2$
$x^2 + 3x + 7$	Subtract to get Diff $= x^2 + 3x + 7$.
$x^2 + 2x$	x into x^2 is x. Multiply x by $x + 2$
$x + 7$	Subtract to get Diff $= x + 7$.
$x + 2$	x into x is 1. Multiply 1 by $x + 2$
5	Subtract to get Diff $=$ 5.

At this point we stop because the degree 0 of 5 is less than the degree 1 of the divisor $x + 2$.

Answer: Quotient is $x^2 + x + 1$. Remainder is 5.

In the simple problem $13 \div 5 = 2$ with Remainder 3, dividend = 13; Divisor = 5, Quotient =2, Remainder = 3 .

They are related by $DQ + R =$dividend, since $5 \cdot 2 + 3 = 13$.

To check the answer to anylong division problem, we must verify $DQ + R =$dividend.

In our problem, this translates to

$(x+2)(x^2+x+1)+5 = x^3+3x^2+3x+7$?

Expand and rewrite the left side:

$$\begin{aligned} &(x+2)(x^2+x+1)+5 \\ =\ & x(x^2+x+1)+2(x^2+x+1)+5 \\ =\ & x^3+x^2+x+2x^2+2x+2+5 \\ =\ & x^3+3x^2+3x+7 = \text{the dividend, as desired.} \end{aligned}$$

We can phrase the problem in a different way.

A fraction of polynomials is called proper if

the degree of the numerator is less than the degree of the denominator.

Example 26: Rewrite $\dfrac{x^3+3x^2+3x+7}{x+2}$ as a polynomial plus a proper fraction of polynomials.

In the simple problem $13 \div 5 = 2$ with Remainder 3,
dividend = 13; Divisor = 5, Quotient =2, Remainder = 3 .

They are related by $DQ + R =$dividend, since $5 \cdot 2 + 3 = 13$.

To check the answer to anylong division problem, we must verify $DQ + R =$dividend.

In our problem, this translates to

$(x + 2)(x^2 + x + 1) + 5 = x^3 + 3x^2 + 3x + 7$?

Expand and rewrite the left side:

$$\begin{aligned}
&(x + 2)(x^2 + x + 1) + 5 \\
&= x(x^2 + x + 1) + 2(x^2 + x + 1) + 5 \\
&= x^3 + x^2 + x + 2x^2 + 2x + 2 + 5 \\
&= x^3 + 3x^2 + 3x + 7 = \text{the dividend, as desired.}
\end{aligned}$$

We can phrase the problem in a different way.

A fraction of polynomials is called proper if

the degree of the numerator is less than
the degree of the denominator.

Example 26: Rewrite $\dfrac{x^3 + 3x^2 + 3x + 7}{x + 2}$ as a polynomial plus a proper fraction of polynomials.

Solution: Do the long division in Example 25. The problem asks for $\dfrac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \dfrac{\text{Remainder}}{\text{Divisor}}$

Answer:

$$\boxed{\frac{x^3 + 3x^2 + 3x + 7}{x + 2} = x^2 + x + 1 + \frac{5}{x + 2}}.$$

1.2.8 Complex fractions

A complex fraction

is a fraction whose numerator or denominator contains one or more fractions, which are often referred to as nested fractions.

Here the word *complex* means *complicated*, and has no connection with complex numbers.

Examples include $\dfrac{\frac{a}{b}}{\frac{4}{5}}$ $\dfrac{1+\frac{1}{5+6}}{\frac{2}{4}+\frac{5}{23}}$ and $\dfrac{\frac{2}{7}+\frac{a}{b}}{7-3\cdot\frac{5}{x}}$

.

How to rewrite a complex fraction

- Multiply both numerator and denominator by the LCD of all the nested fractions.
- Multiply out, collect terms, then factor both numerator and denominator.
- Reduce the resulting fraction.

Example 27: Rewrite $\dfrac{5+\frac{1}{7^2}}{3}$ as a reduced fraction

Solution: Multiply top and bottom by 7^2:

$$\frac{5+\frac{1}{7^2}}{3} = \frac{\left(5+\frac{1}{7^2}\right)7^2}{3\cdot 7^2} = \frac{7^2\cdot 5+1}{7^2\cdot 3} = \frac{49\cdot 5+1}{7^2\cdot 3}$$

$$= \frac{245+1}{7^2\cdot 3} = \frac{246}{7^2\cdot 3} = \frac{41\cdot 2\cdot \cancel{3}}{7^2\cdot \cancel{3}} = \boxed{\frac{82}{49}}$$

Example 28: Rewrite $\dfrac{y+\frac{1}{x^2}}{z}$ as a reduced fraction.

Solution: Multiply top and bottom by x^2:

$$\frac{y+\frac{1}{x^2}}{z} = \frac{\left(y+\frac{1}{x^2}\right)x^2}{zx^2} = \frac{yx^2+\frac{1}{x^2}\cdot x^2}{zx^2} = \boxed{\frac{x^2y+1}{x^2z}}$$

Example 29 : Rewrite $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}}$ as a reduced fraction.

Solution: Multiply numerator and denominator by the LCM of x, y, x^2, y^2, which is x^2y^2. Then

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}} = \frac{\left(\frac{1}{x}+\frac{1}{y}\right)x^2y^2}{\left(\frac{1}{x^2}-\frac{1}{y^2}\right)x^2y^2} = \frac{xy^2+yx^2}{y^2-x^2} = \frac{xy\cancel{(y+x)}}{\cancel{(y+x)}(y-x)} = \boxed{\frac{xy}{y-x}}$$

Example 30: Rewrite $\dfrac{\frac{1}{a+h}-\frac{1}{a}}{h}$ as a reduced fraction.

Solution: Nested fractions are $\frac{1}{a+h}$ and $\frac{1}{a}$.
Multiply top and bottom by their LCD, which is $a(a+h)$.

Multiply by LCD: $$= \frac{a(a+h)\left(\frac{1}{a+h}-\frac{1}{a}\right)}{a(a+h)h}$$

Distributive law: $$= \frac{a(a+h)\left(\frac{1}{a+h}\right)-a(a+h)\left(\frac{1}{a}\right)}{a(a+h)h}$$

Cancel and simplify: $$= \frac{a\cancel{(a+h)}\left(\frac{1}{\cancel{a+h}}\right)-\cancel{a}(a+h)\left(\frac{1}{\cancel{a}}\right)}{a(a+h)h}$$

$$= \frac{a-(a+h)}{a(a+h)h} = \frac{-h}{a(a+h)h} = \frac{-1\cdot\cancel{h}}{a(a+h)\cancel{h}} = \boxed{\frac{-1}{a(a+h)}}$$

Example 30: condensed solution

Do these in your head.

$$a(a+h)\left(\frac{1}{a+h}\right) = a \text{ and}$$

$$a(a+h)\left(\frac{1}{a}\right) = a+h$$

Solution: $$\frac{\frac{1}{a+h}-\frac{1}{a}}{h} = \frac{a(a+h)\left(\frac{1}{a+h}-\frac{1}{a}\right)}{a(a+h)h} = \frac{a-(a+h)}{a(a+h)h} = \frac{-h}{a(a+h)h} = \boxed{\frac{-1}{a(a+h)}}$$

1.2.10 Section 1.2 Examples for Practice with wolframalpha.com

Click on ▸ Wolfram Calculator to find an answer checker. Click on ▸ Wolfram Algebra Examples to see how to check various types of algebra problems.

1. Find each sum by first finding the LCD of the fractions to be added.

a) $\dfrac{3}{100}+\dfrac{5}{200}+\dfrac{7}{800}$

b) $\dfrac{1}{xy}+\dfrac{1}{yz}+\dfrac{1}{xz}$

c) $\dfrac{1}{x^2+2x+1}+\dfrac{1}{x^2-2x+1}+\dfrac{1}{x^2-1}$

d) $\dfrac{x}{x-4}-\dfrac{3}{x+6}$

e) $\dfrac{1}{2x^2+4x+2}+\dfrac{1}{x^2-2x+1}+\dfrac{1}{x^2-1}$

f) $\dfrac{x}{x^2-4}+\dfrac{1}{x-2}$

g) $\dfrac{1}{2-x}+\dfrac{2}{x-2}+\dfrac{x}{x^2-4}$

h) $\dfrac{x}{x^2+x-2}-\dfrac{2}{x^2-5x+4}$

i) $\dfrac{x}{x^2-x-6}-\dfrac{1}{x+2}-\dfrac{2}{x-3}$

j) $\dfrac{3}{2x-4}+\dfrac{1}{8-4x}+\dfrac{16}{4-2x}$

2. For each of the following polynomials

- Try to factor the polynomial by using the methods you learned in other math courses.
- Use the discriminant to decide if the polynomial can be factored. If yes, factor it. If not, explain why.

a) $6x^2+17x+12$	b) $6x^2+18x+12$	c) $9x^2+27x+20$
d) $9x^2+28x+20$	e) $8x^2+18x+9$	f) $8x^2+18x+11$
g) $x^2+30x+200$	h) $x^2-32x-420$	i) $8x^2-19x+12$

3. Multiply or divide as indicated.

a) $\dfrac{x^2-x-6}{x^2+2x}\cdot\dfrac{x^3+x^2}{x^2-2x-3}$

b) $\dfrac{x^2-1}{x^2-4}\cdot\dfrac{2x^3-4x^2+2x}{x^2-4x+4}$

c) $\dfrac{x^2+2x}{x^2-x-6}\div\dfrac{x^3+x^2}{x^2-2x-3}$

d) $\dfrac{x^3-9x^2}{x^2-x-6}\div\dfrac{9-x^2}{x-3}$

4. Find the sum

$$\frac{3}{2x^2+8x+8}+\frac{5}{3x^2(2x+4)^2}+\frac{3}{12x^4+48x^3+48x^2}$$

Section 1.2 Quiz

Ex. 1.2.1: • $\frac{4}{10}+\frac{3}{10}$ • $\frac{x}{ab}+\frac{y}{ba}$

Ex. 1.2.2: • $\frac{12}{14}=$ • $\frac{100}{64}$ • $\frac{450}{1800}=$

Ex. 1.2.3: Factor each blue sum of terms: • ax^8y+bx^6z • x^5+3x^7 • $a^5b^8+a^7b^5$

Ex. 1.2.4: Factor numerator; then reduce: $\frac{a^5b^6+a^6b^5}{a^6b^6}$

Ex. 1.2.5: Reduce: $\frac{(x+1)(x+4)+(x+1)(x+5)}{2x+2}$

Ex. 1.2.6: Reduce each fraction completely: • $\frac{x^2-x}{x^2-x^3}=$ • $\frac{x^2+4x}{x^2+5x}$ • $\frac{x+2}{x^2+5x+6}$ • $\frac{x^4-x^2}{x^4+x^3}$

Ex. 1.2.7: Find $\frac{x^3+x^4}{x^2+7x+12}\cdot\frac{x^3+3x^2}{x^2+6x+5}$

Ex. 1.2.8:Find $\frac{x+3}{4x^2-9}\cdot\frac{2x^2+7x-15}{x^2+7x+12}$

Ex. 1.2.9: Find $\frac{100}{81}\cdot\frac{144}{65}$

Ex. 1.2.10:Perform the indicated divisions: • $\frac{x+3}{4x^2-9}\div\frac{x^2+7x+12}{2x^2+7x-15}$ • $\frac{100}{648}\div\frac{65}{144}$

Ex. 1.2.11: Reduce • $\frac{4-x}{x-4}$ • $\frac{x^2-4x+4}{4-x^2}$

Ex. 1.2.12: Find the LCM
- of $20, 30,$ and 50 ; • of xy^2 and x^2y;
- of x^2+x and x^2+2x; • of $20, 30,$ and 50:;
- of x^3+x^2; x^3+4x^2+4x; and x^2+3x+2.

Ex. 1.2.13: Rewrite $\frac{3}{5}$ as a fraction with denominator 50.

Ex. 1.2.14: Rewrite $\frac{a}{xy^2}$ as a fraction with denominator x^3y^6.

Ex. 1.2.15: Rewrite $\frac{7}{2000}+\frac{11}{3000}$ as a reduced fraction.

Ex. 1.2.16: Rewrite $\frac{2}{35}+\frac{5}{77}$ as a reduced fraction.

Ex. 1.2.17: Find $\frac{3}{20}+\frac{7}{16}+\frac{7}{24}$

Ex. 1.2.18: Is $\frac{211}{240}$ a completely reduced fraction?

Ex. 1.2.19: Find the LCM of x^3y^4 and $x^2y^3z^5$.

Ex. 1.2.20: Add $\frac{a}{x^3}+\frac{b}{x^{10}}$

Ex. 1.2.21: Find $\frac{a}{xy^2}+\frac{b}{x^2y}$

Ex. 1.2.22: Find $\frac{3}{x^2} + \frac{2}{x^2+x}$

Ex. 1.2.23: Find $\frac{x}{(x^2+2x+1)(x+2)} + \frac{3}{(x^2+4x+4)(x+1)}$

Ex. 1.2.24: Find $\frac{x}{x^2+3x+2} + \frac{x+3}{x^2+4x+4}$

Ex. 1.2.25: Use long division with remainder to divide $x^3 + 3x^2 + 3x + 7$ by $x + 2$.

Ex. 1.2.26: Rewrite $\frac{x^3+3x^2+3x+7}{x+2}$ as a polynomial plus a proper fraction of polynomials.

Ex. 1.2.27: Rewrite $\frac{5+\frac{1}{7^2}}{3}$ as a reduced fraction

Ex. 1.2.28: Rewrite $\frac{y+\frac{1}{x^2}}{z}$ as a reduced fraction.

Ex. 1.2.29 : Rewrite $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}}$ as a reduced fraction.

Ex. 1.2.30: Rewrite $\frac{\frac{1}{a+h}-\frac{1}{a}}{h}$ as a reduced fraction.

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10} + \frac{3}{10} =$
- $\frac{x}{ab} + \frac{y}{ba} =$
- $\frac{7}{11} + \frac{2}{11} =$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10} + \frac{3}{10} = \frac{7}{10}$
- $\frac{x}{ab} + \frac{y}{ba} = \frac{x+y}{ab}$
- $\frac{7}{11} + \frac{2}{11} = \frac{9}{11}$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10} + \frac{3}{10} = \frac{7}{10}$
- $\frac{x}{ab} + \frac{y}{ba} = \frac{x+y}{ab}$
- $\frac{7}{11} + \frac{2}{11} = \frac{9}{11}$
- $\frac{3}{ab} + \frac{17}{ba} =$
- $\frac{3}{5} + \frac{7}{5} =$
- $\frac{3x}{3abc} + \frac{3x}{3cab} =$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10} + \frac{3}{10} = \frac{7}{10}$
- $\frac{x}{ab} + \frac{y}{ba} = \frac{x+y}{ab}$
- $\frac{7}{11} + \frac{2}{11} = \frac{9}{11}$
- $\frac{3}{ab} + \frac{17}{ba} = = \frac{20}{ab}$
- $\frac{3}{5} + \frac{7}{5} = \frac{10}{5} = 2$
- $\frac{3x}{3abc} + \frac{3x}{3cab} = \frac{2x}{abc}$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$
- $\frac{x}{ab}+\frac{y}{ba}=\frac{x+y}{ab}$
- $\frac{7}{11}+\frac{2}{11}=\frac{9}{11}$
- $\frac{3}{ab}+\frac{17}{ba}==\frac{20}{ab}$
- $\frac{3}{5}+\frac{7}{5}=\frac{10}{5}=2$
- $\frac{3x}{3abc}+\frac{3x}{3cab}=\frac{2x}{abc}$
- $\frac{3}{2ab}+\frac{17}{2ba}$
- $\frac{17}{16}+\frac{3}{16}=$
- $\frac{x}{yz}+\frac{3x}{yz}=$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$
- $\frac{3}{ab}+\frac{17}{ba}==\frac{20}{ab}$
- $\frac{3}{2ab}+\frac{17}{2ba}=\frac{10}{ab}$
- $\frac{x}{ab}+\frac{y}{ba}=\frac{x+y}{ab}$
- $\frac{3}{5}+\frac{7}{5}=\frac{10}{5}=2$
- $\frac{17}{16}+\frac{3}{16}=\frac{5}{4}$
- $\frac{7}{11}+\frac{2}{11}=\frac{9}{11}$
- $\frac{3x}{3abc}+\frac{3x}{3cab}=\frac{2x}{abc}$
- $\frac{x}{yz}+\frac{3x}{yz}=\frac{4x}{yz}$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$
- $\frac{3}{ab}+\frac{17}{ba}==\frac{20}{ab}$
- $\frac{3}{2ab}+\frac{17}{2ba}=\frac{10}{ab}$
- $\frac{8}{35}+\frac{6}{35}$
- $\frac{x}{ab}+\frac{y}{ba}=\frac{x+y}{ab}$
- $\frac{3}{5}+\frac{7}{5}=\frac{10}{5}=2$
- $\frac{17}{16}+\frac{3}{16}=\frac{5}{4}$
- $\frac{8}{15}+\frac{4}{15}=$
- $\frac{7}{11}+\frac{2}{11}=\frac{9}{11}$
- $\frac{3x}{3abc}+\frac{3x}{3cab}=\frac{2x}{abc}$
- $\frac{x}{yz}+\frac{3x}{yz}=\frac{4x}{yz}$
- $\frac{3x}{2abc}+\frac{x}{2cab}=$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10} + \frac{3}{10} = \frac{7}{10}$
- $\frac{3}{ab} + \frac{17}{ba} = = \frac{20}{ab}$
- $\frac{3}{2ab} + \frac{17}{2ba} = \frac{10}{ab}$
- $\frac{8}{35} + \frac{6}{35} = \frac{2}{5}$
- $\frac{x}{ab} + \frac{y}{ba} = \frac{x+y}{ab}$
- $\frac{3}{5} + \frac{7}{5} = \frac{10}{5} = 2$
- $\frac{17}{16} + \frac{3}{16} = \frac{5}{4}$
- $\frac{8}{15} + \frac{4}{15} = \frac{4}{5}$
- $\frac{7}{11} + \frac{2}{11} = \frac{9}{11}$
- $\frac{3x}{3abc} + \frac{3x}{3cab} = \frac{2x}{abc}$
- $\frac{x}{yz} + \frac{3x}{yz} = \frac{4x}{yz}$
- $\frac{3x}{2abc} + \frac{x}{2cab} = \frac{2x}{abc}$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10} + \frac{3}{10} = \frac{7}{10}$
- $\frac{x}{ab} + \frac{y}{ba} = \frac{x+y}{ab}$
- $\frac{7}{11} + \frac{2}{11} = \frac{9}{11}$
- $\frac{3}{ab} + \frac{17}{ba} = = \frac{20}{ab}$
- $\frac{3}{5} + \frac{7}{5} = \frac{10}{5} = 2$
- $\frac{3x}{3abc} + \frac{3x}{3cab} = \frac{2x}{abc}$
- $\frac{3}{2ab} + \frac{17}{2ba} = \frac{10}{ab}$
- $\frac{17}{16} + \frac{3}{16} = \frac{5}{4}$
- $\frac{x}{yz} + \frac{3x}{yz} = \frac{4x}{yz}$
- $\frac{8}{35} + \frac{6}{35} = \frac{2}{5}$
- $\frac{8}{15} + \frac{4}{15} = \frac{4}{5}$
- $\frac{3x}{2abc} + \frac{x}{2cab} = \frac{2x}{abc}$

Ex. 1.2.2: Rewrite each example as a reduced fraction:

- $\frac{12}{14} =$
- $\frac{100}{64} =$
- $\frac{450}{1800} =$
- $\frac{3}{45} =$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10} + \frac{3}{10} = \frac{7}{10}$
- $\frac{3}{ab} + \frac{17}{ba} = = \frac{20}{ab}$
- $\frac{3}{2ab} + \frac{17}{2ba} = \frac{10}{ab}$
- $\frac{8}{35} + \frac{6}{35} = \frac{2}{5}$
- $\frac{x}{ab} + \frac{y}{ba} = \frac{x+y}{ab}$
- $\frac{3}{5} + \frac{7}{5} = \frac{10}{5} = 2$
- $\frac{17}{16} + \frac{3}{16} = \frac{5}{4}$
- $\frac{8}{15} + \frac{4}{15} = \frac{4}{5}$
- $\frac{7}{11} + \frac{2}{11} = \frac{9}{11}$
- $\frac{3x}{3abc} + \frac{3x}{3cab} = \frac{2x}{abc}$
- $\frac{x}{yz} + \frac{3x}{yz} = \frac{4x}{yz}$
- $\frac{3x}{2abc} + \frac{x}{2cab} = \frac{2x}{abc}$

Ex. 1.2.2: Rewrite each example as a reduced fraction:

- $\frac{12}{14} = \frac{6}{7}$
- $\frac{100}{64} = \frac{25}{16}$
- $\frac{450}{1800} = \frac{1}{4}$
- $\frac{3}{45} = \frac{1}{15}$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$
- $\frac{3}{ab}+\frac{17}{ba}==\frac{20}{ab}$
- $\frac{3}{2ab}+\frac{17}{2ba}=\frac{10}{ab}$
- $\frac{8}{35}+\frac{6}{35}=\frac{2}{5}$
- $\frac{x}{ab}+\frac{y}{ba}=\frac{x+y}{ab}$
- $\frac{3}{5}+\frac{7}{5}=\frac{10}{5}=2$
- $\frac{17}{16}+\frac{3}{16}=\frac{5}{4}$
- $\frac{8}{15}+\frac{4}{15}=\frac{4}{5}$
- $\frac{7}{11}+\frac{2}{11}=\frac{9}{11}$
- $\frac{3x}{3abc}+\frac{3x}{3cab}=\frac{2x}{abc}$
- $\frac{x}{yz}+\frac{3x}{yz}=\frac{4x}{yz}$
- $\frac{3x}{2abc}+\frac{x}{2cab}=\frac{2x}{abc}$

Ex. 1.2.2: Rewrite each example as a reduced fraction:

- $\frac{12}{14}=\frac{6}{7}$
- $\frac{100}{64}=\frac{25}{16}$
- $\frac{450}{1800}=\frac{1}{4}$
- $\frac{3}{45}=\frac{1}{15}$
- $\frac{8}{14}=$
- $\frac{80}{64}=$
- $\frac{-30}{10}=$
- $\frac{45}{35}=$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10} + \frac{3}{10} = \frac{7}{10}$
- $\frac{x}{ab} + \frac{y}{ba} = \frac{x+y}{ab}$
- $\frac{7}{11} + \frac{2}{11} = \frac{9}{11}$
- $\frac{3}{ab} + \frac{17}{ba} = = \frac{20}{ab}$
- $\frac{3}{5} + \frac{7}{5} = \frac{10}{5} = 2$
- $\frac{3x}{3abc} + \frac{3x}{3cab} = \frac{2x}{abc}$
- $\frac{3}{2ab} + \frac{17}{2ba} = \frac{10}{ab}$
- $\frac{17}{16} + \frac{3}{16} = \frac{5}{4}$
- $\frac{x}{yz} + \frac{3x}{yz} = \frac{4x}{yz}$
- $\frac{8}{35} + \frac{6}{35} = \frac{2}{5}$
- $\frac{8}{15} + \frac{4}{15} = \frac{4}{5}$
- $\frac{3x}{2abc} + \frac{x}{2cab} = \frac{2x}{abc}$

Ex. 1.2.2: Rewrite each example as a reduced fraction:

- $\frac{12}{14} = \frac{6}{7}$
- $\frac{100}{64} = \frac{25}{16}$
- $\frac{450}{1800} = \frac{1}{4}$
- $\frac{3}{45} = \frac{1}{15}$
- $\frac{8}{14} = \frac{4}{7}$
- $\frac{80}{64} = \frac{5}{4}$
- $\frac{-30}{10} = -3$
- $\frac{45}{35} = \frac{9}{7}$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10} + \frac{3}{10} = \frac{7}{10}$
- $\frac{3}{ab} + \frac{17}{ba} = = \frac{20}{ab}$
- $\frac{3}{2ab} + \frac{17}{2ba} = \frac{10}{ab}$
- $\frac{8}{35} + \frac{6}{35} = \frac{2}{5}$
- $\frac{x}{ab} + \frac{y}{ba} = \frac{x+y}{ab}$
- $\frac{3}{5} + \frac{7}{5} = \frac{10}{5} = 2$
- $\frac{17}{16} + \frac{3}{16} = \frac{5}{4}$
- $\frac{8}{15} + \frac{4}{15} = \frac{4}{5}$
- $\frac{7}{11} + \frac{2}{11} = \frac{9}{11}$
- $\frac{3x}{3abc} + \frac{3x}{3cab} = \frac{2x}{abc}$
- $\frac{x}{yz} + \frac{3x}{yz} = \frac{4x}{yz}$
- $\frac{3x}{2abc} + \frac{x}{2cab} = \frac{2x}{abc}$

Ex. 1.2.2: Rewrite each example as a reduced fraction:

- $\frac{12}{14} = \frac{6}{7}$
- $\frac{100}{64} = \frac{25}{16}$
- $\frac{450}{1800} = \frac{1}{4}$
- $\frac{3}{45} = \frac{1}{15}$
- $\frac{8}{14} = \frac{4}{7}$
- $\frac{80}{64} = \frac{5}{4}$
- $\frac{-30}{10} = -3$
- $\frac{45}{35} = \frac{9}{7}$
- $\frac{14}{12} =$
- $\frac{64}{100} =$
- $\frac{1800}{450} =$
- $\frac{30}{45} =$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10} + \frac{3}{10} = \frac{7}{10}$
- $\frac{3}{ab} + \frac{17}{ba} = = \frac{20}{ab}$
- $\frac{3}{2ab} + \frac{17}{2ba} = \frac{10}{ab}$
- $\frac{8}{35} + \frac{6}{35} = \frac{2}{5}$
- $\frac{x}{ab} + \frac{y}{ba} = \frac{x+y}{ab}$
- $\frac{3}{5} + \frac{7}{5} = \frac{10}{5} = 2$
- $\frac{17}{16} + \frac{3}{16} = \frac{5}{4}$
- $\frac{8}{15} + \frac{4}{15} = \frac{4}{5}$
- $\frac{7}{11} + \frac{2}{11} = \frac{9}{11}$
- $\frac{3x}{3abc} + \frac{3x}{3cab} = \frac{2x}{abc}$
- $\frac{x}{yz} + \frac{3x}{yz} = \frac{4x}{yz}$
- $\frac{3x}{2abc} + \frac{x}{2cab} = \frac{2x}{abc}$

Ex. 1.2.2: Rewrite each example as a reduced fraction:

- $\frac{12}{14} = \frac{6}{7}$
- $\frac{100}{64} = \frac{25}{16}$
- $\frac{450}{1800} = \frac{1}{4}$
- $\frac{3}{45} = \frac{1}{15}$
- $\frac{14}{12} = \frac{7}{6}$
- $\frac{64}{100} = \frac{16}{25}$
- $\frac{1800}{450} = 4$
- $\frac{30}{45} = \frac{2}{3}$
- $\frac{8}{14} = \frac{4}{7}$
- $\frac{80}{64} = \frac{5}{4}$
- $\frac{-30}{10} = -3$
- $\frac{45}{35} = \frac{9}{7}$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$
- $\frac{x}{ab}+\frac{y}{ba}=\frac{x+y}{ab}$
- $\frac{7}{11}+\frac{2}{11}=\frac{9}{11}$
- $\frac{3}{ab}+\frac{17}{ba}==\frac{20}{ab}$
- $\frac{3}{5}+\frac{7}{5}=\frac{10}{5}=2$
- $\frac{3x}{3abc}+\frac{3x}{3cab}=\frac{2x}{abc}$
- $\frac{3}{2ab}+\frac{17}{2ba}=\frac{10}{ab}$
- $\frac{17}{16}+\frac{3}{16}=\frac{5}{4}$
- $\frac{x}{yz}+\frac{3x}{yz}=\frac{4x}{yz}$
- $\frac{8}{35}+\frac{6}{35}=\frac{2}{5}$
- $\frac{8}{15}+\frac{4}{15}=\frac{4}{5}$
- $\frac{3x}{2abc}+\frac{x}{2cab}=\frac{2x}{abc}$

Ex. 1.2.2: Rewrite each example as a reduced fraction:

- $\frac{12}{14}=\frac{6}{7}$
- $\frac{100}{64}=\frac{25}{16}$
- $\frac{450}{1800}=\frac{1}{4}$
- $\frac{3}{45}=\frac{1}{15}$
- $\frac{8}{14}=\frac{4}{7}$
- $\frac{80}{64}=\frac{5}{4}$
- $\frac{-30}{10}=-3$
- $\frac{45}{35}=\frac{9}{7}$
- $\frac{14}{12}=\frac{7}{6}$
- $\frac{64}{100}=\frac{16}{25}$
- $\frac{1800}{450}=4$
- $\frac{30}{45}=\frac{2}{3}$
- $\frac{14}{8}=$
- $\frac{30}{64}=$
- $\frac{10}{-30}=$
- $\frac{35}{45}=$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$
- $\frac{3}{ab}+\frac{17}{ba}==\frac{20}{ab}$
- $\frac{3}{2ab}+\frac{17}{2ba}=\frac{10}{ab}$
- $\frac{8}{35}+\frac{6}{35}=\frac{2}{5}$
- $\frac{x}{ab}+\frac{y}{ba}=\frac{x+y}{ab}$
- $\frac{3}{5}+\frac{7}{5}=\frac{10}{5}=2$
- $\frac{17}{16}+\frac{3}{16}=\frac{5}{4}$
- $\frac{8}{15}+\frac{4}{15}=\frac{4}{5}$
- $\frac{7}{11}+\frac{2}{11}=\frac{9}{11}$
- $\frac{3x}{3abc}+\frac{3x}{3cab}=\frac{2x}{abc}$
- $\frac{x}{yz}+\frac{3x}{yz}=\frac{4x}{yz}$
- $\frac{3x}{2abc}+\frac{x}{2cab}=\frac{2x}{abc}$

Ex. 1.2.2: Rewrite each example as a reduced fraction:

- $\frac{12}{14}=\frac{6}{7}$
- $\frac{100}{64}=\frac{25}{16}$
- $\frac{450}{1800}=\frac{1}{4}$
- $\frac{3}{45}=\frac{1}{15}$
- $\frac{14}{12}=\frac{7}{6}$
- $\frac{64}{100}=\frac{16}{25}$
- $\frac{1800}{450}=4$
- $\frac{30}{45}=\frac{2}{3}$
- $\frac{8}{14}=\frac{4}{7}$
- $\frac{80}{64}=\frac{5}{4}$
- $\frac{-30}{10}=-3$
- $\frac{45}{35}=\frac{9}{7}$
- $\frac{14}{8}=\frac{7}{4}$
- $\frac{30}{64}=\frac{15}{32}$
- $\frac{10}{-30}=-\frac{1}{3}$
- $\frac{35}{45}=\frac{7}{9}$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$
- $\frac{3}{ab}+\frac{17}{ba}==\frac{20}{ab}$
- $\frac{3}{2ab}+\frac{17}{2ba}=\frac{10}{ab}$
- $\frac{8}{35}+\frac{6}{35}=\frac{2}{5}$
- $\frac{x}{ab}+\frac{y}{ba}=\frac{x+y}{ab}$
- $\frac{3}{5}+\frac{7}{5}=\frac{10}{5}=2$
- $\frac{17}{16}+\frac{3}{16}=\frac{5}{4}$
- $\frac{8}{15}+\frac{4}{15}=\frac{4}{5}$
- $\frac{7}{11}+\frac{2}{11}=\frac{9}{11}$
- $\frac{3x}{3abc}+\frac{3x}{3cab}=\frac{2x}{abc}$
- $\frac{x}{yz}+\frac{3x}{yz}=\frac{4x}{yz}$
- $\frac{3x}{2abc}+\frac{x}{2cab}=\frac{2x}{abc}$

Ex. 1.2.2: Rewrite each example as a reduced fraction:

- $\frac{12}{14}=\frac{6}{7}$
- $\frac{100}{64}=\frac{25}{16}$
- $\frac{450}{1800}=\frac{1}{4}$
- $\frac{3}{45}=\frac{1}{15}$
- $\frac{14}{12}=\frac{7}{6}$
- $\frac{64}{100}=\frac{16}{25}$
- $\frac{1800}{450}=4$
- $\frac{30}{45}=\frac{2}{3}$
- $\frac{8}{14}=\frac{4}{7}$
- $\frac{80}{64}=\frac{5}{4}$
- $\frac{-30}{10}=-3$
- $\frac{45}{35}=\frac{9}{7}$
- $\frac{14}{8}=\frac{7}{4}$
- $\frac{30}{64}=\frac{15}{32}$
- $\frac{10}{-30}=-\frac{1}{3}$
- $\frac{35}{45}=\frac{7}{9}$

Ex. 1.2.3: Factor each blue sum of terms:

- $ax^8y+bx^6z=$
- $x^{15}+3x^7=$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$
- $\frac{x}{ab}+\frac{y}{ba}=\frac{x+y}{ab}$
- $\frac{7}{11}+\frac{2}{11}=\frac{9}{11}$
- $\frac{3}{ab}+\frac{17}{ba}==\frac{20}{ab}$
- $\frac{3}{5}+\frac{7}{5}=\frac{10}{5}=2$
- $\frac{3x}{3abc}+\frac{3x}{3cab}=\frac{2x}{abc}$
- $\frac{3}{2ab}+\frac{17}{2ba}=\frac{10}{ab}$
- $\frac{17}{16}+\frac{3}{16}=\frac{5}{4}$
- $\frac{x}{yz}+\frac{3x}{yz}=\frac{4x}{yz}$
- $\frac{8}{35}+\frac{6}{35}=\frac{2}{5}$
- $\frac{8}{15}+\frac{4}{15}=\frac{4}{5}$
- $\frac{3x}{2abc}+\frac{x}{2cab}=\frac{2x}{abc}$

Ex. 1.2.2: Rewrite each example as a reduced fraction:

- $\frac{12}{14}=\frac{6}{7}$
- $\frac{100}{64}=\frac{25}{16}$
- $\frac{450}{1800}=\frac{1}{4}$
- $\frac{3}{45}=\frac{1}{15}$
- $\frac{8}{14}=\frac{4}{7}$
- $\frac{80}{64}=\frac{5}{4}$
- $\frac{-30}{10}=-3$
- $\frac{45}{35}=\frac{9}{7}$
- $\frac{14}{12}=\frac{7}{6}$
- $\frac{64}{100}=\frac{16}{25}$
- $\frac{1800}{450}=4$
- $\frac{30}{45}=\frac{2}{3}$
- $\frac{14}{8}=\frac{7}{4}$
- $\frac{30}{64}=\frac{15}{32}$
- $\frac{10}{-30}=-\frac{1}{3}$
- $\frac{35}{45}=\frac{7}{9}$

Ex. 1.2.3: Factor each blue sum of terms:

- $ax^8y+bx^6z=x^6(ax^2y+bz)$
- $x^{15}+3x^7=x^7(x^8+3)$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10} + \frac{3}{10} = \frac{7}{10}$
- $\frac{3}{ab} + \frac{17}{ba} = = \frac{20}{ab}$
- $\frac{3}{2ab} + \frac{17}{2ba} = \frac{10}{ab}$
- $\frac{8}{35} + \frac{6}{35} = \frac{2}{5}$
- $\frac{x}{ab} + \frac{y}{ba} = \frac{x+y}{ab}$
- $\frac{3}{5} + \frac{7}{5} = \frac{10}{5} = 2$
- $\frac{17}{16} + \frac{3}{16} = \frac{5}{4}$
- $\frac{8}{15} + \frac{4}{15} = \frac{4}{5}$
- $\frac{7}{11} + \frac{2}{11} = \frac{9}{11}$
- $\frac{3x}{3abc} + \frac{3x}{3cab} = \frac{2x}{abc}$
- $\frac{x}{yz} + \frac{3x}{yz} = \frac{4x}{yz}$
- $\frac{3x}{2abc} + \frac{x}{2cab} = \frac{2x}{abc}$

Ex. 1.2.2: Rewrite each example as a reduced fraction:

- $\frac{12}{14} = \frac{6}{7}$
- $\frac{100}{64} = \frac{25}{16}$
- $\frac{450}{1800} = \frac{1}{4}$
- $\frac{3}{45} = \frac{1}{15}$
- $\frac{14}{12} = \frac{7}{6}$
- $\frac{64}{100} = \frac{16}{25}$
- $\frac{1800}{450} = 4$
- $\frac{30}{45} = \frac{2}{3}$
- $\frac{8}{14} = \frac{4}{7}$
- $\frac{80}{64} = \frac{5}{4}$
- $\frac{-30}{10} = -3$
- $\frac{45}{35} = \frac{9}{7}$
- $\frac{14}{8} = \frac{7}{4}$
- $\frac{30}{64} = \frac{15}{32}$
- $\frac{10}{-30} = -\frac{1}{3}$
- $\frac{35}{45} = \frac{7}{9}$

Ex. 1.2.3: Factor each blue sum of terms:

- $ax^8y + bx^6z = x^6(ax^2y + bz)$
- $ax^4y^8 + bx^6z =$
- $x^{15} + 3x^7 = x^7(x^8 + 3)$
- $x^5 + 3x^7 =$

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$
- $\frac{3}{ab}+\frac{17}{ba}==\frac{20}{ab}$
- $\frac{3}{2ab}+\frac{17}{2ba}=\frac{10}{ab}$
- $\frac{8}{35}+\frac{6}{35}=\frac{2}{5}$
- $\frac{x}{ab}+\frac{y}{ba}=\frac{x+y}{ab}$
- $\frac{3}{5}+\frac{7}{5}=\frac{10}{5}=2$
- $\frac{17}{16}+\frac{3}{16}=\frac{5}{4}$
- $\frac{8}{15}+\frac{4}{15}=\frac{4}{5}$
- $\frac{7}{11}+\frac{2}{11}=\frac{9}{11}$
- $\frac{3x}{3abc}+\frac{3x}{3cab}=\frac{2x}{abc}$
- $\frac{x}{yz}+\frac{3x}{yz}=\frac{4x}{yz}$
- $\frac{3x}{2abc}+\frac{x}{2cab}=\frac{2x}{abc}$

Ex. 1.2.2: Rewrite each example as a reduced fraction:

- $\frac{12}{14}=\frac{6}{7}$
- $\frac{100}{64}=\frac{25}{16}$
- $\frac{450}{1800}=\frac{1}{4}$
- $\frac{3}{45}=\frac{1}{15}$
- $\frac{14}{12}=\frac{7}{6}$
- $\frac{64}{100}=\frac{16}{25}$
- $\frac{1800}{450}=4$
- $\frac{30}{45}=\frac{2}{3}$
- $\frac{8}{14}=\frac{4}{7}$
- $\frac{80}{64}=\frac{5}{4}$
- $\frac{-30}{10}=-3$
- $\frac{45}{35}=\frac{9}{7}$
- $\frac{14}{8}=\frac{7}{4}$
- $\frac{30}{64}=\frac{15}{32}$
- $\frac{10}{-30}=-\frac{1}{3}$
- $\frac{35}{45}=\frac{7}{9}$

Ex. 1.2.3: Factor each blue sum of terms:

- $ax^8y+bx^6z=x^6(ax^2y+bz)$
- $ax^4y^8+bx^6z=x^4(ay^8+bx^2z)$
- $x^{15}+3x^7=x^7(x^8+3)$
- $x^5+3x^7=x^5(1+3x^2)$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10} + \frac{3}{10} = \frac{7}{10}$
- $\frac{3}{ab} + \frac{17}{ba} = = \frac{20}{ab}$
- $\frac{3}{2ab} + \frac{17}{2ba} = \frac{10}{ab}$
- $\frac{8}{35} + \frac{6}{35} = \frac{2}{5}$
- $\frac{x}{ab} + \frac{y}{ba} = \frac{x+y}{ab}$
- $\frac{3}{5} + \frac{7}{5} = \frac{10}{5} = 2$
- $\frac{17}{16} + \frac{3}{16} = \frac{5}{4}$
- $\frac{8}{15} + \frac{4}{15} = \frac{4}{5}$
- $\frac{7}{11} + \frac{2}{11} = \frac{9}{11}$
- $\frac{3x}{3abc} + \frac{3x}{3cab} = \frac{2x}{abc}$
- $\frac{x}{yz} + \frac{3x}{yz} = \frac{4x}{yz}$
- $\frac{3x}{2abc} + \frac{x}{2cab} = \frac{2x}{abc}$

Ex. 1.2.2: Rewrite each example as a reduced fraction:

- $\frac{12}{14} = \frac{6}{7}$
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- $\frac{1800}{450} = 4$
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- $\frac{8}{14} = \frac{4}{7}$
- $\frac{80}{64} = \frac{5}{4}$
- $\frac{-30}{10} = -3$
- $\frac{45}{35} = \frac{9}{7}$
- $\frac{14}{8} = \frac{7}{4}$
- $\frac{30}{64} = \frac{15}{32}$
- $\frac{10}{-30} = -\frac{1}{3}$
- $\frac{35}{45} = \frac{7}{9}$

Ex. 1.2.3: Factor each blue sum of terms:

- $ax^8y + bx^6z = x^6(ax^2y + bz)$
- $ax^4y^8 + bx^6z = x^4(ay^8 + bx^2z)$
- $10xy^2z + 12x^2yz =$
- $x^{15} + 3x^7 = x^7(x^8 + 3)$
- $x^5 + 3x^7 = x^5(1 + 3x^2)$
- $15r^2s + 25rs^2 =$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$
- $\frac{3}{ab}+\frac{17}{ba}==\frac{20}{ab}$
- $\frac{3}{2ab}+\frac{17}{2ba}=\frac{10}{ab}$
- $\frac{8}{35}+\frac{6}{35}=\frac{2}{5}$
- $\frac{x}{ab}+\frac{y}{ba}=\frac{x+y}{ab}$
- $\frac{3}{5}+\frac{7}{5}=\frac{10}{5}=2$
- $\frac{17}{16}+\frac{3}{16}=\frac{5}{4}$
- $\frac{8}{15}+\frac{4}{15}=\frac{4}{5}$
- $\frac{7}{11}+\frac{2}{11}=\frac{9}{11}$
- $\frac{3x}{3abc}+\frac{3x}{3cab}=\frac{2x}{abc}$
- $\frac{x}{yz}+\frac{3x}{yz}=\frac{4x}{yz}$
- $\frac{3x}{2abc}+\frac{x}{2cab}=\frac{2x}{abc}$

Ex. 1.2.2: Rewrite each example as a reduced fraction:

- $\frac{12}{14}=\frac{6}{7}$
- $\frac{100}{64}=\frac{25}{16}$
- $\frac{450}{1800}=\frac{1}{4}$
- $\frac{3}{45}=\frac{1}{15}$
- $\frac{14}{12}=\frac{7}{6}$
- $\frac{64}{100}=\frac{16}{25}$
- $\frac{1800}{450}=4$
- $\frac{30}{45}=\frac{2}{3}$
- $\frac{8}{14}=\frac{4}{7}$
- $\frac{80}{64}=\frac{5}{4}$
- $\frac{-30}{10}=-3$
- $\frac{45}{35}=\frac{9}{7}$
- $\frac{14}{8}=\frac{7}{4}$
- $\frac{30}{64}=\frac{15}{32}$
- $\frac{10}{-30}=-\frac{1}{3}$
- $\frac{35}{45}=\frac{7}{9}$

Ex. 1.2.3: Factor each blue sum of terms:

- $ax^8y+bx^6z = x^6(ax^2y+bz)$
- $ax^4y^8+bx^6z = x^4(ay^8+bx^2z)$
- $10xy^2z+12x^2yz = 2xyz(5y+6x)$
- $x^{15}+3x^7 = x^7(x^8+3)$
- $x^5+3x^7 = x^5(1+3x^2)$
- $15r^2s+25rs^2 = 5rs(3r+5s)$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$
- $\frac{3}{ab}+\frac{17}{ba}==\frac{20}{ab}$
- $\frac{3}{2ab}+\frac{17}{2ba}=\frac{10}{ab}$
- $\frac{8}{35}+\frac{6}{35}=\frac{2}{5}$
- $\frac{x}{ab}+\frac{y}{ba}=\frac{x+y}{ab}$
- $\frac{3}{5}+\frac{7}{5}=\frac{10}{5}=2$
- $\frac{17}{16}+\frac{3}{16}=\frac{5}{4}$
- $\frac{8}{15}+\frac{4}{15}=\frac{4}{5}$
- $\frac{7}{11}+\frac{2}{11}=\frac{9}{11}$
- $\frac{3x}{3abc}+\frac{3x}{3cab}=\frac{2x}{abc}$
- $\frac{x}{yz}+\frac{3x}{yz}=\frac{4x}{yz}$
- $\frac{3x}{2abc}+\frac{x}{2cab}=\frac{2x}{abc}$

Ex. 1.2.2: Rewrite each example as a reduced fraction:

- $\frac{12}{14}=\frac{6}{7}$
- $\frac{100}{64}=\frac{25}{16}$
- $\frac{450}{1800}=\frac{1}{4}$
- $\frac{3}{45}=\frac{1}{15}$
- $\frac{8}{14}=\frac{4}{7}$
- $\frac{80}{64}=\frac{5}{4}$
- $\frac{-30}{10}=-3$
- $\frac{45}{35}=\frac{9}{7}$
- $\frac{14}{12}=\frac{7}{6}$
- $\frac{64}{100}=\frac{16}{25}$
- $\frac{1800}{450}=4$
- $\frac{30}{45}=\frac{2}{3}$
- $\frac{14}{8}=\frac{7}{4}$
- $\frac{30}{64}=\frac{15}{32}$
- $\frac{10}{-30}=-\frac{1}{3}$
- $\frac{35}{45}=\frac{7}{9}$

Ex. 1.2.3: Factor each blue sum of terms:

- $ax^8y+bx^6z = x^6(ax^2y+bz)$
- $ax^4y^8+bx^6z = x^4(ay^8+bx^2z)$
- $10xy^2z+12x^2yz = 2xyz(5y+6x)$
- $a^2c^2bd+ab^2cd^2 =$
- $x^{15}+3x^7 = x^7(x^8+3)$
- $x^5+3x^7 = x^5(1+3x^2)$
- $15r^2s+25rs^2 = 5rs(3r+5s)$
- $4x^5+10x^5 =$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$
- $\frac{x}{ab}+\frac{y}{ba}=\frac{x+y}{ab}$
- $\frac{7}{11}+\frac{2}{11}=\frac{9}{11}$
- $\frac{3}{ab}+\frac{17}{ba}==\frac{20}{ab}$
- $\frac{3}{5}+\frac{7}{5}=\frac{10}{5}=2$
- $\frac{3x}{3abc}+\frac{3x}{3cab}=\frac{2x}{abc}$
- $\frac{3}{2ab}+\frac{17}{2ba}=\frac{10}{ab}$
- $\frac{17}{16}+\frac{3}{16}=\frac{5}{4}$
- $\frac{x}{yz}+\frac{3x}{yz}=\frac{4x}{yz}$
- $\frac{8}{35}+\frac{6}{35}=\frac{2}{5}$
- $\frac{8}{15}+\frac{4}{15}=\frac{4}{5}$
- $\frac{3x}{2abc}+\frac{x}{2cab}=\frac{2x}{abc}$

Ex. 1.2.2: Rewrite each example as a reduced fraction:

- $\frac{12}{14}=\frac{6}{7}$
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- $\frac{450}{1800}=\frac{1}{4}$
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- $\frac{8}{14}=\frac{4}{7}$
- $\frac{80}{64}=\frac{5}{4}$
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- $\frac{45}{35}=\frac{9}{7}$
- $\frac{14}{12}=\frac{7}{6}$
- $\frac{64}{100}=\frac{16}{25}$
- $\frac{1800}{450}=4$
- $\frac{30}{45}=\frac{2}{3}$
- $\frac{14}{8}=\frac{7}{4}$
- $\frac{30}{64}=\frac{15}{32}$
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- $ax^8y+bx^6z = x^6(ax^2y+bz)$
- $x^{15}+3x^7 = x^7(x^8+3)$
- $ax^4y^8+bx^6z = x^4(ay^8+bx^2z)$
- $x^5+3x^7 = x^5(1+3x^2)$
- $10xy^2z+12x^2yz = 2xyz(5y+6x)$
- $15r^2s+25rs^2 = 5rs(3r+5s)$
- $a^2c^2bd+ab^2cd^2 = abcd(ac+bd)$
- $4x^5+10x^5 = 14x^5$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$
- $\frac{3}{ab}+\frac{17}{ba}==\frac{20}{ab}$
- $\frac{3}{2ab}+\frac{17}{2ba}=\frac{10}{ab}$
- $\frac{8}{35}+\frac{6}{35}=\frac{2}{5}$
- $\frac{x}{ab}+\frac{y}{ba}=\frac{x+y}{ab}$
- $\frac{3}{5}+\frac{7}{5}=\frac{10}{5}=2$
- $\frac{17}{16}+\frac{3}{16}=\frac{5}{4}$
- $\frac{8}{15}+\frac{4}{15}=\frac{4}{5}$
- $\frac{7}{11}+\frac{2}{11}=\frac{9}{11}$
- $\frac{3x}{3abc}+\frac{3x}{3cab}=\frac{2x}{abc}$
- $\frac{x}{yz}+\frac{3x}{yz}=\frac{4x}{yz}$
- $\frac{3x}{2abc}+\frac{x}{2cab}=\frac{2x}{abc}$

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- $\frac{30}{45}=\frac{2}{3}$
- $\frac{8}{14}=\frac{4}{7}$
- $\frac{80}{64}=\frac{5}{4}$
- $\frac{-30}{10}=-3$
- $\frac{45}{35}=\frac{9}{7}$
- $\frac{14}{8}=\frac{7}{4}$
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Ex. 1.2.3: Factor each blue sum of terms:

- $ax^8y+bx^6z=x^6(ax^2y+bz)$
- $ax^4y^8+bx^6z=x^4(ay^8+bx^2z)$
- $10xy^2z+12x^2yz=2xyz(5y+6x)$
- $a^2c^2bd+ab^2cd^2=abcd(ac+bd)$
- $a^5b^8+a^7b^5=$
- $x^{15}+3x^7=x^7(x^8+3)$
- $x^5+3x^7=x^5(1+3x^2)$
- $15r^2s+25rs^2=5rs(3r+5s)$
- $4x^5+10x^5=14x^5$
- $4a^5b^8+6a^3b^5=$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10} + \frac{3}{10} = \frac{7}{10}$
- $\frac{3}{ab} + \frac{17}{ba} = = \frac{20}{ab}$
- $\frac{3}{2ab} + \frac{17}{2ba} = \frac{10}{ab}$
- $\frac{8}{35} + \frac{6}{35} = \frac{2}{5}$
- $\frac{x}{ab} + \frac{y}{ba} = \frac{x+y}{ab}$
- $\frac{3}{5} + \frac{7}{5} = \frac{10}{5} = 2$
- $\frac{17}{16} + \frac{3}{16} = \frac{5}{4}$
- $\frac{8}{15} + \frac{4}{15} = \frac{4}{5}$
- $\frac{7}{11} + \frac{2}{11} = \frac{9}{11}$
- $\frac{3x}{3abc} + \frac{3x}{3cab} = \frac{2x}{abc}$
- $\frac{x}{yz} + \frac{3x}{yz} = \frac{4x}{yz}$
- $\frac{3x}{2abc} + \frac{x}{2cab} = \frac{2x}{abc}$

Ex. 1.2.2: Rewrite each example as a reduced fraction:

- $\frac{12}{14} = \frac{6}{7}$
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- $\frac{450}{1800} = \frac{1}{4}$
- $\frac{3}{45} = \frac{1}{15}$
- $\frac{14}{12} = \frac{7}{6}$
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- $\frac{-30}{10} = -3$
- $\frac{45}{35} = \frac{9}{7}$
- $\frac{14}{8} = \frac{7}{4}$
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- $ax^8y + bx^6z = x^6(ax^2y + bz)$
- $ax^4y^8 + bx^6z = x^4(ay^8 + bx^2z)$
- $10xy^2z + 12x^2yz = 2xyz(5y + 6x)$
- $a^2c^2bd + ab^2cd^2 = abcd(ac + bd)$
- $a^5b^8 + a^7b^5 = a^5b^5(b^3 + a^2)$
- $x^{15} + 3x^7 = x^7(x^8 + 3)$
- $x^5 + 3x^7 = x^5(1 + 3x^2)$
- $15r^2s + 25rs^2 = 5rs(3r + 5s)$
- $4x^5 + 10x^5 = 14x^5$
- $4a^5b^8 + 6a^3b^5 = 2a^3b^5(2a^2b^3 + 3)$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$
- $\frac{3}{ab}+\frac{17}{ba}==\frac{20}{ab}$
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- $\frac{17}{16}+\frac{3}{16}=\frac{5}{4}$
- $\frac{8}{15}+\frac{4}{15}=\frac{4}{5}$
- $\frac{7}{11}+\frac{2}{11}=\frac{9}{11}$
- $\frac{3x}{3abc}+\frac{3x}{3cab}=\frac{2x}{abc}$
- $\frac{x}{yz}+\frac{3x}{yz}=\frac{4x}{yz}$
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- $\frac{-30}{10}=-3$
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- $ax^4y^8+bx^6z=x^4(ay^8+bx^2z)$
- $10xy^2z+12x^2yz=2xyz(5y+6x)$
- $a^2c^2bd+ab^2cd^2=abcd(ac+bd)$
- $a^5b^8+a^7b^5=a^5b^5(b^3+a^2)$
- $abc^5+a^7b^5c^2=$
- $x^{15}+3x^7=x^7(x^8+3)$
- $x^5+3x^7=x^5(1+3x^2)$
- $15r^2s+25rs^2=5rs(3r+5s)$
- $4x^5+10x^5=14x^5$
- $4a^5b^8+6a^3b^5=2a^3b^5(2a^2b^3+3)$
- $rst+abc=$

Section 1.2 Review: Fractions

Ex. 1.2.1: Find and reduce the sum

- $\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$
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- $\frac{8}{35}+\frac{6}{35}=\frac{2}{5}$
- $\frac{x}{ab}+\frac{y}{ba}=\frac{x+y}{ab}$
- $\frac{3}{5}+\frac{7}{5}=\frac{10}{5}=2$
- $\frac{17}{16}+\frac{3}{16}=\frac{5}{4}$
- $\frac{8}{15}+\frac{4}{15}=\frac{4}{5}$
- $\frac{7}{11}+\frac{2}{11}=\frac{9}{11}$
- $\frac{3x}{3abc}+\frac{3x}{3cab}=\frac{2x}{abc}$
- $\frac{x}{yz}+\frac{3x}{yz}=\frac{4x}{yz}$
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- $\frac{1800}{450}=4$
- $\frac{30}{45}=\frac{2}{3}$
- $\frac{8}{14}=\frac{4}{7}$
- $\frac{80}{64}=\frac{5}{4}$
- $\frac{-30}{10}=-3$
- $\frac{45}{35}=\frac{9}{7}$
- $\frac{14}{8}=\frac{7}{4}$
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Ex. 1.2.3: Factor each blue sum of terms:

- $ax^8y+bx^6z=x^6(ax^2y+bz)$
- $ax^4y^8+bx^6z=x^4(ay^8+bx^2z)$
- $10xy^2z+12x^2yz=2xyz(5y+6x)$
- $a^2c^2bd+ab^2cd^2=abcd(ac+bd)$
- $a^5b^8+a^7b^5=a^5b^5(b^3+a^2)$
- $abc^5+a^7b^5c^2=abc^2(c^3+a^6b^4)$
- $x^{15}+3x^7=x^7(x^8+3)$
- $x^5+3x^7=x^5(1+3x^2)$
- $15r^2s+25rs^2=5rs(3r+5s)$
- $4x^5+10x^5=14x^5$
- $4a^5b^8+6a^3b^5=2a^3b^5(2a^2b^3+3)$
- $rst+abc=rst+abc$

Ex. 1.2.4: Factor numerator; then reduce:

- $\frac{a^5 b^6 + a^6 b^5}{a^6 b^6} =$
- $\frac{4xy - 6xy^2}{8x^2 y^2} =$
- $\frac{x^3 y^3 + x^4 y^4}{x^5 y^5} =$
- $\frac{a^2 + 2ab + b^2}{ac + cb} =$

Ex. 1.2.4: Factor numerator; then reduce:

- $\frac{a^5 b^6 + a^6 b^5}{a^6 b^6} = \frac{a+b}{ab}$
- $\frac{4xy - 6xy^2}{8x^2 y^2} = \frac{2-3y}{4xy}$
- $\frac{x^3 y^3 + x^4 y^4}{x^5 y^5} = \frac{1+xy}{x^2 y^2}$
- $\frac{a^2 + 2ab + b^2}{ac + cb} = \frac{a+b}{c}$

Ex. 1.2.4: Factor numerator; then reduce:

- $\frac{a^5b^6+a^6b^5}{a^6b^6}=\frac{a+b}{ab}$
- $\frac{4xy-6xy^2}{8x^2y^2}=\frac{2-3y}{4xy}$
- $\frac{x^3y^3+x^4y^4}{x^5y^5}=\frac{1+xy}{x^2y^2}$
- $\frac{a^2+2ab+b^2}{ac+cb}=\frac{a+b}{c}$
- $\frac{ax+ay}{2a}=$
- $\frac{mx+may}{2ma}=$
- $\frac{cy-cx}{c^2x^2-c^2y^2}=$
- $\frac{ar+as}{ra^2}=$

Ex. 1.2.4: Factor numerator; then reduce:

- $\frac{a^5b^6+a^6b^5}{a^6b^6}=\frac{a+b}{ab}$
- $\frac{4xy-6xy^2}{8x^2y^2}=\frac{2-3y}{4xy}$
- $\frac{x^3y^3+x^4y^4}{x^5y^5}=\frac{1+xy}{x^2y^2}$
- $\frac{a^2+2ab+b^2}{ac+cb}=\frac{a+b}{c}$
- $\frac{ax+ay}{2a}=\frac{x+y}{2}$
- $\frac{mx+may}{2ma}=\frac{x+ay}{2a}$
- $\frac{cy-cx}{c^2x^2-c^2y^2}=-\frac{1}{c(x+y)}$
- $\frac{ar+as}{ra^2}=\frac{r+s}{ar}$

Ex. 1.2.4: Factor numerator; then reduce:

- $\frac{a^5 b^6 + a^6 b^5}{a^6 b^6} = \frac{a+b}{ab}$
- $\frac{4xy - 6xy^2}{8x^2 y^2} = \frac{2-3y}{4xy}$
- $\frac{x^3 y^3 + x^4 y^4}{x^5 y^5} = \frac{1+xy}{x^2 y^2}$
- $\frac{a^2 + 2ab + b^2}{ac + cb} = \frac{a+b}{c}$
- $\frac{ax+ay}{2a} = \frac{x+y}{2}$
- $\frac{mx+may}{2ma} = \frac{x+ay}{2a}$
- $\frac{cy-cx}{c^2 x^2 - c^2 y^2} = -\frac{1}{c(x+y)}$
- $\frac{ar+as}{ra^2} = \frac{r+s}{ar}$

Ex. 1.2.5: Reduce:

- $\frac{(x+1)(x+4)+(x+1)(x+5)}{2x+2} =$
- $\frac{(x+3)(x+4)+(2x+6)(x+5)}{3x+9} =$
- $\frac{(b+1)(x+4)+(x+5)(b+1)}{2+2b} =$
- $\frac{(x+1)(3x+4)+(x+1)(3x+5)}{2x+2} =$

Ex. 1.2.4: Factor numerator; then reduce:

- $\frac{a^5b^6+a^6b^5}{a^6b^6}=\frac{a+b}{ab}$
- $\frac{4xy-6xy^2}{8x^2y^2}=\frac{2-3y}{4xy}$
- $\frac{x^3y^3+x^4y^4}{x^5y^5}=\frac{1+xy}{x^2y^2}$
- $\frac{a^2+2ab+b^2}{ac+cb}=\frac{a+b}{c}$
- $\frac{ax+ay}{2a}=\frac{x+y}{2}$
- $\frac{mx+may}{2ma}=\frac{x+ay}{2a}$
- $\frac{cy-cx}{c^2x^2-c^2y^2}=-\frac{1}{c(x+y)}$
- $\frac{ar+as}{ra^2}=\frac{r+s}{ar}$

Ex. 1.2.5: Reduce:

- $\frac{(x+1)(x+4)+(x+1)(x+5)}{2x+2}=\frac{2x+9}{2}$
- $\frac{(x+3)(x+4)+(2x+6)(x+5)}{3x+9}=\frac{3x+14}{3}$
- $\frac{(b+1)(x+4)+(x+5)(b+1)}{2+2b}=\frac{2x+9}{2}$
- $\frac{(x+1)(3x+4)+(x+1)(3x+5)}{2x+2}=\frac{3(2x+3)}{2}$

Ex. 1.2.4: Factor numerator; then reduce:

- $\frac{a^5b^6+a^6b^5}{a^6b^6}=\frac{a+b}{ab}$
- $\frac{4xy-6xy^2}{8x^2y^2}=\frac{2-3y}{4xy}$
- $\frac{x^3y^3+x^4y^4}{x^5y^5}=\frac{1+xy}{x^2y^2}$
- $\frac{a^2+2ab+b^2}{ac+cb}=\frac{a+b}{c}$
- $\frac{ax+ay}{2a}=\frac{x+y}{2}$
- $\frac{mx+may}{2ma}=\frac{x+ay}{2a}$
- $\frac{cy-cx}{c^2x^2-c^2y^2}=-\frac{1}{c(x+y)}$
- $\frac{ar+as}{ra^2}=\frac{r+s}{ar}$

Ex. 1.2.5: Reduce:

- $\frac{(x+1)(x+4)+(x+1)(x+5)}{2x+2}=\frac{2x+9}{2}$
- $\frac{(x+3)(x+4)+(2x+6)(x+5)}{3x+9}=\frac{3x+14}{3}$
- $\frac{(b+1)(x+4)+(x+5)(b+1)}{2+2b}=\frac{2x+9}{2}$
- $\frac{(x+1)(3x+4)+(x+1)(3x+5)}{2x+2}=\frac{3(2x+3)}{2}$

Ex. 1.2.6: Reduce each fraction completely:

- $\frac{x^2-x}{x^2-x^3}=$
- $\frac{x^2-4x}{x^2-16}=$
- $\frac{x-9}{x^2-13x+36}=$
- $\frac{x^2-7x+12}{x^2-4x}=$

Ex. 1.2.4: Factor numerator; then reduce:

- $\frac{a^5b^6+a^6b^5}{a^6b^6}=\frac{a+b}{ab}$
- $\frac{4xy-6xy^2}{8x^2y^2}=\frac{2-3y}{4xy}$
- $\frac{x^3y^3+x^4y^4}{x^5y^5}=\frac{1+xy}{x^2y^2}$
- $\frac{a^2+2ab+b^2}{ac+cb}=\frac{a+b}{c}$
- $\frac{ax+ay}{2a}=\frac{x+y}{2}$
- $\frac{mx+may}{2ma}=\frac{x+ay}{2a}$
- $\frac{cy-cx}{c^2x^2-c^2y^2}=-\frac{1}{c(x+y)}$
- $\frac{ar+as}{ra^2}=\frac{r+s}{ar}$

Ex. 1.2.5: Reduce:

- $\frac{(x+1)(x+4)+(x+1)(x+5)}{2x+2}=\frac{2x+9}{2}$
- $\frac{(x+3)(x+4)+(2x+6)(x+5)}{3x+9}=\frac{3x+14}{3}$
- $\frac{(b+1)(x+4)+(x+5)(b+1)}{2+2b}=\frac{2x+9}{2}$
- $\frac{(x+1)(3x+4)+(x+1)(3x+5)}{2x+2}=\frac{3(2x+3)}{2}$

Ex. 1.2.6: Reduce each fraction completely:

- $\frac{x^2-x}{x^2-x^3}=-\frac{1}{x}$
- $\frac{x^2-4x}{x^2-16}=\frac{x}{x+4}$
- $\frac{x-9}{x^2-13x+36}=\frac{1}{x-4}$
- $\frac{x^2-7x+12}{x^2-4x}=\frac{x-3}{x}$

Ex. 1.2.4: Factor numerator; then reduce:

- $\frac{a^5b^6+a^6b^5}{a^6b^6}=\frac{a+b}{ab}$
- $\frac{4xy-6xy^2}{8x^2y^2}=\frac{2-3y}{4xy}$
- $\frac{x^3y^3+x^4y^4}{x^5y^5}=\frac{1+xy}{x^2y^2}$
- $\frac{a^2+2ab+b^2}{ac+cb}=\frac{a+b}{c}$
- $\frac{ax+ay}{2a}=\frac{x+y}{2}$
- $\frac{mx+may}{2ma}=\frac{x+ay}{2a}$
- $\frac{cy-cx}{c^2x^2-c^2y^2}=-\frac{1}{c(x+y)}$
- $\frac{ar+as}{ra^2}=\frac{r+s}{ar}$

Ex. 1.2.5: Reduce:

- $\frac{(x+1)(x+4)+(x+1)(x+5)}{2x+2}=\frac{2x+9}{2}$
- $\frac{(x+3)(x+4)+(2x+6)(x+5)}{3x+9}=\frac{3x+14}{3}$
- $\frac{(b+1)(x+4)+(x+5)(b+1)}{2+2b}=\frac{2x+9}{2}$
- $\frac{(x+1)(3x+4)+(x+1)(3x+5)}{2x+2}=\frac{3(2x+3)}{2}$

Ex. 1.2.6: Reduce each fraction completely:

- $\frac{x^2-x}{x^2-x^3}=-\frac{1}{x}$
- $\frac{x^2-4x}{x^2-16}=\frac{x}{x+4}$
- $\frac{x-9}{x^2-13x+36}=\frac{1}{x-4}$
- $\frac{x^2-7x+12}{x^2-4x}=\frac{x-3}{x}$

Ex. 1.2.7: Rewrite as a reduced fraction

- $\frac{x^3+x^4}{x^2+7x+12}\cdot\frac{x^3+3x^2}{x^2+6x+5}=$
- $\frac{(x-4)^2}{x^2+7x+12}\cdot\frac{x^3+3x^2}{x^2-7x+12}=$
- $\frac{4x^3+x^4}{x^2+4x}\cdot\frac{x^3+3x^2}{x^2-16}=$
- $\frac{x^3+6x^2}{x^2-9x+20}\cdot\frac{x^3-5x^2}{x^2+8x+12}=$

Ex. 1.2.4: Factor numerator; then reduce:

- $\frac{a^5b^6+a^6b^5}{a^6b^6}=\frac{a+b}{ab}$
- $\frac{4xy-6xy^2}{8x^2y^2}=\frac{2-3y}{4xy}$
- $\frac{x^3y^3+x^4y^4}{x^5y^5}=\frac{1+xy}{x^2y^2}$
- $\frac{a^2+2ab+b^2}{ac+cb}=\frac{a+b}{c}$
- $\frac{ax+ay}{2a}=\frac{x+y}{2}$
- $\frac{mx+may}{2ma}=\frac{x+ay}{2a}$
- $\frac{cy-cx}{c^2x^2-c^2y^2}=-\frac{1}{c(x+y)}$
- $\frac{ar+as}{ra^2}=\frac{r+s}{ar}$

Ex. 1.2.5: Reduce:

- $\frac{(x+1)(x+4)+(x+1)(x+5)}{2x+2}=\frac{2x+9}{2}$
- $\frac{(x+3)(x+4)+(2x+6)(x+5)}{3x+9}=\frac{3x+14}{3}$
- $\frac{(b+1)(x+4)+(x+5)(b+1)}{2+2b}=\frac{2x+9}{2}$
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Ex. 1.2.7: Rewrite as a reduced fraction

- $\frac{x^3+x^4}{x^2+7x+12}\cdot\frac{x^3+3x^2}{x^2+6x+5}=\frac{x^5}{(x+4)(x+5)}$
- $\frac{(x-4)^2}{x^2+7x+12}\cdot\frac{x^3+3x^2}{x^2-7x+12}=\frac{(x-4)x^2}{(x-3)(x+4)}$
- $\frac{4x^3+x^4}{x^2+4x}\cdot\frac{x^3+3x^2}{x^2-16}=\frac{(x+3)x^4}{(x+4)(x-4)}$
- $\frac{x^3+6x^2}{x^2-9x+20}\cdot\frac{x^3-5x^2}{x^2+8x+12}=\frac{x^4}{(x-4)(x+2)}$

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- $\frac{4x^3+x^4}{x^2+4x}\cdot\frac{x^3+3x^2}{x^2-16}=\frac{(x+3)x^4}{(x+4)(x-4)}$
- $\frac{x^3+6x^2}{x^2-9x+20}\cdot\frac{x^3-5x^2}{x^2+8x+12}=\frac{x^4}{(x-4)(x+2)}$
- $\frac{(2x+1)^5(x+4)^6+(2x+1)^6(x+4)^5}{(2x+1)^3(x+4)^6}=$
- $\frac{(x+1)^5(x+2)^6+(x+1)^6(x+2)^5}{(x+1)^6(x+4)^6}=$
- $\frac{(x+3)^5(x+4)^6+(x+3)^6(x+4)^5}{(x+3)^{12}(x+4)^6}=$
- $\frac{(a+4)^5(b+4)^6+(a+4)^6(b+4)^5}{(b+1)^6(a+4)^6}=$

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- $\frac{(x-4)^2}{x^2+7x+12}\cdot\frac{x^3+3x^2}{x^2-7x+12}=\frac{(x-4)x^2}{(x-3)(x+4)}$
- $\frac{4x^3+x^4}{x^2+4x}\cdot\frac{x^3+3x^2}{x^2-16}=\frac{(x+3)x^4}{(x+4)(x-4)}$
- $\frac{x^3+6x^2}{x^2-9x+20}\cdot\frac{x^3-5x^2}{x^2+8x+12}=\frac{x^4}{(x-4)(x+2)}$
- $\frac{(2x+1)^5(x+4)^6+(2x+1)^6(x+4)^5}{(2x+1)^3(x+4)^6}=\frac{(2x+1)^2(3x+5)}{x+4}$
- $\frac{(x+1)^5(x+2)^6+(x+1)^6(x+2)^5}{(x+1)^6(x+4)^6}=\frac{(x+2)^5(2x+3)}{(x+1)(x+4)^6}$
- $\frac{(x+3)^5(x+4)^6+(x+3)^6(x+4)^5}{(x+3)^{12}(x+4)^6}=\frac{2x+7}{(x+4)(x+3)^7}$
- $\frac{(a+4)^5(b+4)^6+(a+4)^6(b+4)^5}{(b+1)^6(a+4)^6}=\frac{(b+4)^5(a+b+8)}{(a+4)(b+1)^6}$

Ex. 1.2.8: Find

- $\frac{x+3}{4x^2-9} \cdot \frac{2x^2+7x-15}{x^2+7x+12} =$
- $\frac{x+5}{4x^2-9} \cdot \frac{4x^2+12x+9}{x^2+9x+20} =$
- $\frac{x-5}{4x^2-9} \cdot \frac{4x^2-12x+9}{x^2-7x+10} =$
- $\frac{x+6}{4x+10} \cdot \frac{2x^2+5x}{x^2+8x+12} =$
- $\frac{x-5}{4x^2-9} \cdot \frac{4x^2-12x+9}{x^2-7x+10} =$
- $\frac{x+6}{4x+10} \cdot \frac{2x^2+5x}{x^2+8x+12} =$

Ex. 1.2.8: Find

- $\frac{x+3}{4x^2-9} \cdot \frac{2x^2+7x-15}{x^2+7x+12} = \frac{x+5}{(x+4)(2x+3)}$
- $\frac{x+5}{4x^2-9} \cdot \frac{4x^2+12x+9}{x^2+9x+20} = \frac{2x+3}{(2x-3)(x+4)}$
- $\frac{x-5}{4x^2-9} \cdot \frac{4x^2-12x+9}{x^2-7x+10} = \frac{2x-3}{(x-2)(2x+3)}$
- $\frac{x+6}{4x+10} \cdot \frac{2x^2+5x}{x^2+8x+12} = \frac{x}{2(x+2)}$
- $\frac{x-5}{4x^2-9} \cdot \frac{4x^2-12x+9}{x^2-7x+10} = \frac{2x-3}{(x-2)(2x+3)}$
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- $\frac{x+6}{4x+10} \cdot \frac{2x^2+5x}{x^2+8x+12} = \frac{x}{2(x+2)}$
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- $\frac{x+6}{4x+10} \cdot \frac{2x^2+5x}{x^2+8x+12} = \frac{x}{2(x+2)}$

Ex. 1.2.9: Find

- $\frac{100}{81} \cdot \frac{144}{65} =$
- $\frac{60}{75} \cdot \frac{750}{600} =$
- $\frac{15}{8} \cdot \frac{144}{75} =$
- $\frac{77}{81} \cdot \frac{18}{121} =$

Ex. 1.2.8: Find

- $\frac{x+3}{4x^2-9} \cdot \frac{2x^2+7x-15}{x^2+7x+12} = \frac{x+5}{(x+4)(2x+3)}$
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- $\frac{x-5}{4x^2-9} \cdot \frac{4x^2-12x+9}{x^2-7x+10} = \frac{2x-3}{(x-2)(2x+3)}$
- $\frac{x+6}{4x+10} \cdot \frac{2x^2+5x}{x^2+8x+12} = \frac{x}{2(x+2)}$
- $\frac{x-5}{4x^2-9} \cdot \frac{4x^2-12x+9}{x^2-7x+10} = \frac{2x-3}{(x-2)(2x+3)}$
- $\frac{x+6}{4x+10} \cdot \frac{2x^2+5x}{x^2+8x+12} = \frac{x}{2(x+2)}$

Ex. 1.2.9: Find

- $\frac{100}{81} \cdot \frac{144}{65} = \frac{320}{117}$
- $\frac{60}{75} \cdot \frac{750}{600} = 1$
- $\frac{15}{8} \cdot \frac{144}{75} = \frac{18}{5}$
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- $\frac{100}{81} \cdot \frac{144}{65} = \frac{320}{117}$
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- $\frac{77}{81} \cdot \frac{18}{121} = \frac{14}{99}$

Ex. 1.2.10: Divide
- $\frac{x+3}{4x^2-9} \div \frac{x^2+7x+12}{2x^2+7x-15} =$
- $\frac{x+5}{4x^2-9} \div \frac{x^2+9x+20}{4x^2+12x+9} =$
- $\frac{x-5}{4x^2-9} \div \frac{x^2-7x+10}{4x^2-12x+9} =$
- $\frac{x+6}{4x+10} \div \frac{x^2+8x+12}{2x^2+5x} =$

Ex. 1.2.8: Find
- $\frac{x+3}{4x^2-9} \cdot \frac{2x^2+7x-15}{x^2+7x+12} = \frac{x+5}{(x+4)(2x+3)}$
- $\frac{x+5}{4x^2-9} \cdot \frac{4x^2+12x+9}{x^2+9x+20} = \frac{2x+3}{(2x-3)(x+4)}$
- $\frac{x-5}{4x^2-9} \cdot \frac{4x^2-12x+9}{x^2-7x+10} = \frac{2x-3}{(x-2)(2x+3)}$
- $\frac{x+6}{4x+10} \cdot \frac{2x^2+5x}{x^2+8x+12} = \frac{x}{2(x+2)}$
- $\frac{x-5}{4x^2-9} \cdot \frac{4x^2-12x+9}{x^2-7x+10} = \frac{2x-3}{(x-2)(2x+3)}$
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- $\frac{x+6}{4x+10} \div \frac{x^2+8x+12}{2x^2+5x} = \frac{x}{2(x+2)}$

Ex. 1.2.11: Reduce

- $\frac{4-x}{x-4} =$
- $\frac{x^2-4x+4}{4-x^2} =$
- $\frac{4-x}{8-2x} =$
- $\frac{x^2-4x+4}{x^2-5x+6} =$
- $\frac{12-4x}{3x-9} =$
- $\frac{x^2-5x-36}{16-x^2} =$
- $\frac{40-10x}{x-4} =$
- $\frac{x^2-x-20}{3x^2+27x+60} =$

Ex. 1.2.8: Find
- $\frac{x+3}{4x^2-9}\cdot\frac{2x^2+7x-15}{x^2+7x+12}=\frac{x+5}{(x+4)(2x+3)}$
- $\frac{x+5}{4x^2-9}\cdot\frac{4x^2+12x+9}{x^2+9x+20}=\frac{2x+3}{(2x-3)(x+4)}$
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- $\frac{x-5}{4x^2-9}\div\frac{x^2-7x+10}{4x^2-12x+9}=\frac{2x-3}{(x-2)(2x+3)}$
- $\frac{x+6}{4x+10}\div\frac{x^2+8x+12}{2x^2+5x}=\frac{x}{2(x+2)}$

Ex. 1.2.11: Reduce
- $\frac{4-x}{x-4}=-1$
- $\frac{x^2-4x+4}{4-x^2}=\frac{2-x}{x+2}$
- $\frac{4-x}{8-2x}=\frac{1}{2}$
- $\frac{x^2-4x+4}{x^2-5x+6}=\frac{x-2}{x-3}$
- $\frac{12-4x}{3x-9}=-\frac{4}{3}$
- $\frac{x^2-5x-36}{16-x^2}=\frac{9-x}{x-4}$
- $\frac{40-10x}{x-4}=-10$
- $\frac{x^2-x-20}{3x^2+27x+60}=\frac{x-5}{3x+15}$

Ex. 1.2.8: Find
- $\frac{x+3}{4x^2-9} \cdot \frac{2x^2+7x-15}{x^2+7x+12} = \frac{x+5}{(x+4)(2x+3)}$
- $\frac{x+5}{4x^2-9} \cdot \frac{4x^2+12x+9}{x^2+9x+20} = \frac{2x+3}{(2x-3)(x+4)}$
- $\frac{x-5}{4x^2-9} \cdot \frac{4x^2-12x+9}{x^2-7x+10} = \frac{2x-3}{(x-2)(2x+3)}$
- $\frac{x+6}{4x+10} \cdot \frac{2x^2+5x}{x^2+8x+12} = \frac{x}{2(x+2)}$
- $\frac{x-5}{4x^2-9} \cdot \frac{4x^2-12x+9}{x^2-7x+10} = \frac{2x-3}{(x-2)(2x+3)}$
- $\frac{x+6}{4x+10} \cdot \frac{2x^2+5x}{x^2+8x+12} = \frac{x}{2(x+2)}$

Ex. 1.2.9: Find
- $\frac{100}{81} \cdot \frac{144}{65} = \frac{320}{117}$
- $\frac{60}{75} \cdot \frac{750}{600} = 1$
- $\frac{15}{8} \cdot \frac{144}{75} = \frac{18}{5}$
- $\frac{77}{81} \cdot \frac{18}{121} = \frac{14}{99}$

Ex. 1.2.10: Divide
- $\frac{x+3}{4x^2-9} \div \frac{x^2+7x+12}{2x^2+7x-15} = \frac{x+5}{(x+4)(2x+3)}$
- $\frac{x+5}{4x^2-9} \div \frac{x^2+9x+20}{4x^2+12x+9} = \frac{2x+3}{(2x-3)(x+4)}$
- $\frac{x-5}{4x^2-9} \div \frac{x^2-7x+10}{4x^2-12x+9} = \frac{2x-3}{(x-2)(2x+3)}$
- $\frac{x+6}{4x+10} \div \frac{x^2+8x+12}{2x^2+5x} = \frac{x}{2(x+2)}$

Ex. 1.2.11: Reduce
- $\frac{4-x}{x-4} = -1$
- $\frac{x^2-4x+4}{4-x^2} = \frac{2-x}{x+2}$
- $\frac{4-x}{8-2x} = \frac{1}{2}$
- $\frac{x^2-4x+4}{x^2-5x+6} = \frac{x-2}{x-3}$
- $\frac{12-4x}{3x-9} = -\frac{4}{3}$
- $\frac{x^2-5x-36}{16-x^2} = \frac{9-x}{x-4}$
- $\frac{40-10x}{x-4} = -10$
- $\frac{x^2-x-20}{3x^2+27x+60} = \frac{x-5}{3x+15}$

Ex. 1.2.12: Find LCM:
- 35 and 77 =
- 42 and 140 =
- 24 and 36 =
- 36 and 54 =

Ex. 1.2.8: Find
- $\frac{x+3}{4x^2-9}\cdot\frac{2x^2+7x-15}{x^2+7x+12}=\frac{x+5}{(x+4)(2x+3)}$
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- $\frac{x+3}{4x^2-9}\div\frac{x^2+7x+12}{2x^2+7x-15}=\frac{x+5}{(x+4)(2x+3)}$
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- $\frac{3}{5}$ as a fraction with denominator 50 $=\frac{30}{50}$
- $\frac{-3}{5}$ as a fraction with denominator 20 $=\frac{-12}{20}$
- $\frac{7}{50}$ as a fraction with denominator 150 $=\frac{21}{150}$
- $\frac{3}{4}$ as a fraction with denominator 32. $=\frac{24}{32}$

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Ex. 1.2.14: Rewrite the fraction as requested :

- $\frac{a}{xy^2}$ with denominator $x^3y^6 =$
- $\frac{b}{xyz}$ with denominator $x^3y^6z^5 =$
- $\frac{c}{x^2y^2z^2}$ with denominator $x^3y^3z^3 =$
- $\frac{a}{x+y}$ with denominator $xab + yab =$

Ex. 1.2.8: Find
- $\frac{x+3}{4x^2-9}\cdot\frac{2x^2+7x-15}{x^2+7x+12}=\frac{x+5}{(x+4)(2x+3)}$
- $\frac{x+5}{4x^2-9}\cdot\frac{4x^2+12x+9}{x^2+9x+20}=\frac{2x+3}{(2x-3)(x+4)}$
- $\frac{x-5}{4x^2-9}\cdot\frac{4x^2-12x+9}{x^2-7x+10}=\frac{2x-3}{(x-2)(2x+3)}$
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Ex. 1.2.14: Rewrite the fraction as requested :

- $\frac{a}{xy^2}$ with denominator $x^3y^6=\frac{ax^2y^4}{x^3y^6}$
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- $\frac{a}{x+y}$ with denominator $xab+yab=\frac{a^2b}{xab+yab}$

Ex. 1.2.15: Rewrite as a reduced fraction
- $\frac{7}{2000}+\frac{11}{3000}=$
- $\frac{7}{200}+\frac{11}{3000}=$
- $\frac{7}{2000}+\frac{11}{30}=$
- $\frac{7}{50}+\frac{11}{35}=$

Ex. 1.2.8: Find

- $\frac{x+3}{4x^2-9}\cdot\frac{2x^2+7x-15}{x^2+7x+12}=\frac{x+5}{(x+4)(2x+3)}$
- $\frac{x+5}{4x^2-9}\cdot\frac{4x^2+12x+9}{x^2+9x+20}=\frac{2x+3}{(2x-3)(x+4)}$
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- $\frac{x^2-4x+4}{x^2-5x+6}=\frac{x-2}{x-3}$
- $\frac{12-4x}{3x-9}=-\frac{4}{3}$
- $\frac{x^2-5x-36}{16-x^2}=\frac{9-x}{x-4}$
- $\frac{40-10x}{x-4}=-10$
- $\frac{x^2-x-20}{3x^2+27x+60}=\frac{x-5}{3x+15}$

Ex. 1.2.12: Find LCM:

- 35 and 77 $=385$
- 42 and 140 $=420$
- 24 and 36 $=72$
- 36 and 54 $=108$

Ex. 1.2.13: Rewrite

- $\frac{3}{5}$ as a fraction with denominator 50 $=\frac{30}{50}$
- $\frac{-3}{5}$ as a fraction with denominator 20 $=\frac{-12}{20}$
- $\frac{7}{50}$ as a fraction with denominator 150 $=\frac{21}{150}$
- $\frac{3}{4}$ as a fraction with denominator 32. $=\frac{24}{32}$

Ex. 1.2.14: Rewrite the fraction as requested :

- $\frac{a}{xy^2}$ with denominator $x^3y^6=\frac{ax^2y^4}{x^3y^6}$
- $\frac{b}{xyz}$ with denominator $x^3y^6z^5=\frac{bx^2y^5z^4}{x^3y^6z^5}$
- $\frac{c}{x^2y^2z^2}$ with denominator $x^3y^3z^3=\frac{cxyz}{x^3y^3z^3}$
- $\frac{a}{x+y}$ with denominator $xab+yab=\frac{a^2b}{xab+yab}$

Ex. 1.2.15: Rewrite as a reduced fraction

- $\frac{7}{2000}+\frac{11}{3000}=\frac{43}{6000}$
- $\frac{7}{200}+\frac{11}{3000}=\frac{29}{750}$
- $\frac{7}{2000}+\frac{11}{30}=\frac{2221}{6000}$
- $\frac{7}{50}+\frac{11}{35}=\frac{159}{350}$

Ex. 1.2.16: Rewrite as a reduced fraction:

- $\frac{2}{35} + \frac{5}{77} =$
- $\frac{3}{8} + \frac{5}{16} + \frac{1}{20} =$
- $\frac{3}{160} + \frac{5}{24} =$
- $\frac{3}{200} + \frac{7}{50} - \frac{7}{8} =$

Ex. 1.2.16: Rewrite as a reduced fraction:

- $\frac{2}{35} + \frac{5}{77} = \frac{47}{385}$
- $\frac{3}{8} + \frac{5}{16} + \frac{1}{20} =$
- $\frac{3}{160} + \frac{5}{24} = \frac{109}{480}$
- $\frac{3}{200} + \frac{7}{50} - \frac{7}{8} =$

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- $\frac{2}{35} + \frac{5}{77} = \frac{47}{385}$
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Ex. 1.2.17: Rewrite as a reduced fraction:

- $\frac{3}{20} + \frac{7}{16} + \frac{7}{24} =$
- $\frac{2}{15} + \frac{5}{25} =$
- $-\frac{3}{7} + \frac{-5}{28} + \frac{-7}{24} =$
- $\frac{3}{160} + \frac{5}{120} =$

Ex. 1.2.16: Rewrite as a reduced fraction:

- $\frac{2}{35}+\frac{5}{77}=\frac{47}{385}$
- $\frac{3}{8}+\frac{5}{16}+\frac{1}{20}=\frac{59}{80}$
- $\frac{3}{160}+\frac{5}{24}=\frac{109}{480}$
- $\frac{3}{200}+\frac{7}{50}-\frac{7}{8}=-\frac{18}{25}$

Ex. 1.2.17: Rewrite as a reduced fraction:

- $\frac{3}{20}+\frac{7}{16}+\frac{7}{24}=\frac{211}{240}$
- $\frac{2}{15}+\frac{5}{25}=\frac{1}{3}$
- $-\frac{3}{7}+\frac{-5}{28}+\frac{-7}{24}=-\frac{151}{168}$
- $\frac{3}{160}+\frac{5}{120}=\frac{29}{480}$

Ex. 1.2.16: Rewrite as a reduced fraction:

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- $\frac{3}{160}+\frac{5}{120}=\frac{29}{480}$

Ex. 1.2.18: Is the fraction completely reduced?

- $\frac{211}{240}$?
- $\frac{375}{221}$?
- $\frac{77}{121}$?
- $\frac{143}{220}$?

Ex. 1.2.16: Rewrite as a reduced fraction:

- $\frac{2}{35}+\frac{5}{77}=\frac{47}{385}$
- $\frac{3}{8}+\frac{5}{16}+\frac{1}{20}=\frac{59}{80}$
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Ex. 1.2.18: Is the fraction completely reduced?

- $\frac{211}{240}$? $\Rightarrow$ Yes
- $\frac{375}{221}$? $\Rightarrow$ Yes
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- $\frac{143}{220}$? $\Rightarrow$ No

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- $\frac{143}{220}$? $\Rightarrow$ No

Ex. 1.2.19: Find the LCM of

- x^3y^4 and $x^2y^3z^5 \Rightarrow$
- $12x^3z^4$ and $10x^2y^3z^5 \Rightarrow$
- $30x^3y^4z^8$ and $66x^2y^3z^5 \Rightarrow$
- $9ab^3$ and $30abcdx^2$ $:\Rightarrow$

Ex. 1.2.16: Rewrite as a reduced fraction:

- $\frac{2}{35}+\frac{5}{77}=\frac{47}{385}$
- $\frac{3}{8}+\frac{5}{16}+\frac{1}{20}=\frac{59}{80}$
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- $\frac{143}{220}$? $\Rightarrow$ No

Ex. 1.2.19: Find the LCM of

- x^3y^4 and $x^2y^3z^5 \Rightarrow \ x^3y^4z^5$
- $12x^3z^4$ and $10x^2y^3z^5 \Rightarrow \ 60x^3y^3z^5$
- $30x^3y^4z^8$ and $66x^2y^3z^5 \Rightarrow \ 330x^3y^4z^8$
- $9ab^3$ and $30abcdx^2 :\Rightarrow \ 90ab^3cdx^2$

Ex. 1.2.16: Rewrite as a reduced fraction:

- $\frac{2}{35}+\frac{5}{77}=\frac{47}{385}$
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- $12x^3z^4$ and $10x^2y^3z^5 \Rightarrow\ 60x^3y^3z^5$
- $30x^3y^4z^8$ and $66x^2y^3z^5 \Rightarrow\ 330x^3y^4z^8$
- $9ab^3$ and $30abcdx^2 :\Rightarrow\ 90ab^3cdx^2$

Ex. 1.2.20: Rewrite as a reduced fraction

- $\frac{a}{x^3}+\frac{b}{x^{10}}=$
- $\frac{a}{a^3b^6a^3}+\frac{b}{a^{10}}=$
- $\frac{ar}{40x^3}-\frac{b}{12ax^{10}}=$
- $\frac{ax}{bx^3}-\frac{b}{ax^{10}}=$

Ex. 1.2.16: Rewrite as a reduced fraction:

- $\frac{2}{35} + \frac{5}{77} = \frac{47}{385}$
- $\frac{3}{8} + \frac{5}{16} + \frac{1}{20} = \frac{59}{80}$
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Ex. 1.2.18: Is the fraction completely reduced?

- $\frac{211}{240}$? $\Rightarrow$ Yes
- $\frac{375}{221}$? $\Rightarrow$ Yes
- $\frac{77}{121}$? $\Rightarrow$ No:
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Ex. 1.2.19: Find the LCM of

- x^3y^4 and $x^2y^3z^5 \Rightarrow \ x^3y^4z^5$
- $12x^3z^4$ and $10x^2y^3z^5 \Rightarrow \ 60x^3y^3z^5$
- $30x^3y^4z^8$ and $66x^2y^3z^5 \Rightarrow \ 330x^3y^4z^8$
- $9ab^3$ and $30abcdx^2$ $:\Rightarrow \ 90ab^3cdx^2$

Ex. 1.2.20: Rewrite as a reduced fraction

- $\frac{a}{x^3} + \frac{b}{x^{10}} = \frac{ax^7+b}{x^{10}}$
- $\frac{a}{a^3b^6a^3} + \frac{b}{a^{10}} = \frac{a^5+b^7}{a^{10}b^6}$
- $\frac{ar}{40x^3} - \frac{b}{12ax^{10}} = \frac{3a^2rx^7-10b}{120ax^{10}}$
- $\frac{ax}{bx^3} - \frac{b}{ax^{10}} = \frac{b^2-a^2x^8}{abx^{10}}$

Ex. 1.2.16: Rewrite as a reduced fraction:

- $\frac{2}{35}+\frac{5}{77}=\frac{47}{385}$
- $\frac{3}{8}+\frac{5}{16}+\frac{1}{20}=\frac{59}{80}$
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- $\frac{211}{240}$? $\Rightarrow$ Yes
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Ex. 1.2.19: Find the LCM of

- x^3y^4 and $x^2y^3z^5 \Rightarrow\ x^3y^4z^5$
- $12x^3z^4$ and $10x^2y^3z^5 \Rightarrow\ 60x^3y^3z^5$
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- $9ab^3$ and $30abcdx^2 :\Rightarrow\ 90ab^3cdx^2$

Ex. 1.2.20: Rewrite as a reduced fraction

- $\frac{a}{x^3}+\frac{b}{x^{10}}=\frac{ax^7+b}{x^{10}}$
- $\frac{a}{a^3b^6a^3}+\frac{b}{a^{10}}=\frac{a^5+b^7}{a^{10}b^6}$
- $\frac{ar}{40x^3}-\frac{b}{12ax^{10}}=\frac{3a^2rx^7-10b}{120ax^{10}}$
- $\frac{ax}{bx^3}-\frac{b}{ax^{10}}=\frac{b^2-a^2x^8}{abx^{10}}$

Ex. 1.2.21: Rewrite as a reduced fraction :

- $\frac{a}{xy^2}+\frac{b}{x^2y}=$
- $\frac{3}{x^2}+\frac{2}{x^2+x}=$
- $\frac{3}{x^2+2x+1}-7\frac{2}{x^2+x}=$
- $\frac{3}{ax^2}-\frac{2}{ax^2+axa}=$

Ex. 1.2.16: Rewrite as a reduced fraction:

- $\frac{2}{35}+\frac{5}{77}=\frac{47}{385}$
- $\frac{3}{8}+\frac{5}{16}+\frac{1}{20}=\frac{59}{80}$
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Ex. 1.2.17: Rewrite as a reduced fraction:

- $\frac{3}{20}+\frac{7}{16}+\frac{7}{24}=\frac{211}{240}$
- $\frac{2}{15}+\frac{5}{25}=\frac{1}{3}$
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- $\frac{211}{240}$? $\Rightarrow$ Yes
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- x^3y^4 and $x^2y^3z^5 \Rightarrow\ x^3y^4z^5$
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- $9ab^3$ and $30abcdx^2 :\Rightarrow\ 90ab^3cdx^2$

Ex. 1.2.20: Rewrite as a reduced fraction

- $\frac{a}{x^3}+\frac{b}{x^{10}}=\frac{ax^7+b}{x^{10}}$
- $\frac{a}{a^3b^6a^3}+\frac{b}{a^{10}}=\frac{a^5+b^7}{a^{10}b^6}$
- $\frac{ar}{40x^3}-\frac{b}{12ax^{10}}=\frac{3a^2rx^7-10b}{120ax^{10}}$
- $\frac{ax}{bx^3}-\frac{b}{ax^{10}}=\frac{b^2-a^2x^8}{abx^{10}}$

Ex. 1.2.21: Rewrite as a reduced fraction :

- $\frac{a}{xy^2}+\frac{b}{x^2y}=\frac{ax+by}{x^2y^2}$
- $\frac{3}{x^2}+\frac{2}{x^2+x}=\frac{5x+3}{x^2(x+1)}$
- $\frac{3}{x^2+2x+1}-7\frac{2}{x^2+x}=\frac{-11x-14}{x(x+1)^2}$
- $\frac{3}{ax^2}-\frac{2}{ax^2+axa}=\frac{3a+x}{ax^2(a+x)}$

Ex. 1.2.16: Rewrite as a reduced fraction:

- $\frac{2}{35} + \frac{5}{77} = \frac{47}{385}$
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- $9ab^3$ and $30abcdx^2$ $:\Rightarrow\ 90ab^3cdx^2$

Ex. 1.2.20: Rewrite as a reduced fraction

- $\frac{a}{x^3} + \frac{b}{x^{10}} = \frac{ax^7+b}{x^{10}}$
- $\frac{a}{a^3b^6a^3} + \frac{b}{a^{10}} = \frac{a^5+b^7}{a^{10}b^6}$
- $\frac{ar}{40x^3} - \frac{b}{12ax^{10}} = \frac{3a^2rx^7-10b}{120ax^{10}}$
- $\frac{ax}{bx^3} - \frac{b}{ax^{10}} = \frac{b^2-a^2x^8}{abx^{10}}$

Ex. 1.2.21: Rewrite as a reduced fraction :

- $\frac{a}{xy^2} + \frac{b}{x^2y} = \frac{ax+by}{x^2y^2}$
- $\frac{3}{x^2} + \frac{2}{x^2+x} = \frac{5x+3}{x^2(x+1)}$
- $\frac{3}{x^2+2x+1} - 7\frac{2}{x^2+x} = \frac{-11x-14}{x(x+1)^2}$
- $\frac{3}{ax^2} - \frac{2}{ax^2+axa} = \frac{3a+x}{ax^2(a+x)}$

Ex. 1.2.22: Rewrite as a reduced fraction

- $\frac{3}{x^2} + \frac{2}{x^2+x} =$
- $\frac{3}{x^2-1} - \frac{2}{x^2+x} =$
- $\frac{3}{x^3+4x} - \frac{2}{x^3-x^2} =$
- $\frac{3}{b^2-4} - \frac{2}{b^2-2b} =$

Ex. 1.2.16: Rewrite as a reduced fraction:
- $\frac{2}{35}+\frac{5}{77}=\frac{47}{385}$
- $\frac{3}{8}+\frac{5}{16}+\frac{1}{20}=\frac{59}{80}$
- $\frac{3}{160}+\frac{5}{24}=\frac{109}{480}$
- $\frac{3}{200}+\frac{7}{50}-\frac{7}{8}=-\frac{18}{25}$

Ex. 1.2.17: Rewrite as a reduced fraction:
- $\frac{3}{20}+\frac{7}{16}+\frac{7}{24}=\frac{211}{240}$
- $\frac{2}{15}+\frac{5}{25}=\frac{1}{3}$
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- $\frac{211}{240}$? $\Rightarrow$ Yes
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- x^3y^4 and $x^2y^3z^5 \Rightarrow\ x^3y^4z^5$
- $12x^3z^4$ and $10x^2y^3z^5 \Rightarrow\ 60x^3y^3z^5$
- $30x^3y^4z^8$ and $66x^2y^3z^5 \Rightarrow\ 330x^3y^4z^8$
- $9ab^3$ and $30abcdx^2 :\Rightarrow\ 90ab^3cdx^2$

Ex. 1.2.20: Rewrite as a reduced fraction
- $\frac{a}{x^3}+\frac{b}{x^{10}}=\frac{ax^7+b}{x^{10}}$
- $\frac{a}{a^3b^6a^3}+\frac{b}{a^{10}}=\frac{a^5+b^7}{a^{10}b^6}$
- $\frac{ar}{40x^3}-\frac{b}{12ax^{10}}=\frac{3a^2rx^7-10b}{120ax^{10}}$
- $\frac{ax}{bx^3}-\frac{b}{ax^{10}}=\frac{b^2-a^2x^8}{abx^{10}}$

Ex. 1.2.21: Rewrite as a reduced fraction :
- $\frac{a}{xy^2}+\frac{b}{x^2y}=\frac{ax+by}{x^2y^2}$
- $\frac{3}{x^2}+\frac{2}{x^2+x}=\frac{5x+3}{x^2(x+1)}$
- $\frac{3}{x^2+2x+1}-7\frac{2}{x^2+x}=\frac{-11x-14}{x(x+1)^2}$
- $\frac{3}{ax^2}-\frac{2}{ax^2+axa}=\frac{3a+x}{ax^2(a+x)}$

Ex. 1.2.22: Rewrite as a reduced fraction
- $\frac{3}{x^2}+\frac{2}{x^2+x}=\frac{5x+3}{x^2(x+1)}$
- $\frac{3}{x^2-1}-\frac{2}{x^2+x}=\frac{x+2}{(x-1)x(x+1)}$
- $\frac{3}{x^3+4x}-\frac{2}{x^3-x^2}=\frac{x^2-3x-8}{(x-1)x^2(x^2+4)}$
- $\frac{3}{b^2-4}-\frac{2}{b^2-2b}=\frac{b-4}{(b-2)b(b+2)}$

Ex. 1.2.23: Rewrite as a reduced fraction:

- $\frac{x}{(x^2+2x+1)(x+2)} + \frac{3}{(x^2+4x+4)(x+1)} =$
- $\frac{x}{(x^2-2x-15)} + \frac{6-x}{x^2-11x+30} =$
- $\frac{x}{(x^2+3x+2)(x+2)} + \frac{3}{(x^2+4x+4)(x-1)} =$
- $\frac{x}{x^2+3x-10} - \frac{3}{x^2-4x+4} =$

Ex. 1.2.23: Rewrite as a reduced fraction:

- $\frac{x}{(x^2+2x+1)(x+2)}+\frac{3}{(x^2+4x+4)(x+1)}=\frac{x^2+5x+3}{(x+1)^2(x+2)^2}$
- $\frac{x}{(x^2+3x+2)(x+2)}+\frac{3}{(x^2+4x+4)(x-1)}=\frac{3}{(x-1)(x+1)(x+2)^2}$
- $\frac{x}{(x^2-2x-15)}+\frac{6-x}{x^2-11x+30}=-\frac{3}{(x-5)(x+3)}$
- $\frac{x}{x^2+3x-10}-\frac{3}{x^2-4x+4}=\frac{x^2-5x-15}{(x-2)^2(x+5))}$

Ex. 1.2.23: Rewrite as a reduced fraction:

- $\frac{x}{(x^2+2x+1)(x+2)} + \frac{3}{(x^2+4x+4)(x+1)} = \frac{x^2+5x+3}{(x+1)^2(x+2)^2}$
- $\frac{x}{(x^2+3x+2)(x+2)} + \frac{3}{(x^2+4x+4)(x-1)} = \frac{3}{(x-1)(x+1)(x+2)^2}$
- $\frac{x}{(x^2-2x-15)} + \frac{6-x}{x^2-11x+30} = -\frac{3}{(x-5)(x+3)}$
- $\frac{x}{x^2+3x-10} - \frac{3}{x^2-4x+4} = \frac{x^2-5x-15}{(x-2)^2(x+5))}$

Ex. 1.2.24: Rewrite as a reduced fraction:

- $\frac{x}{x^2+3x+2} + \frac{x+3}{x^2+4x+4} =$
- $\frac{x}{x^2-5x+6} - \frac{x+3}{x^2-6x+8} =$
- $\frac{x}{x^2-4} + \frac{x+3}{x^2+7x+10} =$
- $\frac{x}{x^2+3x+2} \frac{x+3}{x^2+7x+6} =$

Ex. 1.2.23: Rewrite as a reduced fraction:

- $\frac{x}{(x^2+2x+1)(x+2)}+\frac{3}{(x^2+4x+4)(x+1)}=\frac{x^2+5x+3}{(x+1)^2(x+2)^2}$
- $\frac{x}{(x^2+3x+2)(x+2)}+\frac{3}{(x^2+4x+4)(x-1)}=\frac{3}{(x-1)(x+1)(x+2)^2}$
- $\frac{x}{(x^2-2x-15)}+\frac{6-x}{x^2-11x+30}=-\frac{3}{(x-5)(x+3)}$
- $\frac{x}{x^2+3x-10}-\frac{3}{x^2-4x+4}=\frac{x^2-5x-15}{(x-2)^2(x+5))}$

Ex. 1.2.24: Rewrite as a reduced fraction:

- $\frac{x}{x^2+3x+2}+\frac{x+3}{x^2+4x+4}=\frac{2x^2+6x+3}{(x+1)(x+2)^2}$
- $\frac{x}{x^2-5x+6}-\frac{x+3}{x^2-6x+8}=\frac{9-4x}{(x-4)(x-3)(x-2)}$
- $\frac{x}{x^2-4}+\frac{x+3}{x^2+7x+10}=\frac{2(x^2+3x-3)}{(x-2)(x+2)(x+5)}$
- $\frac{x}{x^2+3x+2}\,\frac{x+3}{x^2+7x+6}=\frac{x(x+3)}{(x+2)(x+6)(x+1)^2}$

Ex. 1.2.23: Rewrite as a reduced fraction:

- $\frac{x}{(x^2+2x+1)(x+2)} + \frac{3}{(x^2+4x+4)(x+1)} = \frac{x^2+5x+3}{(x+1)^2(x+2)^2}$
- $\frac{x}{(x^2+3x+2)(x+2)} + \frac{3}{(x^2+4x+4)(x-1)} = \frac{3}{(x-1)(x+1)(x+2)^2}$
- $\frac{x}{(x^2-2x-15)} + \frac{6-x}{x^2-11x+30} = -\frac{3}{(x-5)(x+3)}$
- $\frac{x}{x^2+3x-10} - \frac{3}{x^2-4x+4} = \frac{x^2-5x-15}{(x-2)^2(x+5))}$

Ex. 1.2.24: Rewrite as a reduced fraction:

- $\frac{x}{x^2+3x+2} + \frac{x+3}{x^2+4x+4} = \frac{2x^2+6x+3}{(x+1)(x+2)^2}$
- $\frac{x}{x^2-5x+6} - \frac{x+3}{x^2-6x+8} = \frac{9-4x}{(x-4)(x-3)(x-2)}$
- $\frac{x}{x^2-4} + \frac{x+3}{x^2+7x+10} = \frac{2(x^2+3x-3)}{(x-2)(x+2)(x+5)}$
- $\frac{x}{x^2+3x+2} \frac{x+3}{x^2+7x+6} = \frac{x(x+3)}{(x+2)(x+6)(x+1)^2}$

Ex. 1.2.25: Use long division with Quotient Q and Remainder R to divide

- $x^3 + 3x^2 + 3x + 7$ by $x + 2$ $\Rightarrow$
- $x^3 + 3x^2 + 3x + 7$ by $x - 2$ $\Rightarrow$
- $x^3 + 7$ by $2 - x$ $\Rightarrow$
- $2x^3 + 3x + 7$ by $2x + 1$ $\Rightarrow$

Ex. 1.2.23: Rewrite as a reduced fraction:

- $\frac{x}{(x^2+2x+1)(x+2)} + \frac{3}{(x^2+4x+4)(x+1)} = \frac{x^2+5x+3}{(x+1)^2(x+2)^2}$
- $\frac{x}{(x^2+3x+2)(x+2)} + \frac{3}{(x^2+4x+4)(x-1)} = \frac{3}{(x-1)(x+1)(x+2)^2}$
- $\frac{x}{(x^2-2x-15)} + \frac{6-x}{x^2-11x+30} = -\frac{3}{(x-5)(x+3)}$
- $\frac{x}{x^2+3x-10} - \frac{3}{x^2-4x+4} = \frac{x^2-5x-15}{(x-2)^2(x+5))}$

Ex. 1.2.24: Rewrite as a reduced fraction:

- $\frac{x}{x^2+3x+2} + \frac{x+3}{x^2+4x+4} = \frac{2x^2+6x+3}{(x+1)(x+2)^2}$
- $\frac{x}{x^2-5x+6} - \frac{x+3}{x^2-6x+8} = \frac{9-4x}{(x-4)(x-3)(x-2)}$
- $\frac{x}{x^2-4} + \frac{x+3}{x^2+7x+10} = \frac{2(x^2+3x-3)}{(x-2)(x+2)(x+5)}$
- $\frac{x}{x^2+3x+2}\,\frac{x+3}{x^2+7x+6} = \frac{x(x+3)}{(x+2)(x+6)(x+1)^2}$

Ex. 1.2.25: Use long division with Quotient Q and Remainder R to divide

- x^3+3x^2+3x+7 by $x+2$ $\Rightarrow Q = x^2+x+1;\ R=5$
- x^3+3x^2+3x+7 by $x-2$ $\Rightarrow Q = x^2+5x+13;\ R=33$
- x^3+7 by $2-x$ $\Rightarrow Q = -x^2-2x-4;\ R=15$
- $2x^3+3x+7$ by $2x+1$ $\Rightarrow Q = x^2-\frac{1}{2}x+\frac{7}{4};\ R=\frac{21}{4}$

Ex. 1.2.23: Rewrite as a reduced fraction:

- $\frac{x}{(x^2+2x+1)(x+2)} + \frac{3}{(x^2+4x+4)(x+1)} = \frac{x^2+5x+3}{(x+1)^2(x+2)^2}$
- $\frac{x}{(x^2+3x+2)(x+2)} + \frac{3}{(x^2+4x+4)(x-1)} = \frac{3}{(x-1)(x+1)(x+2)^2}$
- $\frac{x}{(x^2-2x-15)} + \frac{6-x}{x^2-11x+30} = -\frac{3}{(x-5)(x+3)}$
- $\frac{x}{x^2+3x-10} - \frac{3}{x^2-4x+4} = \frac{x^2-5x-15}{(x-2)^2(x+5))}$

Ex. 1.2.24: Rewrite as a reduced fraction:

- $\frac{x}{x^2+3x+2} + \frac{x+3}{x^2+4x+4} = \frac{2x^2+6x+3}{(x+1)(x+2)^2}$
- $\frac{x}{x^2-5x+6} - \frac{x+3}{x^2-6x+8} = \frac{9-4x}{(x-4)(x-3)(x-2)}$
- $\frac{x}{x^2-4} + \frac{x+3}{x^2+7x+10} = \frac{2(x^2+3x-3)}{(x-2)(x+2)(x+5)}$
- $\frac{x}{x^2+3x+2} \frac{x+3}{x^2+7x+6} = \frac{x(x+3)}{(x+2)(x+6)(x+1)^2}$

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- x^3+3x^2+3x+7 by $x+2$ $\Rightarrow Q = x^2+x+1; R = 5$
- x^3+3x^2+3x+7 by $x-2$ $\Rightarrow Q = x^2+5x+13; R = 33$
- x^3+7 by $2-x$ $\Rightarrow Q = -x^2-2x-4; R = 15$
- $2x^3+3x+7$ by $2x+1$ $\Rightarrow Q = x^2-\frac{1}{2}x+\frac{7}{4}; R = \frac{21}{4}$

Ex. 1.2.26: Rewrite as a polynomial plus a proper fraction of polynomials:

- x^3+3x^2+3x+7 by $x+2$ $=$
- x^3+3x^2+3x+7 by $x-2$ $=$
- x^3+7 by $2-x$ $=$
- $2x^3+3x+7$ by $2x+1$ $=$

Ex. 1.2.23: Rewrite as a reduced fraction:

- $\frac{x}{(x^2+2x+1)(x+2)} + \frac{3}{(x^2+4x+4)(x+1)} = \frac{x^2+5x+3}{(x+1)^2(x+2)^2}$
- $\frac{x}{(x^2+3x+2)(x+2)} + \frac{3}{(x^2+4x+4)(x-1)} = \frac{3}{(x-1)(x+1)(x+2)^2}$
- $\frac{x}{(x^2-2x-15)} + \frac{6-x}{x^2-11x+30} = -\frac{3}{(x-5)(x+3)}$
- $\frac{x}{x^2+3x-10} - \frac{3}{x^2-4x+4} = \frac{x^2-5x-15}{(x-2)^2(x+5))}$

Ex. 1.2.24: Rewrite as a reduced fraction:

- $\frac{x}{x^2+3x+2} + \frac{x+3}{x^2+4x+4} = \frac{2x^2+6x+3}{(x+1)(x+2)^2}$
- $\frac{x}{x^2-5x+6} - \frac{x+3}{x^2-6x+8} = \frac{9-4x}{(x-4)(x-3)(x-2)}$
- $\frac{x}{x^2-4} + \frac{x+3}{x^2+7x+10} = \frac{2(x^2+3x-3)}{(x-2)(x+2)(x+5)}$
- $\frac{x}{x^2+3x+2} \; \frac{x+3}{x^2+7x+6} = \frac{x(x+3)}{(x+2)(x+6)(x+1)^2}$

Ex. 1.2.25: Use long division with Quotient Q and Remainder R to divide

- $x^3 + 3x^2 + 3x + 7$ by $x + 2$ $\Rightarrow Q = x^2 + x + 1; R = 5$
- $x^3 + 3x^2 + 3x + 7$ by $x - 2$ $\Rightarrow Q = x^2 + 5x + 13; R = 33$
- $x^3 + 7$ by $2 - x$ $\Rightarrow Q = -x^2 - 2x - 4; R = 15$
- $2x^3 + 3x + 7$ by $2x + 1$ $\Rightarrow Q = x^2 - \frac{1}{2}x + \frac{7}{4}; R = \frac{21}{4}$

Ex. 1.2.26: Rewrite as a polynomial plus a proper fraction of polynomials:

- $x^3 + 3x^2 + 3x + 7$ by $x + 2$ $= x^2 + x + 1 + \frac{5}{x+2}$
- $x^3 + 3x^2 + 3x + 7$ by $x - 2$ $= x^2 + 5x + 13 + \frac{33}{x-2}$
- $x^3 + 7$ by $2 - x$ $= -x^2 - 2x - 4 + \frac{15}{2-x}$
- $2x^3 + 3x + 7$ by $2x + 1$ $= x^2 - \frac{1}{2}x + \frac{7}{4} + \frac{21}{8x+4}$

Ex. 1.2.23: Rewrite as a reduced fraction:

- $\frac{x}{(x^2+2x+1)(x+2)} + \frac{3}{(x^2+4x+4)(x+1)} = \frac{x^2+5x+3}{(x+1)^2(x+2)^2}$
- $\frac{x}{(x^2+3x+2)(x+2)} + \frac{3}{(x^2+4x+4)(x-1)} = \frac{3}{(x-1)(x+1)(x+2)^2}$
- $\frac{x}{(x^2-2x-15)} + \frac{6-x}{x^2-11x+30} = -\frac{3}{(x-5)(x+3)}$
- $\frac{x}{x^2+3x-10} - \frac{3}{x^2-4x+4} = \frac{x^2-5x-15}{(x-2)^2(x+5))}$

Ex. 1.2.24: Rewrite as a reduced fraction:

- $\frac{x}{x^2+3x+2} + \frac{x+3}{x^2+4x+4} = \frac{2x^2+6x+3}{(x+1)(x+2)^2}$
- $\frac{x}{x^2-5x+6} - \frac{x+3}{x^2-6x+8} = \frac{9-4x}{(x-4)(x-3)(x-2)}$
- $\frac{x}{x^2-4} + \frac{x+3}{x^2+7x+10} = \frac{2(x^2+3x-3)}{(x-2)(x+2)(x+5)}$
- $\frac{x}{x^2+3x+2} \frac{x+3}{x^2+7x+6} = \frac{x(x+3)}{(x+2)(x+6)(x+1)^2}$

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- $x^3 + 3x^2 + 3x + 7$ by $x + 2$ $\Rightarrow Q = x^2 + x + 1; R = 5$
- $x^3 + 3x^2 + 3x + 7$ by $x - 2$ $\Rightarrow Q = x^2 + 5x + 13; R = 33$
- $x^3 + 7$ by $2 - x$ $\Rightarrow Q = -x^2 - 2x - 4; R = 15$
- $2x^3 + 3x + 7$ by $2x + 1$ $\Rightarrow Q = x^2 - \frac{1}{2}x + \frac{7}{4}; R = \frac{21}{4}$

Ex. 1.2.26: Rewrite as a polynomial plus a proper fraction of polynomials:

- $x^3 + 3x^2 + 3x + 7$ by $x + 2$ $= x^2 + x + 1 + \frac{5}{x+2}$
- $x^3 + 3x^2 + 3x + 7$ by $x - 2$ $= x^2 + 5x + 13 + \frac{33}{x-2}$
- $x^3 + 7$ by $2 - x$ $= -x^2 - 2x - 4 + \frac{15}{2-x}$
- $2x^3 + 3x + 7$ by $2x + 1$ $= x^2 - \frac{1}{2}x + \frac{7}{4} + \frac{21}{8x+4}$

Ex. 1.2.27: Rewrite as a reduced fraction:

- $\frac{5+\frac{1}{7^2}}{3} =$
- $\frac{5-\frac{3}{7^2}}{3} =$
- $\frac{35+\frac{1}{20}}{5} =$
- $\frac{5-\frac{1}{10}}{\frac{5}{3}} =$

Ex. 1.2.23: Rewrite as a reduced fraction:

- $\frac{x}{(x^2+2x+1)(x+2)} + \frac{3}{(x^2+4x+4)(x+1)} = \frac{x^2+5x+3}{(x+1)^2(x+2)^2}$
- $\frac{x}{(x^2+3x+2)(x+2)} + \frac{3}{(x^2+4x+4)(x-1)} = \frac{3}{(x-1)(x+1)(x+2)^2}$
- $\frac{x}{(x^2-2x-15)} + \frac{6-x}{x^2-11x+30} = -\frac{3}{(x-5)(x+3)}$
- $\frac{x}{x^2+3x-10} - \frac{3}{x^2-4x+4} = \frac{x^2-5x-15}{(x-2)^2(x+5))}$

Ex. 1.2.24: Rewrite as a reduced fraction:

- $\frac{x}{x^2+3x+2} + \frac{x+3}{x^2+4x+4} = \frac{2x^2+6x+3}{(x+1)(x+2)^2}$
- $\frac{x}{x^2-5x+6} - \frac{x+3}{x^2-6x+8} = \frac{9-4x}{(x-4)(x-3)(x-2)}$
- $\frac{x}{x^2-4} + \frac{x+3}{x^2+7x+10} = \frac{2(x^2+3x-3)}{(x-2)(x+2)(x+5)}$
- $\frac{x}{x^2+3x+2} \frac{x+3}{x^2+7x+6} = \frac{x(x+3)}{(x+2)(x+6)(x+1)^2}$

Ex. 1.2.25: Use long division with Quotient Q and Remainder R to divide

- $x^3 + 3x^2 + 3x + 7$ by $x + 2$ $\Rightarrow Q = x^2 + x + 1; R = 5$
- $x^3 + 3x^2 + 3x + 7$ by $x - 2$ $\Rightarrow Q = x^2 + 5x + 13; R = 33$
- $x^3 + 7$ by $2 - x$ $\Rightarrow Q = -x^2 - 2x - 4; R = 15$
- $2x^3 + 3x + 7$ by $2x + 1$ $\Rightarrow Q = x^2 - \frac{1}{2}x + \frac{7}{4}; R = \frac{21}{4}$

Ex. 1.2.26: Rewrite as a polynomial plus a proper fraction of polynomials:

- $x^3 + 3x^2 + 3x + 7$ by $x + 2$ $= x^2 + x + 1 + \frac{5}{x+2}$
- $x^3 + 3x^2 + 3x + 7$ by $x - 2$ $= x^2 + 5x + 13 + \frac{33}{x-2}$
- $x^3 + 7$ by $2 - x$ $= -x^2 - 2x - 4 + \frac{15}{2-x}$
- $2x^3 + 3x + 7$ by $2x + 1$ $= x^2 - \frac{1}{2}x + \frac{7}{4} + \frac{21}{8x+4}$

Ex. 1.2.27: Rewrite as a reduced fraction:

- $\frac{5+\frac{1}{7^2}}{3} = \frac{82}{49}$
- $\frac{5-\frac{3}{7^2}}{3} = \frac{242}{147}$
- $\frac{35+\frac{1}{20}}{5} = \frac{701}{100}$
- $\frac{5-\frac{1}{10}}{\frac{5}{3}} = \frac{147}{50}$

Ex. 1.2.28: Rewrite as a reduced fraction:

- $\dfrac{y + \frac{1}{x^2}}{z} =$
- $\dfrac{x + \frac{1}{x^2 z}}{xz^2} =$
- $\dfrac{3a - \frac{1}{xz^2}}{az} =$
- $\dfrac{\frac{y}{z} + \frac{1}{x^2}}{yz} =$

Ex. 1.2.28: Rewrite as a reduced fraction:

- $\dfrac{y + \frac{1}{x^2}}{z} = \dfrac{x^2 y + 1}{x^2 z}$
- $\dfrac{x + \frac{1}{x^2 z}}{xz^2} = \dfrac{x^3 z + 1}{x^3 z^3}$
- $\dfrac{3a - \frac{1}{xz^2}}{az} = \dfrac{3axz^2 - 1}{axz^3}$
- $\dfrac{\frac{y}{z} + \frac{1}{x^2}}{yz} = \dfrac{x^2 y + z}{x^2 yz^2}$

Ex. 1.2.28: Rewrite as a reduced fraction:

- $\dfrac{y+\frac{1}{x^2}}{z} = \dfrac{x^2y+1}{x^2z}$
- $\dfrac{x+\frac{1}{x^2z}}{xz^2} = \dfrac{x^3z+1}{x^3z^3}$
- $\dfrac{3a-\frac{1}{xz^2}}{az} = \dfrac{3axz^2-1}{axz^3}$
- $\dfrac{\frac{y}{z}+\frac{1}{x^2}}{yz} = \dfrac{x^2y+z}{x^2yz^2}$

Ex. 1.2.29: Rewrite as a reduced fraction:

- $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}} =$
- $\dfrac{\frac{3}{a}-\frac{4}{8y}}{\frac{1}{ay^2}-\frac{1}{y^2}} =$
- $\dfrac{\frac{1}{x^2}+\frac{1}{y^2}}{\frac{1}{x^2}-\frac{1}{y^2}} =$
- $\dfrac{\frac{z}{x}+\frac{1}{z}}{\frac{1}{z^2}-\frac{1}{z}} =$

Ex. 1.2.28: Rewrite as a reduced fraction:

- $\dfrac{y+\frac{1}{x^2}}{z} = \dfrac{x^2y+1}{x^2z}$
- $\dfrac{x+\frac{1}{x^2z}}{xz^2} = \dfrac{x^3z+1}{x^3z^3}$
- $\dfrac{3a-\frac{1}{xz^2}}{az} = \dfrac{3axz^2-1}{axz^3}$
- $\dfrac{\frac{y}{z}+\frac{1}{x^2}}{yz} = \dfrac{x^2y+z}{x^2yz^2}$

Ex. 1.2.29: Rewrite as a reduced fraction:

- $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}} = \dfrac{xy}{y-x}$
- $\dfrac{\frac{3}{a}-\frac{4}{8y}}{\frac{1}{ay^2}-\frac{1}{y^2}} = \dfrac{y(6y-a)}{2(1-a)}$
- $\dfrac{\frac{1}{x^2}+\frac{1}{y^2}}{\frac{1}{x^2}-\frac{1}{y^2}} = \dfrac{-x^2-y^2}{x^2-y^2}$
- $\dfrac{\frac{z}{x}+\frac{1}{z}}{\frac{1}{z^2}-\frac{1}{z}} = \dfrac{z(x+z^2)}{x(1-z)}$

Ex. 1.2.28: Rewrite as a reduced fraction:

- $\dfrac{y+\frac{1}{x^2}}{z} = \dfrac{x^2y+1}{x^2z}$
- $\dfrac{x+\frac{1}{x^2z}}{xz^2} = \dfrac{x^3z+1}{x^3z^3}$
- $\dfrac{3a-\frac{1}{xz^2}}{az} = \dfrac{3axz^2-1}{axz^3}$
- $\dfrac{\frac{y}{z}+\frac{1}{x^2}}{yz} = \dfrac{x^2y+z}{x^2yz^2}$

Ex. 1.2.29: Rewrite as a reduced fraction:

- $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}} = \dfrac{xy}{y-x}$
- $\dfrac{\frac{3}{a}-\frac{4}{8y}}{\frac{1}{ay^2}-\frac{1}{y^2}} = \dfrac{y(6y-a)}{2(1-a)}$
- $\dfrac{\frac{1}{x^2}+\frac{1}{y^2}}{\frac{1}{x^2}-\frac{1}{y^2}} = \dfrac{-x^2-y^2}{x^2-y^2}$
- $\dfrac{\frac{z}{x}+\frac{1}{z}}{\frac{1}{z^2}-\frac{1}{z}} = \dfrac{z(x+z^2)}{x(1-z)}$

Ex. 1.2.30: Rewrite as a reduced fraction:

- $\dfrac{\frac{1}{a+h}-\frac{1}{a}}{h} =$
- $\dfrac{\frac{1}{(a+h)^2}-\frac{1}{a^2}}{h} =$
- $\dfrac{\frac{1}{(x+h)^3}-\frac{1}{x^3}}{h} =$
- $\dfrac{\frac{1}{2x+2h+1}-\frac{1}{2x+1}}{h} =$

Ex. 1.2.28: Rewrite as a reduced fraction:

- $\dfrac{y+\frac{1}{x^2}}{z} = \dfrac{x^2y+1}{x^2z}$
- $\dfrac{x+\frac{1}{x^2z}}{xz^2} = \dfrac{x^3z+1}{x^3z^3}$
- $\dfrac{3a-\frac{1}{xz^2}}{az} = \dfrac{3axz^2-1}{axz^3}$
- $\dfrac{\frac{y}{z}+\frac{1}{x^2}}{yz} = \dfrac{x^2y+z}{x^2yz^2}$

Ex. 1.2.29: Rewrite as a reduced fraction:

- $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}} = \dfrac{xy}{y-x}$
- $\dfrac{\frac{3}{a}-\frac{4}{8y}}{\frac{1}{ay^2}-\frac{1}{y^2}} = \dfrac{y(6y-a)}{2(1-a)}$
- $\dfrac{\frac{1}{x^2}+\frac{1}{y^2}}{\frac{1}{x^2}-\frac{1}{y^2}} = \dfrac{-x^2-y^2}{x^2-y^2}$
- $\dfrac{\frac{z}{x}+\frac{1}{z}}{\frac{1}{z^2}-\frac{1}{z}} = \dfrac{z(x+z^2)}{x(1-z)}$

Ex. 1.2.30: Rewrite as a reduced fraction:

- $\dfrac{\frac{1}{a+h}-\frac{1}{a}}{h} = \frac{-1}{a(a+h)}$
- $\dfrac{\frac{1}{(a+h)^2}-\frac{1}{a^2}}{h} = \frac{-2a-h}{a^2(a+h)^2}$
- $\dfrac{\frac{1}{(x+h)^3}-\frac{1}{x^3}}{h} = \frac{-h^2-3hx-3x^2}{x^3(h+x)^3}$
- $\dfrac{\frac{1}{2x+2h+1}-\frac{1}{2x+1}}{h} = \frac{-2}{(2x+1)(2h+2x+1)}$

- 1.3.1: Rational and negative exponents
- 1.3.2: Roots and radicals
- 1.3.3: Square and n^{th} root identities
- 1.3.4: Converting between fractions, radicals, and negative powers
- Section 1.3 Review

Section 1.3 Preview: Definitions/Theorems

- Definition 1.3.1: A rational number is a fraction with integer numerator and denominator
- Definition 1.3.2: Power rules for rational exponents
- Definition 1.3.3: For any real number K, the number of solutions of $x^2 = K$ is
- Definition 1.3.4: For any real number K, the equation $x^3 = K$ has one solution.
- Definition 1.3.5: Even roots and radicals
- Definition 1.3.6: Odd roots and radicals
- Definition 1.3.7: Notation: Fractional powers are roots.

Section 1.3 Preview: Procedures

- Procedure 1.3.1: To rewrite a fraction without negative exponents
- Procedure 1.3.2: To simplify radicals, apply power rules to fractional exponents:
- Procedure 1.3.3: To simplify square roots of positive integers
- Procedure 1.3.4: To *rationalize* a fraction
- Procedure 1.3.5: To simplify n^{th} roots of an integer
- Procedure 1.3.6: To simplify n^{th} roots of a monomial

1.3.1 Rational and negative exponents

Power rules stated earlier for positive integer exponents also work for negative and rational exponents!
Reminder: a fraction has nonzero denominator.

Definition: A rational number is

a fraction with integer numerator and denominator

In this section, m and n are integers.

Negative exponents: The product of powers rule $A^{m+n} = A^m A^n$ with $n = -m$ says that $A^m A^{-m} = A^{m+-m} = A^0 = 1$ provided $A \neq 0$.

Dividing by A^m gives $\boxed{A^{-m} = \frac{1}{A^m}}$

Some examples rewritten without negative exponents:
$A^{-1} = \frac{1}{A}$ $\quad A^{-2} = \frac{1}{A^2}$ $\quad A^{-3} = \frac{1}{A^3}$

Rational exponents: Let n be a positive integer. The power to a power rule $A^{mn} = (A^m)^n$ with $m = \frac{1}{n}$ says $(A^{\frac{1}{n}})^n = A^{(\frac{1}{n}\cdot n)} = A^1 = A$. In other words, $A^{\frac{1}{n}}$ raised to the power n is equal to A.

$$\boxed{A^{\frac{m}{n}} = A^{\frac{1}{n}\cdot m} = \left(A^{\frac{1}{n}}\right)^m = 1 \text{ if } m = n}$$

If we take $A = 9$ and $n = 2$, then $(9^{\frac{1}{2}})^2 = 9$. Since $3^2 = 9$ but also $(-3)^2 = 9$, it seems that $9^{\frac{1}{2}}$ might have two values. We will clarify this in the next section.

Power rules for rational exponents

- Product of powers: $A^m A^n = A^{m+n}$
- Power of product: $(AB)^m = A^m B^m$
- Power of quotient $\left(\frac{A}{B}\right)^m = \frac{A^m}{B^m}$
- Negation of power: $A^{-m} = \frac{1}{A^m}$
- Power of power: $(A^m)^n = A^{mn}$
- Quotient of power: $\frac{A^m}{A^n} = A^m \frac{1}{A^n} = A^{m-n}$

Example 1: Reduce each fraction by cancelling common powers of x from numerator and denominator.

- $\frac{ax^{10}}{bx^{\frac{3}{2}}} = \frac{ax^{10-\frac{3}{2}}}{b} = \boxed{\frac{ax^{\frac{17}{2}}}{b}}$
- $\frac{ax^{-\frac{4}{3}}}{bx^{10}} = \frac{a}{bx^{10-(-\frac{4}{3})}} = \boxed{\frac{a}{bx^{\frac{34}{3}}}}$

Example 2: Reduce each fraction completely:

- $\dfrac{x^3}{x^{\frac{10}{3}}} = x^{3-\frac{10}{3}} = x^{-\frac{1}{3}} = \boxed{\dfrac{1}{x^{\frac{1}{3}}}}$
- $\dfrac{x^{-\frac{3}{4}}}{x^3} = x^{-\frac{3}{4}-3} = x^{-\frac{15}{4}} = \boxed{\dfrac{1}{x^{\frac{15}{4}}}}$
- $\dfrac{x^{10}y^2}{x^4 y^{\frac{7}{2}}} = \dfrac{x^{10-4}}{y^{\frac{7}{2}-2}} = \boxed{\dfrac{x^6}{y^{\frac{3}{2}}}}$

Example 3: Factor $x^{\frac{1}{2}} + x^{\frac{5}{2}}$

Solution: The power of x with the lowest exponent is $x^{1/2}$. Factor it out using $x^{\frac{5}{2}} = x^{\frac{1}{2}} x^{\frac{5}{2}-\frac{1}{2}}$

$$x^{\frac{1}{2}} + x^{\frac{5}{2}} = x^{\frac{1}{2}}(1 + x^{\frac{5}{2}-\frac{1}{2}}) = \boxed{x^{\frac{1}{2}}(1 + x^2)}$$

Example 4: Rewrite $\dfrac{x^{\frac{1}{2}} + x^{\frac{5}{2}}}{x^{\frac{3}{2}}}$ as a reduced fraction with positive integer exponents.

Solution:The last example factored the numerator:

$$\frac{x^{\frac{1}{2}} + x^{\frac{5}{2}}}{x^{\frac{3}{2}}} = \frac{x^{\frac{1}{2}}(1 + x^2)}{x^{\frac{3}{2}}} = \frac{(1 + x^2)}{x^{\frac{3}{2}-\frac{1}{2}}} = \boxed{\frac{1 + x^2}{x}}$$

Example 5: Rewrite $\dfrac{x^{\frac{1}{2}} + x^{\frac{5}{2}}}{x^{\frac{3}{2}}}$ as a sum of powers of x.

Solution: Separate into two fractions

$$\frac{x^{\frac{1}{2}} + x^{\frac{5}{2}}}{x^{\frac{3}{2}}} = \frac{x^{\frac{1}{2}}}{x^{\frac{3}{2}}} + \frac{x^{\frac{5}{2}}}{x^{\frac{3}{2}}} \qquad \text{Now apply } \frac{x^m}{x^n} = x^{m-n}.$$

$$= x^{\frac{1}{2}-\frac{3}{2}} + x^{\frac{5}{2}-\frac{3}{2}} = x^{-1} + x^1 = \boxed{x^{-1} + x}$$

How to rewrite a fraction without negative exponents

- $\dfrac{x^{-m}E}{F} = \dfrac{E}{x^m F}$
- $\dfrac{E}{x^{-m}F} = \dfrac{x^m E}{F}$

Both identities follow immediately if the left fraction is multiplied by $\frac{x^m}{x^m}$.

Basic examples:

$$\frac{x^{-12}}{y^{-3}} = \frac{y^3}{x^{12}} \text{ and } \frac{x^{-5}}{y^{-3}x^{-4}} = \frac{y^3}{x^5 x^{-4}} = \frac{y^3}{x}$$

Be careful: Don't use the above identities unless the negative power is a factor of numerator or denominator.

For example, $\dfrac{x^{-3} + B}{C} = \dfrac{B}{x^3 + C}$ is a false statement.

Example 6:

Rewrite $\dfrac{x^{-3}y^5}{z^3}$ without negative exponents.

Solution: There are two methods. First, the numerator is x^{-3} times y^5 and so x^{-3} is a factor of the numerator.

Therefore $\dfrac{x^{-3}y^5}{z^3} = \dfrac{y^5}{x^3 z^3}$

OR, multiply by $\frac{x^3}{x^3}$:

$$\frac{x^3}{x^3} \cdot \frac{x^{-3}y^5}{z^3} = \frac{x^3 x^{-3} y^5}{x^3 z^3} = \boxed{\frac{y^5}{x^3 z^3}}$$

Example 7: Rewrite $\dfrac{x^{-2}+xy}{y^3}$ without negative exponents.

Solution: Numerator and denominator have no common factor, so the only method is to multiply by $\frac{x^2}{x^2}$:

$$\frac{x^2}{x^2} \cdot \frac{x^{-2}+xy}{y^3} = \frac{x^2(x^{-2}+xy)}{x^2 y^3}$$

$$= \frac{x^2 x^{-2} + x^2 xy}{x^2 y^3} = \boxed{\frac{1+x^3 y}{x^2 y^3}}$$

Be careful: Since neither x nor y is a factor of the numerator, the boxed answer cannot be rewritten by canceling.

Example 8: Rewrite $\dfrac{a+6}{c^{-2}d^8}$ without negative exponents.

Solution:

Same idea as Example 7: $\dfrac{c^2}{c^2} \cdot \dfrac{a+6}{c^{-2}d^8} = \dfrac{c^2(a+6)}{c^2 c^{-2} d^8} = \boxed{\dfrac{c^2(a+6)}{d^8}}$

Be careful: Parentheses matter! $\dfrac{c^2 a+6}{d^8}$ is wrong.

Example 9: Rewrite without parentheses: $\left(\dfrac{2a^2}{b^3 c^4}\right)^2$

Use $\left(\dfrac{A}{B}\right)^m = \dfrac{A^m}{B^m}$ with $A = 2a^2$ and $B = b^3 c^4$: $= \dfrac{(2a^2)^2}{(b^3 c^4)^2}$

Use $(AB)^2 = A^2 B^2$ to rewrite numerator. $= \dfrac{2^2 (a^2)^2}{(b^3 c^4)^2}$

Similarly, rewrite the denominator $= \dfrac{2^2 (a^2)^2}{(b^3)^2 (c^4)^2}$

For each power of a power, multiply exponents: $= \boxed{\dfrac{4a^4}{b^6 c^8}}$

With practice, you should be able to skip the middle steps.

Example 10: Rewrite as a reduced fraction without negative exponents: $\left(\frac{2a^{-1}b}{a^2b^{-3}}\right)^{-3}$

First rewrite the inside fraction without negative exponents Then use the identity $\left(\frac{E}{F}\right)^{-n} = \left(\frac{F}{E}\right)^{n} = \frac{F^n}{E^n}$ with

$E = 2b^4$ and $F = a^3$ to get: $\left(\frac{2a^{-1}b}{a^2b^{-3}}\right)^{-3} = \left(\frac{2b^3b}{aa^2}\right)^{-3} = \left(\frac{2b^4}{a^3}\right)^{-3} = \frac{(a^3)^3}{(2b^4)^3} = \frac{a^9}{2^3(b^4)^3} = \boxed{\frac{a^9}{8b^{12}}}$

A slightly harder method: Use $\left(\frac{A}{B}\right)^m = \frac{A^m}{B^m}$ with $n = -3$ to get $\left(\frac{2a^{-1}b}{a^2b^{-3}}\right)^{-3} = \frac{(2a^{-1}b)^{-3}}{(a^2b^{-3})^{-3}}$

Use $(ABC)^n = A^nB^nC^n$ with $n = -3$;

$(AB)^n = A^nB^n$ with $A = a^2, B = b^{-3}, m = -3$ $\quad = \frac{2^{-3}(a^{-1})^{-3}b^{-3}}{(a^2)^{-3}(b^{-3})^{-3}}$

Use $(A^m)^n = A^{mn}$ in several places: $\quad = \frac{2^{-3}a^3b^{-3}}{a^{-6}b^9}$

Flip powers with negative exponents: $\quad = \frac{a^6a^3}{2^3b^3b^9}$

Use $(A^m)^n = A^{mn}$ in numerator and denominator. $\quad = \boxed{\frac{a^9}{8b^{12}}}$

Example 11: Rewrite $\left(\frac{2x^{\frac{3}{4}}}{y^{\frac{1}{3}}}\right)^3 \left(\frac{y^4}{x^{-\frac{1}{2}}}\right)$ without parentheses or negative exponents:

Solution: Use $\left(\frac{A}{B}\right)^3 = \frac{A^3}{B^3}$: $= \frac{\left(2x^{\frac{3}{4}}\right)^3}{\left(y^{\frac{1}{3}}\right)^3} \cdot \frac{y^4 x^{\frac{1}{2}}}{1}$

Use $(A^m)^n = A^{mn}$ twice: $= \frac{2^3 x^{\frac{9}{4}}}{y} \cdot \frac{y^4 x^{\frac{1}{2}}}{1}$

Multiply and cancel $= \frac{8x^{\frac{9}{4}} x^{\frac{1}{2}} \cdot y^3 \not{y}}{\not{y} \cdot 1} = 8x^{\frac{9}{4}+\frac{1}{2}} y^3 = \boxed{8x^{\frac{11}{4}} y^3 \text{ or } 8x^2 \sqrt[4]{x^3} y^3}$

Example 12: $\left(\frac{3x^3}{y^{\frac{1}{2}}}\right)^2 \left(\frac{x^{\frac{4}{3}}}{y^{-\frac{2}{3}}}\right)^3$ as a reduced fraction without negative exponents.

Solution: Remove negative power $= \left(\frac{3x^3}{y^{\frac{1}{2}}}\right)^2 \left(x^{\frac{4}{3}} y^{\frac{2}{3}}\right)^3$

Use $\left(\frac{A}{B}\right)^m = \frac{A^m}{B^m}$ $= \frac{(3x^3)^2}{\left(y^{(\frac{1}{2}}\right)^2}) \left(x^{\frac{4}{3}}\right)^3 \left(y^{\frac{2}{3}}\right)^3$

and $(AB)^m = A^m B^m$ $= \frac{3^2 (x^3)^2}{y^{\frac{1}{2} \cdot 2}} \cdot x^{(\frac{4}{3} \cdot 3)} y^{(\frac{2}{3} \cdot 3)} = \frac{9x^6}{y^1} \frac{x^4 y^2}{1} = \frac{9x^6 x^4 y^2}{y} = \boxed{9x^{10} y}$

1.3.2 Roots and radicals

For any real number K, the equation $x^2 = K$ has

- two solutions $x = \pm\sqrt{K}$ if $K > 0$,
- one solution $x = 0$ if if $K = 0$,
- no real solutions if $K < 0$.

- The solution of $x^2 = 3$ is $x = \pm\sqrt{3}$.
- $x = \sqrt{3}$ is the *positive square root* of 3
- $x = -\sqrt{3}$ is the *negative square root* of 3.
- The solution of $x^2 = 25$ is $x = \pm\sqrt{25} = \pm 5$
- $x^2 = -3$ has no real solutions, since

the square of any real number is positive.

For any real number K, the equation

- $x^3 = K$ has one solution $x = \sqrt[3]{K}$
- $x^3 = -K$ has one solution $x = -\sqrt[3]{K}$

$\sqrt[3]{8} = 2$ is read "the 3rd (or cube) root of 8 is 2."
The solution of

- $x^3 = 8$ is $x = \sqrt[3]{8} = 2$ since $2^3 = 8$
- $x^3 = -8$ is $x = -\sqrt[3]{8} = -2$ since $(-2)^3 = -8$
- of $x^3 = 5$ is $x = \sqrt[3]{5}$.
- of $x^3 = -5$ is $x = -\sqrt[3]{5}$.

Reminder: $|x|$ is the absolute value of x:
$|7| = |-7| = 7$.

Even roots and radicals

- For *all* real numbers x, $\sqrt{x^2} = |x|$.
- If $x < 0$, then $\sqrt{x}$ is undefined and $\sqrt{x^2} = -x = |x|$.

Assume $n \geq 2$ is even and x is a real number.

- If $x < 0$, $\sqrt[n]{x}$ is undefined and $\sqrt[n]{x^n} = -x = |x|$.
- If $x \geq 0$, $\sqrt[n]{x}$ is defined and
  - $(\sqrt[n]{x})^n = (-\sqrt[n]{x})^n = \sqrt[n]{x^n} = x$.
  - $\sqrt[n]{x}$ is the positive n^{th} root of x.
  - $-\sqrt[n]{x}$ is the negative n^{th} root of x.

Example 13: Rewrite each root as an integer.

- $\sqrt[6]{64} = 2$ because $64 = 8 \cdot 8 = 2^3 \cdot 2^3 = 2^{3+3} = 2^6$.
- The solution of $x^6 = 64$ is $x = \pm\sqrt[6]{64} = \pm 2$.
- $\sqrt[6]{-64}$ is undefined.
- $\sqrt[6]{1,000,000} = 10$ because $1,000,000 = 10^6$

Odd roots and radicals: If x is any real number

- $\sqrt[3]{x}$ is the cube root of x and $\sqrt[3]{x^3} = (\sqrt[3]{x})^3 = x$.
- If $n \geq 3$ is odd
  - $\sqrt[n]{x}$ is defined and has the same sign as x.
  - $\sqrt[n]{x^n} = (\sqrt[n]{x})^n = x$

Example 13 continued:

- $\sqrt[5]{32} = 2$ because $2^5 = 32$.
- The solution of $x^5 = 32$ is $x = \sqrt[5]{32} = 2$.
- $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$.
- The solution of $x^5 = -32$ is $x = \sqrt[5]{-32} = -2$.

Notation: Fractional powers are roots.

- If $n \geq 2$ is an integer, $x^{\frac{1}{n}} = \sqrt[n]{x}$
- If $m, n \geq 1$ are integers with no common factor, $x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m = \sqrt[n]{x^m}$

Above and elsewhere, the equals sign means: both sides are defined and equal or both sides are undefined.

- $x = 5^{\frac{1}{2}}$ means $x^2 = 5$ and $x > 0$. Thus $x = \sqrt{5}$
- $x = 5^{\frac{1}{7}}$ means $x^7 = 5$. Thus $x = \sqrt[7]{5}$

- Reminder: $\frac{3}{7} = 3 \cdot \frac{1}{7} = \frac{1}{7} \cdot 3$
- $5^{\frac{3}{7}} = 5^{3 \cdot \frac{1}{7}} = (5^3)^{1/7} = \sqrt[7]{5^3} = \sqrt[7]{125}$
- $5^{\frac{3}{7}} = 5^{\frac{1}{7} \cdot 3} = (5^{\frac{1}{7}})^3 = (\sqrt[7]{5})^3$
- $5^{\frac{7}{3}} = 5^{2+\frac{1}{3}} = 5^2 5^{\frac{1}{3}} = 25\sqrt[3]{5})$

To simplify radicals, apply power rules to fractional exponents:

- $\sqrt{xy} = \sqrt{x}\sqrt{y}$ if $x \geq 0$ and $y \geq 0$.
- $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$ if $x \geq 0$ and $y > 0$.

  For any integer $n > 0$ and real numbers x, y:
- $(xy)^{\frac{1}{n}} = \sqrt[n]{xy} = \sqrt[n]{x} \cdot \sqrt[n]{y}$ if $\sqrt[n]{x}$ and $\sqrt[n]{y}$ are both defined.
- $\left(\frac{x}{y}\right)^{\frac{1}{n}} = \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$ if $\sqrt[n]{x}$ and $\sqrt[n]{y}$ are both defined and $y \neq 0$.

Advice: Don't memorize these formulas. Just compute rational powers $x^{\frac{m}{n}}$ by applying the rule $(x^s)^t = x^{st}$ with $s = m$ and $t = \frac{1}{n}$.

1.3.3 Square and n^{th} root identities

Be careful: If neither x nor y equals 0:

- $\sqrt{x+y}$ IS NOT equal to $\sqrt{x}+\sqrt{y}$.
- $(x+y)^2$ IS NOT equal to x^2+y^2.
- If $n>1$
  $\sqrt[n]{x+y}$ IS NOT equal to $\sqrt[n]{x}+\sqrt[n]{y}$.
- If $n\neq 1$,
  $(x+y)^n$ IS NOT equal to x^n+y^n.

To simplify square roots of positive integers

- Factor the integer
- Use $\sqrt{AB}=\sqrt{A}\sqrt{B}$ to factor out perfect squares until the integer remaining under the radical sign doesn't have any factor that is a perfect square.
- In your answer, write the integer to the left of the radical.

Example 14: Simplify each of the following:

- $\sqrt{9}=\sqrt{3^2}=3$
- $\sqrt{18}=\sqrt{2\cdot 3^2}=\sqrt{2}\sqrt{3^2}=\sqrt{2}\cdot 3=3\sqrt{2}$
- $\sqrt{45}=\sqrt{9\cdot 5}=\sqrt{9}\sqrt{5}=3\sqrt{5}$
- $\sqrt{108}=\sqrt{4\cdot 27}=\sqrt{4}\sqrt{27}=2\sqrt{9}\cdot\sqrt{3}$
  $=2\cdot 3\cdot\sqrt{3}=6\sqrt{3}$
- $\sqrt{1000}=\sqrt{100\cdot 10}=\sqrt{100}\sqrt{10}=10\sqrt{10}$

To *rationalize* a fraction with denominator

- $a+\sqrt{b}$, multiply top and bottom by $a-\sqrt{b}$ or
- $a-\sqrt{b}$, multiply top and bottom by $a+\sqrt{b}$
  to get a fraction with denominator a^2-b.

Examples:

- $\frac{3}{5+\sqrt{7}}=\frac{3}{5+\sqrt{7}}\cdot\frac{5-\sqrt{7}}{5-\sqrt{7}}=\frac{3(5-\sqrt{7})}{5^2-7}=\frac{3(5-\sqrt{7})}{18}=\frac{5-\sqrt{7}}{6}$
- $\frac{3}{5-\sqrt{7}}=\frac{3}{5-\sqrt{7}}\cdot\frac{5+\sqrt{7}}{5+\sqrt{7}}=\frac{3(5+\sqrt{7})}{5^2-7}=\frac{3(5+\sqrt{7})}{18}=\frac{5+\sqrt{7}}{6}$
- $\frac{3}{\sqrt{7}}=\frac{3}{\sqrt{7}}\frac{\sqrt{7}}{\sqrt{7}}=\frac{3\sqrt{7}}{7}$

Example 15: Rationalize each fraction's denominator:

- $\frac{3}{\sqrt{12}}=\frac{3}{\sqrt{4}\sqrt{3}}=\frac{3}{2\sqrt{3}}=\frac{3\sqrt{3}}{2\sqrt{3}\sqrt{3}}=\frac{\cancel{3}\sqrt{3}}{2\cdot\cancel{3}}=\boxed{\frac{\sqrt{3}}{2}}$
- $\sqrt{\frac{12}{5}}=\frac{\sqrt{12}}{\sqrt{5}}=\frac{\sqrt{4\cdot 3}}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}}=\frac{\sqrt{4}\sqrt{3}\sqrt{5}}{5}=\boxed{\frac{2\sqrt{15}}{5}}$
- $\frac{12}{3-\sqrt{5}}=\frac{12}{3-\sqrt{5}}\cdot\frac{3+\sqrt{5}}{3+\sqrt{5}}=\frac{12(3+\sqrt{5})}{3^2-(\sqrt{5})^2}$
  $=\frac{12(3+\sqrt{5})}{9-5}=\frac{3\cdot\cancel{4}(3+\sqrt{5})}{1\cdot\cancel{4}}=\boxed{3(3+\sqrt{5})}$

To simplify n^{th} roots of an integer x

Factor out perfect n^{th} powers from x until the integer left under the radical sign doesn't doesn't have any factor that is a perfect n^{th} power.

To do this, you may need to factor x completely into powers of distinct primes. The strategy varies with each problem.

Example 16: Simplify each of the following:

- $\sqrt[3]{27000} = \sqrt[3]{27 \cdot 1000} = \sqrt[3]{27} \cdot \sqrt[3]{1000}$
  $= \sqrt[3]{3^3} \cdot \sqrt[3]{10^3} = 3 \cdot 10 = 30.$
- $\sqrt[3]{250} = \sqrt[3]{25 \cdot 10} = \sqrt[3]{5 \cdot 5 \cdot 5 \cdot 2} = \sqrt[3]{5^3} \cdot \sqrt[3]{2} = 5\sqrt[3]{2}$
- $\sqrt[4]{80} = \sqrt[4]{16 \cdot 5} = \sqrt[4]{16}\sqrt[4]{5} = \sqrt[4]{2^4}\sqrt[4]{5} = 2\sqrt[4]{5}$
- $\sqrt[5]{64} = \sqrt[5]{2^6} = \sqrt[5]{2^5 \cdot 2} = \sqrt[5]{2^5}\sqrt[5]{2} = 2\sqrt[5]{2}$
- $\sqrt{125} + \sqrt{45} = \sqrt{25 \cdot 5} + \sqrt{9 \cdot 5} = \sqrt{25}\sqrt{5} + \sqrt{9}\sqrt{5}$
  $= 5\sqrt{5} + 3\sqrt{5} = 8\sqrt{5}$
- $\sqrt{100} + \sqrt{200} = 10 + \sqrt{10^2 \cdot 2} = 10 + 10\sqrt{2}$
  $= 10(1 + \sqrt{2})$

The same idea works for simplifying the n^{th} root of a monomial. However, if n is even, letters must represent positive numbers.

To simplify the n^{th} root of a monomial

- Rewrite the monomial as a product of roots of letter powers or prime number powers.
- To simplify each factor $\sqrt[n]{x^m}$, factor out the largest possible power of x^n from x^m to get $x^m = (x^n)^k x^r$ where $0 \leq k < r$. Here k is the quotient and r is the remainder when you divide m by n. Then
- $\sqrt[n]{x^m} = \sqrt[n]{(x^n)^k x^r} = \sqrt[n]{(x^k)^n}\sqrt[n]{x^r} = x^k\sqrt[n]{x^r}$

Example 17: Simplify each of the following :

- $\sqrt[3]{128} = \sqrt[3]{64 \cdot 2} = \sqrt[3]{64} \cdot \sqrt[3]{2} = 4\sqrt[3]{2}.$
- $\sqrt[3]{x^{10}} = \sqrt[3]{x^9 x} = \sqrt[3]{(x^3)^3}\sqrt[3]{x} = x^3 \cdot \sqrt[3]{x}$
- $\sqrt[3]{y^8} = \sqrt[3]{y^6 y^2} = \sqrt[3]{(y^2)^3}\sqrt[3]{xy^2} = y^2\sqrt[3]{y^2}$

Now multiply the last 3 bullets to rewrite

- $\sqrt[3]{128x^{10}y^8}$ as $4\sqrt[3]{2} \cdot x^3\sqrt[3]{x} \cdot y^2\sqrt[3]{y^2}$
  $= 4x^3y^2\sqrt[3]{2xy^2}$
- $\sqrt{x^2y^3} + \sqrt{x^3y^2}$
  $= \sqrt{x^2} \cdot \sqrt{y^2 \cdot y} + \sqrt{x^2 \cdot x} \cdot \sqrt{y^2}$
  $= xy\sqrt{y} + x\sqrt{x}y = xy(\sqrt{y} + \sqrt{x})$
  provided $x \geq 0$ and $y \geq 0$.

1.3.4 Converting between radicals, fractions, and negative powers

We have seen that $x^{-\frac{1}{2}} = \dfrac{1}{x^{1/2}} = \dfrac{1}{\sqrt{x}}$

You may have to convert between these three formats, since each has some virtue:

- $x^{-1/2}$ is not a fraction.
- $\dfrac{1}{x^{1/2}}$ is written without negative exponents.
- $\dfrac{1}{\sqrt{x}}$ is written without fractional exponents.

If the answer format is not specified, your answer's format should match the question's.

The easiest conversion is between radical notation and fractional exponents.

Example 18: Rewrite $x\sqrt{x}$ without radical signs.

Solution: $x\sqrt{x} = x^1 \cdot x^{1/2} = x^{1+1/2} = \boxed{x^{3/2}}$

If you start with an improper fraction exponent, convert it to a mixed number, as in

Example 19: Rewrite $x^{5/2}$ without fractional exponents:

Solution: $x^{5/2} = x^{2+1/2} = x^2 \cdot x^{1/2} = \boxed{x^2\sqrt{x}}$

Example 20: Rewrite $(x+\sqrt{x})^2$ without radical signs or parentheses.

Solution: $(x+\sqrt{x})^2 = (x+\sqrt{x})(x+\sqrt{x})$
$= x^2 + x\sqrt{x} + \sqrt{x}\cdot x + (\sqrt{x})^2 = x^2 + 2x\sqrt{x} + (\sqrt{x})^2$
$= \boxed{x^2 + 2x^{\frac{3}{2}} + x}$

Example 21:

Rewrite $\dfrac{1+x^2\sqrt{x}}{x^3}$ as a sum of powers of x.

Solution: First get rid of the radical:

$x^2\sqrt{x} = x^2x^{1/2} = x^{2+1/2} = x^{5/2}$. Therefore

$$\frac{1+x^2\sqrt{x}}{x^3} = \frac{1+x^{5/2}}{x^3} = \frac{1}{x^3} + \frac{x^{5/2}}{x^3}$$
$$= x^{-3} + x^{5/2-3} = \boxed{x^{-3} + x^{-1/2}}$$

Example 22:

Rewrite the last answer with neither fractional nor negative exponents.

Solution: $x^{-3} + x^{-1/2} = \dfrac{1}{x^3} + \dfrac{1}{x^{1/2}} = \boxed{\dfrac{1}{x^3} + \dfrac{1}{\sqrt{x}}}$

Example 23 : Factor completely. $12x^{\frac{5}{2}} - 36x^{\frac{3}{2}} + 24x^{\frac{1}{2}}$

This problem requires factoring a sum of powers of x with fractional exponents.
Factor out the power x^a with the lowest exponent and use $x^b = x^a x^{b-a}$.

Solution:

Factor out GCF of $12, 36, 24$:	$= 12(x^{\frac{5}{2}} - 3x^{\frac{3}{2}} + 2x^{\frac{1}{2}})$
Factor out lowest power of x:	$= 12x^{\frac{1}{2}}(x^{\frac{5}{2}-\frac{1}{2}} - 3x^{\frac{3}{2}-\frac{1}{2}} + 2x^{\frac{1}{2}-\frac{1}{2}})$
Rewrite the exponents:	$= 12x^{\frac{1}{2}}(x^2 - 3x^1 + 2x^0)$
Make it neater:	$= 12x^{\frac{1}{2}}(x^2 - 3x + 2)$
Factor the remaining part:	$= \boxed{12x^{\frac{1}{2}}(x-1)(x-2)}$

Since the original problem included fractional powers, don't change $x^{\frac{1}{2}}$ to $\sqrt{x}$.

Section 1.3 Quiz

Reduce each fraction completely:

Ex. 1.3.1: • $\frac{ax^{10}}{bx^{\frac{3}{2}}}$ • $\frac{ax^{-\frac{4}{3}}}{bx^{10}}$

Ex. 1.3.2: • $\frac{x^3}{x^{10/3}}$ • $\frac{x^{-3/4}}{x^3}$ • $\frac{x^{10}y^2}{x^4y^{7/2}}$

Ex. 1.3.3: Factor $x^{\frac{1}{2}} + x^{\frac{5}{2}}$.

Ex. 1.3.4: Reduce the fraction $\frac{x^{\frac{1}{2}}+x^{\frac{5}{2}}}{x^{\frac{3}{2}}}$.

Ex. 1.3.5: Rewrite $\frac{x^{\frac{1}{2}}+x^{\frac{5}{2}}}{x^{\frac{3}{2}}}$ as a sum of powers of x.

Rewrite the next 3 fractions without negative exponents:

Ex. 1.3.6: Rewrite $\frac{x^{-3}y^5}{z^3}$

Ex. 1.3.7: Rewrite $\frac{x^{-2}+xy}{y^3}$.

Ex. 1.3.8: Rewrite $\frac{a+6}{c^{-2}d^8}$.

Rewrite each of the next 4 problems as a reduced fraction without parentheses or negative exponents:

Ex. 1.3.9: $\left(\frac{2a^2}{b^3c^4}\right)^2$ Ex. 1.3.10: $\left(\frac{2a^{-1}b}{a^2b^{-3}}\right)^{-3}$

Ex. 1.3.11: $\left(\frac{2x^{3/4}}{y^{1/3}}\right)^3\left(\frac{y^4}{x^{-1/2}}\right)$

Ex. 1.3.12: $\left(\frac{3x^3}{y^{1/2}}\right)^2\left(\frac{x^{4/3}}{y^{-2/3}}\right)^3$

Ex. 1.3.13: Rewrite each root as an integer:
• $\sqrt[6]{64}$ • $\sqrt[6]{-64}$ • $\sqrt[5]{-32}$

Rewrite each expression as a radical:
• $5^{\frac{1}{2}}$ • $5^{\frac{1}{7}}$ • • $5^{\frac{3}{7}}$

Ex. 1.3.14: Simplify each of the following:
• $\sqrt{9}$ • $\sqrt{18}$ • $\sqrt{45}$ • $\sqrt{108}$

Ex. 1.3.15: Rationalize each fraction's denominator:
• $\frac{3}{\sqrt{12}}$ • $\sqrt{\frac{12}{5}}$ • $\frac{12}{3-\sqrt{5}}$

Ex. 1.3.16: Simplify each n^{th} root:
• $\sqrt[3]{27000}$ • $\sqrt[3]{250}$ • $\sqrt[4]{80}$ • $\sqrt[5]{64}$
• $\sqrt{125}+\sqrt{45}$ • $\sqrt{100}+\sqrt{200}$

Ex. 1.3.17: Simplify each n^{th} root: • $\sqrt[3]{x10}$ • $\sqrt[3]{y^8}$
• $\sqrt[3]{128}$ • $\sqrt[3]{128x^{10}y^8}$ • $\sqrt{x^2y^3}+\sqrt{x^3y^2}$

Ex. 1.3.18: Rewrite $x\sqrt{x}$ without radical signs.

Rewrite without fractional exponents:

Ex. 1.3.19: $x^{5/2}$ Ex. 1.3.20: $(x+\sqrt{x})^2$

Ex. 1.3.21: Rewrite $\frac{1+x^2\sqrt{x}}{x^3}$ as a sum of powers of x.

Ex. 1.3.22: Rewrite the last answer with neither fractional nor negative exponents.

Ex. 1.3.23: Factor $12x^{5/2}-36x^{3/2}+24x^{1/2}$

Section 1.3 Review: Powers, roots, and exponents

Ex. 1.3.1: Reduce each fraction completely so that no negative exponents appear in the answer.

- $\frac{ax^{10}}{bx^{3/2}} =$
- $\frac{ax^{-4/3}}{bx^{10}} =$
- $\frac{ax^{5}}{bx^{3/2}} =$
- $\frac{ax^{-1/3}}{bx^{-2}} =$

Section 1.3 Review: Powers, roots, and exponents

Ex. 1.3.1: Reduce each fraction completely so that no negative exponents appear in the answer.

- $\frac{ax^{10}}{bx^{3/2}} = \frac{ax^{17/2}}{b}$
- $\frac{ax^{-4/3}}{bx^{10}} = \frac{a}{bx^{34/3}}$
- $\frac{ax^{5}}{bx^{3/2}} = \frac{ax^{7/2}}{b}$
- $\frac{ax^{-1/3}}{bx^{-2}} = \frac{ax^{5/3}}{b}$

Section 1.3 Review: Powers, roots, and exponents

Ex. 1.3.1: Reduce each fraction completely so that no negative exponents appear in the answer.

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- $\frac{ax^{5}}{bx^{3/2}} = \frac{ax^{7/2}}{b}$
- $\frac{ax^{-1/3}}{bx^{-2}} = \frac{ax^{5/3}}{b}$
- $\frac{ab^{4}}{bx^{2/3}} =$
- $\frac{a^{1/2}x^{-2/5}}{a^{-3/2}x^{4}} =$
- $\frac{abcx^{10}}{a^{2}b^{2}x^{3/2}} =$
- $\frac{ax^{-4/3}}{bx^{2/3}} =$

Section 1.3 Review: Powers, roots, and exponents

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- $\frac{ax^{-1/3}}{bx^{-2}} = \frac{ax^{5/3}}{b}$
- $\frac{ab^{4}}{bx^{2/3}} = \frac{ab^{3}}{x^{2/3}}$
- $\frac{a^{1/2}x^{-2/5}}{a^{-3/2}x^{4}} = \frac{a^{2}}{bx^{22/5}}$
- $\frac{abcx^{10}}{a^{2}b^{2}x^{3/2}} = \frac{cx^{17/2}}{ab}$
- $\frac{ax^{-4/3}}{bx^{2/3}} = \frac{a}{bx^{2}}$

Section 1.3 Review: Powers, roots, and exponents

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- $\frac{ax^{-4/3}}{bx^{2/3}} = \frac{a}{bx^{2}}$

Ex. 1.3.2: Reduce each fraction completely so that no negative exponents appear in the answer.

- $\frac{x^{3}}{x^{10/3}} =$
- $\frac{x^{-3/4}}{5x^{-3}} =$
- $\frac{x^{10}y^{2}}{x^{4}y^{7/2}} =$
- $\frac{x^{3}a}{x^{10/3}a^{3}} =$

Section 1.3 Review: Powers, roots, and exponents

Ex. 1.3.1: Reduce each fraction completely so that no negative exponents appear in the answer.

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- $\frac{abcx^{10}}{a^{2}b^{2}x^{3/2}} = \frac{cx^{17/2}}{ab}$
- $\frac{ax^{-4/3}}{bx^{2/3}} = \frac{a}{bx^{2}}$

Ex. 1.3.2: Reduce each fraction completely so that no negative exponents appear in the answer.

- $\frac{x^{3}}{x^{10/3}} = \frac{1}{x^{1/3}}$
- $\frac{x^{-3/4}}{5x^{-3}} = \frac{x^{9/4}}{5}$
- $\frac{x^{10}y^{2}}{x^{4}y^{7/2}} = \frac{x^{6}}{y^{3/2}}$
- $\frac{x^{3}a}{x^{10/3}a^{3}} = \frac{1}{x^{1/3}a^{2}}$

Section 1.3 Review: Powers, roots, and exponents

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- $\frac{ab^{4}}{bx^{2/3}} = \frac{ab^{3}}{x^{2/3}}$
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- $\frac{x^{10}y^{2}}{x^{4}y^{7/2}} = \frac{x^{6}}{y^{3/2}}$
- $\frac{x^{3}a}{x^{10/3}a^{3}} = \frac{1}{x^{1/3}a^{2}}$
- $\frac{x^{-3/4}}{x^{3}} =$
- $\frac{s^{10}y^{2}}{s^{4}y^{7/2}} =$
- $\frac{y^{3}}{y^{1/5}} =$
- $\frac{6y^{-5/2}}{9xy^{1/2}} =$

Section 1.3 Review: Powers, roots, and exponents

Ex. 1.3.1: Reduce each fraction completely so that no negative exponents appear in the answer.

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Ex. 1.3.2: Reduce each fraction completely so that no negative exponents appear in the answer.

- $\frac{x^{3}}{x^{10/3}} = \frac{1}{x^{1/3}}$
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- $\frac{x^{3}a}{x^{10/3}a^{3}} = \frac{1}{x^{1/3}a^{2}}$
- $\frac{x^{-3/4}}{x^{3}} = \frac{1}{x^{15/4}}$
- $\frac{s^{10}y^{2}}{s^{4}y^{7/2}} = \frac{s^{6}}{y^{3/2}}$
- $\frac{y^{3}}{y^{1/5}} = y^{14/5}$
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Section 1.3 Review: Powers, roots, and exponents

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- $\frac{x^{3}a}{x^{10/3}a^{3}} = \frac{1}{x^{1/3}a^{2}}$
- $\frac{x^{-3/4}}{x^{3}} = \frac{1}{x^{15/4}}$
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- $\frac{y^{3}}{y^{1/5}} = y^{14/5}$
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- $\frac{a^{10}b^{1/3}}{b^{7/3}a^{4}} =$
- $\frac{8b^{1/3}}{16b^{10/3}} =$
- $\frac{16x^{1/2}}{20x^{3/2}} =$
- $\frac{x^{10}y^{1/3}}{x^{4}y^{7/3}} =$

Section 1.3 Review: Powers, roots, and exponents

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- $\frac{y^{3}}{y^{1/5}} = y^{14/5}$
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- $\frac{16x^{1/2}}{20x^{3/2}} = \frac{4}{5x}$
- $\frac{x^{10}y^{1/3}}{x^{4}y^{7/3}} = \frac{x^{6}}{y^{2}}$

Section 1.3 Review: Powers, roots, and exponents

Ex. 1.3.1: Reduce each fraction completely so that no negative exponents appear in the answer.

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- $\frac{a^{1/2}x^{-2/5}}{a^{-3/2}x^{4}} = \frac{a^{2}}{bx^{22/5}}$
- $\frac{abcx^{10}}{a^{2}b^{2}x^{3/2}} = \frac{cx^{17/2}}{ab}$
- $\frac{ax^{-4/3}}{bx^{2/3}} = \frac{a}{bx^{2}}$

Ex. 1.3.2: Reduce each fraction completely so that no negative exponents appear in the answer.

- $\frac{x^{3}}{x^{10/3}} = \frac{1}{x^{1/3}}$
- $\frac{x^{-3/4}}{5x^{-3}} = \frac{x^{9/4}}{5}$
- $\frac{x^{10}y^{2}}{x^{4}y^{7/2}} = \frac{x^{6}}{y^{3/2}}$
- $\frac{x^{3}a}{x^{10/3}a^{3}} = \frac{1}{x^{1/3}a^{2}}$
- $\frac{x^{-3/4}}{x^{3}} = \frac{1}{x^{15/4}}$
- $\frac{s^{10}y^{2}}{s^{4}y^{7/2}} = \frac{s^{6}}{y^{3/2}}$
- $\frac{y^{3}}{y^{1/5}} = y^{14/5}$
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- $\frac{a^{10}b^{1/3}}{b^{7/3}a^{4}} = \frac{a^{6}}{b^{2}}$
- $\frac{8b^{1/3}}{16b^{10/3}} = \frac{1}{2b^{3}}$
- $\frac{16x^{1/2}}{20x^{3/2}} = \frac{4}{5x}$
- $\frac{x^{10}y^{1/3}}{x^{4}y^{7/3}} = \frac{x^{6}}{y^{2}}$

Ex. 1.3.3: Factor

- $x^{1/2} + x^{5/2} =$
- $3x^{3/2} + 6x^{5/2} =$
- $5x^{2} + 2x^{20/3} =$
- $63x^{1/2} + 12x^{5/2} =$

Section 1.3 Review: Powers, roots, and exponents

Ex. 1.3.1: Reduce each fraction completely so that no negative exponents appear in the answer.

- $\frac{ax^{10}}{bx^{3/2}} = \frac{ax^{17/2}}{b}$
- $\frac{ax^{-4/3}}{bx^{10}} = \frac{a}{bx^{34/3}}$
- $\frac{ax^{5}}{bx^{3/2}} = \frac{ax^{7/2}}{b}$
- $\frac{ax^{-1/3}}{bx^{-2}} = \frac{ax^{5/3}}{b}$
- $\frac{ab^{4}}{bx^{2/3}} = \frac{ab^{3}}{x^{2/3}}$
- $\frac{a^{1/2}x^{-2/5}}{a^{-3/2}x^{4}} = \frac{a^{2}}{bx^{22/5}}$
- $\frac{abcx^{10}}{a^{2}b^{2}x^{3/2}} = \frac{cx^{17/2}}{ab}$
- $\frac{ax^{-4/3}}{bx^{2/3}} = \frac{a}{bx^{2}}$

Ex. 1.3.2: Reduce each fraction completely so that no negative exponents appear in the answer.

- $\frac{x^{3}}{x^{10/3}} = \frac{1}{x^{1/3}}$
- $\frac{x^{-3/4}}{5x^{-3}} = \frac{x^{9/4}}{5}$
- $\frac{x^{10}y^{2}}{x^{4}y^{7/2}} = \frac{x^{6}}{y^{3/2}}$
- $\frac{x^{3}a}{x^{10/3}a^{3}} = \frac{1}{x^{1/3}a^{2}}$
- $\frac{x^{-3/4}}{x^{3}} = \frac{1}{x^{15/4}}$
- $\frac{s^{10}y^{2}}{s^{4}y^{7/2}} = \frac{s^{6}}{y^{3/2}}$
- $\frac{y^{3}}{y^{1/5}} = y^{14/5}$
- $\frac{6y^{-5/2}}{9xy^{1/2}} = \frac{2}{3xy^{3}}$
- $\frac{a^{10}b^{1/3}}{b^{7/3}a^{4}} = \frac{a^{6}}{b^{2}}$
- $\frac{8b^{1/3}}{16b^{10/3}} = \frac{1}{2b^{3}}$
- $\frac{16x^{1/2}}{20x^{3/2}} = \frac{4}{5x}$
- $\frac{x^{10}y^{1/3}}{x^{4}y^{7/3}} = \frac{x^{6}}{y^{2}}$

Ex. 1.3.3: Factor

- $x^{1/2} + x^{5/2} =$
- $3x^{3/2} + 6x^{5/2} =$
- $5x^{2} + 2x^{20/3} =$
- $63x^{1/2} + 12x^{5/2} =$

Section 1.3 Review: Powers, roots, and exponents

Ex. 1.3.1: Reduce each fraction completely so that no negative exponents appear in the answer.

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- $\frac{ax^{-4/3}}{bx^{10}} = \frac{a}{bx^{34/3}}$
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- $\frac{ax^{-1/3}}{bx^{-2}} = \frac{ax^{5/3}}{b}$
- $\frac{ab^{4}}{bx^{2/3}} = \frac{ab^{3}}{x^{2/3}}$
- $\frac{a^{1/2}x^{-2/5}}{a^{-3/2}x^{4}} = \frac{a^{2}}{bx^{22/5}}$
- $\frac{abcx^{10}}{a^{2}b^{2}x^{3/2}} = \frac{cx^{17/2}}{ab}$
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- $\frac{x^{10}y^{2}}{x^{4}y^{7/2}} = \frac{x^{6}}{y^{3/2}}$
- $\frac{x^{3}a}{x^{10/3}a^{3}} = \frac{1}{x^{1/3}a^{2}}$
- $\frac{x^{-3/4}}{x^{3}} = \frac{1}{x^{15/4}}$
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- $\frac{x^{10}y^{1/3}}{x^{4}y^{7/3}} = \frac{x^{6}}{y^{2}}$

Ex. 1.3.3: Factor

- $x^{1/2} + x^{5/2} = x^{1/2}(1 + x^{2})$
- $3x^{3/2} + 6x^{5/2} = 3x^{3/2}(1 + 2x)$
- $5x^{2} + 2x^{20/3} = x^{2}(5 + 2x^{14/3})$
- $63x^{1/2} + 12x^{5/2} = 3x^{1/2}(21 + 4x^{2})$

Ex. 1.3.4: Rewrite as a reduced fraction with positive integer exponents:

- $\frac{x^{1/2}+x^{5/2}}{x^{3/2}} =$
- $\frac{a^{3/2}+a^{5/2}}{a^{3/2}} =$
- $\frac{x^{1/2}}{x^{3/2}+x^{5/2}} =$
- $\frac{x^{7/4}-x^{3/4}}{x^{3/4}+x^{7/4}} =$

Section 1.3 Review: Powers, roots, and exponents

Ex. 1.3.1: Reduce each fraction completely so that no negative exponents appear in the answer.

- $\frac{ax^{10}}{bx^{3/2}} = \frac{ax^{17/2}}{b}$
- $\frac{ax^{-4/3}}{bx^{10}} = \frac{a}{bx^{34/3}}$
- $\frac{ax^5}{bx^{3/2}} = \frac{ax^{7/2}}{b}$
- $\frac{ax^{-1/3}}{bx^{-2}} = \frac{ax^{5/3}}{b}$
- $\frac{ab^4}{bx^{2/3}} = \frac{ab^3}{x^{2/3}}$
- $\frac{a^{1/2}x^{-2/5}}{a^{-3/2}x^4} = \frac{a^2}{bx^{22/5}}$
- $\frac{abcx^{10}}{a^2b^2x^{3/2}} = \frac{cx^{17/2}}{ab}$
- $\frac{ax^{-4/3}}{bx^{2/3}} = \frac{a}{bx^2}$

Ex. 1.3.2: Reduce each fraction completely so that no negative exponents appear in the answer.

- $\frac{x^3}{x^{10/3}} = \frac{1}{x^{1/3}}$
- $\frac{x^{-3/4}}{5x^{-3}} = \frac{x^{9/4}}{5}$
- $\frac{x^{10}y^2}{x^4y^{7/2}} = \frac{x^6}{y^{3/2}}$
- $\frac{x^3a}{x^{10/3}a^3} = \frac{1}{x^{1/3}a^2}$
- $\frac{x^{-3/4}}{x^3} = \frac{1}{x^{15/4}}$
- $\frac{s^{10}y^2}{s^4y^{7/2}} = \frac{s^6}{y^{3/2}}$
- $\frac{y^3}{y^{1/5}} = y^{14/5}$
- $\frac{6y^{-5/2}}{9xy^{1/2}} = \frac{2}{3xy^3}$
- $\frac{a^{10}b^{1/3}}{b^{7/3}a^4} = \frac{a^6}{b^2}$
- $\frac{8b^{1/3}}{16b^{10/3}} = \frac{1}{2b^3}$
- $\frac{16x^{1/2}}{20x^{3/2}} = \frac{4}{5x}$
- $\frac{x^{10}y^{1/3}}{x^4y^{7/3}} = \frac{x^6}{y^2}$

Ex. 1.3.3: Factor

- $x^{1/2} + x^{5/2} = x^{1/2}(1 + x^2)$
- $3x^{3/2} + 6x^{5/2} = 3x^{3/2}(1 + 2x)$
- $5x^2 + 2x^{20/3} = x^2(5 + 2x^{14/3})$
- $63x^{1/2} + 12x^{5/2} = 3x^{1/2}(21 + 4x^2)$

Ex. 1.3.4: Rewrite as a reduced fraction with positive integer exponents:

- $\frac{x^{1/2}+x^{5/2}}{x^{3/2}} = \frac{1+x^2}{x}$
- $\frac{a^{3/2}+a^{5/2}}{a^{3/2}} = 1 + a$
- $\frac{x^{1/2}}{x^{3/2}+x^{5/2}} = \frac{1}{x+x^2}$
- $\frac{x^{7/4}-x^{3/4}}{x^{3/4}+x^{7/4}} = \frac{x-1}{x+1}$

Section 1.3 Review: Powers, roots, and exponents

Ex. 1.3.1: Reduce each fraction completely so that no negative exponents appear in the answer.

- $\frac{ax^{10}}{bx^{3/2}} = \frac{ax^{17/2}}{b}$
- $\frac{ax^{-4/3}}{bx^{10}} = \frac{a}{bx^{34/3}}$
- $\frac{ax^5}{bx^{3/2}} = \frac{ax^{7/2}}{b}$
- $\frac{ax^{-1/3}}{bx^{-2}} = \frac{ax^{5/3}}{b}$
- $\frac{ab^4}{bx^{2/3}} = \frac{ab^3}{x^{2/3}}$
- $\frac{a^{1/2}x^{-2/5}}{a^{-3/2}x^4} = \frac{a^2}{bx^{22/5}}$
- $\frac{abcx^{10}}{a^2b^2x^{3/2}} = \frac{cx^{17/2}}{ab}$
- $\frac{ax^{-4/3}}{bx^{2/3}} = \frac{a}{bx^2}$

Ex. 1.3.2: Reduce each fraction completely so that no negative exponents appear in the answer.

- $\frac{x^3}{x^{10/3}} = \frac{1}{x^{1/3}}$
- $\frac{x^{-3/4}}{5x^{-3}} = \frac{x^{9/4}}{5}$
- $\frac{x^{10}y^2}{x^4y^{7/2}} = \frac{x^6}{y^{3/2}}$
- $\frac{x^3a}{x^{10/3}a^3} = \frac{1}{x^{1/3}a^2}$
- $\frac{x^{-3/4}}{x^3} = \frac{1}{x^{15/4}}$
- $\frac{s^{10}y^2}{s^4y^{7/2}} = \frac{s^6}{y^{3/2}}$
- $\frac{y^3}{y^{1/5}} = y^{14/5}$
- $\frac{6y^{-5/2}}{9xy^{1/2}} = \frac{2}{3xy^3}$
- $\frac{a^{10}b^{1/3}}{b^{7/3}a^4} = \frac{a^6}{b^2}$
- $\frac{8b^{1/3}}{16b^{10/3}} = \frac{1}{2b^3}$
- $\frac{16x^{1/2}}{20x^{3/2}} = \frac{4}{5x}$
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Ex. 1.3.3: Factor

- $x^{1/2} + x^{5/2} = x^{1/2}(1 + x^2)$
- $3x^{3/2} + 6x^{5/2} = 3x^{3/2}(1 + 2x)$
- $5x^2 + 2x^{20/3} = x^2(5 + 2x^{14/3})$
- $63x^{1/2} + 12x^{5/2} = 3x^{1/2}(21 + 4x^2)$

Ex. 1.3.4: Rewrite as a reduced fraction with positive integer exponents:

- $\frac{x^{1/2}+x^{5/2}}{x^{3/2}} = \frac{1+x^2}{x}$
- $\frac{a^{3/2}+a^{5/2}}{a^{3/2}} = 1 + a$
- $\frac{x^{1/2}}{x^{3/2}+x^{5/2}} = \frac{1}{x+x^2}$
- $\frac{x^{7/4}-x^{3/4}}{x^{3/4}+x^{7/4}} = \frac{x-1}{x+1}$

Ex. 1.3.5: Rewrite as a sum of powers of x:

- $\frac{x^{\frac{1}{2}}+x^{\frac{5}{2}}}{x^{\frac{3}{2}}} =$
- $\frac{x^{-3}+x^{\frac{5}{2}}}{x^2} =$
- $\frac{x^{-\frac{1}{3}}+x^{\frac{2}{3}}}{x^3} =$
- $\frac{x^8+x^5}{x^5} =$

Section 1.3 Review: Powers, roots, and exponents

Ex. 1.3.1: Reduce each fraction completely so that no negative exponents appear in the answer.

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- $\frac{ab^{4}}{bx^{2/3}} = \frac{ab^{3}}{x^{2/3}}$
- $\frac{a^{1/2}x^{-2/5}}{a^{-3/2}x^{4}} = \frac{a^{2}}{bx^{22/5}}$
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- $\frac{ax^{-4/3}}{bx^{2/3}} = \frac{a}{bx^{2}}$

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- $\frac{x^{3}}{x^{10/3}} = \frac{1}{x^{1/3}}$
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- $\frac{x^{3}a}{x^{10/3}a^{3}} = \frac{1}{x^{1/3}a^{2}}$
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Ex. 1.3.5: Rewrite as a sum of powers of x:

- $\frac{x^{\frac{1}{2}}+x^{\frac{5}{2}}}{x^{\frac{3}{2}}} = x^{-1} + x$
- $\frac{x^{-3}+x^{\frac{5}{2}}}{x^{2}} = x^{-5} + x^{\frac{1}{2}}$
- $\frac{x^{-\frac{1}{3}}+x^{\frac{2}{3}}}{x^{3}} = x^{-\frac{10}{3}} + x^{-\frac{7}{3}}$
- $\frac{x^{8}+x^{5}}{x^{5}} = x^3 + 1$

Section 1.3 Review: Powers, roots, and exponents

Ex. 1.3.1: Reduce each fraction completely so that no negative exponents appear in the answer.

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- $\frac{a^{1/2}x^{-2/5}}{a^{-3/2}x^{4}} = \frac{a^{2}}{bx^{22/5}}$
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- $\frac{ax^{-4/3}}{bx^{2/3}} = \frac{a}{bx^{2}}$

Ex. 1.3.2: Reduce each fraction completely so that no negative exponents appear in the answer.

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- $\frac{x^{10}y^{2}}{x^{4}y^{7/2}} = \frac{x^{6}}{y^{3/2}}$
- $\frac{x^{3}a}{x^{10/3}a^{3}} = \frac{1}{x^{1/3}a^{2}}$
- $\frac{x^{-3/4}}{x^{3}} = \frac{1}{x^{15/4}}$
- $\frac{s^{10}y^{2}}{s^{4}y^{7/2}} = \frac{s^{6}}{y^{3/2}}$
- $\frac{y^{3}}{y^{1/5}} = y^{14/5}$
- $\frac{6y^{-5/2}}{9xy^{1/2}} = \frac{2}{3xy^{3}}$
- $\frac{a^{10}b^{1/3}}{b^{7/3}a^{4}} = \frac{a^{6}}{b^{2}}$
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- $\frac{x^{-\frac{1}{3}}+x^{\frac{2}{3}}}{x^{3}} = x^{-\frac{10}{3}} + x^{-\frac{7}{3}}$
- $\frac{x^{8}+x^{5}}{x^{5}} = x^{3} + 1$

Ex. 1.3.6: Rewrite as a reduced fraction without negative exponents:

- $\frac{x^{-3}y^{5}}{z^{3}} =$
- $\frac{x^{-10}y^{-5}}{x^{3}} =$
- $\frac{x^{-3}y^{-5}}{z^{-3}} =$
- $\frac{12x^{-3}y^{5}}{8x^{7}y^{2}} =$

Section 1.3 Review: Powers, roots, and exponents

Ex. 1.3.1: Reduce each fraction completely so that no negative exponents appear in the answer.

- $\frac{ax^{10}}{bx^{3/2}} = \frac{ax^{17/2}}{b}$
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- $\frac{ax^{5}}{bx^{3/2}} = \frac{ax^{7/2}}{b}$
- $\frac{ax^{-1/3}}{bx^{-2}} = \frac{ax^{5/3}}{b}$
- $\frac{ab^{4}}{bx^{2/3}} = \frac{ab^{3}}{x^{2/3}}$
- $\frac{a^{1/2}x^{-2/5}}{a^{-3/2}x^{4}} = \frac{a^{2}}{bx^{22/5}}$
- $\frac{abcx^{10}}{a^2b^2x^{3/2}} = \frac{cx^{17/2}}{ab}$
- $\frac{ax^{-4/3}}{bx^{2/3}} = \frac{a}{bx^{2}}$

Ex. 1.3.2: Reduce each fraction completely so that no negative exponents appear in the answer.

- $\frac{x^3}{x^{10/3}} = \frac{1}{x^{1/3}}$
- $\frac{x^{-3/4}}{5x^{-3}} = \frac{x^{9/4}}{5}$
- $\frac{x^{10}y^2}{x^4y^{7/2}} = \frac{x^6}{y^{3/2}}$
- $\frac{x^3a}{x^{10/3}a^3} = \frac{1}{x^{1/3}a^2}$
- $\frac{x^{-3/4}}{x^3} = \frac{1}{x^{15/4}}$
- $\frac{s^{10}y^2}{s^4y^{7/2}} = \frac{s^6}{y^{3/2}}$
- $\frac{y^3}{y^{1/5}} = y^{14/5}$
- $\frac{6y^{-5/2}}{9xy^{1/2}} = \frac{2}{3xy^3}$
- $\frac{a^{10}b^{1/3}}{b^{7/3}a^4} = \frac{a^6}{b^2}$
- $\frac{8b^{1/3}}{16b^{10/3}} = \frac{1}{2b^3}$
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- $x^{1/2} + x^{5/2} = x^{1/2}(1 + x^2)$
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Ex. 1.3.4: Rewrite as a reduced fraction with positive integer exponents:

- $\frac{x^{1/2}+x^{5/2}}{x^{3/2}} = \frac{1+x^2}{x}$
- $\frac{a^{3/2}+a^{5/2}}{a^{3/2}} = 1 + a$
- $\frac{x^{1/2}}{x^{3/2}+x^{5/2}} = \frac{1}{x+x^2}$
- $\frac{x^{7/4}-x^{3/4}}{x^{3/4}+x^{7/4}} = \frac{x-1}{x+1}$

Ex. 1.3.5: Rewrite as a sum of powers of x:

- $\frac{x^{\frac{1}{2}}+x^{\frac{5}{2}}}{x^{\frac{3}{2}}} = x^{-1} + x$
- $\frac{x^{-3}+x^{\frac{5}{2}}}{x^2} = x^{-5} + x^{\frac{1}{2}}$
- $\frac{x^{-\frac{1}{3}}+x^{\frac{2}{3}}}{x^3} = x^{-\frac{10}{3}} + x^{-\frac{7}{3}}$
- $\frac{x^8+x^5}{x^5} = x^3 + 1$

Ex. 1.3.6: Rewrite as a reduced fraction without negative exponents:

- $\frac{x^{-3}y^5}{z^3} = \frac{y^5}{x^3z^3}$
- $\frac{x^{-10}y^{-5}}{x^3} = \frac{1}{x^{13}y^5}$
- $\frac{x^{-3}y^{-5}}{z^{-3}} = \frac{z^3}{x^3y^5}$
- $\frac{12x^{-3}y^5}{8x^7y^2} = \frac{3y^3}{2x^{10}}$

Ex. 1.3.7: Rewrite without negative exponents:

- $\frac{x^{-2}+xy}{y^3}=$
- $\frac{x^{-2}+x}{y^3+x^{-2}}=$
- $\frac{x^{-2}+xy}{y^{-3}+1}=$
- $\frac{x^{-2}+xy}{y^{-2}}=$

Ex. 1.3.7: Rewrite without negative exponents:

- $\frac{x^{-2}+xy}{y^3} = \frac{1+x^3y}{x^2y^3}$
- $\frac{x^{-2}+x}{y^3+x^{-2}} = \frac{1+x^3}{x^2y^3+1}$
- $\frac{x^{-2}+xy}{y^{-3}+1} = \frac{y^3+x^3y^3}{x^2+x^2y^3}$
- $\frac{x^{-2}+xy}{y^{-2}} = \frac{y^2+x^3y^3}{x^2}$

Ex. 1.3.7: Rewrite without negative exponents:

- $\frac{x^{-2}+xy}{y^3} = \frac{1+x^3y}{x^2y^3}$
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- $\frac{x^{-2}+xy}{y^{-2}} = \frac{y^2+x^3y^3}{x^2}$

Ex. 1.3.8: Rewrite without negative exponents:

- $\frac{a+6}{c^{-2}d^8} =$
- $\frac{a+6}{c^{-2}b^{-3}a} =$
- $\frac{a+6}{a^{-1}+6} =$
- $\frac{(a+6)^{-2}}{c^{-2}} =$

Ex. 1.3.7: Rewrite without negative exponents:

- $\frac{x^{-2}+xy}{y^3} = \frac{1+x^3y}{x^2y^3}$
- $\frac{x^{-2}+x}{y^3+x^{-2}} = \frac{1+x^3}{x^2y^3+1}$
- $\frac{x^{-2}+xy}{y^{-3}+1} = \frac{y^3+x^3y^3}{x^2+x^2y^3}$
- $\frac{x^{-2}+xy}{y^{-2}} = \frac{y^2+x^3y^3}{x^2}$

Ex. 1.3.8: Rewrite without negative exponents:

- $\frac{a+6}{c^{-2}d^8} = \frac{c^2(a+6)}{d^8}$
- $\frac{a+6}{c^{-2}b^{-3}a} = \frac{c^2b^3(a+6)}{a}$
- $\frac{a+6}{a^{-1}+6} = \frac{a(a+6)}{1+6a}$
- $\frac{(a+6)^{-2}}{c^{-2}} = \frac{c^2}{(a+6)^2}$

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- $\frac{(a+6)^{-2}}{c^{-2}}=\ \frac{c^2}{(a+6)^2}$

Rewrite the fractions in the next 4 problems as a reduced fraction without parentheses or negative exponents:

Ex. 1.3.9:

- $\left(\frac{2a^2}{b^3c^4}\right)^2=$
- $\left(\frac{2abc}{3x^3y}\right)^3=$
- $\left(\frac{2a^4}{b^3cd}\right)^2=$
- $\left(\frac{3a^2}{2bcd}\right)^3=$

Ex. 1.3.7: Rewrite without negative exponents:

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- $\frac{(a+6)^{-2}}{c^{-2}} = \frac{c^2}{(a+6)^2}$

Rewrite the fractions in the next 4 problems as a reduced fraction without parentheses or negative exponents:

Ex. 1.3.9:

- $\left(\frac{2a^2}{b^3c^4}\right)^2 = \frac{4a^4}{b^6c^8}$
- $\left(\frac{2abc}{3x^3y}\right)^3 = \frac{8a^3b^3c^3}{27x^9y^3}$
- $\left(\frac{2a^4}{b^3cd}\right)^2 = \frac{4a^8}{b^6c^2d^2}$
- $\left(\frac{3a^2}{2bcd}\right)^3 = \frac{27a^6}{8b^3c^3d^3}$

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Ex. 1.3.10:
- $\left(\frac{2a^{-1}b}{a^2b^{-3}}\right)^{-3} =$
- $\left(\frac{2a^{-1}}{c^3b^2}\right)^{-3} =$
- $\left(\frac{a^{-1}b}{2b^{-1}c^{-1}}\right)^{-3} =$
- $\left(\frac{2a^{-1}b}{c^{-1}d^3}\right)^{-2} =$

Ex. 1.3.7: Rewrite without negative exponents:

- $\frac{x^{-2}+xy}{y^3} = \frac{1+x^3y}{x^2y^3}$
- $\frac{x^{-2}+x}{y^3+x^{-2}} = \frac{1+x^3}{x^2y^3+1}$
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- $\frac{a+6}{c^{-2}b^{-3}a} = \frac{c^2b^3(a+6)}{a}$
- $\frac{a+6}{a^{-1}+6} = \frac{a(a+6)}{1+6a}$
- $\frac{(a+6)^{-2}}{c^{-2}} = \frac{c^2}{(a+6)^2}$

Rewrite the fractions in the next 4 problems as a reduced fraction without parentheses or negative exponents:

Ex. 1.3.9:

- $\left(\frac{2a^2}{b^3c^4}\right)^2 = \frac{4a^4}{b^6c^8}$
- $\left(\frac{2abc}{3x^3y}\right)^3 = \frac{8a^3b^3c^3}{27x^9y^3}$
- $\left(\frac{2a^4}{b^3cd}\right)^2 = \frac{4a^8}{b^6c^2d^2}$
- $\left(\frac{3a^2}{2bcd}\right)^3 = \frac{27a^6}{8b^3c^3d^3}$

Ex. 1.3.10:

- $\left(\frac{2a^{-1}b}{a^2b^{-3}}\right)^{-3} = \frac{a^9}{8b^{12}}$
- $\left(\frac{2a^{-1}}{c^3b^2}\right)^{-3} = \frac{a^3b^6c^9}{8}$
- $\left(\frac{a^{-1}b}{2b^{-1}c^{-1}}\right)^{-3} = \frac{8a^3}{b^6c^3}$
- $\left(\frac{2a^{-1}b}{c^{-1}d^3}\right)^{-2} = \frac{a^2d^6}{4b^2c^2}$

Ex. 1.3.7: Rewrite without negative exponents:

- $\frac{x^{-2}+xy}{y^3} = \frac{1+x^3y}{x^2y^3}$
- $\frac{x^{-2}+x}{y^3+x^{-2}} = \frac{1+x^3}{x^2y^3+1}$
- $\frac{x^{-2}+xy}{y^{-3}+1} = \frac{y^3+x^3y^3}{x^2+x^2y^3}$
- $\frac{x^{-2}+xy}{y^{-2}} = \frac{y^2+x^3y^3}{x^2}$

Ex. 1.3.8: Rewrite without negative exponents:

- $\frac{a+6}{c^{-2}d^8} = \frac{c^2(a+6)}{d^8}$
- $\frac{a+6}{c^{-2}b^{-3}a} = \frac{c^2b^3(a+6)}{a}$
- $\frac{a+6}{a^{-1}+6} = \frac{a(a+6)}{1+6a}$
- $\frac{(a+6)^{-2}}{c^{-2}} = \frac{c^2}{(a+6)^2}$

Rewrite the fractions in the next 4 problems as a reduced fraction without parentheses or negative exponents:

Ex. 1.3.9:
- $\left(\frac{2a^2}{b^3c^4}\right)^2 = \frac{4a^4}{b^6c^8}$
- $\left(\frac{2abc}{3x^3y}\right)^3 = \frac{8a^3b^3c^3}{27x^9y^3}$
- $\left(\frac{2a^4}{b^3cd}\right)^2 = \frac{4a^8}{b^6c^2d^2}$
- $\left(\frac{3a^2}{2bcd}\right)^3 = \frac{27a^6}{8b^3c^3d^3}$

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- $\left(\frac{2a^{-1}}{c^3b^2}\right)^{-3} = \frac{a^3b^6c^9}{8}$
- $\left(\frac{a^{-1}b}{2b^{-1}c^{-1}}\right)^{-3} = \frac{8a^3}{b^6c^3}$
- $\left(\frac{2a^{-1}b}{c^{-1}d^3}\right)^{-2} = \frac{a^2d^6}{4b^2c^2}$

Ex. 1.3.11:
- $\left(\frac{2x^{3/4}}{y^{1/3}}\right)^3\left(\frac{y^4}{x^{-1/2}}\right) =$
- $\left(\frac{2x^{3/4}}{y^{1/3}}\right)^2\left(\frac{y^4}{x^{-1/2}}\right)^{-1} =$
- $\left(\frac{2x^{-3/4}}{y^{-1/3}}\right)^2\left(\frac{y^{-1}}{x^{-1/2}}\right) =$
- $\left(\frac{x^{3/4}}{2y^{1/3}}\right)^{-2}\left(\frac{y^{2/3}}{x^{1/2}}\right)^2 =$

Ex. 1.3.7: Rewrite without negative exponents:

- $\frac{x^{-2}+xy}{y^3}=\frac{1+x^3y}{x^2y^3}$
- $\frac{x^{-2}+x}{y^3+x^{-2}}=\frac{1+x^3}{x^2y^3+1}$
- $\frac{x^{-2}+xy}{y^{-3}+1}=\frac{y^3+x^3y^3}{x^2+x^2y^3}$
- $\frac{x^{-2}+xy}{y^{-2}}=\frac{y^2+x^3y^3}{x^2}$

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- $\frac{a+6}{c^{-2}d^8}=\frac{c^2(a+6)}{d^8}$
- $\frac{a+6}{c^{-2}b^{-3}a}=\frac{c^2b^3(a+6)}{a}$
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- $\left(\frac{a^{-1}b}{2b^{-1}c^{-1}}\right)^{-3}=\frac{8a^3}{b^6c^3}$
- $\left(\frac{2a^{-1}b}{c^{-1}d^3}\right)^{-2}=\frac{a^2d^6}{4b^2c^2}$

Ex. 1.3.11:

- $\left(\frac{2x^{3/4}}{y^{1/3}}\right)^3\left(\frac{y^4}{x^{-1/2}}\right)=8x^{11/4}y^3=8x^2\sqrt[4]{x^3}y^3$
- $\left(\frac{2x^{3/4}}{y^{1/3}}\right)^2\left(\frac{y^4}{x^{-1/2}}\right)^{-1}=\frac{4x}{y^{14/3}}=\frac{4x}{y^4\sqrt[3]{y^2}}$
- $\left(\frac{2x^{-3/4}}{y^{-1/3}}\right)^2\left(\frac{y^{-1}}{x^{-1/2}}\right)=\frac{4}{xy^{1/3}}=\frac{4}{x\sqrt[3]{y}}$
- $\left(\frac{x^{3/4}}{2y^{1/3}}\right)^{-2}\left(\frac{y^{2/3}}{x^{1/2}}\right)^2=\frac{4y^2}{x^{5/2}}=\frac{4y^2}{x^2\sqrt{x}}$

Ex. 1.3.7: Rewrite without negative exponents:

- $\frac{x^{-2}+xy}{y^3}=\frac{1+x^3y}{x^2y^3}$
- $\frac{x^{-2}+x}{y^3+x^{-2}}=\frac{1+x^3}{x^2y^3+1}$
- $\frac{x^{-2}+xy}{y^{-3}+1}=\frac{y^3+x^3y^3}{x^2+x^2y^3}$
- $\frac{x^{-2}+xy}{y^{-2}}=\frac{y^2+x^3y^3}{x^2}$

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- $\frac{a+6}{c^{-2}d^8}=\frac{c^2(a+6)}{d^8}$
- $\frac{a+6}{c^{-2}b^{-3}a}=\frac{c^2b^3(a+6)}{a}$
- $\frac{a+6}{a^{-1}+6}=\frac{a(a+6)}{1+6a}$
- $\frac{(a+6)^{-2}}{c^{-2}}=\frac{c^2}{(a+6)^2}$

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- $\left(\frac{2a^4}{b^3cd}\right)^2=\frac{4a^8}{b^6c^2d^2}$
- $\left(\frac{3a^2}{2bcd}\right)^3=\frac{27a^6}{8b^3c^3d^3}$

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- $\left(\frac{2a^{-1}}{c^3b^2}\right)^{-3}=\frac{a^3b^6c^9}{8}$
- $\left(\frac{a^{-1}b}{2b^{-1}c^{-1}}\right)^{-3}=\frac{8a^3}{b^6c^3}$
- $\left(\frac{2a^{-1}b}{c^{-1}d^3}\right)^{-2}=\frac{a^2d^6}{4b^2c^2}$

Ex. 1.3.11:

- $\left(\frac{2x^{3/4}}{y^{1/3}}\right)^3\left(\frac{y^4}{x^{-1/2}}\right)=8x^{11/4}y^3=8x^2\sqrt[4]{x^3}y^3$
- $\left(\frac{2x^{3/4}}{y^{1/3}}\right)^2\left(\frac{y^4}{x^{-1/2}}\right)^{-1}=\frac{4x}{y^{14/3}}=\frac{4x}{y^4\sqrt[3]{y^2}}$
- $\left(\frac{2x^{-3/4}}{y^{-1/3}}\right)^2\left(\frac{y^{-1}}{x^{-1/2}}\right)=\frac{4}{xy^{1/3}}=\frac{4}{x\sqrt[3]{y}}$
- $\left(\frac{x^{3/4}}{2y^{1/3}}\right)^{-2}\left(\frac{y^{2/3}}{x^{1/2}}\right)^2=\frac{4y^2}{x^{5/2}}=\frac{4y^2}{x^2\sqrt{x}}$

Ex. 1.3.12:

- $\left(\frac{3x^3}{y^{1/2}}\right)^2\left(\frac{x^{4/3}}{y^{-2/3}}\right)^3=$
- $\left(\frac{3x^2}{y^2}\right)^3\left(\frac{x}{\sqrt{y}}\right)^2=$
- $\left(3\frac{x^{-1/2}}{y^2}\right)^3\left(\frac{x^{-2}}{y^{2/3}}\right)^3=$
- $\left(\frac{y^{-3/2}x^{-1/2}}{z}\right)^6\left(\frac{x^{-2}}{z^{2/3}}\right)^3=$

Ex. 1.3.7: Rewrite without negative exponents:

- $\frac{x^{-2}+xy}{y^3} = \frac{1+x^3y}{x^2y^3}$
- $\frac{x^{-2}+x}{y^3+x^{-2}} = \frac{1+x^3}{x^2y^3+1}$
- $\frac{x^{-2}+xy}{y^{-3}+1} = \frac{y^3+x^3y^3}{x^2+x^2y^3}$
- $\frac{x^{-2}+xy}{y^{-2}} = \frac{y^2+x^3y^3}{x^2}$

Ex. 1.3.8: Rewrite without negative exponents:

- $\frac{a+6}{c^{-2}d^8} = \frac{c^2(a+6)}{d^8}$
- $\frac{a+6}{c^{-2}b^{-3}a} = \frac{c^2b^3(a+6)}{a}$
- $\frac{a+6}{a^{-1}+6} = \frac{a(a+6)}{1+6a}$
- $\frac{(a+6)^{-2}}{c^{-2}} = \frac{c^2}{(a+6)^2}$

Rewrite the fractions in the next 4 problems as a reduced fraction without parentheses or negative exponents:

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- $\left(\frac{2a^2}{b^3c^4}\right)^2 = \frac{4a^4}{b^6c^8}$
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- $\left(\frac{2a^4}{b^3cd}\right)^2 = \frac{4a^8}{b^6c^2d^2}$
- $\left(\frac{3a^2}{2bcd}\right)^3 = \frac{27a^6}{8b^3c^3d^3}$

Ex. 1.3.10:
- $\left(\frac{2a^{-1}b}{a^2b^{-3}}\right)^{-3} = \frac{a^9}{8b^{12}}$
- $\left(\frac{2a^{-1}}{c^3b^2}\right)^{-3} = \frac{a^3b^6c^9}{8}$
- $\left(\frac{a^{-1}b}{2b^{-1}c^{-1}}\right)^{-3} = \frac{8a^3}{b^6c^3}$
- $\left(\frac{2a^{-1}b}{c^{-1}d^3}\right)^{-2} = \frac{a^2d^6}{4b^2c^2}$

Ex. 1.3.11:
- $\left(\frac{2x^{3/4}}{y^{1/3}}\right)^3\left(\frac{y^4}{x^{-1/2}}\right) = 8x^{11/4}y^3 = 8x^2\sqrt[4]{x^3}y^3$
- $\left(\frac{2x^{3/4}}{y^{1/3}}\right)^2\left(\frac{y^4}{x^{-1/2}}\right)^{-1} = \frac{4x}{y^{14/3}} = \frac{4x}{y^4\sqrt[3]{y^2}}$
- $\left(\frac{2x^{-3/4}}{y^{-1/3}}\right)^2\left(\frac{y^{-1}}{x^{-1/2}}\right) = \frac{4}{xy^{1/3}} = \frac{4}{x\sqrt[3]{y}}$
- $\left(\frac{x^{3/4}}{2y^{1/3}}\right)^{-2}\left(\frac{y^{2/3}}{x^{1/2}}\right)^2 = \frac{4y^2}{x^{5/2}} = \frac{4y^2}{x^2\sqrt{x}}$

Ex. 1.3.12:
- $\left(\frac{3x^3}{y^{1/2}}\right)^2\left(\frac{x^{4/3}}{y^{-2/3}}\right)^3 = 9x^{10}y$
- $\left(\frac{3x^2}{y^2}\right)^3\left(\frac{x}{\sqrt{y}}\right)^2 = \frac{27x^8}{y^7}$
- $\left(3\frac{x^{-1/2}}{y^2}\right)^3\left(\frac{x^{-2}}{y^{2/3}}\right)^3 = \frac{27}{x^{15/2}y^8} = \frac{27}{x^7\sqrt{x}y^8}$
- $\left(\frac{y^{-3/2}x^{-1/2}}{z}\right)^6\left(\frac{x^{-2}}{z^{2/3}}\right)^3 = \frac{1}{x^9y^9z^8}$

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}=$
- $\sqrt[6]{1,000,000}=$
- $\sqrt[5]{32}=$
- $\sqrt[5]{-32}=$
- $\sqrt[6]{-64}=$
- $\sqrt[3]{1,000,000}=$
- $\sqrt[5]{1}=$
- $\sqrt[5]{-1}=$

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}=2$
- $\sqrt[6]{1,000,000}=10$
- $\sqrt[5]{32}=2$
- $\sqrt[5]{-32}=-2$
- $\sqrt[6]{-64}=$Undefined
- $\sqrt[3]{1,000,000}=100$
- $\sqrt[5]{1}=1$
- $\sqrt[5]{-1}=-1$

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}=2$
- $\sqrt[6]{-64}=$Undefined
- $\sqrt[4]{81}=$
- $\sqrt[3]{27}=$
- $\sqrt[6]{1,000,000}=10$
- $\sqrt[3]{1,000,000}=100$
- $\sqrt{1,000,000}=$
- $\sqrt[3]{8,000,000}=$
- $\sqrt[5]{32}=2$
- $\sqrt[5]{1}=1$
- $\sqrt[4]{16}\ =$
- $\sqrt[3]{125}=$
- $\sqrt[5]{-32}=-2$
- $\sqrt[5]{-1}=-1$
- $\sqrt[3]{54}=$
- $\sqrt[5]{-64}=$

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}=2$
- $\sqrt[6]{-64}=$Undefined
- $\sqrt[4]{81}=3$
- $\sqrt[3]{27}=3$
- $\sqrt[6]{1,000,000}=10$
- $\sqrt[3]{1,000,000}=100$
- $\sqrt{1,000,000}=1000$
- $\sqrt[3]{8,000,000}=200$
- $\sqrt[5]{32}=2$
- $\sqrt[5]{1}=1$
- $\sqrt[4]{16}=2$
- $\sqrt[3]{125}=5$
- $\sqrt[5]{-32}=-2$
- $\sqrt[5]{-1}=-1$
- $\sqrt[3]{54}=\sqrt[3]{27\cdot 2}=3\sqrt[3]{2}$
- $\sqrt[5]{-64}=\sqrt[5]{-32\cdot 2}=-2\sqrt[5]{2}$

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}$=2
- $\sqrt[6]{-64}$=Undefined
- $\sqrt[4]{81}$=3
- $\sqrt[3]{27}$=3
- $\sqrt[6]{1,000,000}$=10
- $\sqrt[3]{1,000,000}$=100
- $\sqrt{1,000,000}$=1000
- $\sqrt[3]{8,000,000}$=200
- $\sqrt[5]{32}$=2
- $\sqrt[5]{1}$=1
- $\sqrt[4]{16}$ =2
- $\sqrt[3]{125}$=5
- $\sqrt[5]{-32}$=−2
- $\sqrt[5]{-1}$=−1
- $\sqrt[3]{54}$= $\sqrt[3]{27 \cdot 2} = 3\sqrt[3]{2}$
- $\sqrt[5]{-64}$= $\sqrt[5]{-32 \cdot 2} = -2\sqrt[5]{2}$

Solve each equation for x:

- $x^5 = -32 \quad \Rightarrow$
- $x^5 = 32 \quad \Rightarrow$
- $x^6 = 64 \quad \Rightarrow$
- $x^6 = -64 \quad \Rightarrow$

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}=2$
- $\sqrt[6]{-64}=$Undefined
- $\sqrt[4]{81}=3$
- $\sqrt[3]{27}=3$
- $\sqrt[6]{1,000,000}=10$
- $\sqrt[3]{1,000,000}=100$
- $\sqrt{1,000,000}=1000$
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- $\sqrt[5]{32}=2$
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- $\sqrt[3]{54}=\sqrt[3]{27\cdot 2}=3\sqrt[3]{2}$
- $\sqrt[5]{-64}=\sqrt[5]{-32\cdot 2}=-2\sqrt[5]{2}$

Solve each equation for x:

- $x^5=-32 \quad \Rightarrow x=\sqrt[5]{-32}=-2$
- $x^5=32 \quad \Rightarrow x=\sqrt[5]{32}=2$
- $x^6=64 \quad \Rightarrow x=\pm\sqrt[6]{64}=\pm 2$
- $x^6=-64 \quad \Rightarrow$ No solution

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}$=2
- $\sqrt[6]{-64}$=Undefined
- $\sqrt[4]{81}$=3
- $\sqrt[3]{27}$=3
- $\sqrt[6]{1,000,000}$=10
- $\sqrt[3]{1,000,000}$=100
- $\sqrt{1,000,000}$=1000
- $\sqrt[3]{8,000,000}$=200
- $\sqrt[5]{32}$=2
- $\sqrt[5]{1}$=1
- $\sqrt[4]{16}$ =2
- $\sqrt[3]{125}$=5
- $\sqrt[5]{-32}$=−2
- $\sqrt[5]{-1}$=−1
- $\sqrt[3]{54}= \sqrt[3]{27 \cdot 2} = 3\sqrt[3]{2}$
- $\sqrt[5]{-64}= \sqrt[5]{-32 \cdot 2} = -2\sqrt[5]{2}$

Solve each equation for x:

- $x^5 = -32 \quad \Rightarrow x = \sqrt[5]{-32} = -2$
- $x^5 = 32 \quad \Rightarrow x = \sqrt[5]{32} = 2$
- $x^4 = 16 \quad \Rightarrow$
- $x^2 = 1 \quad \Rightarrow$
- $x^6 = 64 \quad \Rightarrow x = \pm\sqrt[6]{64} = \pm 2$
- $x^6 = -64 \quad \Rightarrow$ No solution
- $x^2 = 64 \quad \Rightarrow$
- $x^4 = -191 \quad \Rightarrow$

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}=2$
- $\sqrt[6]{-64}=$Undefined
- $\sqrt[4]{81}=3$
- $\sqrt[3]{27}=3$
- $\sqrt[6]{1,000,000}=10$
- $\sqrt[3]{1,000,000}=100$
- $\sqrt{1,000,000}=1000$
- $\sqrt[3]{8,000,000}=200$
- $\sqrt[5]{32}=2$
- $\sqrt[5]{1}=1$
- $\sqrt[4]{16}=2$
- $\sqrt[3]{125}=5$
- $\sqrt[5]{-32}=-2$
- $\sqrt[5]{-1}=-1$
- $\sqrt[3]{54}=\sqrt[3]{27\cdot 2}=3\sqrt[3]{2}$
- $\sqrt[5]{-64}=\sqrt[5]{-32\cdot 2}=-2\sqrt[5]{2}$

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- $x^5=-32 \quad \Rightarrow x=\sqrt[5]{-32}=-2$
- $x^5=32 \quad \Rightarrow x=\sqrt[5]{32}=2$
- $x^4=16 \quad \Rightarrow x=\pm 2$
- $x^2=1 \quad \Rightarrow x=\pm 1$
- $x^6=64 \quad \Rightarrow x=\pm\sqrt[6]{64}=\pm 2$
- $x^6=-64 \quad \Rightarrow$ No solution
- $x^2=64 \quad \Rightarrow x=\pm 8$
- $x^4=-191 \quad \Rightarrow$ No solution

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}=2$
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- $\sqrt[3]{27}=3$
- $\sqrt[6]{1,000,000}=10$
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- $\sqrt[3]{8,000,000}=200$
- $\sqrt[5]{32}=2$
- $\sqrt[5]{1}=1$
- $\sqrt[4]{16}=2$
- $\sqrt[3]{125}=5$
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- $x^5=32 \quad \Rightarrow x=\sqrt[5]{32}=2$
- $x^4=16 \quad \Rightarrow x=\pm 2$
- $x^2=1 \quad \Rightarrow x=\pm 1$
- $x^3=-27 \quad \Rightarrow$
- $x^5=96 \quad \Rightarrow$
- $x^6=64 \quad \Rightarrow x=\pm\sqrt[6]{64}=\pm 2$
- $x^6=-64 \quad \Rightarrow$ No solution
- $x^2=64 \quad \Rightarrow x=\pm 8$
- $x^4=-191 \quad \Rightarrow$ No solution
- $x^4=64 \quad \Rightarrow$
- $x^6=-64 \quad \Rightarrow$.

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- $\sqrt[4]{81}=3$
- $\sqrt[3]{27}=3$
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- $\sqrt[5]{1}=1$
- $\sqrt[4]{16}=2$
- $\sqrt[3]{125}=5$
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Solve each equation for x:

- $x^5=-32 \quad \Rightarrow x=\sqrt[5]{-32}=-2$
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- $x^4=16 \quad \Rightarrow x=\pm 2$
- $x^2=1 \quad \Rightarrow x=\pm 1$
- $x^3=-27 \quad \Rightarrow x=-3$
- $x^5=96 \quad \Rightarrow x=\sqrt[5]{32\cdot 3}=2\sqrt[5]{3}$
- $x^6=64 \quad \Rightarrow x=\pm\sqrt[6]{64}=\pm 2$
- $x^6=-64 \quad \Rightarrow$ No solution
- $x^2=64 \quad \Rightarrow x=\pm 8$
- $x^4=-191 \quad \Rightarrow$ No solution
- $x^4=64 \quad \Rightarrow x=\pm\sqrt[4]{64}=\pm 2\sqrt{2}$
- $x^6=-64 \quad \Rightarrow$ No solution.

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}=2$
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- $\sqrt[4]{81}=3$
- $\sqrt[3]{27}=3$
- $\sqrt[6]{1,000,000}=10$
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- $\sqrt[5]{-1}=-1$
- $\sqrt[3]{54}=\sqrt[3]{27\cdot 2}=3\sqrt[3]{2}$
- $\sqrt[5]{-64}=\sqrt[5]{-32\cdot 2}=-2\sqrt[5]{2}$

Solve each equation for x:

- $x^5=-32 \quad\Rightarrow x=\sqrt[5]{-32}=-2$
- $x^5=32 \quad\Rightarrow x=\sqrt[5]{32}=2$
- $x^4=16 \quad\Rightarrow x=\pm 2$
- $x^2=1 \quad\Rightarrow x=\pm 1$
- $x^3=-27 \quad\Rightarrow x=-3$
- $x^5=96 \quad\Rightarrow x=\sqrt[5]{32\cdot 3}=2\sqrt[5]{3}$
- $x^5=-32 \quad\Rightarrow$
- $x^2=32 \quad\Rightarrow$
- $x^6=64 \quad\Rightarrow x=\pm\sqrt[6]{64}=\pm 2$
- $x^6=-64 \quad\Rightarrow$ No solution
- $x^2=64 \quad\Rightarrow x=\pm 8$
- $x^4=-191 \quad\Rightarrow$ No solution
- $x^4=64 \quad\Rightarrow x=\pm\sqrt[4]{64}=\pm 2\sqrt{2}$
- $x^6=-64 \quad\Rightarrow$ No solution.
- $x^3=125 \quad\Rightarrow$
- $x^6=0 \quad\Rightarrow$

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}$=2
- $\sqrt[6]{-64}$=Undefined
- $\sqrt[4]{81}$=3
- $\sqrt[3]{27}$=3
- $\sqrt[6]{1,000,000}$=10
- $\sqrt[3]{1,000,000}$=100
- $\sqrt{1,000,000}$=1000
- $\sqrt[3]{8,000,000}$=200
- $\sqrt[5]{32}$=2
- $\sqrt[5]{1}$=1
- $\sqrt[4]{16}$ =2
- $\sqrt[3]{125}$=5
- $\sqrt[5]{-32}$=−2
- $\sqrt[5]{-1}$=−1
- $\sqrt[3]{54}$= $\sqrt[3]{27 \cdot 2} = 3\sqrt[3]{2}$
- $\sqrt[5]{-64}$= $\sqrt[5]{-32 \cdot 2} = -2\sqrt[5]{2}$

Solve each equation for x:

- $x^5 = -32 \quad \Rightarrow x = \sqrt[5]{-32} = -2$
- $x^5 = 32 \quad \Rightarrow x = \sqrt[5]{32} = 2$
- $x^4 = 16 \quad \Rightarrow x = \pm 2$
- $x^2 = 1 \quad \Rightarrow x = \pm 1$
- $x^3 = -27 \quad \Rightarrow x = -3$
- $x^5 = 96 \quad \Rightarrow x = \sqrt[5]{32 \cdot 3} = 2\sqrt[5]{3}$
- $x^5 = -32 \quad \Rightarrow x = \sqrt[5]{-32} = -2$
- $x^2 = 32 \quad \Rightarrow x = \pm\sqrt{32} = \pm\sqrt{16 \cdot 2} = \pm 4\sqrt{2}$
- $x^6 = 64 \quad \Rightarrow x = \pm\sqrt[6]{64} = \pm 2$
- $x^6 = -64 \quad \Rightarrow$ No solution
- $x^2 = 64 \quad \Rightarrow x = \pm 8$
- $x^4 = -191 \quad \Rightarrow$ No solution
- $x^4 = 64 \quad \Rightarrow x = \pm\sqrt[4]{64} = \pm 2\sqrt{2}$
- $x^6 = -64 \quad \Rightarrow$ No solution.
- $x^3 = 125 \quad \Rightarrow x = 5$
- $x^6 = 0 \quad \Rightarrow x = 0$

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}=2$
- $\sqrt[6]{-64}=$Undefined
- $\sqrt[4]{81}=3$
- $\sqrt[3]{27}=3$
- $\sqrt[6]{1,000,000}=10$
- $\sqrt[3]{1,000,000}=100$
- $\sqrt{1,000,000}=1000$
- $\sqrt[3]{8,000,000}=200$
- $\sqrt[5]{32}=2$
- $\sqrt[5]{1}=1$
- $\sqrt[4]{16}=2$
- $\sqrt[3]{125}=5$
- $\sqrt[5]{-32}=-2$
- $\sqrt[5]{-1}=-1$
- $\sqrt[3]{54}=\sqrt[3]{27\cdot 2}=3\sqrt[3]{2}$
- $\sqrt[5]{-64}=\sqrt[5]{-32\cdot 2}=-2\sqrt[5]{2}$

Solve each equation for x:

- $x^5=-32 \Rightarrow x=\sqrt[5]{-32}=-2$
- $x^5=32 \Rightarrow x=\sqrt[5]{32}=2$
- $x^4=16 \Rightarrow x=\pm 2$
- $x^2=1 \Rightarrow x=\pm 1$
- $x^3=-27 \Rightarrow x=-3$
- $x^5=96 \Rightarrow x=\sqrt[5]{32\cdot 3}=2\sqrt[5]{3}$
- $x^5=-32 \Rightarrow x=\sqrt[5]{-32}=-2$
- $x^2=32 \Rightarrow x=\pm\sqrt{32}=\pm\sqrt{16\cdot 2}=\pm 4\sqrt{2}$
- $x^6=64 \Rightarrow x=\pm\sqrt[6]{64}=\pm 2$
- $x^6=-64 \Rightarrow$ No solution
- $x^2=64 \Rightarrow x=\pm 8$
- $x^4=-191 \Rightarrow$ No solution
- $x^4=64 \Rightarrow x=\pm\sqrt[4]{64}=\pm 2\sqrt{2}$
- $x^6=-64 \Rightarrow$ No solution.
- $x^3=125 \Rightarrow x=5$
- $x^6=0 \Rightarrow x=0$

Rewrite each expression using only radical signs, not fractional powers:

- $5^{\frac{1}{2}} =$
- $5^{-\frac{3}{2}} =$
- $5^{\frac{1}{7}} =$
- $5^{-\frac{8}{7}} =$
- $5^{\frac{3}{7}} =$
- $5^{-\frac{3}{7}} =$
- $5^{\frac{7}{3}} =$
- $5^{\frac{5}{3}} =$

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}=2$
- $\sqrt[6]{-64}=$Undefined
- $\sqrt[4]{81}=3$
- $\sqrt[3]{27}=3$
- $\sqrt[6]{1,000,000}=10$
- $\sqrt[3]{1,000,000}=100$
- $\sqrt{1,000,000}=1000$
- $\sqrt[3]{8,000,000}=200$
- $\sqrt[5]{32}=2$
- $\sqrt[5]{1}=1$
- $\sqrt[4]{16}=2$
- $\sqrt[3]{125}=5$
- $\sqrt[5]{-32}=-2$
- $\sqrt[5]{-1}=-1$
- $\sqrt[3]{54}=\sqrt[3]{27\cdot 2}=3\sqrt[3]{2}$
- $\sqrt[5]{-64}=\sqrt[5]{-32\cdot 2}=-2\sqrt[5]{2}$

Solve each equation for x:

- $x^5=-32 \Rightarrow x=\sqrt[5]{-32}=-2$
- $x^5=32 \Rightarrow x=\sqrt[5]{32}=2$
- $x^4=16 \Rightarrow x=\pm 2$
- $x^2=1 \Rightarrow x=\pm 1$
- $x^3=-27 \Rightarrow x=-3$
- $x^5=96 \Rightarrow x=\sqrt[5]{32\cdot 3}=2\sqrt[5]{3}$
- $x^5=-32 \Rightarrow x=\sqrt[5]{-32}=-2$
- $x^2=32 \Rightarrow x=\pm\sqrt{32}=\pm\sqrt{16\cdot 2}=\pm 4\sqrt{2}$
- $x^6=64 \Rightarrow x=\pm\sqrt[6]{64}=\pm 2$
- $x^6=-64 \Rightarrow$ No solution
- $x^2=64 \Rightarrow x=\pm 8$
- $x^4=-191 \Rightarrow$ No solution
- $x^4=64 \Rightarrow x=\pm\sqrt[4]{64}=\pm 2\sqrt{2}$
- $x^6=-64 \Rightarrow$ No solution.
- $x^3=125 \Rightarrow x=5$
- $x^6=0 \Rightarrow x=0$

Rewrite each expression using only radical signs, not fractional powers:

- $5^{\frac{1}{2}} = \sqrt{5}$
- $5^{-\frac{3}{2}} = \frac{1}{5\sqrt{5}}$
- $5^{\frac{1}{7}} = \sqrt[7]{5}$
- $5^{-\frac{8}{7}} = \frac{1}{5\sqrt[7]{5}}$
- $5^{\frac{3}{7}} = \sqrt[7]{5^3}$
- $5^{-\frac{3}{7}} = \frac{1}{\sqrt[7]{5^3}}$
- $5^{\frac{7}{3}} = 5^2\sqrt[3]{5}$
- $5^{\frac{5}{3}} = 5\sqrt[3]{5^2}$

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}=2$
- $\sqrt[6]{-64}=$Undefined
- $\sqrt[4]{81}=3$
- $\sqrt[3]{27}=3$
- $\sqrt[6]{1,000,000}=10$
- $\sqrt[3]{1,000,000}=100$
- $\sqrt{1,000,000}=1000$
- $\sqrt[3]{8,000,000}=200$
- $\sqrt[5]{32}=2$
- $\sqrt[5]{1}=1$
- $\sqrt[4]{16}=2$
- $\sqrt[3]{125}=5$
- $\sqrt[5]{-32}=-2$
- $\sqrt[5]{-1}=-1$
- $\sqrt[3]{54}=\sqrt[3]{27\cdot 2}=3\sqrt[3]{2}$
- $\sqrt[5]{-64}=\sqrt[5]{-32\cdot 2}=-2\sqrt[5]{2}$

Solve each equation for x:

- $x^5=-32 \quad \Rightarrow x=\sqrt[5]{-32}=-2$
- $x^5=32 \quad \Rightarrow x=\sqrt[5]{32}=2$
- $x^4=16 \quad \Rightarrow x=\pm 2$
- $x^2=1 \quad \Rightarrow x=\pm 1$
- $x^3=-27 \quad \Rightarrow x=-3$
- $x^5=96 \quad \Rightarrow x=\sqrt[5]{32\cdot 3}=2\sqrt[5]{3}$
- $x^5=-32 \quad \Rightarrow x=\sqrt[5]{-32}=-2$
- $x^2=32 \quad \Rightarrow x=\pm\sqrt{32}=\pm\sqrt{16\cdot 2}=\pm 4\sqrt{2}$
- $x^6=64 \quad \Rightarrow x=\pm\sqrt[6]{64}=\pm 2$
- $x^6=-64 \quad \Rightarrow$ No solution
- $x^2=64 \quad \Rightarrow x=\pm 8$
- $x^4=-191 \quad \Rightarrow$ No solution
- $x^4=64 \quad \Rightarrow x=\pm\sqrt[4]{64}=\pm 2\sqrt{2}$
- $x^6=-64 \quad \Rightarrow$ No solution.
- $x^3=125 \quad \Rightarrow x=5$
- $x^6=0 \quad \Rightarrow x=0$

Rewrite each expression using only radical signs, not fractional powers:

- $5^{\frac{1}{2}} = \sqrt{5}$
- $5^{-\frac{3}{2}} = \frac{1}{5\sqrt{5}}$
- $20^{\frac{1}{2}} =$
- $125^{\frac{1}{2}} =$
- $5^{\frac{1}{7}} = \sqrt[7]{5}$
- $5^{-\frac{8}{7}} = \frac{1}{5\sqrt[7]{5}}$
- $5^{-\frac{5}{2}} =$
- $5^{-\frac{10}{3}} =$
- $5^{\frac{3}{7}} = \sqrt[7]{5^3}$
- $5^{-\frac{3}{7}} = \frac{1}{\sqrt[7]{5^3}}$
- $5^{\frac{2}{7}} =$
- $5^{-\frac{12}{7}} =$
- $5^{\frac{7}{3}} = 5^2\sqrt[3]{5}$
- $5^{\frac{5}{3}} = 5\sqrt[3]{5^2}$
- $5^{-\frac{7}{3}} =$
- $5^{-\frac{33}{10}} =$

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}=2$
- $\sqrt[6]{-64}=$Undefined
- $\sqrt[4]{81}=3$
- $\sqrt[3]{27}=3$
- $\sqrt[6]{1,000,000}=10$
- $\sqrt[3]{1,000,000}=100$
- $\sqrt{1,000,000}=1000$
- $\sqrt[3]{8,000,000}=200$
- $\sqrt[5]{32}=2$
- $\sqrt[5]{1}=1$
- $\sqrt[4]{16}=2$
- $\sqrt[3]{125}=5$
- $\sqrt[5]{-32}=-2$
- $\sqrt[5]{-1}=-1$
- $\sqrt[3]{54}=\sqrt[3]{27\cdot 2}=3\sqrt[3]{2}$
- $\sqrt[5]{-64}=\sqrt[5]{-32\cdot 2}=-2\sqrt[5]{2}$

Solve each equation for x:

- $x^5=-32 \Rightarrow x=\sqrt[5]{-32}=-2$
- $x^5=32 \Rightarrow x=\sqrt[5]{32}=2$
- $x^4=16 \Rightarrow x=\pm 2$
- $x^2=1 \Rightarrow x=\pm 1$
- $x^3=-27 \Rightarrow x=-3$
- $x^5=96 \Rightarrow x=\sqrt[5]{32\cdot 3}=2\sqrt[5]{3}$
- $x^5=-32 \Rightarrow x=\sqrt[5]{-32}=-2$
- $x^2=32 \Rightarrow x=\pm\sqrt{32}=\pm\sqrt{16\cdot 2}=\pm 4\sqrt{2}$
- $x^6=64 \Rightarrow x=\pm\sqrt[6]{64}=\pm 2$
- $x^6=-64 \Rightarrow$ No solution
- $x^2=64 \Rightarrow x=\pm 8$
- $x^4=-191 \Rightarrow$ No solution
- $x^4=64 \Rightarrow x=\pm\sqrt[4]{64}=\pm 2\sqrt{2}$
- $x^6=-64 \Rightarrow$ No solution.
- $x^3=125 \Rightarrow x=5$
- $x^6=0 \Rightarrow x=0$

Rewrite each expression using only radical signs, not fractional powers:

- $5^{\frac{1}{2}}=\sqrt{5}$
- $5^{-\frac{3}{2}}=\frac{1}{5\sqrt{5}}$
- $20^{\frac{1}{2}}=2\sqrt{5}$
- $125^{\frac{1}{2}}=5\sqrt{5}$
- $5^{\frac{1}{7}}=\sqrt[7]{5}$
- $5^{-\frac{8}{7}}=\frac{1}{5\sqrt[7]{5}}$
- $5^{-\frac{5}{2}}=\frac{1}{25\sqrt{5}}$
- $5^{-\frac{10}{3}}=\frac{1}{5^3\sqrt[3]{5}}$
- $5^{\frac{3}{7}}=\sqrt[7]{5^3}$
- $5^{-\frac{3}{7}}=\frac{1}{\sqrt[7]{5^3}}$
- $5^{\frac{2}{7}}=\sqrt[7]{5^2}$
- $5^{-\frac{12}{7}}=\frac{1}{5\sqrt[7]{5^5}}$
- $5^{\frac{7}{3}}=5^2\sqrt[3]{5}$
- $5^{\frac{5}{3}}=5\sqrt[3]{5^2}$
- $5^{-\frac{7}{3}}=\frac{1}{5^2\sqrt[3]{5}}$
- $5^{-\frac{33}{10}}=\frac{1}{5^3\sqrt[10]{5^3}}$

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}=2$
- $\sqrt[6]{-64}=$Undefined
- $\sqrt[4]{81}=3$
- $\sqrt[3]{27}=3$
- $\sqrt[6]{1,000,000}=10$
- $\sqrt[3]{1,000,000}=100$
- $\sqrt{1,000,000}=1000$
- $\sqrt[3]{8,000,000}=200$
- $\sqrt[5]{32}=2$
- $\sqrt[5]{1}=1$
- $\sqrt[4]{16}=2$
- $\sqrt[3]{125}=5$
- $\sqrt[5]{-32}=-2$
- $\sqrt[5]{-1}=-1$
- $\sqrt[3]{54}=\sqrt[3]{27\cdot 2}=3\sqrt[3]{2}$
- $\sqrt[5]{-64}=\sqrt[5]{-32\cdot 2}=-2\sqrt[5]{2}$

Solve each equation for x:

- $x^5=-32 \quad \Rightarrow x=\sqrt[5]{-32}=-2$
- $x^5=32 \quad \Rightarrow x=\sqrt[5]{32}=2$
- $x^4=16 \quad \Rightarrow x=\pm 2$
- $x^2=1 \quad \Rightarrow x=\pm 1$
- $x^3=-27 \quad \Rightarrow x=-3$
- $x^5=96 \quad \Rightarrow x=\sqrt[5]{32\cdot 3}=2\sqrt[5]{3}$
- $x^5=-32 \quad \Rightarrow x=\sqrt[5]{-32}=-2$
- $x^2=32 \quad \Rightarrow x=\pm\sqrt{32}=\pm\sqrt{16\cdot 2}=\pm 4\sqrt{2}$
- $x^6=64 \quad \Rightarrow x=\pm\sqrt[6]{64}=\pm 2$
- $x^6=-64 \quad \Rightarrow$ No solution
- $x^2=64 \quad \Rightarrow x=\pm 8$
- $x^4=-191 \quad \Rightarrow$ No solution
- $x^4=64 \quad \Rightarrow x=\pm\sqrt[4]{64}=\pm 2\sqrt{2}$
- $x^6=-64 \quad \Rightarrow$ No solution.
- $x^3=125 \quad \Rightarrow x=5$
- $x^6=0 \quad \Rightarrow x=0$

Rewrite each expression using only radical signs, not fractional powers:

- $5^{\frac{1}{2}}=\sqrt{5}$
- $5^{-\frac{3}{2}}=\frac{1}{5\sqrt{5}}$
- $20^{\frac{1}{2}}=2\sqrt{5}$
- $125^{\frac{1}{2}}=5\sqrt{5}$
- $5^{\frac{1}{7}}=\sqrt[7]{5}$
- $5^{-\frac{8}{7}}=\frac{1}{5\sqrt[7]{5}}$
- $5^{-\frac{5}{2}}=\frac{1}{25\sqrt{5}}$
- $5^{-\frac{10}{3}}=\frac{1}{5^3\sqrt[3]{5}}$
- $5^{\frac{3}{7}}=\sqrt[7]{5^3}$
- $5^{-\frac{3}{7}}=\frac{1}{\sqrt[7]{5^3}}$
- $5^{\frac{2}{7}}=\sqrt[7]{5^2}$
- $5^{-\frac{12}{7}}=\frac{1}{5\sqrt[7]{5^5}}$
- $5^{\frac{7}{3}}=5^2\sqrt[3]{5}$
- $5^{\frac{5}{3}}=5\sqrt[3]{5^2}$
- $5^{-\frac{7}{3}}=\frac{1}{5^2\sqrt[3]{5}}$
- $5^{-\frac{33}{10}}=\frac{1}{5^3\sqrt[10]{5^3}}$

Ex. 1.3.14: Simplify each of the following by extracting all integer squares from the radical sign :

- $\sqrt{9}=$
- $\sqrt{18}=$
- $\sqrt{45}=$
- $\sqrt{108}=$

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}=2$
- $\sqrt[6]{1,000,000}=10$
- $\sqrt[5]{32}=2$
- $\sqrt[5]{-32}=-2$
- $\sqrt[6]{-64}=$Undefined
- $\sqrt[3]{1,000,000}=100$
- $\sqrt[5]{1}=1$
- $\sqrt[5]{-1}=-1$
- $\sqrt[4]{81}=3$
- $\sqrt{1,000,000}=1000$
- $\sqrt[4]{16}=2$
- $\sqrt[3]{54}=\sqrt[3]{27\cdot 2}=3\sqrt[3]{2}$
- $\sqrt[3]{27}=3$
- $\sqrt[3]{8,000,000}=200$
- $\sqrt[3]{125}=5$
- $\sqrt[5]{-64}=\sqrt[5]{-32\cdot 2}=-2\sqrt[5]{2}$

Solve each equation for x:

- $x^5=-32 \Rightarrow x=\sqrt[5]{-32}=-2$
- $x^5=32 \Rightarrow x=\sqrt[5]{32}=2$
- $x^4=16 \Rightarrow x=\pm 2$
- $x^2=1 \Rightarrow x=\pm 1$
- $x^3=-27 \Rightarrow x=-3$
- $x^5=96 \Rightarrow x=\sqrt[5]{32\cdot 3}=2\sqrt[5]{3}$
- $x^5=-32 \Rightarrow x=\sqrt[5]{-32}=-2$
- $x^2=32 \Rightarrow x=\pm\sqrt{32}=\pm\sqrt{16\cdot 2}=\pm 4\sqrt{2}$
- $x^6=64 \Rightarrow x=\pm\sqrt[6]{64}=\pm 2$
- $x^6=-64 \Rightarrow$ No solution
- $x^2=64 \Rightarrow x=\pm 8$
- $x^4=-191 \Rightarrow$ No solution
- $x^4=64 \Rightarrow x=\pm\sqrt[4]{64}=\pm 2\sqrt{2}$
- $x^6=-64 \Rightarrow$ No solution.
- $x^3=125 \Rightarrow x=5$
- $x^6=0 \Rightarrow x=0$

Rewrite each expression using only radical signs, not fractional powers:

- $5^{\frac{1}{2}}=\sqrt{5}$
- $5^{\frac{1}{7}}=\sqrt[7]{5}$
- $5^{\frac{3}{7}}=\sqrt[7]{5^3}$
- $5^{\frac{7}{3}}=5^2\sqrt[3]{5}$
- $5^{-\frac{3}{2}}=\frac{1}{5\sqrt{5}}$
- $5^{-\frac{8}{7}}=\frac{1}{5\sqrt[7]{5}}$
- $5^{-\frac{3}{7}}=\frac{1}{\sqrt[7]{5^3}}$
- $5^{\frac{5}{3}}=5\sqrt[3]{5^2}$
- $20^{\frac{1}{2}}=2\sqrt{5}$
- $5^{-\frac{5}{2}}=\frac{1}{25\sqrt{5}}$
- $5^{\frac{2}{7}}=\sqrt[7]{5^2}$
- $5^{-\frac{7}{3}}=\frac{1}{5^2\sqrt[3]{5}}$
- $125^{\frac{1}{2}}=5\sqrt{5}$
- $5^{-\frac{10}{3}}=\frac{1}{5^3\sqrt[3]{5}}$
- $5^{-\frac{12}{7}}=\frac{1}{5\sqrt[7]{5^5}}$
- $5^{-\frac{33}{10}}=\frac{1}{5^3\sqrt[10]{5^3}}$

Ex. 1.3.14: Simplify each of the following by extracting all integer squares from the radical sign :

- $\sqrt{9}=3$
- $\sqrt{18}=3\sqrt{2}$
- $\sqrt{45}=3\sqrt{5}$
- $\sqrt{108}=6\sqrt{3}$

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}=2$
- $\sqrt[6]{-64}=$Undefined
- $\sqrt[4]{81}=3$
- $\sqrt[3]{27}=3$
- $\sqrt[6]{1,000,000}=10$
- $\sqrt[3]{1,000,000}=100$
- $\sqrt{1,000,000}=1000$
- $\sqrt[3]{8,000,000}=200$
- $\sqrt[5]{32}=2$
- $\sqrt[5]{1}=1$
- $\sqrt[4]{16}=2$
- $\sqrt[3]{125}=5$
- $\sqrt[5]{-32}=-2$
- $\sqrt[5]{-1}=-1$
- $\sqrt[3]{54}=\sqrt[3]{27\cdot 2}=3\sqrt[3]{2}$
- $\sqrt[5]{-64}=\sqrt[5]{-32\cdot 2}=-2\sqrt[5]{2}$

Solve each equation for x:

- $x^5=-32 \Rightarrow x=\sqrt[5]{-32}=-2$
- $x^5=32 \Rightarrow x=\sqrt[5]{32}=2$
- $x^4=16 \Rightarrow x=\pm 2$
- $x^2=1 \Rightarrow x=\pm 1$
- $x^3=-27 \Rightarrow x=-3$
- $x^5=96 \Rightarrow x=\sqrt[5]{32\cdot 3}=2\sqrt[5]{3}$
- $x^5=-32 \Rightarrow x=\sqrt[5]{-32}=-2$
- $x^2=32 \Rightarrow x=\pm\sqrt{32}=\pm\sqrt{16\cdot 2}=\pm 4\sqrt{2}$
- $x^6=64 \Rightarrow x=\pm\sqrt[6]{64}=\pm 2$
- $x^6=-64 \Rightarrow$ No solution
- $x^2=64 \Rightarrow x=\pm 8$
- $x^4=-191 \Rightarrow$ No solution
- $x^4=64 \Rightarrow x=\pm\sqrt[4]{64}=\pm 2\sqrt{2}$
- $x^6=-64 \Rightarrow$ No solution.
- $x^3=125 \Rightarrow x=5$
- $x^6=0 \Rightarrow x=0$

Rewrite each expression using only radical signs, not fractional powers:

- $5^{\frac{1}{2}}=\sqrt{5}$
- $5^{-\frac{3}{2}}=\frac{1}{5\sqrt{5}}$
- $20^{\frac{1}{2}}=2\sqrt{5}$
- $125^{\frac{1}{2}}=5\sqrt{5}$
- $5^{\frac{1}{7}}=\sqrt[7]{5}$
- $5^{-\frac{8}{7}}=\frac{1}{5\sqrt[7]{5}}$
- $5^{-\frac{5}{2}}=\frac{1}{25\sqrt{5}}$
- $5^{-\frac{10}{3}}=\frac{1}{5^3\sqrt[3]{5}}$
- $5^{\frac{3}{7}}=\sqrt[7]{5^3}$
- $5^{-\frac{3}{7}}=\frac{1}{\sqrt[7]{5^3}}$
- $5^{\frac{2}{7}}=\sqrt[7]{5^2}$
- $5^{-\frac{12}{7}}=\frac{1}{5\sqrt[7]{5^5}}$
- $5^{\frac{7}{3}}=5^2\sqrt[3]{5}$
- $5^{\frac{5}{3}}=5\sqrt[3]{5^2}$
- $5^{-\frac{7}{3}}=\frac{1}{5^2\sqrt[3]{5}}$
- $5^{-\frac{33}{10}}=\frac{1}{5^3\sqrt[10]{5^3}}$

Ex. 1.3.14: Simplify each of the following by extracting all integer squares from the radical sign :

- $\sqrt{9}=3$
- $\sqrt{1000}=$
- $\sqrt{18}=3\sqrt{2}$
- $\sqrt{45}\sqrt{5}=$
- $\sqrt{45}=3\sqrt{5}$
- $\sqrt{108}\sqrt{1000}=$
- $\sqrt{108}=6\sqrt{3}$
- $\sqrt{45}\sqrt{1000}=$

Ex1.3.13: Rewrite each root by extracting maximum even powers.

- $\sqrt[6]{64}=2$
- $\sqrt[6]{-64}=$Undefined
- $\sqrt[4]{81}=3$
- $\sqrt[3]{27}=3$
- $\sqrt[6]{1,000,000}=10$
- $\sqrt[3]{1,000,000}=100$
- $\sqrt{1,000,000}=1000$
- $\sqrt[3]{8,000,000}=200$
- $\sqrt[5]{32}=2$
- $\sqrt[5]{1}=1$
- $\sqrt[4]{16}=2$
- $\sqrt[3]{125}=5$
- $\sqrt[5]{-32}=-2$
- $\sqrt[5]{-1}=-1$
- $\sqrt[3]{54}=\sqrt[3]{27\cdot 2}=3\sqrt[3]{2}$
- $\sqrt[5]{-64}=\sqrt[5]{-32\cdot 2}=-2\sqrt[5]{2}$

Solve each equation for x:

- $x^5=-32 \Rightarrow x=\sqrt[5]{-32}=-2$
- $x^5=32 \Rightarrow x=\sqrt[5]{32}=2$
- $x^4=16 \Rightarrow x=\pm 2$
- $x^2=1 \Rightarrow x=\pm 1$
- $x^3=-27 \Rightarrow x=-3$
- $x^5=96 \Rightarrow x=\sqrt[5]{32\cdot 3}=2\sqrt[5]{3}$
- $x^5=-32 \Rightarrow x=\sqrt[5]{-32}=-2$
- $x^2=32 \Rightarrow x=\pm\sqrt{32}=\pm\sqrt{16\cdot 2}=\pm 4\sqrt{2}$
- $x^6=64 \Rightarrow x=\pm\sqrt[6]{64}=\pm 2$
- $x^6=-64 \Rightarrow$ No solution
- $x^2=64 \Rightarrow x=\pm 8$
- $x^4=-191 \Rightarrow$ No solution
- $x^4=64 \Rightarrow x=\pm\sqrt[4]{64}=\pm 2\sqrt{2}$
- $x^6=-64 \Rightarrow$ No solution.
- $x^3=125 \Rightarrow x=5$
- $x^6=0 \Rightarrow x=0$

Rewrite each expression using only radical signs, not fractional powers:

- $5^{\frac{1}{2}}=\sqrt{5}$
- $5^{-\frac{3}{2}}=\frac{1}{5\sqrt{5}}$
- $20^{\frac{1}{2}}=2\sqrt{5}$
- $125^{\frac{1}{2}}=5\sqrt{5}$
- $5^{\frac{1}{7}}=\sqrt[7]{5}$
- $5^{-\frac{8}{7}}=\frac{1}{5\sqrt[7]{5}}$
- $5^{-\frac{5}{2}}=\frac{1}{25\sqrt{5}}$
- $5^{-\frac{10}{3}}=\frac{1}{5^3\sqrt[3]{5}}$
- $5^{\frac{3}{7}}=\sqrt[7]{5^3}$
- $5^{-\frac{3}{7}}=\frac{1}{\sqrt[7]{5^3}}$
- $5^{\frac{2}{7}}=\sqrt[7]{5^2}$
- $5^{-\frac{12}{7}}=\frac{1}{5\sqrt[7]{5^5}}$
- $5^{\frac{7}{3}}=5^2\sqrt[3]{5}$
- $5^{\frac{5}{3}}=5\sqrt[3]{5^2}$
- $5^{-\frac{7}{3}}=\frac{1}{5^2\sqrt[3]{5}}$
- $5^{-\frac{33}{10}}=\frac{1}{5^3\sqrt[10]{5^3}}$

Ex. 1.3.14: Simplify each of the following by extracting all integer squares from the radical sign :

- $\sqrt{9}=3$
- $\sqrt{1000}=10\sqrt{10}$
- $\sqrt{18}=3\sqrt{2}$
- $\sqrt{45}\sqrt{5}=15$
- $\sqrt{45}=3\sqrt{5}$
- $\sqrt{108}\sqrt{1000}=60\sqrt{30}$
- $\sqrt{108}=6\sqrt{3}$
- $\sqrt{45}\sqrt{1000}=150\sqrt{2}$

Ex. 1.3.15: Rationalize each fraction's denominator:

- $\frac{3}{\sqrt{12}} =$
- $\sqrt{\frac{12}{5}} =$
- $\frac{12}{3-\sqrt{5}} =$

Ex. 1.3.15: Rationalize each fraction's denominator:

- $\frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2}$
- $\sqrt{\frac{12}{5}} = \frac{2\sqrt{15}}{5}$
- $\frac{12}{3-\sqrt{5}} = 3(3+\sqrt{5})$

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- $\sqrt{\frac{12}{5}} = \frac{2\sqrt{15}}{5}$
- $\frac{12}{3-\sqrt{5}} = 3(3+\sqrt{5})$
- $\frac{5}{\sqrt{60}} =$
- $\frac{2}{\sqrt{40}} =$
- $\frac{12}{4+\sqrt{5}} =$

Ex. 1.3.15: Rationalize each fraction's denominator:

- $\frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2}$
- $\sqrt{\frac{12}{5}} = \frac{2\sqrt{15}}{5}$
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- $\frac{2}{\sqrt{60}} =$
- $\frac{2}{\sqrt{10}} =$
- $\frac{10}{5+\sqrt{5}} =$

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- $\frac{2}{\sqrt{40}} = \frac{\sqrt{10}}{10}$
- $\frac{12}{4+\sqrt{5}} = \frac{12(4-\sqrt{5})}{11}$
- $\frac{2}{\sqrt{60}} = \frac{\sqrt{15}}{15}$
- $\frac{2}{\sqrt{10}} = \frac{\sqrt{10}}{5}$
- $\frac{10}{5+\sqrt{5}} = \frac{(5-\sqrt{5})}{2}$

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- $\frac{2}{\sqrt{60}} = \frac{\sqrt{15}}{15}$
- $\frac{2}{\sqrt{10}} = \frac{\sqrt{10}}{5}$
- $\frac{10}{5+\sqrt{5}} = \frac{(5-\sqrt{5})}{2}$
- $\frac{5}{\sqrt{30}} =$
- $\frac{8}{\sqrt{10}} =$
- $\frac{2}{\sqrt{5}-3} =$

Ex. 1.3.15: Rationalize each fraction's denominator:

- $\frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2}$
- $\frac{5}{\sqrt{60}} = \frac{\sqrt{15}}{6}$
- $\frac{2}{\sqrt{60}} = \frac{\sqrt{15}}{15}$
- $\frac{5}{\sqrt{30}} = \frac{\sqrt{30}}{6}$
- $\sqrt{\frac{12}{5}} = \frac{2\sqrt{15}}{5}$
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- $\frac{2}{\sqrt{10}} = \frac{\sqrt{10}}{5}$
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- $\frac{12}{3-\sqrt{5}} = 3(3+\sqrt{5})$
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Ex. 1.3.16: Simplify each n^{th} root or write ND if it is not defined:

- $\sqrt[3]{27000} =$
- $\sqrt[3]{250} =$
- $\sqrt[4]{80} =$
- $\sqrt[5]{64} =$
- $\sqrt{125} + \sqrt{45} =$
- $\sqrt[3]{2700} =$
- $\sqrt{100} + \sqrt{200} =$
- $\sqrt[3]{250} =$

Ex. 1.3.15: Rationalize each fraction's denominator:

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- $\frac{2}{\sqrt{60}} = \frac{\sqrt{15}}{15}$
- $\frac{2}{\sqrt{10}} = \frac{\sqrt{10}}{5}$
- $\frac{10}{5+\sqrt{5}} = \frac{(5-\sqrt{5})}{2}$
- $\frac{5}{\sqrt{30}} = \frac{\sqrt{30}}{6}$
- $\frac{8}{\sqrt{10}} = \frac{2\sqrt{5}}{25}$
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Ex. 1.3.16: Simplify each n^{th} root or write ND if it is not defined:

- $\sqrt[3]{27000} = 30$
- $\sqrt[3]{250} = 5\sqrt[3]{2}$
- $\sqrt[4]{80} = 2\sqrt[4]{5}$
- $\sqrt[5]{64} = 2\sqrt[5]{2}$
- $\sqrt{125} + \sqrt{45} = 8\sqrt{5}$
- $\sqrt[3]{2700} = 3\sqrt[3]{100}$
- $\sqrt{100} + \sqrt{200} = 10 + 10\sqrt{2}$
- $\sqrt[3]{250} = 5\sqrt{2}$

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- $\sqrt[3]{2700} = 3\sqrt[3]{100}$
- $\sqrt{100} + \sqrt{200} = 10 + 10\sqrt{2}$
- $\sqrt[3]{250} = 5\sqrt{2}$
- $\sqrt[4]{80} =$
- $\sqrt[5]{64} =$
- $\sqrt{125} + \sqrt{125} =$
- $\sqrt{160} =$
- $\sqrt{8000} =$
- $\sqrt[3]{80} =$
- $\sqrt[4]{160} =$
- $\sqrt[3]{640} =$

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- $\sqrt[3]{80} = 2\sqrt{10}$
- $\sqrt[4]{160} = 2\sqrt[4]{10}$
- $\sqrt[3]{640} = 4\sqrt[3]{10}$
- $\sqrt{12} + \sqrt{75} =$
- $\sqrt{50} + \sqrt{32} =$
- $\sqrt[3]{-27000} =$
- $\sqrt[3]{-250} =$

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- $\sqrt[3]{640} = 4\sqrt[3]{10}$
- $\sqrt{12} + \sqrt{75} = 7\sqrt{3}$
- $\sqrt{50} + \sqrt{32} = 9\sqrt{2}$
- $\sqrt[3]{-27000} = -30$
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- $\sqrt[4]{64} + \sqrt[4]{4} =$

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- $\sqrt[3]{-27000} = -30$
- $\sqrt[3]{-250} = -5\sqrt{2}$
- $\sqrt[5]{-160} = -2\sqrt[5]{5}$
- $\sqrt[3]{-81} = -3\sqrt[3]{3}$
- $\sqrt{-125} + \sqrt{45} =$ ND
- $\sqrt[4]{64} + \sqrt[4]{4} = 3\sqrt{2}$

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- $\sqrt{-125} + \sqrt{45} =$ ND
- $\sqrt[4]{64} + \sqrt[4]{4} = 3\sqrt{2}$

Ex. 1.3.17: Simplify each n^{th} root: Assume x, y, a, b are positive:

- $\sqrt[3]{128} =$
- $\sqrt[3]{x^{10}} =$
- $\sqrt[3]{y^8} =$
- $\sqrt[3]{128x^{10}y^8} =$

Ex. 1.3.15: Rationalize each fraction's denominator:

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- $\frac{2}{\sqrt{60}} = \frac{\sqrt{15}}{15}$
- $\frac{2}{\sqrt{10}} = \frac{\sqrt{10}}{5}$
- $\frac{10}{5+\sqrt{5}} = \frac{(5-\sqrt{5})}{2}$
- $\frac{5}{\sqrt{30}} = \frac{\sqrt{30}}{6}$
- $\frac{8}{\sqrt{10}} = \frac{2\sqrt{5}}{25}$
- $\frac{2}{\sqrt{5}-3} = -\frac{(3+\sqrt{5})}{2}$

Ex. 1.3.16: Simplify each n^{th} root or write ND if it is not defined:

- $\sqrt[3]{27000} = 30$
- $\sqrt[3]{250} = 5\sqrt[3]{2}$
- $\sqrt[4]{80} = 2\sqrt[4]{5}$
- $\sqrt[5]{64} = 2\sqrt[5]{2}$
- $\sqrt{125} + \sqrt{45} = 8\sqrt{5}$
- $\sqrt[3]{2700} = 3\sqrt[3]{100}$
- $\sqrt{100} + \sqrt{200} = 10 + 10\sqrt{2}$
- $\sqrt[3]{250} = 5\sqrt{2}$
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- $\sqrt{125} + \sqrt{125} = 10\sqrt{5}$
- $\sqrt{160} = 4\sqrt{10}$
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- $\sqrt[3]{80} = 2\sqrt{10}$
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- $\sqrt[3]{640} = 4\sqrt[3]{10}$
- $\sqrt{12} + \sqrt{75} = 7\sqrt{3}$
- $\sqrt{50} + \sqrt{32} = 9\sqrt{2}$
- $\sqrt[3]{-27000} = -30$
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- $\sqrt[3]{y^{64}} = y^{21}\sqrt[3]{y}$
- $\sqrt[3]{32x^4} = 2x\sqrt[3]{4x}$
- $\sqrt[3]{x^6y^8} = x^2y^2\sqrt[3]{y^2}$
- $\sqrt[3]{a^4b^4} = ab\sqrt[3]{ab}$
- $\sqrt[4]{y^8} = y^2$
- $\sqrt[3]{250x^3y^4} = 5xy\sqrt[3]{2y}$
- $\sqrt{54x^2y^2} = 3xy\sqrt{6}$
- $\sqrt{8x^{10}} = 2\sqrt{2}x^5$
- $\sqrt[3]{81} = 3\sqrt[3]{3}$
- $\sqrt[3]{40} = 2\sqrt[3]{5}$
- $\sqrt[4]{128x^{10}} = 2x^2\sqrt[4]{8x^2}$
- $\sqrt{x^2y^3} + \sqrt{x^3y^2} =$
- $\sqrt{x^3y^3} + \sqrt{4x^3y^3} =$
- $\sqrt[3]{8x^3y^3} + \sqrt{8x^3y^3} =$
- $\sqrt[4]{x^5y^5} + \sqrt{x^4y^4} =$
- $2\sqrt{40} + \sqrt{160} =$
- $\sqrt{x^2y^3} - \sqrt{x^3y^2} =$

Ex. 1.3.15: Rationalize each fraction's denominator:

- $\frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2}$
- $\sqrt{\frac{12}{5}} = \frac{2\sqrt{15}}{5}$
- $\frac{12}{3-\sqrt{5}} = 3(3+\sqrt{5})$
- $\frac{5}{\sqrt{60}} = \frac{\sqrt{15}}{6}$
- $\frac{2}{\sqrt{40}} = \frac{\sqrt{10}}{10}$
- $\frac{12}{4+\sqrt{5}} = \frac{12(4-\sqrt{5})}{11}$
- $\frac{2}{\sqrt{60}} = \frac{\sqrt{15}}{15}$
- $\frac{2}{\sqrt{10}} = \frac{\sqrt{10}}{5}$
- $\frac{10}{5+\sqrt{5}} = \frac{(5-\sqrt{5})}{2}$
- $\frac{5}{\sqrt{30}} = \frac{\sqrt{30}}{6}$
- $\frac{8}{\sqrt{10}} = \frac{2\sqrt{5}}{25}$
- $\frac{2}{\sqrt{5}-3} = -\frac{(3+\sqrt{5})}{2}$

Ex. 1.3.16: Simplify each n^{th} root or write ND if it is not defined:

- $\sqrt[3]{27000} = 30$
- $\sqrt[3]{250} = 5\sqrt[3]{2}$
- $\sqrt[4]{80} = 2\sqrt[4]{5}$
- $\sqrt[5]{64} = 2\sqrt[5]{2}$
- $\sqrt{125} + \sqrt{45} = 8\sqrt{5}$
- $\sqrt[3]{2700} = 3\sqrt[3]{100}$
- $\sqrt{100} + \sqrt{200} = 10 + 10\sqrt{2}$
- $\sqrt[3]{250} = 5\sqrt{2}$
- $\sqrt[4]{80} = 2\sqrt[4]{5}$
- $\sqrt[5]{64} = 2\sqrt[5]{2}$
- $\sqrt{125} + \sqrt{125} = 10\sqrt{5}$
- $\sqrt{160} = 4\sqrt{10}$
- $\sqrt{8000} = 40\sqrt{5}$
- $\sqrt[3]{80} = 2\sqrt{10}$
- $\sqrt[4]{160} = 2\sqrt[4]{10}$
- $\sqrt[3]{640} = 4\sqrt[3]{10}$
- $\sqrt{12} + \sqrt{75} = 7\sqrt{3}$
- $\sqrt{50} + \sqrt{32} = 9\sqrt{2}$
- $\sqrt[3]{-27000} = -30$
- $\sqrt[3]{-250} = -5\sqrt{2}$
- $\sqrt[5]{-160} = -2\sqrt[5]{5}$
- $\sqrt[3]{-81} = -3\sqrt[3]{3}$
- $\sqrt{-125} + \sqrt{45} =$ ND
- $\sqrt[4]{64} + \sqrt[4]{4} = 3\sqrt{2}$

Ex. 1.3.17: Simplify each n^{th} root: Assume x, y, a, b are positive:

- $\sqrt[3]{128} = 4\sqrt[3]{2}$
- $\sqrt[3]{x^{10}} = x^3\sqrt[3]{x}$
- $\sqrt[3]{y^8} = y^2\sqrt[3]{y^2}$
- $\sqrt[3]{128x^{10}y^8} = 4x^3y^2\sqrt[3]{2xy^2}$
- $\sqrt[3]{x^{100}} = x^{33}\sqrt[3]{x}$
- $\sqrt[3]{y^{64}} = y^{21}\sqrt[3]{y}$
- $\sqrt[3]{32x^4} = 2x\sqrt[3]{4x}$
- $\sqrt[3]{x^6y^8} = x^2y^2\sqrt[3]{y^2}$
- $\sqrt[3]{a^4b^4} = ab\sqrt[3]{ab}$
- $\sqrt[4]{y^8} = y^2$
- $\sqrt[3]{250x^3y^4} = 5xy\sqrt[3]{2y}$
- $\sqrt{54x^2y^2} = 3xy\sqrt{6}$
- $\sqrt{8x^{10}} = 2\sqrt{2}x^5$
- $\sqrt[3]{81} = 3\sqrt[3]{3}$
- $\sqrt[3]{40} = 2\sqrt[3]{5}$
- $\sqrt[4]{128x^{10}} = 2x^2\sqrt[4]{8x^2}$
- $\sqrt{x^2y^3} + \sqrt{x^3y^2} = xy(\sqrt{x} + \sqrt{y})$
- $\sqrt{x^3y^3} + \sqrt{4x^3y^3} = 3xy\sqrt{xy}$
- $\sqrt[3]{8x^3y^3} + \sqrt{8x^3y^3} = 2xy(1 + \sqrt{2xy})$
- $\sqrt[4]{x^5y^5} + \sqrt{x^4y^4} = xy(1 + \sqrt[4]{xy})$
- $2\sqrt{40} + \sqrt{160} = 8\sqrt{10}$
- $\sqrt{x^2y^3} - \sqrt{x^3y^2} = xy(\sqrt{y} - \sqrt{x})$

Ex. 1.3.18: Rewrite without radical signs:

- $x\sqrt{x} =$
- $x^3\sqrt[3]{x} =$
- $x^8\sqrt[8]{x} =$
- $\sqrt{x}x\sqrt{x} =$

Ex. 1.3.18: Rewrite without radical signs:

- $x\sqrt{x} = x^{\frac{3}{2}}$
- $x^3\sqrt[3]{x} = x^{\frac{10}{3}}$
- $x^8\sqrt[8]{x} = x^{\frac{65}{8}}$
- $\sqrt{x}x\sqrt{x} = x^2$

Ex. 1.3.18: Rewrite without radical signs:

- $x\sqrt{x} = x^{\frac{3}{2}}$
- $x^3\sqrt[3]{x} = x^{\frac{10}{3}}$
- $x^8\sqrt[8]{x} = x^{\frac{65}{8}}$
- $\sqrt{x}x\sqrt{x} = x^2$

Ex. 1.3.19: Rewrite without fractional exponents:

- $x^{5/2} =$
- $x^{5/3} =$
- $x^{15/2} =$
- $x^{5/8} =$

Ex. 1.3.18: Rewrite without radical signs:

- $x\sqrt{x} = x^{\frac{3}{2}}$
- $x^3\sqrt[3]{x} = x^{\frac{10}{3}}$
- $x^8\sqrt[8]{x} = x^{\frac{65}{8}}$
- $\sqrt{x}x\sqrt{x} = x^2$

Ex. 1.3.19: Rewrite without fractional exponents:

- $x^{5/2} = x^2\sqrt{x}$
- $x^{5/3} = x\sqrt[3]{x^2}$
- $x^{15/2} = x^7\sqrt{x}$
- $x^{5/8} = \sqrt[8]{x^5}$

Ex. 1.3.18: Rewrite without radical signs:

- $x\sqrt{x} = x^{\frac{3}{2}}$ • $x^3\sqrt[3]{x} = x^{\frac{10}{3}}$ • $x^8\sqrt[8]{x} = x^{\frac{65}{8}}$ • $\sqrt{x}x\sqrt{x} = x^2$

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- $x^{5/2} = x^2\sqrt{x}$ • $x^{5/3} = x\sqrt[3]{x^2}$ • $x^{15/2} = x^7\sqrt{x}$ • $x^{5/8} = \sqrt[8]{x^5}$

Ex. 1.3.20: Rewrite without radical signs or parentheses:

- $(x + \sqrt{x})^2 =$
- $(x^2 + \sqrt{2x})^2 =$
- $(x + 4\sqrt{x})^2 =$
- $(x + \sqrt{x^3})^2 =$

Ex. 1.3.18: Rewrite without radical signs:

- $x\sqrt{x} = x^{\frac{3}{2}}$ • $x^3\sqrt[3]{x} = x^{\frac{10}{3}}$ • $x^8\sqrt[8]{x} = x^{\frac{65}{8}}$ • $\sqrt{x}x\sqrt{x} = x^2$

Ex. 1.3.19: Rewrite without fractional exponents:

- $x^{5/2} = x^2\sqrt{x}$ • $x^{5/3} = x\sqrt[3]{x^2}$ • $x^{15/2} = x^7\sqrt{x}$ • $x^{5/8} = \sqrt[8]{x^5}$

Ex. 1.3.20: Rewrite without radical signs or parentheses:

- $(x + \sqrt{x})^2 = x^2 + 2x^{\frac{3}{2}} + x$
- $(x^2 + \sqrt{2x})^2 = x^4 + 2x^2(2x)^{\frac{1}{2}} + 2x$
- $(x + 4\sqrt{x})^2 = x^2 + 8x^{\frac{3}{2}} + 16x$
- $(x + \sqrt{x^3})^2 = x^2 + 2x^{\frac{5}{2}} + x^3$

Ex. 1.3.18: Rewrite without radical signs:

- $x\sqrt{x} = x^{\frac{3}{2}}$ • $x^3\sqrt[3]{x} = x^{\frac{10}{3}}$ • $x^8\sqrt[8]{x} = x^{\frac{65}{8}}$ • $\sqrt{x}x\sqrt{x} = x^2$

Ex. 1.3.19: Rewrite without fractional exponents:

- $x^{5/2} = x^2\sqrt{x}$ • $x^{5/3} = x\sqrt[3]{x^2}$ • $x^{15/2} = x^7\sqrt{x}$ • $x^{5/8} = \sqrt[8]{x^5}$

Ex. 1.3.20: Rewrite without radical signs or parentheses:

- $(x+\sqrt{x})^2 = x^2 + 2x^{\frac{3}{2}} + x$
- $(x^2+\sqrt{2x})^2 = x^4 + 2x^2(2x)^{\frac{1}{2}} + 2x$
- $(x+4\sqrt{x})^2 = x^2 + 8x^{\frac{3}{2}} + 16x$
- $(x+\sqrt{x^3})^2 = x^2 + 2x^{\frac{5}{2}} + x^3$

Ex. 1.3.21: Rewrite as a sum of powers of x:

- $\frac{1+x^2\sqrt{x}}{x^3} =$
- $\frac{x^3+x^2\sqrt{x}}{\sqrt{x}} =$
- $\frac{1+x^2\sqrt[3]{x}}{x^3} =$
- $\frac{\sqrt{x}+x^2\sqrt{x}}{x^3} =$

Ex. 1.3.18: Rewrite without radical signs:

- $x\sqrt{x} = x^{\frac{3}{2}}$
- $x^3\sqrt[3]{x} = x^{\frac{10}{3}}$
- $x^8\sqrt[8]{x} = x^{\frac{65}{8}}$
- $\sqrt{x}x\sqrt{x} = x^2$

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- $x^{5/3} = x\sqrt[3]{x^2}$
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- $(x+4\sqrt{x})^2 = x^2 + 8x^{\frac{3}{2}} + 16x$
- $(x+\sqrt{x^3})^2 = x^2 + 2x^{\frac{5}{2}} + x^3$

Ex. 1.3.21: Rewrite as a sum of powers of x:

- $\frac{1+x^2\sqrt{x}}{x^3} = x^{-3} + x^{-\frac{1}{2}}$
- $\frac{x^3+x^2\sqrt{x}}{\sqrt{x}} = x^{\frac{5}{2}} + x^2$
- $\frac{1+x^2\sqrt[3]{x}}{x^3} = x^{-3} + x^{-\frac{2}{3}}$
- $\frac{\sqrt{x}+x^2\sqrt{x}}{x^3} = x^{-\frac{5}{2}} + x^{-\frac{1}{2}}$

Ex. 1.3.18: Rewrite without radical signs:

- $x\sqrt{x} = x^{\frac{3}{2}}$ • $x^3\sqrt[3]{x} = x^{\frac{10}{3}}$ • $x^8\sqrt[8]{x} = x^{\frac{65}{8}}$ • $\sqrt{x}x\sqrt{x} = x^2$

Ex. 1.3.19: Rewrite without fractional exponents:

- $x^{5/2} = x^2\sqrt{x}$ • $x^{5/3} = x\sqrt[3]{x^2}$ • $x^{15/2} = x^7\sqrt{x}$ • $x^{5/8} = \sqrt[8]{x^5}$

Ex. 1.3.20: Rewrite without radical signs or parentheses:

- $(x+\sqrt{x})^2 = x^2 + 2x^{\frac{3}{2}} + x$
- $(x^2+\sqrt{2x})^2 = x^4 + 2x^2(2x)^{\frac{1}{2}} + 2x$
- $(x+4\sqrt{x})^2 = x^2 + 8x^{\frac{3}{2}} + 16x$
- $(x+\sqrt{x^3})^2 = x^2 + 2x^{\frac{5}{2}} + x^3$

Ex. 1.3.21: Rewrite as a sum of powers of x:

- $\frac{1+x^2\sqrt{x}}{x^3} = x^{-3} + x^{-\frac{1}{2}}$
- $\frac{x^3+x^2\sqrt{x}}{\sqrt{x}} = x^{\frac{5}{2}} + x^2$
- $\frac{1+x^2\sqrt[3]{x}}{x^3} = x^{-3} + x^{-\frac{2}{3}}$
- $\frac{\sqrt{x}+x^2\sqrt{x}}{x^3} = x^{-\frac{5}{2}} + x^{-\frac{1}{2}}$

Ex. 1.3.22: Rewrite with neither fractional nor negative exponents.

- $\frac{1+x^2\sqrt{x}}{x^3} =$
- $\frac{x^3+x^2\sqrt{x}}{\sqrt{x}} =$
- $\frac{1+x^2\sqrt[3]{x}}{x^3} =$
- $\frac{\sqrt{x}+x^2\sqrt[3]{x}}{x^3} =$

Ex. 1.3.18: Rewrite without radical signs:

- $x\sqrt{x} = x^{\frac{3}{2}}$
- $x^3\sqrt[3]{x} = x^{\frac{10}{3}}$
- $x^8\sqrt[8]{x} = x^{\frac{65}{8}}$
- $\sqrt{x}x\sqrt{x} = x^2$

Ex. 1.3.19: Rewrite without fractional exponents:

- $x^{5/2} = x^2\sqrt{x}$
- $x^{5/3} = x\sqrt[3]{x^2}$
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- $(x^2+\sqrt{2x})^2 = x^4 + 2x^2(2x)^{\frac{1}{2}} + 2x$
- $(x+4\sqrt{x})^2 = x^2 + 8x^{\frac{3}{2}} + 16x$
- $(x+\sqrt{x^3})^2 = x^2 + 2x^{\frac{5}{2}} + x^3$

Ex. 1.3.21: Rewrite as a sum of powers of x:

- $\frac{1+x^2\sqrt{x}}{x^3} = x^{-3} + x^{-\frac{1}{2}}$
- $\frac{x^3+x^2\sqrt{x}}{\sqrt{x}} = x^{\frac{5}{2}} + x^2$
- $\frac{1+x^2\sqrt[3]{x}}{x^3} = x^{-3} + x^{-\frac{2}{3}}$
- $\frac{\sqrt{x}+x^2\sqrt{x}}{x^3} = x^{-\frac{5}{2}} + x^{-\frac{1}{2}}$

Ex. 1.3.22: Rewrite with neither fractional nor negative exponents.

- $\frac{1+x^2\sqrt{x}}{x^3} = \frac{1}{x^3} + \frac{\sqrt{x}}{x}$
- $\frac{x^3+x^2\sqrt{x}}{\sqrt{x}} = x^2\sqrt{x} + x^2$
- $\frac{1+x^2\sqrt[3]{x}}{x^3} = \frac{1}{x^3} + \frac{\sqrt[3]{x}}{x}$
- $\frac{\sqrt{x}+x^2\sqrt[3]{x}}{x^3} = \frac{\sqrt{x}}{x^3} + \frac{\sqrt[3]{x}}{x}$

Ex. 1.3.18: Rewrite without radical signs:

- $x\sqrt{x} = x^{\frac{3}{2}}$ • $x^3\sqrt[3]{x} = x^{\frac{10}{3}}$ • $x^8\sqrt[8]{x} = x^{\frac{65}{8}}$ • $\sqrt{x}x\sqrt{x} = x^2$

Ex. 1.3.19: Rewrite without fractional exponents:

- $x^{5/2} = x^2\sqrt{x}$ • $x^{5/3} = x\sqrt[3]{x^2}$ • $x^{15/2} = x^7\sqrt{x}$ • $x^{5/8} = \sqrt[8]{x^5}$

Ex. 1.3.20: Rewrite without radical signs or parentheses:

- $(x+\sqrt{x})^2 = x^2 + 2x^{\frac{3}{2}} + x$
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Ex. 1.3.21: Rewrite as a sum of powers of x:

- $\frac{1+x^2\sqrt{x}}{x^3} = x^{-3} + x^{-\frac{1}{2}}$
- $\frac{x^3+x^2\sqrt{x}}{\sqrt{x}} = x^{\frac{5}{2}} + x^2$
- $\frac{1+x^2\sqrt[3]{x}}{x^3} = x^{-3} + x^{-\frac{2}{3}}$
- $\frac{\sqrt{x}+x^2\sqrt{x}}{x^3} = x^{-\frac{5}{2}} + x^{-\frac{1}{2}}$

Ex. 1.3.22: Rewrite with neither fractional nor negative exponents.

- $\frac{1+x^2\sqrt{x}}{x^3} = \frac{1}{x^3} + \frac{\sqrt{x}}{x}$
- $\frac{x^3+x^2\sqrt{x}}{\sqrt{x}} = x^2\sqrt{x} + x^2$
- $\frac{1+x^2\sqrt[3]{x}}{x^3} = \frac{1}{x^3} + \frac{\sqrt[3]{x}}{x}$
- $\frac{\sqrt{x}+x^2\sqrt[3]{x}}{x^3} = \frac{\sqrt{x}}{x^3} + \frac{\sqrt[3]{x}}{x}$

Ex. 1.3.23: Factor completely: leave a fractional root in your answer.

- $12x^{5/2} - 36x^{3/2} + 24x^{1/2} =$
- $4x^{5/2} + 20x^{3/2} + 24x^{1/2} =$
- $3x^{5/2} - 12x^{3/2} + 3x^{1/2} =$
- $12x^{7/3} - 12x^{4/3} - 144x^{1/3} =$

Ex. 1.3.18: Rewrite without radical signs:

- $x\sqrt{x} = x^{\frac{3}{2}}$ • $x^3\sqrt[3]{x} = x^{\frac{10}{3}}$ • $x^8\sqrt[8]{x} = x^{\frac{65}{8}}$ • $\sqrt{x}x\sqrt{x} = x^2$

Ex. 1.3.19: Rewrite without fractional exponents:

- $x^{5/2} = x^2\sqrt{x}$ • $x^{5/3} = x\sqrt[3]{x^2}$ • $x^{15/2} = x^7\sqrt{x}$ • $x^{5/8} = \sqrt[8]{x^5}$

Ex. 1.3.20: Rewrite without radical signs or parentheses:

- $(x+\sqrt{x})^2 = x^2 + 2x^{\frac{3}{2}} + x$
- $(x^2+\sqrt{2x})^2 = x^4 + 2x^2(2x)^{\frac{1}{2}} + 2x$
- $(x+4\sqrt{x})^2 = x^2 + 8x^{\frac{3}{2}} + 16x$
- $(x+\sqrt{x^3})^2 = x^2 + 2x^{\frac{5}{2}} + x^3$

Ex. 1.3.21: Rewrite as a sum of powers of x:

- $\frac{1+x^2\sqrt{x}}{x^3} = x^{-3} + x^{-\frac{1}{2}}$
- $\frac{x^3+x^2\sqrt{x}}{\sqrt{x}} = x^{\frac{5}{2}} + x^2$
- $\frac{1+x^2\sqrt[3]{x}}{x^3} = x^{-3} + x^{-\frac{2}{3}}$
- $\frac{\sqrt{x}+x^2\sqrt{x}}{x^3} = x^{-\frac{5}{2}} + x^{-\frac{1}{2}}$

Ex. 1.3.22: Rewrite with neither fractional nor negative exponents.

- $\frac{1+x^2\sqrt{x}}{x^3} = \frac{1}{x^3} + \frac{\sqrt{x}}{x}$
- $\frac{x^3+x^2\sqrt{x}}{\sqrt{x}} = x^2\sqrt{x} + x^2$
- $\frac{1+x^2\sqrt[3]{x}}{x^3} = \frac{1}{x^3} + \frac{\sqrt[3]{x}}{x}$
- $\frac{\sqrt{x}+x^2\sqrt[3]{x}}{x^3} = \frac{\sqrt{x}}{x^3} + \frac{\sqrt[3]{x}}{x}$

Ex. 1.3.23: Factor completely: leave a fractional root in your answer.

- $12x^{5/2} - 36x^{3/2} + 24x^{1/2} = 12x^{\frac{1}{2}}(x-2)(x-1)$
- $4x^{5/2} + 20x^{3/2} + 24x^{1/2} = 4x^{\frac{1}{2}}(x+2)(x+3)$
- $3x^{5/2} - 12x^{3/2} + 3x^{1/2} = 3x^{\frac{1}{2}}(x^2-4x+1)$
- $12x^{7/3} - 12x^{4/3} - 144x^{1/3} = 12x^{\frac{1}{3}}(x-4)(x+3)$

- 1.4.1: How precalculus can save the dolphins
- 1.4.2: Animation for a range of oil spill rates
- 1.4.3: How calculus can save the deer
- Section 1.4 Review

1.4.1 How precalculus can save the dolphins

At high noon, an oil tanker explosion releases an oil slick on the surface of the ocean. A dolphin located d meters (m) from the tanker immediately swims away from the tanker in a straight line at a speed of v meters per second (m/sec). The spilled oil volume grows at constant rate K cubic meters (m^3) per second and forms an expanding cylinder whose height is one meter at all times.

- For what minimum value of K will the oil overtake the dolphin?
- When and where will this happen?

Solution: At time t,
Let $R(t)$ be the radius of the oil slick: this is how far the oil has spread in every direction from the tanker.
Let $D(t)$ be the dolphin's distance from the tanker.

- If $R(t) < D(t)$, the oil slick has not reached the dolphin.
- If $R(t) = D(t)$, the oil just reaches the dolphin.
- If $R(t) > D(t)$, the oil reaches beyond the dolphin. This is bad for the dolphin.

To find $R(t)$, compute the oil slick's volume V at time t in two ways:

- Oil volume is $V = Kt$, the oil spilled per second times the number of seconds.
- Oil cylinder volume is $V = \pi r^2 h = \pi R(t)^2$ since its height is 1 meter.

Solving $Kt = \pi R(t)^2$ for $R(t)$ yields $\boxed{R(t) = \sqrt{\frac{Kt}{\pi}}}$.

Next find $D(t)$, the dolphin's distance from the tanker at time t. That distance starts out as d at time $t = 0$ and increases at the rate of v m/sec. Therefore $D(t)$ is given by the simple linear function $\boxed{D(t) = vt + d}$.

The oil first touches the dolphin at time t when $R(t) = D(t)$. At that time $\sqrt{\frac{Kt}{\pi}} = vt + d$, and so $\frac{Kt}{\pi} = (vt + d)^2$. We can rewrite this equation (try it!) as $v^2 t^2 + \left(2vd - \frac{K}{\pi}\right) t + d^2 = 0$. This is a quadratic equation $at^2 + bt + c = 0$ where $a = v^2, b = (2vd - \frac{K}{\pi})$, and $c = d^2$.

According to the quadratic formula, the solutions are

$t = \frac{-b \pm \sqrt{D}}{2a}$ where $D = b^2 - 4ac$.

- If $D = 0$, there is one solution $t = -\frac{b}{2a}$.
- $D < 0$ there are no solutions.
- $D > 0$ there are two solutions.

To see which solution happens, work out
$D = b^2 - 4ac = (2vd - \frac{K}{\pi})^2 - 4v^2 d^2$

$= 4v^2 d^2 - 4vd\frac{K}{\pi} + \frac{K^2}{\pi^2} - 4v^2 d^2$

$= \frac{K^2}{\pi^2} - 4vd\frac{K}{\pi} = \frac{K^2 - 4\pi vdK}{\pi^2}$

$= \frac{K(K - 4\pi vd)}{\pi^2} = D$. Since $K/\pi^2 > 0$,

D and $K - 4\pi vd$ have the same sign.

There are three cases:

- If $K < 4\pi vd$ then $D < 0$: no solutions.
- If $K > 4\pi vd$ then $D > 0$: has two solutions.
- If $K = 4\pi vd$ then $D = 0$: one solution.

In the last case, the oil touches the dolphin at one instant of time

$t = -\frac{b}{2a} = \frac{\frac{K}{\pi} - 2vd}{2v^2} = \frac{4vd - 2vd}{2v^2} = \frac{d}{v}$ seconds,
when the dolphin's distance from the tanker is
$vt + d = v \cdot \frac{d}{v} + d = 2d$ meters.

From the dolphin's viewpoint:

- If $K < 4\pi vd$, the oil slick radius $R(t)$ is less than the dolphin's distance $D(t)$ from the tanker at all times t, and the dolphin escapes.
- If $K = 4\pi vd$ then $R(t) = D(t)$ at exactly one time, $t = \frac{d}{v}$ seconds. The dolphin gets a bit oily but will probably be OK.
- If $K > 4\pi vd$ then $R(t) > D(t)$ for an interval of time, and the dolphin will be smothered by the oil.

Since the dolphin starts out $d = 2$ m from the tanker and swims away at speed $v = 1$ m/sec:
Answer:

- the minimum oil spill rate for catching the dolphin is $K = 4\pi vd = 8\pi$ m^3/sec. and
- at this spill rate, the oil just catches the dolphin at distance $vt + d = t + 2 = 4$ meters from the oil tanker, at time $t = \frac{d}{v} = 2$ seconds.

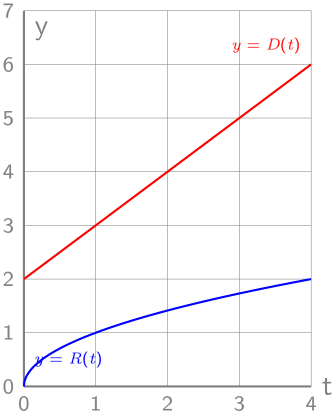
The animation on the next slides illustrates how the oil pursues the dolphin. Move between slides by using the mouse wheel or the up/down arrow keys. Press and hold the down-arrow key to speed up the action.

Timeline for the critical oil spill rate

If the oil spill rate is $1\pi \mathrm{m}^3$/sec, then the oil slick radius at time t is $R(t) = \sqrt{1t}$ m.

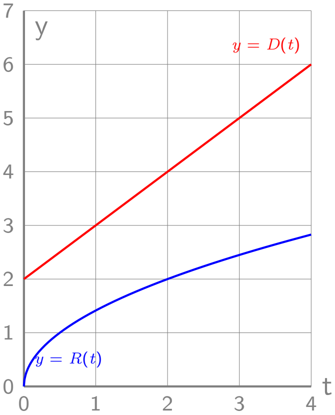
The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.

The oil never touches the dolphin.



Timeline for the critical oil spill rate

If the oil spill rate is 2πm^3/sec, then the oil slick radius at time t is $R(t) = \sqrt{2t}$ m.
The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.
The oil never touches the dolphin.

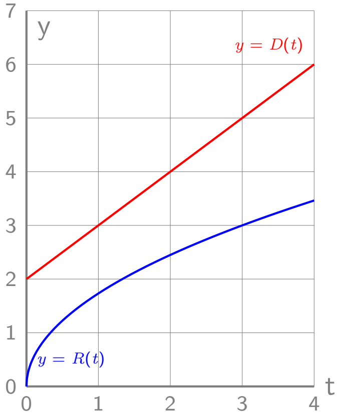


Timeline for the critical oil spill rate

If the oil spill rate is $3\pi\text{m}^3/\text{sec}$, then the oil slick radius at time t is $R(t) = \sqrt{3t}$ m.

The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.

The oil never touches the dolphin.

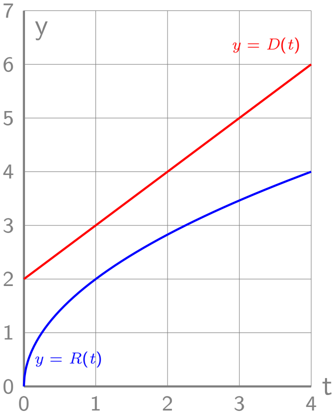


Timeline for the critical oil spill rate

If the oil spill rate is $4\pi\text{m}^3/\text{sec}$, then the oil slick radius at time t is $R(t) = \sqrt{4t}$ m.

The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.

The oil never touches the dolphin.

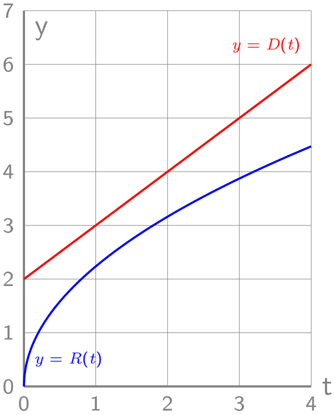


Timeline for the critical oil spill rate

If the oil spill rate is 5πm^3/sec, then the oil slick radius at time t is $R(t) = \sqrt{5t}$ m.

The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.

The oil never touches the dolphin.

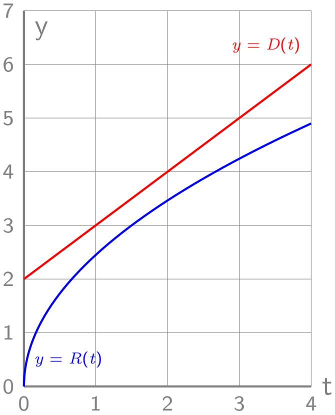


Timeline for the critical oil spill rate

If the oil spill rate is $6\pi \text{m}^3/\text{sec}$, then the oil slick radius at time t is $R(t) = \sqrt{6t}$ m.

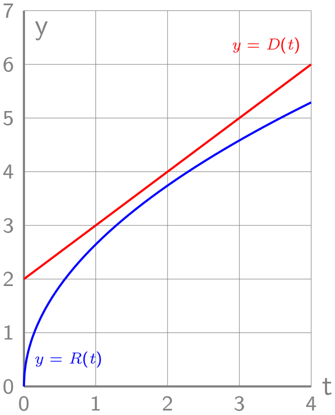
The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.

The oil never touches the dolphin.



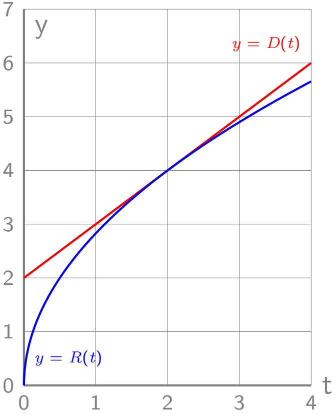
Timeline for the critical oil spill rate

If the oil spill rate is $7\pi \text{m}^3$/sec, then the oil slick radius at time t is $R(t) = \sqrt{7t}$ m.
The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.
The oil never touches the dolphin.



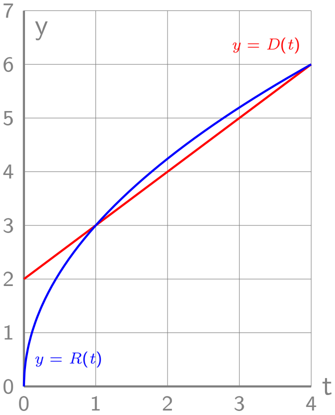
Timeline for the critical oil spill rate

If the oil spill rate is $8\pi\text{m}^3/\text{sec}$, then the oil slick radius at time t is $R(t) = \sqrt{8t}$ m.
The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.
The oil touches the dolphin at exactly one time $t = 2$ seconds.



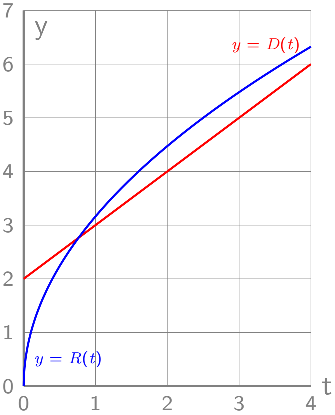
Timeline for the critical oil spill rate

If the oil spill rate is $9\pi\text{m}^3$/sec, then the oil slick radius at time t is $R(t) = \sqrt{9t}$ m.
The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.
The oil covers the dolphin for an interval of time that includes $t = 2$ seconds.



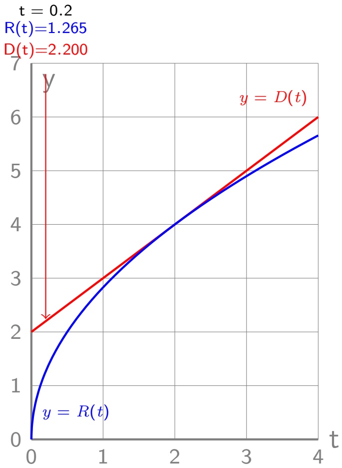
Timeline for the critical oil spill rate

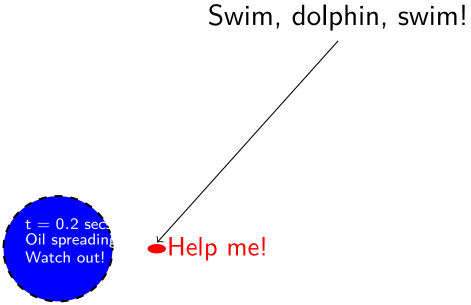
If the oil spill rate is $10\pi\text{m}^3$/sec, then the oil slick radius at time t is $R(t) = \sqrt{10t}$ m.
The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.
The oil covers the dolphin for an interval of time that includes $t = 2$ seconds.



Timeline for the critical oil spill rate

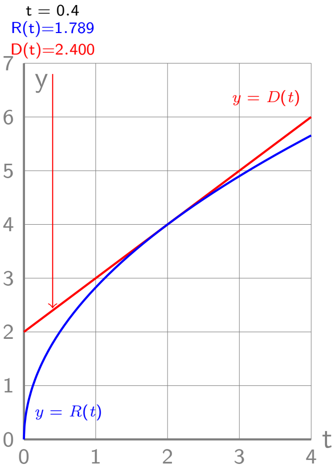
How the oil spill *almost* smothers the dolphin at the critical oil flow rate
$K = 8\pi$ m^3/sec. Oil slick radius at time t is $R(t) = \sqrt{8t}$ meters.
The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.
$R(t) = D(t)$ (the oil slick just touches the dolphin) at only one time $t = 2$ seconds.
The vertical distance between the curves shows how close the oil spill is to the dolphin at time t.

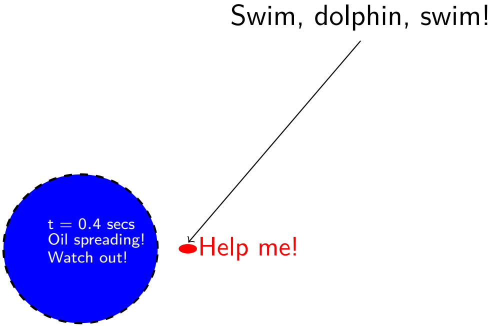




Timeline for the critical oil spill rate

How the oil spill *almost* smothers the dolphin at the critical oil flow rate
$K = 8\pi$ m^3/sec. Oil slick radius at time t is $R(t) = \sqrt{8t}$ meters.
The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.
$R(t) = D(t)$ (the oil slick just touches the dolphin) at only one time $t = 2$ seconds.
The vertical distance between the curves shows how close the oil spill is to the dolphin at time t.

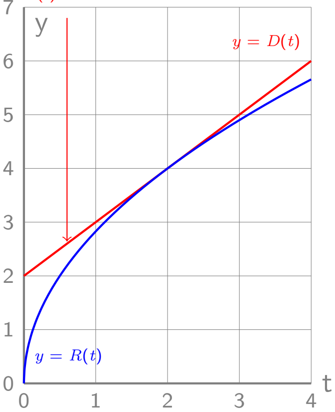


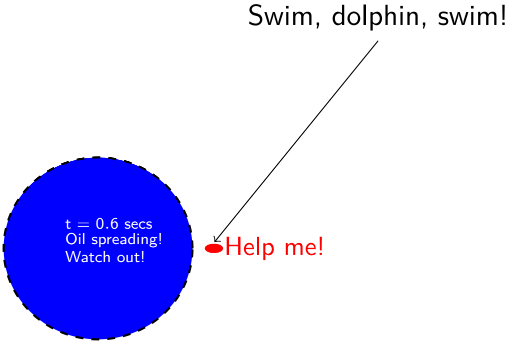


Timeline for the critical oil spill rate

How the oil spill *almost* smothers the dolphin at the critical oil flow rate $K = 8\pi$ m^3/sec. Oil slick radius at time t is $R(t) = \sqrt{8t}$ meters.
The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.
$R(t) = D(t)$ (the oil slick just touches the dolphin) at only one time $t = 2$ seconds.
The vertical distance between the curves shows how close the oil spill is to the dolphin at time t.

t = 0.6
R(t)=2.191
D(t)=2.600





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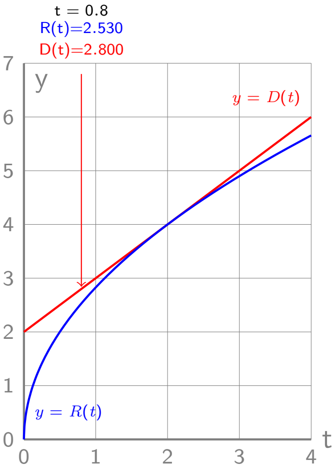

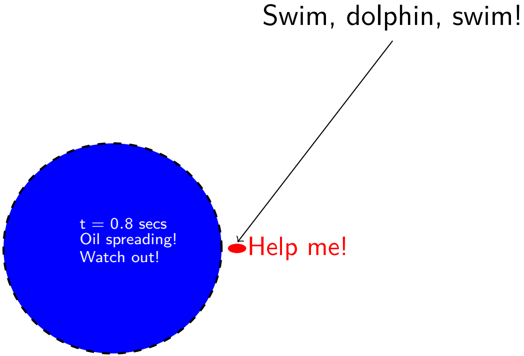

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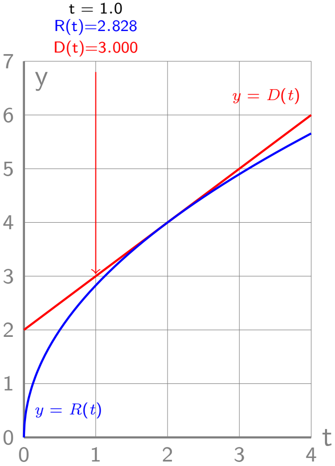

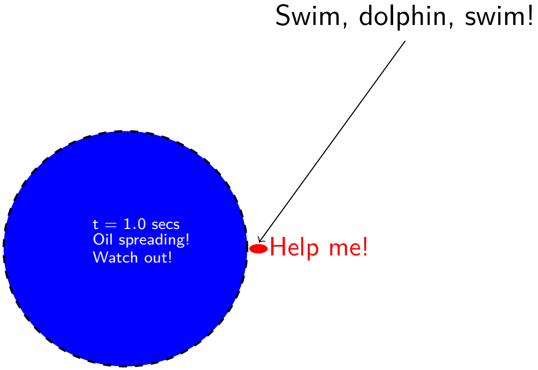

Timeline for the critical oil spill rate

How the oil spill *almost* smothers the dolphin at the critical oil flow rate $K = 8\pi$ m^3/sec. Oil slick radius at time t is $R(t) = \sqrt{8t}$ meters.
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$R(t) = D(t)$ (the oil slick just touches the dolphin) at only one time $t = 2$ seconds.
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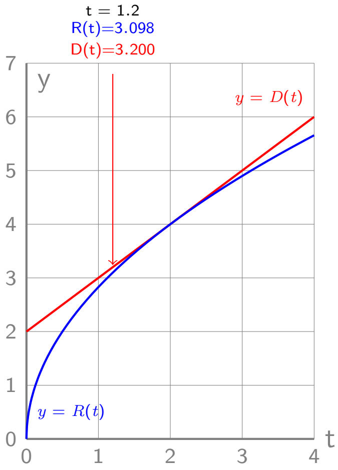

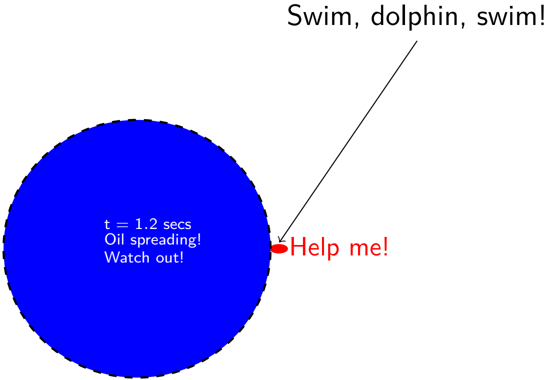

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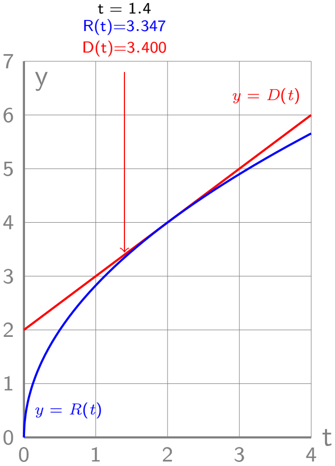

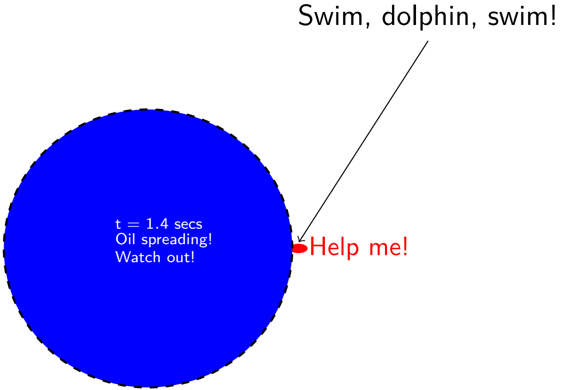

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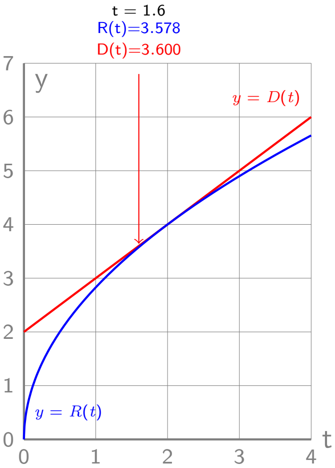
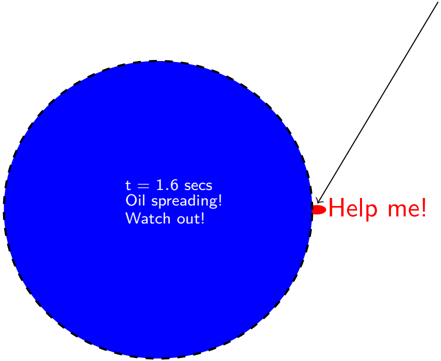


Swim, dolphin, swim!



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The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.
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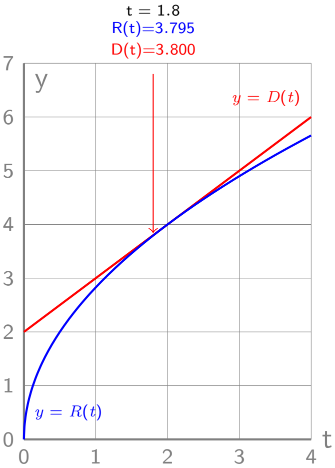

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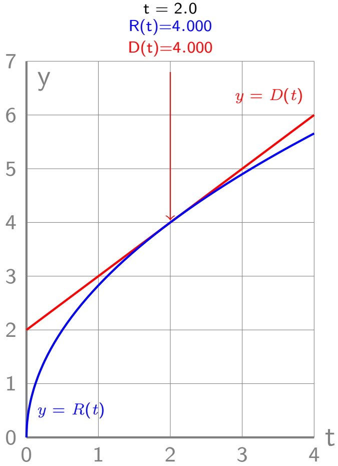

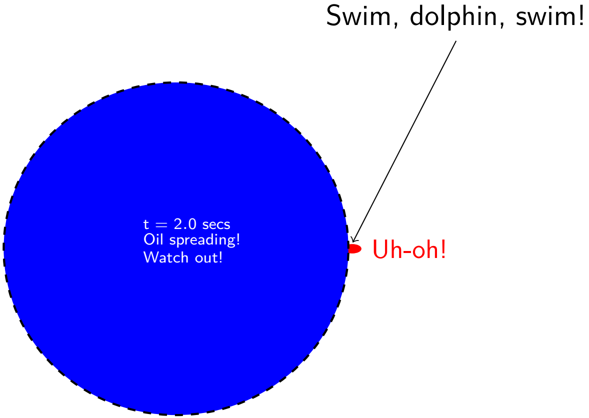

Timeline for the critical oil spill rate

How the oil spill *almost* smothers the dolphin at the critical oil flow rate $K = 8\pi$ m^3/sec. Oil slick radius at time t is $R(t) = \sqrt{8t}$ meters.
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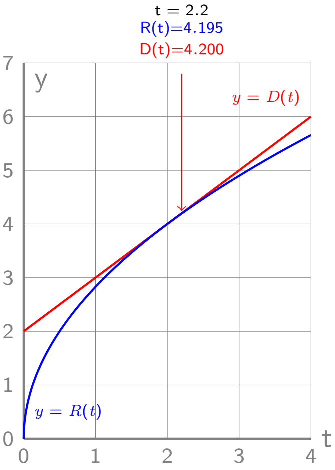

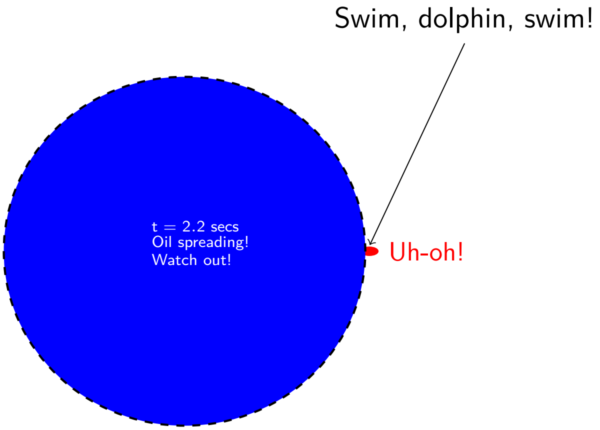

Timeline for the critical oil spill rate

How the oil spill *almost* smothers the dolphin at the critical oil flow rate $K = 8\pi$ m^3/sec. Oil slick radius at time t is $R(t) = \sqrt{8t}$ meters.
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$R(t) = D(t)$ (the oil slick just touches the dolphin) at only one time $t = 2$ seconds.
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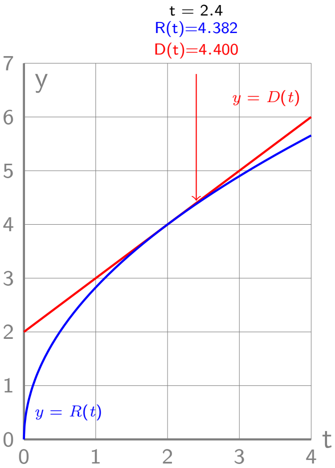

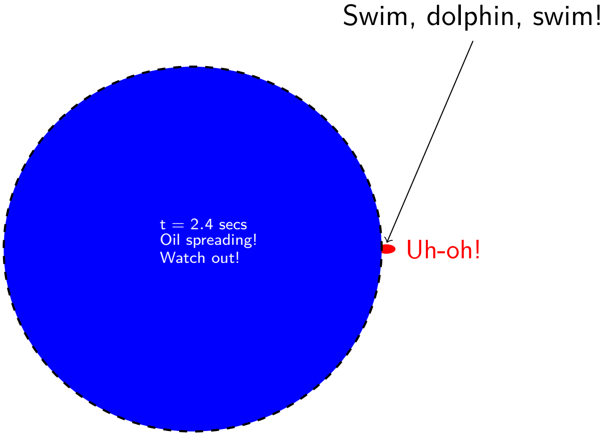

Timeline for the critical oil spill rate

How the oil spill *almost* smothers the dolphin at the critical oil flow rate
$K = 8\pi$ m^3/sec. Oil slick radius at time t is $R(t) = \sqrt{8t}$ meters.
The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.
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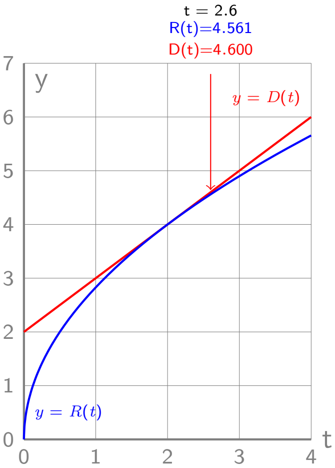
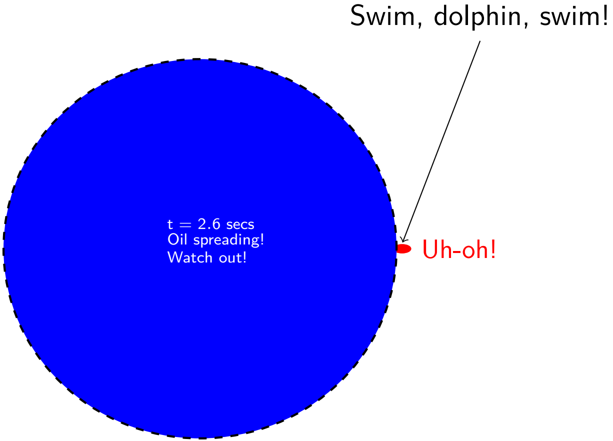
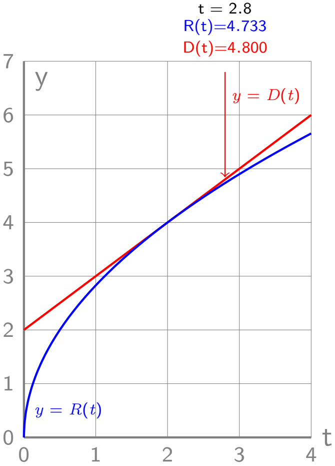
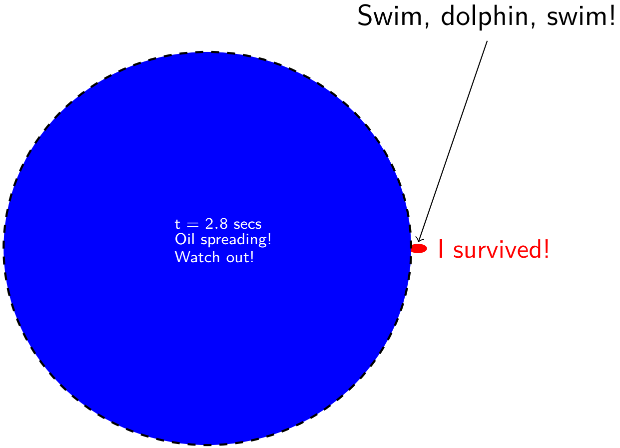




Timeline for the critical oil spill rate

How the oil spill *almost* smothers the dolphin at the critical oil flow rate $K = 8\pi$ m^3/sec. Oil slick radius at time t is $R(t) = \sqrt{8t}$ meters.
The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.
$R(t) = D(t)$ (the oil slick just touches the dolphin) at only one time $t = 2$ seconds.
The vertical distance between the curves shows how close the oil spill is to the dolphin at time t.





Timeline for the critical oil spill rate

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The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.
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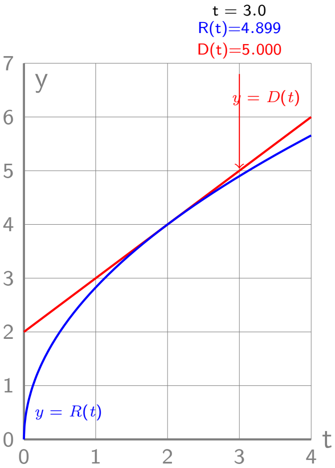

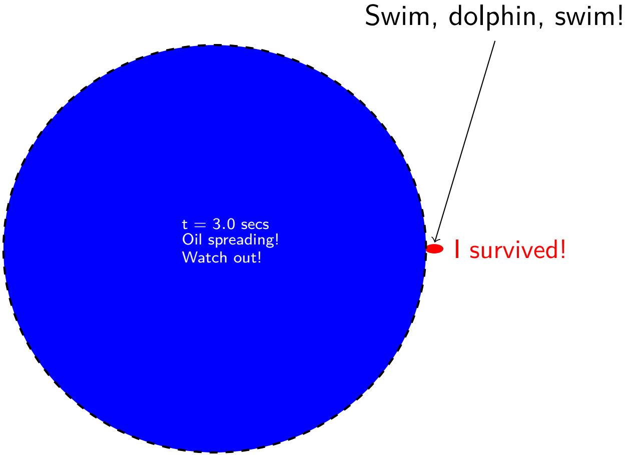

Timeline for the critical oil spill rate

How the oil spill *almost* smothers the dolphin at the critical oil flow rate $K = 8\pi$ m^3/sec. Oil slick radius at time t is $R(t) = \sqrt{8t}$ meters.
The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.
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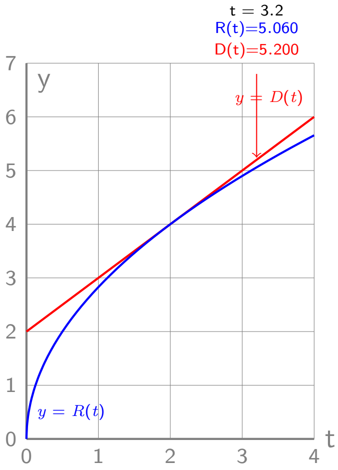

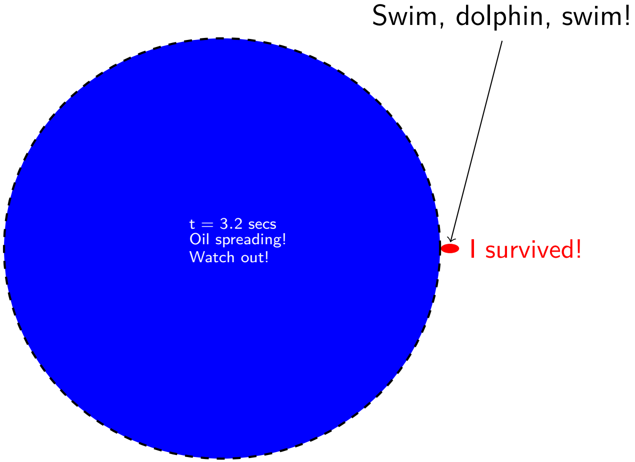

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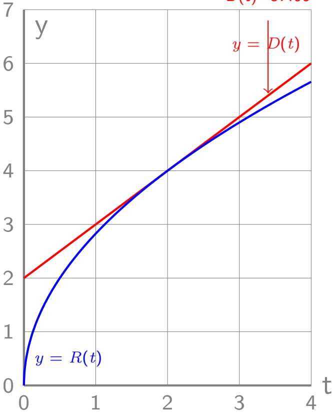
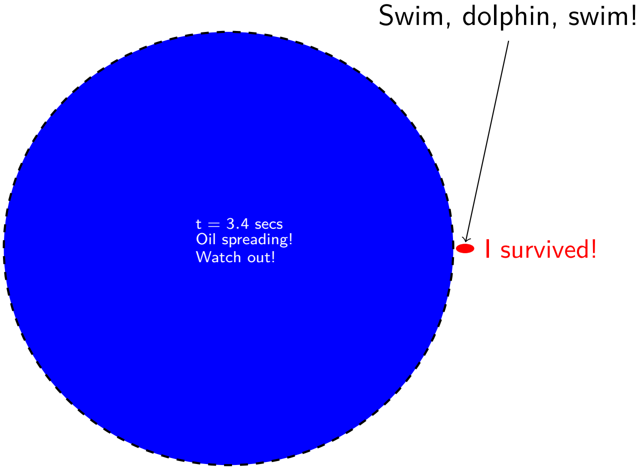
$R(t) = D(t)$ (the oil slick just touches the dolphin) at only one time $t = 2$ seconds.
The vertical distance between the curves shows how close the oil spill is to the dolphin at time t.

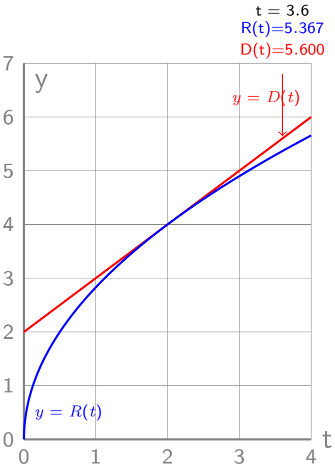
t = 3.4
R(t)=5.215
D(t)=5.400

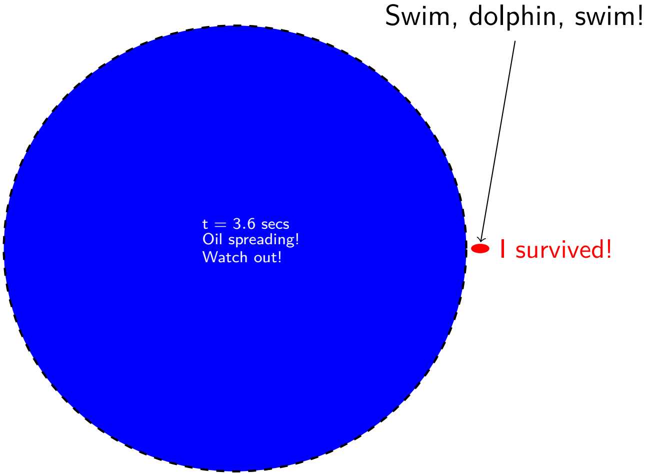




Timeline for the critical oil spill rate

How the oil spill *almost* smothers the dolphin at the critical oil flow rate $K = 8\pi$ m^3/sec. Oil slick radius at time t is $R(t) = \sqrt{8t}$ meters.
The dolphin's distance from the tanker at time t is $D(t) = vt + d = t + 2$ meters.
$R(t) = D(t)$ (the oil slick just touches the dolphin) at only one time $t = 2$ seconds.
The vertical distance between the curves shows how close the oil spill is to the dolphin at time t.





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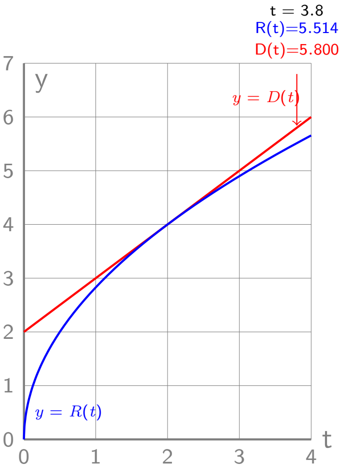

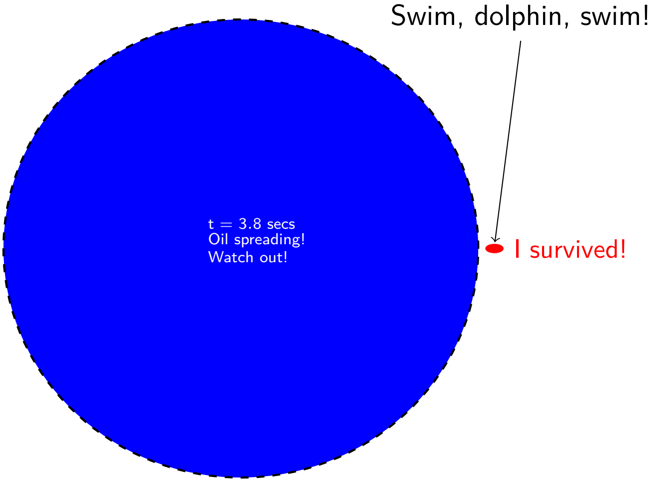

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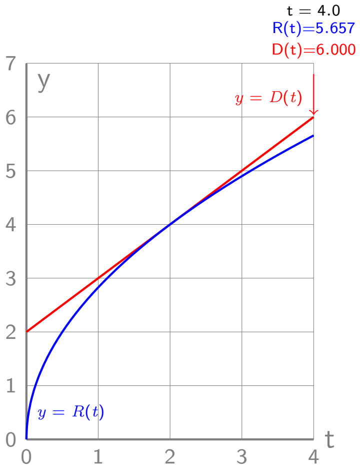
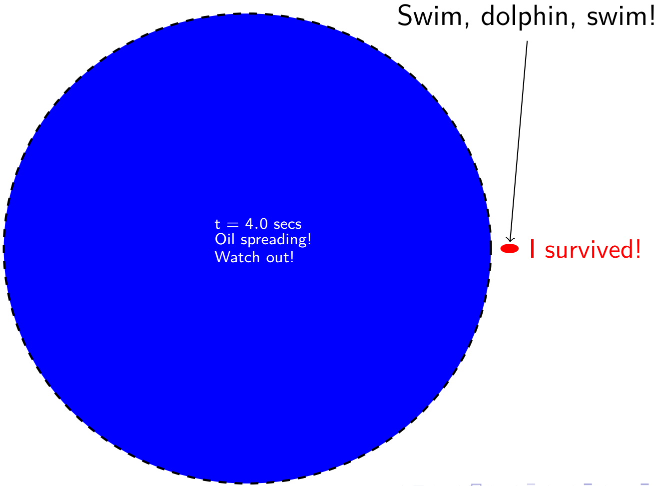




1.4.3 How calculus can save the deer by analyzing a degree 3 polynomial

At high noon, a volcano erupts. A deer standing d m from the (center of the circular base of the) volcano immediately runs away at a speed of v m/sec.

Lava is emitted by the volcano at a rate of $K\text{m}^3/\text{sec}$ and forms an expanding cone whose radius and height stay equal at all times.

- For what minimum value of K will the lava cone overtake the deer?
- When and where will this happen?

Solution:

At time t, the deer's distance from the volcano is $D(t) = vt + d$. The cone's volume is $Kt = \frac{\pi}{3}r^2h = \frac{\pi}{3}r^3$ while its radius is $R(t) = \sqrt[3]{\frac{3Kt}{\pi}}$. The lava cone catches the deer at time t when $D(t) = R(t)$. At that moment

$\sqrt[3]{\frac{3Kt}{\pi}} = vt + d$, i.e. $(vt+d)^3 = \frac{3Kt}{\pi}$

To determine if this equation has a solution for $t > 0$, use calculus as follows. Let $f(t) = R(t)^3 - D(t)^3$. The edge of the lava cone touches the deer when $R(t) = D(t)$, i.e. when $f(t) = 0$.

The lava just manages to overtake the deer if $R(t) < D(t)$ at all other times. Then $f(t)$ has a local maximum and so $f'(t) = 3v(vt+d)^2 - \frac{3K}{\pi} = 0$ as well. Thus $3v(vt+d)^2 = \frac{3K}{\pi}$ and so $vt + d = \pm\alpha$, where we have set $\alpha = \sqrt{\frac{K}{v\pi}}.$ Since $t > 0$, so is $vt + d$ and thus $t = \frac{\alpha - d}{v}$. At this time

$$f(t) = \alpha^3 - \frac{3K}{\pi}\frac{(\alpha - d)}{v} = \alpha^3 - 3\left(\frac{K}{v\pi}\right)(\alpha - d)$$

$$= \alpha^3 - 3\left(\alpha^2\right)(\alpha - d) = -2\alpha^3 + 3\alpha^2 d = \alpha^2(3d - 2\alpha)$$

Set $f(t) = 0$ to find $\alpha = \frac{3d}{2}$. It follows that

- the time to overtake is $t = \frac{\alpha - d}{v} = \frac{d}{2v}$,
- the deer's distance from the volcano's center is $vt + d = \alpha = \frac{3d}{2}$, and $K = \pi v\alpha^2 = \frac{9}{4}\pi v d^2$

Answer:

- The deer will be overtaken if the lava production rate rate is at least $\frac{9}{4}\pi v d^2 \frac{\text{m}^3}{\text{sec}}$. Otherwise the deer will escape.
- When the lava flows at precisely that rate, the deer will be overtaken after $\frac{d}{2v}$ seconds, at distance $\frac{d}{2}$ m. from its starting point.

Until and when the lava reaches the deer, their distance apart at time t is

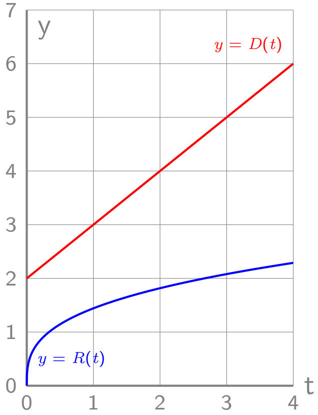
$$D(t) - R(t) = vt + d - \left(\sqrt[3]{\frac{3Kt}{\pi}}\right) = vt + d - 3\left(\sqrt[3]{\frac{vd^2 t}{4}}\right).$$

Example: Suppose the deer starts off $d = 40$ meters from the volcano's center and scurries away at $v = 10\,\frac{\text{m}^3}{\text{sec}}$.

- To overtake the deer, the lava production rate K must be at least $\frac{9}{4}\pi v d^2 = 36,000\pi\,\frac{\text{m}^3}{\text{sec}}$. Set $K = 36,000\pi;\ v = 10;$ and $d = 40$ in the above equation to see that:
- the deer is $D(t) - R(t) = 10t + 40 - 30\sqrt[3]{4t}$ m. away from the advancing lava at time t, and
- the deer gets caught after 2 seconds, when it is 20 m. from its starting point.

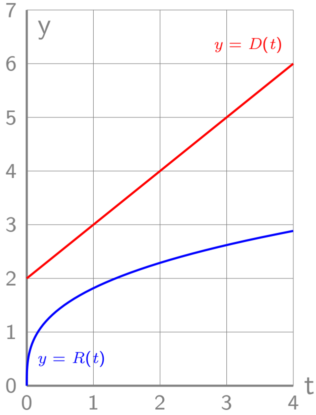
Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

If the lava flow rate is 1π m^3/sec, the lavacone radius at time t is $R(t) = \sqrt[3]{3 \cdot 1t}$ m^3/sec.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
The lava never touches the deer.



Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

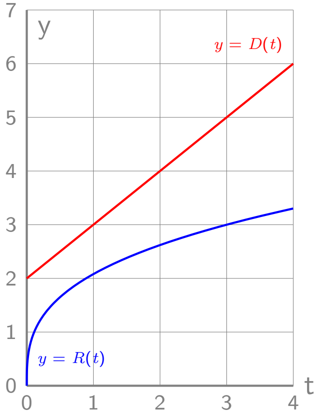
If the lava flow rate is 2π m^3/sec, the lavacone radius at time t is $R(t) = \sqrt[3]{3 \cdot 2t}$ m^3/sec.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
The lava never touches the deer.



Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

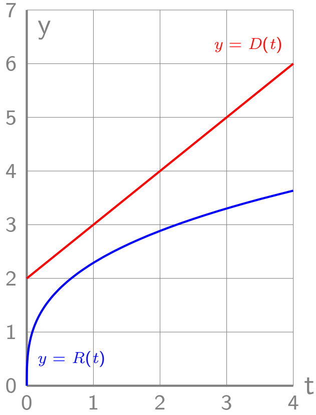
If the lava flow rate is 3π m^3/sec, the lavacone radius at time t is $R(t) = \sqrt[3]{3 \cdot 3t}$ m^3/sec.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.

The lava never touches the deer.



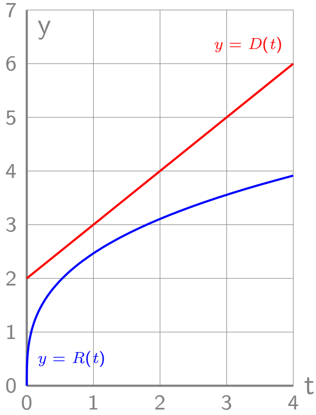
Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

If the lava flow rate is 4π m^3/sec, the lavacone radius at time t is $R(t) = \sqrt[3]{3 \cdot 4t}$ m^3/sec.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
The lava never touches the deer.



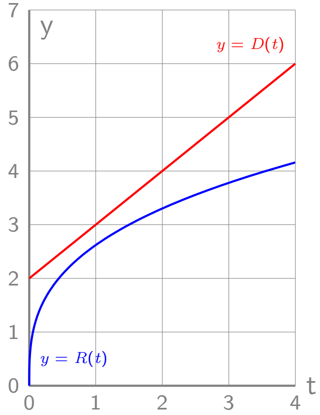
Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

If the lava flow rate is 5π m^3/sec, the lavacone radius at time t is $R(t) = \sqrt[3]{3 \cdot 5t}$ m^3/sec.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
The lava never touches the deer.



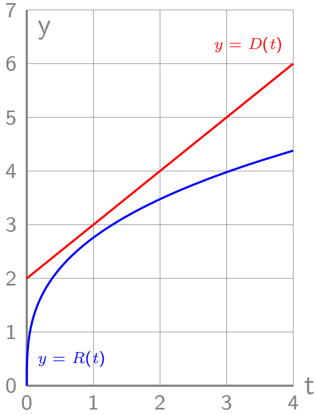
Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

If the lava flow rate is 6π m^3/sec, the lavacone radius at time t is $R(t) = \sqrt[3]{3 \cdot 6t}$ m^3/sec.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
The lava never touches the deer.



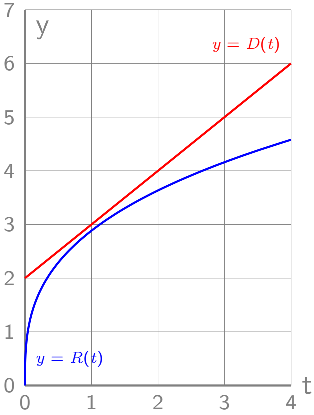
Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

If the lava flow rate is 7π m^3/sec, the lavacone radius at time t is $R(t) = \sqrt[3]{3 \cdot 7t}$ m^3/sec.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
The lava never touches the deer.



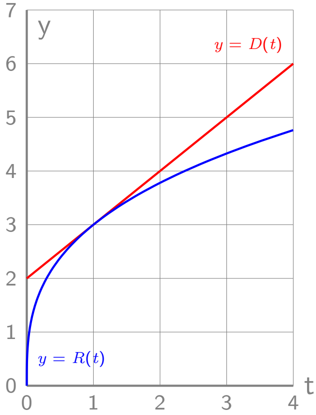
Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

If the lava flow rate is 8π m^3/sec, the lavacone radius at time t is $R(t) = \sqrt[3]{3 \cdot 8t}$ m^3/sec.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
The lava never touches the deer.



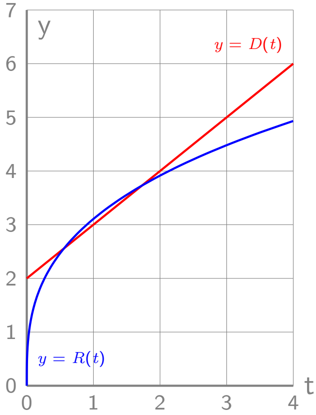
Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

If the lava flow rate is 9π m^3/sec, the lavacone radius at time t is $R(t) = \sqrt[3]{3 \cdot 9t}$ m^3/sec.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
The lava touches the deer at exactly one time $t = 1$ second.



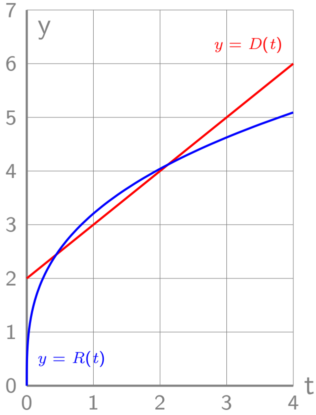
Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

If the lava flow rate is 10π m^3/sec, the lavacone radius at time t is $R(t) = \sqrt[3]{3 \cdot 10t}$ m^3/sec.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
The lava reaches and goes past the deer for $.55 \leq t \leq 1.73$.



Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

If the lava flow rate is 11π m^3/sec, the lavacone radius at time t is $R(t) = \sqrt[3]{3 \cdot 11t}$ m^3/sec.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
The lava reaches and goes past the deer for $.44 \leq t \leq 2.12$.

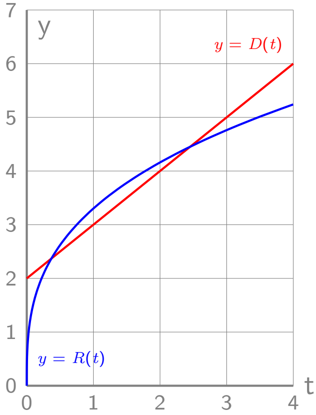


Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

If the lava flow rate is 12π m^3/sec, the lavacone radius at time t is $R(t) = \sqrt[3]{3 \cdot 12t}$ m^3/sec.

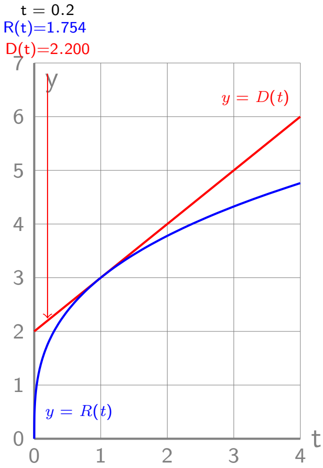
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.

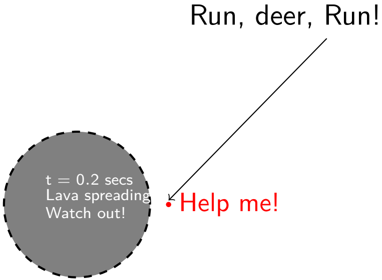
The lava reaches and goes past the deer for $.37 \leq t \leq 2.45$.



Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

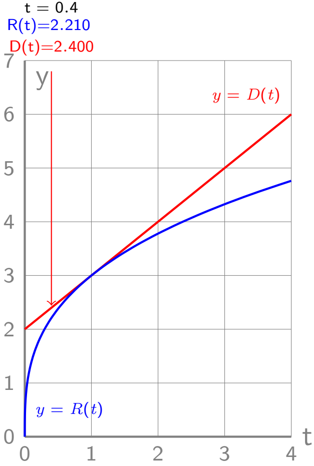
Critical flow rate is $K = 9\pi$m^3/sec. Lava cone radius at time t is $R(t) = 3\sqrt[3]{t}$ meters.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
$R(t) = D(t)$ (the lava cone just touches the deer) only one time, at $t = 1$ second.
The vertical distance between the curves shows how close the lava is to the deer at time t.

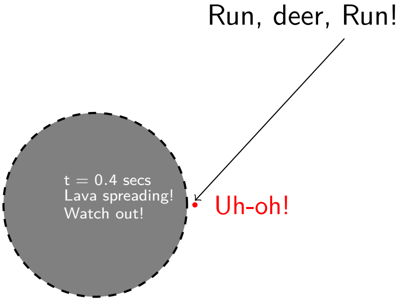




Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

Critical flow rate is $K = 9\pi$m^3/sec. Lava cone radius at time t is $R(t) = 3\sqrt[3]{t}$ meters.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
$R(t) = D(t)$ (the lava cone just touches the deer) only one time, at $t = 1$ second.
The vertical distance between the curves shows how close the lava is to the deer at time t.





Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

Critical flow rate is $K = 9\pi \text{m}^3/\text{sec}$. Lava cone radius at time t is $R(t) = 3\sqrt[3]{t}$ meters.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
$R(t) = D(t)$ (the lava cone just touches the deer) only one time, at $t = 1$ second.
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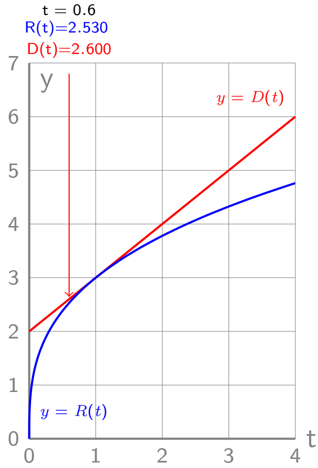

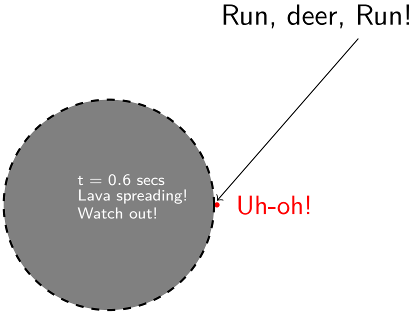

Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

Critical flow rate is $K = 9\pi$m^3/sec. Lava cone radius at time t is $R(t) = 3\sqrt[3]{t}$ meters.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
$R(t) = D(t)$ (the lava cone just touches the deer) only one time, at $t = 1$ second.
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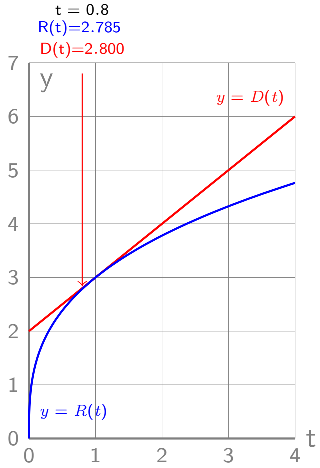
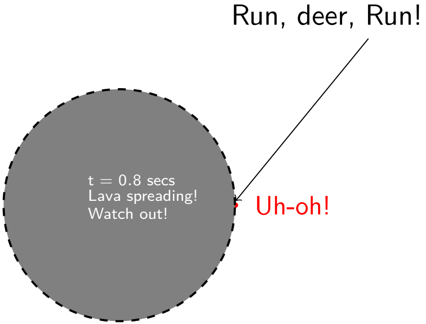
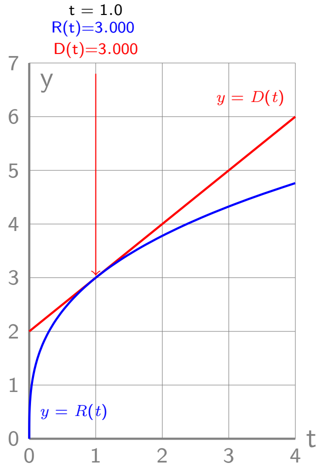
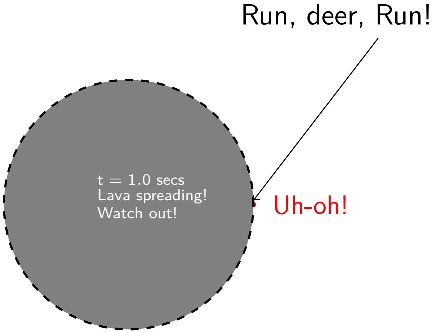




Animation for a range of lava flow rates: $\pi \le K \le 12\pi$ m^3/sec

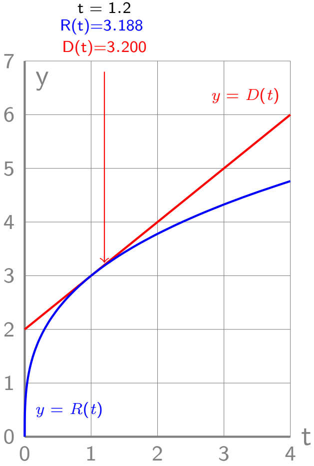
Critical flow rate is $K = 9\pi\text{m}^3$/sec. Lava cone radius at time t is $R(t) = 3\sqrt[3]{t}$ meters.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
$R(t) = D(t)$ (the lava cone just touches the deer) only one time, at $t = 1$ second.
The vertical distance between the curves shows how close the lava is to the deer at time t.

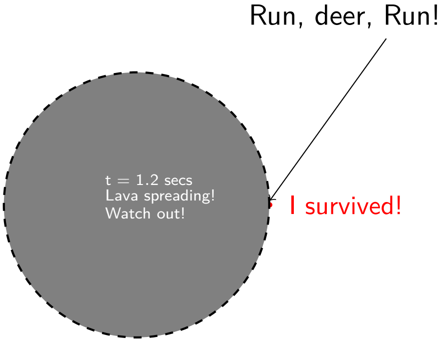




Animation for a range of lava flow rates: $\pi \le K \le 12\pi$ m^3/sec

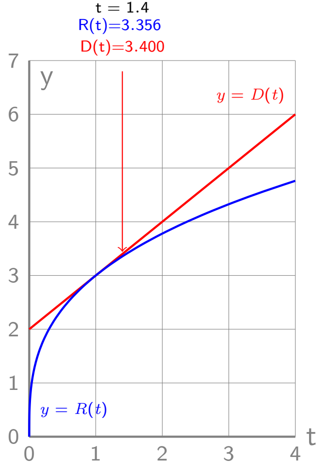
Critical flow rate is $K = 9\pi$m^3/sec. Lava cone radius at time t is $R(t) = 3\sqrt[3]{t}$ meters.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
$R(t) = D(t)$ (the lava cone just touches the deer) only one time, at $t = 1$ second.
The vertical distance between the curves shows how close the lava is to the deer at time t.

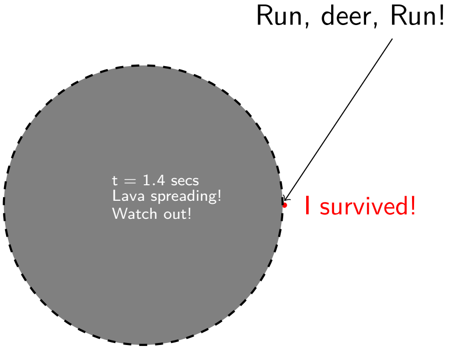




Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

Critical flow rate is $K = 9\pi \text{m}^3$/sec. Lava cone radius at time t is $R(t) = 3\sqrt[3]{t}$ meters.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
$R(t) = D(t)$ (the lava cone just touches the deer) only one time, at $t = 1$ second.
The vertical distance between the curves shows how close the lava is to the deer at time t.





Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

Critical flow rate is $K = 9\pi \text{m}^3/\text{sec}$. Lava cone radius at time t is $R(t) = 3\sqrt[3]{t}$ meters.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
$R(t) = D(t)$ (the lava cone just touches the deer) only one time, at $t = 1$ second.
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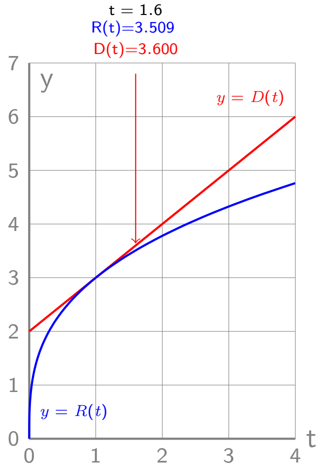

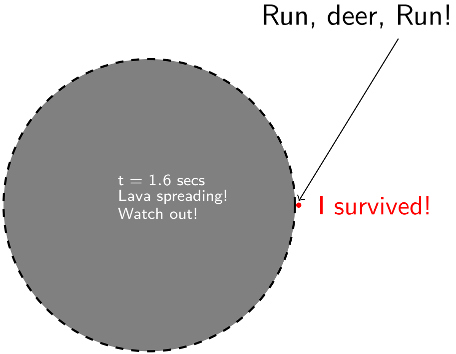

Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

Critical flow rate is $K = 9\pi$m^3/sec. Lava cone radius at time t is $R(t) = 3\sqrt[3]{t}$ meters.
The deer's distance from the volcano at time t is $D(t) = vt + d = t + 2$ meters.
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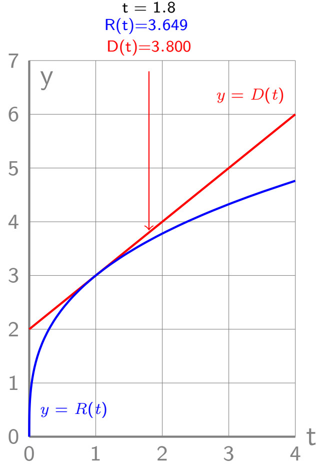

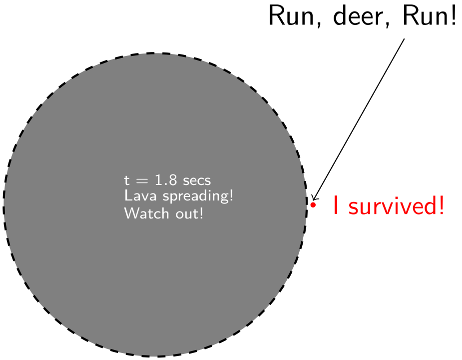

Animation for a range of lava flow rates: $\pi \leq K \leq 12\pi$ m^3/sec

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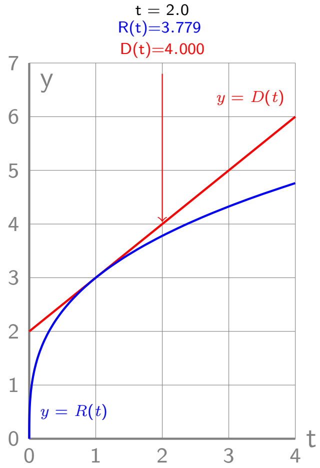

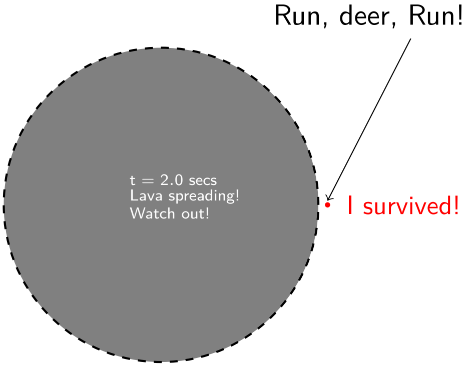

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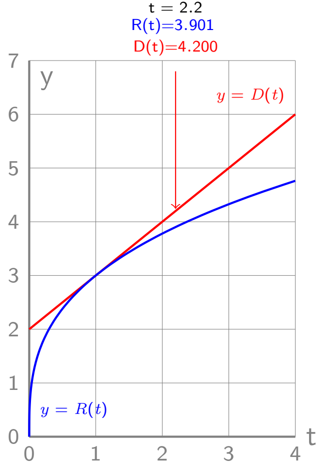

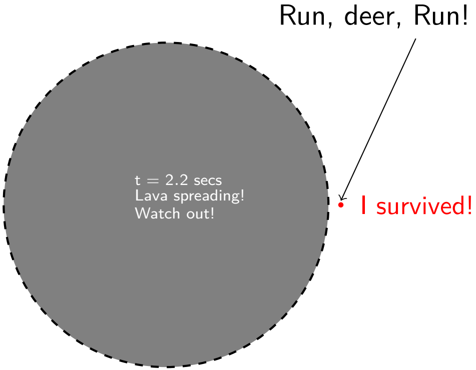

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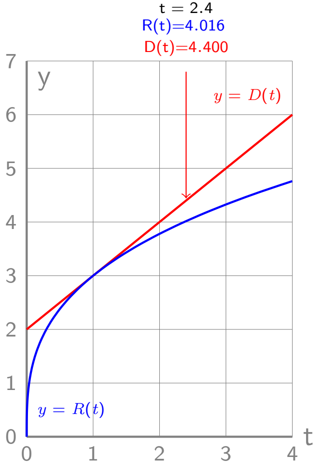

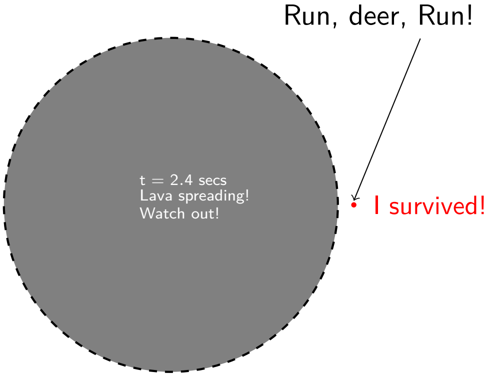

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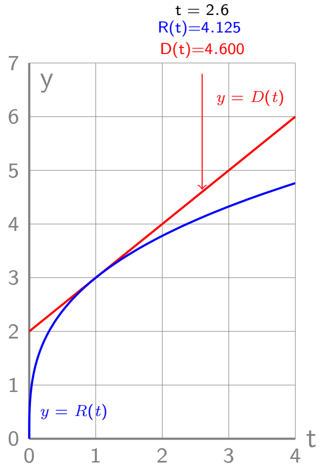

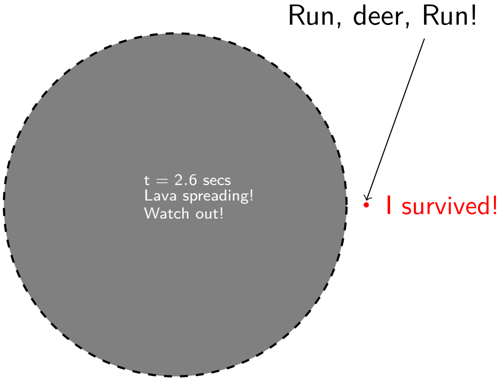

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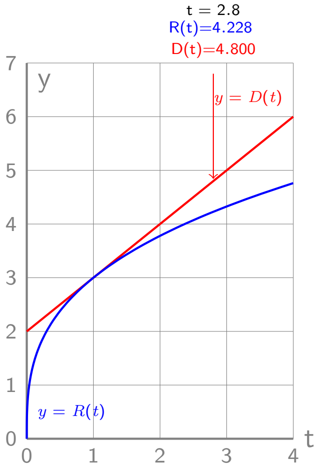

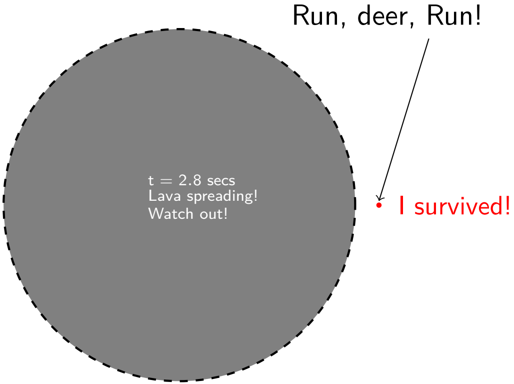

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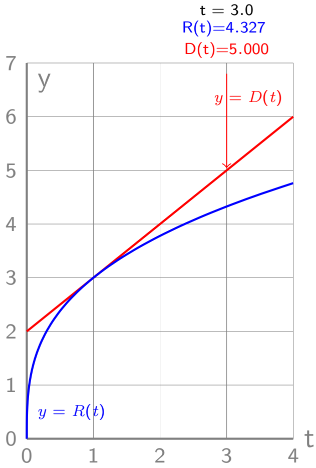

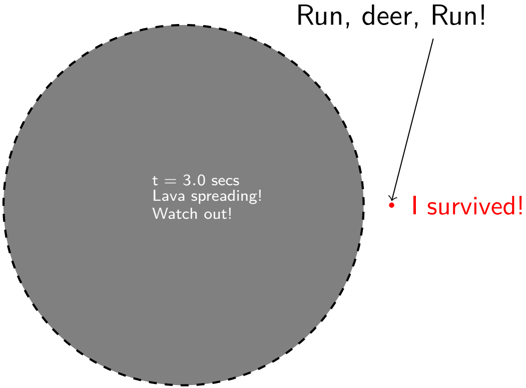

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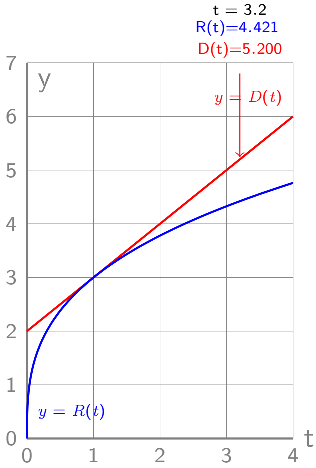

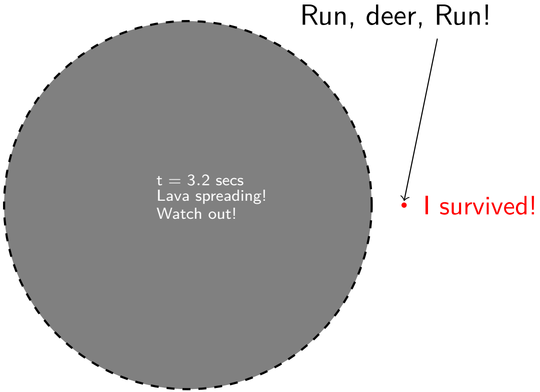

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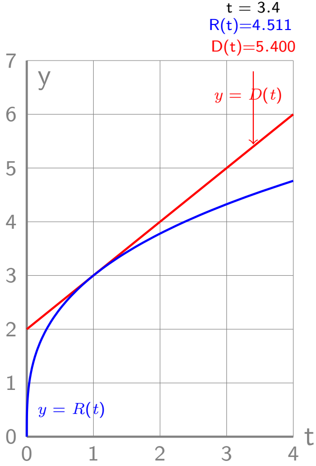

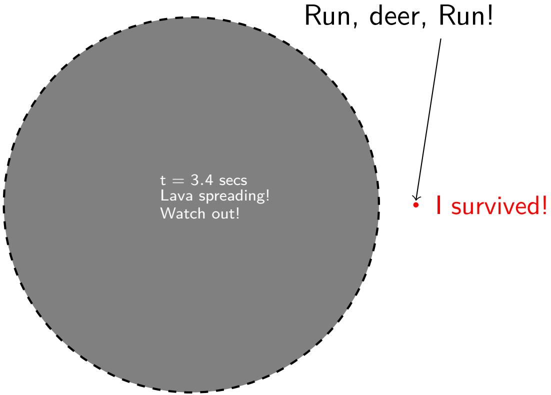

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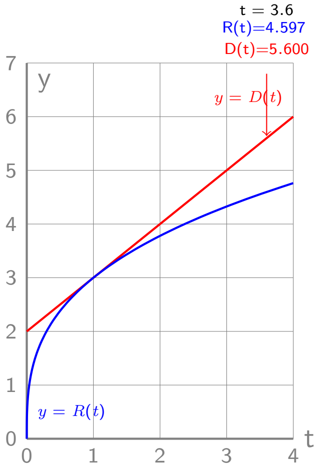

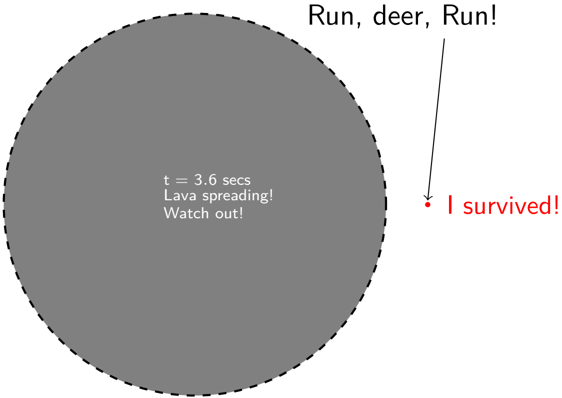

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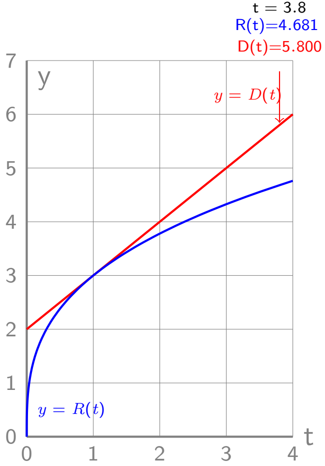

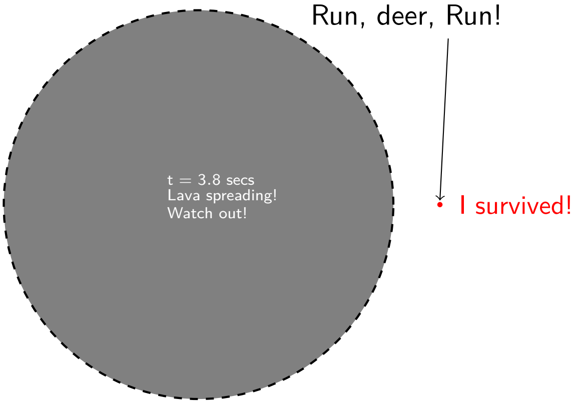

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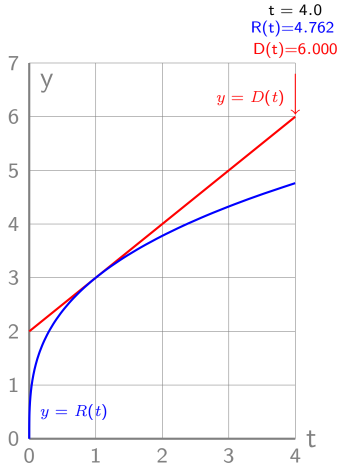

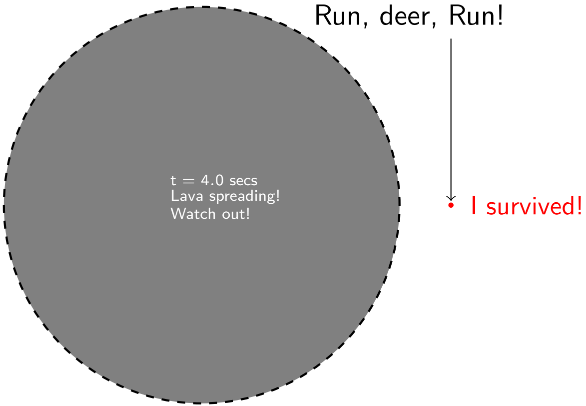

1.4.4 Section 1.4 Exercises

Can you think of any nonstandard modeling projects in engineering?

Section 1.4 Review: Modeling real-life problems

Can you think of any nonstandard modeling projects in engineering?

- 1.5.1: Polynomial equations
- 1.5.2: Quadratic equations
- 1.5.3: Rational equations
- 1.5.4: Equations with radicals
- 1.5.5: Complex numbers solve quadratic equations
- 1.5.6: Hidden quadratic equations
- Section 1.5 Review

Section 1.5 Preview: Definitions and Theorems

- Definition 1.5.1: Zero Product Rule (ZPR)
- Definition 1.5.2: The roots of a polynomial P in x are
- Definition 1.5.3: A Quadratic Equation in x is
- Definition 1.5.4: A degree 2 (quadratic) polynomial is
- Definition 1.5.5: Factor, cofactor, prime number, prime power factorization
- Definition 1.5.6: The degree of a polynomial equation $E = F$ is
- Definition 1.5.7: The Greatest Common Factor of a set of integers is
- Definition 1.5.8: The Greatest Monomial Factor (GMF) of a polynomial is
- Definition 1.5.9: The GMF of a polynomial in x with nonzero constant term is
- Definition 1.5.10: The GMF of a polynomial with no constant term is
- Definition 1.5.11: Some special factoring rules
- Definition 1.5.12: An equation is rational if
- Definition 1.5.13: Definition of complex numbers
- Definition 1.5.14: The quadratic equation $ax^2 + bx + c = 0$ has two complex solutions if

Section 1.5 Preview: Procedures

- Procedure 1.5.1: To solve $x^n = K$ with $n \geq 2$ an integer
- Procedure 1.5.2: To solve other equations $P(x) = Q(x)$
- Procedure 1.5.3: Never solve an equation by trial and error.
- Procedure 1.5.4: To solve $ax^2 + c = 0$
- Procedure 1.5.5: To solve the quadratic equation $ax^2 + bx + c = 0$
- Procedure 1.5.6: To factor the quadratic polynomial $P = ax^2 + bx + c$
- Procedure 1.5.7: To find the prime factorization of an integer $N > 0$.
- Procedure 1.5.8: To find the GCF of integers
- Procedure 1.5.9: To factor a polynomial in one letter
- Procedure 1.5.10: To solve $(x + k)^2 = K$
- Procedure 1.5.11: To complete the square with $x^2 + Bx = K$
- Procedure 1.5.12: To solve $ax^2 + bx + c = 0$ by completing the square
- Procedure 1.5.13: To solve a rational equation
- Procedure 1.5.14: To solve an equation with radicals
- Procedure 1.5.15: To do arithmetic with complex numbers
- Procedure 1.5.16: To recognize hidden quadratic equations

Using the Zero Product Rule to solve equations

In Section 1.1.7 we showed how to solve linear (degree 1) equations. Now we discuss degree 2 or higher equations.

To solve $x^n = K$ with $n \geq 2$ an integer:

- $x = \sqrt[n]{K}$ if n is odd.
- $x = \pm\sqrt[n]{K}$ if n is even and and $K \geq 0$.
- There is no real solution if n is even and $K < 0$.

Examples of equations that can be rewritten as $x^2 = K$:

Example 1: Solve $x^2 = 4$.
Solution: Rewriting as $x^2 - 4 = 0$ and factoring works, but wastes time. The solution of $x^2 = 4$ is $x = \pm\sqrt{4} = \pm 2$.

$\boxed{x = -2;\ x = 2}$

Be careful: If you say the solution of $x^2 = 4$ is $x = 2$, you deserve zero partial credit.

Example 2: Solve $x^2 = -4$.
Solution: Don't bother to rewrite as $x^2 + 4 = 0$. There is no real number whose square is -4.

No real solutions

Later we'll see that $x^2 = -4$ has solutions $x = \pm 2i$ that are complex numbers.

Example 3: Solve $(x + k)^2 = K$. Assume $K \geq 0$.
Solution: Use the idea of the above example to see that $x + k = \pm\sqrt{K}$ and so

$\boxed{x = -k \pm \sqrt{K}}$

A basic tool for solving higher degree equations is the

Zero Product Rule (ZPR)

Let A and B be algebra expressions or numbers.
If $AB = 0$, then $A = 0$ or $B = 0$.

To solve an equation $P(x) = Q(x)$ that can't be rewritten as $x^n = K$,

- Rewrite $P(x) = Q(x)$ as $P(x) - Q(x) = 0$
- Factor $P(x) - Q(x)$.
- Set each factor to zero and solve for x. By ZPR, these solutions together are all solutions of $P(x) = Q(x)$

Using this result requires factoring polynomials, as covered in the next section. For an overview, here is a list of examples, many of which are familiar to you.

Example 4: Solve each equation by using ZPR.

- Solve $x^2 + 3x + 2 = 0$
  Factor: $(x + 2)(x + 1) = 0$
  Zero Product Rule $x + 2 = 0$ or $x + 1 = 0$
  Solve each part $\boxed{x = -2 \text{ or } x = -1}$

- Solve $x = x^3$
  One side: $x^3 - x = 0$
  Factor: $x(x^2 - 1) = x(x - 1)(x + 1) = 0$
  Zero Product Rule $x = 0$ or $x - 1 = 0$ or $x + 1 = 0$
  Solve each part $\boxed{x = 0 \text{ or } x = 1 \text{ or } x = -1}$

- Solve $x^6 - x^4 = 0$
  Factor: $x^4(x^2 - 1) = x^4(x + 1)(x - 1) = 0$
  Zero Product Rule $x^4 = 0$ or $x - 1 = 0$ or $x + 1 = 0$
  Solve each part $\boxed{x = 0 \text{ or } x = -1 \text{ or } x = 1}$

- Solve $4x^6 = 12x^5 - 12x^4$
  One side $4x^6 - 12x^5 - 12x^4 = 0$
  Factor: $4x^4(x^2 - 3x - 3) = 0$
  Zero Product Rule $4x^4 = 0$ or $x^2 - 3x - 3 = 0$
  Solve each part $x = 0; x^2 - 3x - 3 = 0$
  This requires the Quadratic formula. Read on!

- Solve $x = 7x^2$.
  Rewrite: $7x^2 - x = 0$
  Factor: $x(7x - 1) = 0$
  Set each factor to 0:
  $x = 0; \quad 7x - 1 = 0$ $\boxed{x = 0; x = \frac{1}{7}}$

- Solve $x^4 + 3x^2 = 4x^3$.
  Rewrite as $x^4 - 4x^3 + 3x^2 = 0$
  Factor: $x^2(x^2 - 4x + 3) = x^2(x - 3)(x - 1) = 0$.
  Set each factor to 0:
  $x^2 = 0$ if $x = 0$;
  $x - 3 = 0$ if $x = 3$;
  $x - 1 = 0$ if $x = 1$. $\boxed{x = 0;\ x = 1;\ x = 3}$

A standard trick question: guessing $x = 1$ misses the second solution.

Example 5: Solve $x^2 = x$.
Rewrite: $x^2 - x = 0$
Factor: $x(x-1) = 0$
Set each factor to 0 : $x = 0; x - 1 = 0$ $\boxed{x = 0; x = 1}$

Never solve an equation by trial and error.

- Don't solve $x^2 = x$ by writing $x = \pm\sqrt{x}$. The solutions must be numbers.
- Don't solve $x^2 = x$ by dividing both sides by x. If you do, you will miss the solution $x = 0$.
- if you say that the solution of $x^2 = x$ is $x = 1$, you deserve zero partial credit OR
- if you say that the solution is $x = 0$ you deserve zero partial credit.

  Reason: *Solve* means find *all* solutions.
- Even if you guessed both $x = 0$ and $x = 1$, you haven't shown that you found all solutions. However
- Using the ZPR proves that $x = 0; x = 1$ are all of the solutions.

Example 6: Solve $x^4 - 20 = 0$.
Rewrite as: $x^4 = 20$
$n = 4$ is even; $K = 20$ is positive
Therefore $x = \pm\sqrt[n]{K}$: $\boxed{x = \pm\sqrt[4]{20}}$

Example 7: Solve $x^4 - 25 = 0$.
Method 1:
Rewrite as : $x^4 = 25$, so $\boxed{x = \pm\sqrt[4]{25}}$

Method 2:
Factor: $x^4 - 25 = (x^2 + 5)(x^2 - 5) = 0$.
Set each factor to 0
$x^2 + 5 = 0$ gives $x^2 = -5$, no real solution.
$x^2 - 5 = 0$ gives $x^2 = 5$, so $\boxed{x = \pm\sqrt{5}}$

To check that the two methods give the same answer, simplify $\sqrt[4]{25}$ by factoring 25.

Definition: The roots of a polynomial P in x

are the solutions for x of the equation $P = 0$.

The Quadratic Formula

Definition: If a, b, c are real numbers and $a \neq 0$, then $ax^2 + bx + c = 0$ is a Quadratic Equation in x.

Its solutions are given by the

Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Since a polynomial $ax^2 + bx + c$ with integer coefficients doesn't factor if $D = b^2 - 4ac$ is not a perfect square, we need a different tool to find its roots.

Start with the easy case $b = 0$. Then $ax^2 + c = 0$ so $x^2 = -\dfrac{a}{c}$, which we will call K, which will be negative if $\frac{a}{c} > 0$.

In this case, we need to use the imaginary number i, which satisfies $i^2 = -1$.

How to solve $ax^2 + c = 0$

If $a \neq 0$ there are two

- real solutions $x = \pm\sqrt{\frac{-a}{c}}$ If $\dfrac{a}{c} < 0$
- complex solutions $x = \pm\sqrt{\frac{a}{c}}\,i$ if $\dfrac{a}{c} > 0$

To solve the quadratic equation $ax^2 + bx + c = 0$

- $D = b^2 - 4ac$ is the equation's *discriminant*
- If $D > 0$, there are two different real solutions $x = r_1 = \dfrac{-b + \sqrt{D}}{2a}$ and $x = r_2 = \dfrac{-b - \sqrt{D}}{2a}$, usually abbreviated $x = \dfrac{-b \pm \sqrt{D}}{2a}$.
- If $D = 0$, there is one real solution $x = r_1 = r_2 = \dfrac{-b}{2a}$.
- If $D < 0$, there are no *real* solutions. The two complex solutions are $x = \dfrac{-b \pm \sqrt{-D}\; i}{2a}$.
- In all cases: $ax^2 + bx + c$ factors as $a(x - r_1)(x - r_2)$

Example 8: Solve each equation:

- $2x^2 + 5x + 1 = 0$:

$D = (5)^2 - 4 \cdot 2 \cdot 1 = 23$

$x = \frac{-5 \pm \sqrt{23}}{4}$

- $2x^2 + 2x + 1 = 0$: $D = (2)^2 - 4 \cdot 2 \cdot 1 = -4$

$x = \frac{-2 \pm \sqrt{-4}}{4} = \frac{-2 \pm 2i}{4} =$ $x = -\frac{1}{2} \pm \frac{i}{2}$

Using the Quadratic Formula to factor degree 2 polynomials

Reminder of earlier definitions:

If $a \neq 0$, the expression $ax^2 + bx + c$ is

- a degree 2 polynomial if a, b, c are any real numbers;
- a quadratic polynomial if a, b, c are integers.

Here is a method for factoring quadratic polynomials that is totally automatic and avoids trial and error.

To factor the quadratic polynomial $P = ax^2 + bx + c$

- Use the Quadratic Formula to find the roots r_1 and r_2 of $ax^2 + bx + c = 0$.
- Then $P = a(x - r_1)(x - r_2)$.
- If a, b, c are integers and $D = b^2 - 4ac$ is a perfect square, rewrite the previous formula by factoring a so as to cancel denominators in r_1 and r_2. This rewrites P as a product of factors with integer coefficients.

Example 9: Factor $x^2 + bx + c = 6x^2 + 13x + 6$.
Solution: $D = b^2 - 4ac = 25$ is a perfect square. Thus $ax^2 + bx + c = a(x - r_1)(x - r_2)$ where
$r_1 = \frac{-13+\sqrt{25}}{12} = \frac{-13+5}{12} = -\frac{2}{3}$;
$r_2 = \frac{-13-\sqrt{25}}{12} = \frac{-13-5}{12} = -\frac{3}{2}$. Therefore
$6x^2 + 13x + 6 = a\left(x - r_1\right)\left(x - r_2\right) =$
$6\left(x + \frac{2}{3}\right)\left(x + \frac{3}{2}\right)$
$= 3\left(x + \frac{2}{3}\right) \cdot 2\left(x + \frac{3}{2}\right) = \boxed{(3x + 2)(2x + 3)}$

Example 10 : Factor $12x^2 + 44x + 35$.
Here $ax^2 + bx + c = 12x^2 + 44x + 35$, so $D = b^2 - 4ac = 44^2 - 12 \cdot 35 = 256 = 16^2$ is a perfect square. Thus $\sqrt{D} = 16$ and so $12x^2 + 44x + 35$ can be factored. Now $ax^2 + bx + c = a(x - r_1)(x - r_2)$ where
$r_1 = \frac{-44+16}{24} = -\frac{7}{6}$; and $r_2 = \frac{-44-16}{24} = -\frac{5}{2}$.
So $12x^2 + 44x + 35$
$= a\left(x - r_1\right)\left(x - r_2\right)$
$= 12\left(x + \frac{7}{6}\right)\left(x + \frac{5}{2}\right)$
$= 6\left(x + \frac{7}{6}\right) \cdot 2\left(x + \frac{5}{2}\right) = \boxed{(6x + 7)(2x + 5)}$

For comparison, solve this equation by using the trial and error AC method in Example 1.1.19:

Exercise: Factor each polynomial below, or explain why it does not factor by writing
“D = ... is not a PS” (perfect square).

a) $x^2 + 25x + 100$
b) $x^2 - 25$
c) $x^2 - 26$
d) $2x^2 + 3x + 1$
e) $2x^2 + 7x + 4$
f) $4x^2 + 100x + 400$
g) $5x^2 + 3x - 11$
h) $10x^2 + 19x + 6$
i) $x^2 - 10x + 16$
j) $18t^2 + 60t + 48$
k) $10t^2 - 90$
l) $-12t^2 - 48t - 48$
m) $6t^2 - 57t - 30$
n) $12k^2 + 26k + 12$
o) $25x - x^3$
p) $2x^4 + 10x^3 + 12x^2$
q) $x^5 + 2x^4 + x^3$

Please do all the problems before you check your answers ⇒.

Exercise: Factor each polynomial below, or explain why it does not factor by writing
"D = ... is not a PS" (perfect square).

a) $x^2 + 25x + 100$
b) $x^2 - 25$
c) $x^2 - 26$
d) $2x^2 + 3x + 1$
e) $2x^2 + 7x + 4$
f) $4x^2 + 100x + 400$
g) $5x^2 + 3x - 11$
h) $10x^2 + 19x + 6$
i) $x^2 - 10x + 16$
j) $18t^2 + 60t + 48$
k) $10t^2 - 90$
l) $-12t^2 - 48t - 48$
m) $6t^2 - 57t - 30$
n) $12k^2 + 26k + 12$
o) $25x - x^3$
p) $2x^4 + 10x^3 + 12x^2$
q) $x^5 + 2x^4 + x^3$

Please do all the problems before you check your answers ⇒.

a) $(x + 5)(x + 20)$
b) $(x + 5)(x - 5)$
c) $D = 104$ is not a PS
d) $(2x + 1)(x + 1)$
e) $D = 17$ is not a PS
f) $4(x + 5)(x + 20)$
g) $D = 229$ is not a PS
h) $(2x + 3)(5x + 2)$
i) $(x - 2)(x - 8)$
j) $6(t + 2)(3t + 4)$
k) $10(t + 3)(t - 3)$
l) $-12(t + 2)(t + 2)$
m) $3(t - 10)(2t + 1)$
n) $2(2k + 3)(3k + 2)$
o) $x(5 + x)(5 - x)$
p) $2x^2(x + 3)(x + 2)$
q) $x^3(x + 1)^2$

Using factoring to solve higher degree equations

In Section 1.5.1, we noted that solving equations requires the Zero Product Rule, which in turn requires factoring polynomials. Here are the details.

Addition and multiplication are very different.
Every positive integer can be obtained by adding 1's.
For example, $5 = 1 + 1 + 1 + 1 + 1$.

It's harder to obtain numbers by multiplying. The smallest collection of numbers whose products yield all integers bigger than 1 are called prime numbers.

Factoring definitions: Let K, A, B be positive integers.

- A is a factor of K if $K = AB$ for some number B.
- $K = AB$ is a factorization of K.
  A is the cofactor of B and B is the cofactor of A.
- A is a prime number if A has exactly two different factorizations: $A = A \cdot 1$ and $A = 1 \cdot A$.
- To factor K completely, write K as a product of primes. For example, $9000 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 5$.
- The prime power factorization of A rewrites A as a product of powers of distinct primes that increase from left to right: $9000 = 2^3 3^2 5^3$.

These definitions apply equally well to polynomials in x with positive leading coefficient, provided "increasing order" means that degrees of the polynomials in a prime power factorization stay the same or increase as you read from left to right.

The prime numbers less than 100 are
$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41,$
$43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.$

1 is not prime since it has only one factorization $1 = 1 \cdot 1$.

It's easy to prove that there are an infinite number of primes. However, other simple problems involving prime numbers are among the most difficult in mathematics.

For example, *prime pairs* are prime numbers that are consecutive odd integers. For centuries, mathematicians have tried, so far without success, to figure out whether or not there are infinitely many prime pairs!

- The first 3 prime pairs are $(3, 5), (17, 19)$, and $(29, 31)$.
- Every prime pair other than $(3, 5)$ has the form $(6n - 1, 6n + 1)$ for some positive integer n.
- A large known prime pair, with $388, 342$ digits, is $2, 996, 863, 034, 895 \cdot 2^{1,290,000} \pm 1$.

Exercise: Find five pairs of the form $(6n - 1, 6n + 1)$ with $n > 5$ that are not prime pairs.

Definition: The degree of a polynomial equation $E = F$

is the highest power of x that remains after rewriting $E - F$ as a polynomial.

Every quadratic equation $ax^2 + bx + c = 0$ has degree 2 since $a \neq 0$.

Example 11: The degree of $x^2 = x$ is 2, the highest power of x in $x^2 - x$.

Example 12: Let $E = x^3 + 3x^2$ and $F = 2x^3 - 5x^2 + x^3 + 12$. Find the degree of the equation $E = F$.
Solution: $E - F = x^3 + 3x^2 - 2x^3 + 5x^2 - x^3 - 12$
Collect terms to get $E - F = 8x^2 - 12$.
Answer: The degree of equation $E = F$ is 2.

We have shown how to solve all degree 1 (linear) and degree 2 (quadratic) equations.

Advanced algebra texts show
- how to solve all polynomial equations of degree 3 or 4.
- that there is no general formula for solving equations of degree 5 or higher.

The next column reviews how to factor integers. Then we show that similar techniques allow factoring a limited class of polynomials and solving some equations with degree 3 or higher.

How to find the prime factorization of an integer $N > 0$.

- Factor N.
- Factor each of the factors you just found.
- Keep on going until all factors are prime.
- Rewrite N as a product of powers of distinct increasing primes.

Example 13:

$$360 = 36 \cdot 10 = 6 \cdot 6 \cdot 2 \cdot 5 = 2 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 5 = \boxed{2^3 3^2 \cdot 5}$$

Definition: The Greatest Common Factor

(GCF) of a set of integers is the largest integer that is a factor of all of them.

How to find the GCF of integers

- Find the prime power factorizations of each integer.
- For each prime that appears in every factorization, write down the lowest power that appears.
- The GCF is the product of those powers.
- Check your answer: the cofactors of the GCF should have no common factor.

Example 14: Find the GCF of $72, 180$, and 1200.

Solution: $72 = 2^3 3^2; 180 = 2^2 3^2 \cdot 5; 1200 = 2^3 \cdot 3 \cdot 5^2$

2 appears in all 3 factorizations: lowest power 2^2.

3 appears in all 3 factorizations: lowest power $3^1 = 3$.

5 is missing from the factorization of 72.

The GCF is the product of lowest powers $\boxed{2^2 \cdot 3 = 12}$

Check:
- $72 = 12 \cdot 5; 180 = 12 \cdot 15; 1200 = 12 \cdot 100$ *and*
- $5; 12; 100$ have no common factor.

The Greatest Monomial Factor (GMF) of a polynomial P

is the monomial with highest degree and largest leading coefficient that is a factor of all of the terms of P.

The GMF of a polynomial in x with nonzero constant term is the GCF of its coefficients.

Examples: Factor out the GMF and color it red:

- $2x + 4 = 2(x + 2)$: GMF $= 2$.
- $3x^2 + 6x + 12 = 3(x^2 + 2x + 4)$: GMF $= 3$.
- $60x^2 + 40x + 1000 = 10(6x^2 + 4x + 100)$ $= 10 \cdot 2(3x^2 + 2x + 50) = 20(3x^2 + 2x + 50)$
- Faster: $60x^2 + 40x + 1000$ $= \boxed{20\ (3x^2 + 2x + 50)}$: GMF $= 20$.

No further factoring is possible, since $D = b^2 - 4ac = 2^2 - 4 \cdot 3 \cdot 50$ is negative and is therefore not a perfect square.

The GMF of a polynomial with no constant term is the GCF of its coefficients times the product of the lowest powers of letters that appear in every term.

Examples: Factor out the GMF and color it red:

- $2x^8 + 4x^6 + 6x^7$: GCF of coefficients $2, 4, 6$ is 2.
  $= 2(x^8 + 2x^6 + 3x^7)$: Lowest x-power is x^6.
  $= 2x^6(x^{8-6} + 2x^{6-6} + 3x^{7-6})$ Factor out x^6.
  $= 2x^6(x^2 + 2x^0 + 3x^1)$
  $= 2x^6\ (x^2 + 3x + 2) = \boxed{2x^6\ (x + 1)(x + 2))}$
- $100x^3 + 32x^2 - 64x^5$ GCF of coeffs is 4.
  $= 4(25x^3 + 8x^2 - 16x^5)$ Lowest x-power is x^2.
  $= 4x^2(25x + 8 - 16x^3) = \boxed{4x^2(-16x^3 + 25x + 8)}$
  or, if you factor out -1, $\boxed{-4x^2\ (16x^3 - 25x - 8)}$
- $20x^5y^7 + 30x^3y^{10}z + 45x^2y^8z^2$
  $= 5 \cdot (4x^5y^7 + 6x^3y^{10}z + 9x^2y^8z^2)$
  $= \boxed{5x^2y^7(4x^3 + 6xy^3z + 9yz^2)}$

In the second example, the cofactor $16x^3 - 25x - 8$ cannot be factored, since $D = (-25)^2 - 4(16)(-8) = 1137$ is not a perfect square.
That's because there is no decimal whole number whose square ends with $2, 3, 7$, or 8. Why not?

To factor a polynomial in one letter:

- Factor out the GMF of the polynomial.
- If the remaining polynomial is quadratic (degree 2), apply the factoring methods in the last section.
- If the remaining polynomial has degree > 2, look for a hidden quadratic equation, as will be discussed in Section 1.5.6.
- If that doesn't work, take a more advanced math course!

Example 15: Factor each blue polynomial:

- $5x^3 + x^4 + 6x^2 = x^4 + 5x^3 + 6x^2 = x^2(x^2 + 5x + 6) = \boxed{x^2(x+2)(x+3)}$
- $x^5 + 3x^4 - x^3 = \boxed{x^3(x^2 + 3x - 1)}$.

This is the final answer: $x^2 + 3x - 1 = ax^2 + bx + c$ doesn't factor further because $b^2 - 4ac = 13$ is not a perfect square.

- $-8x^3 + 6x^4 - x^5 = -x^5 + 6x^4 - 8x^3 = -x^3(x^2 - 6x + 8) = \boxed{-x^3(x-4)(x-2)}$

Some special factoring rules] For any expressions A and B:

- $A^2 - B^2 = (A+B)(A-B)$ (difference of squares)
- $A^2 + 2AB + B^2 = (A+B)^2$ (square of a sum)
- $A^2 - 2AB + B^2 = (A-B)^2$ (square of a difference)

Example 16:
Factor each polynomial by using a special factorization.

	A	B	$A^2 - B^2$	$= (A+B)(A-B)$
$x^2 - 9$	x	3	$x^2 - 3^2$	$= (x+3)(x-3)$
$4x^2 - 9$	$2x$	3	$(2x)^2 - 3^2$	$= (2x+3)(2x-3)$
$9 - x^2$	3	x	$3^2 - x^2$	$= (3+x)(3-x)$
$25 - 4x^2$	5	$2x$	$5^2 - (2x)^2$	$= (5+2x)(5-2x)$

$A^2 + 2AB + B^2$	A	B		
$x^2 + 6x + 9$	x	3	$(A+B)^2$	$= (x+3)^2$
$4x^2 - 12x + 9$	$2x$	-3	$(A+B)^2$	$= (2x-3)^2$

One final note: completing the square is another way to solve quadratic equations.

How to solve $(x+k)^2 = K$

Since $x+k = \pm\sqrt{K}$ the solution is $x = -k \pm \sqrt{K}$

To complete the square with $x^2 + Bx = K$

Add $\frac{B^2}{4}$ to both sides to get

$x^2 + Bx + \frac{B^2}{4} = (x + \frac{B}{2})^2 = K + \frac{B^2}{4}$. Then

$x = -\frac{B}{2} \pm \sqrt{K + \frac{B^2}{4}}$

Example 17:
Solve $2x^2 + 12x + 3 = 0$ by completing the square.
Check your answer by using the quadratic formula.
Solution: To complete the square:

- Divide by $a = 2$ to get $x^2 + 6x + \frac{3}{2} = 0$.
- Rewrite as $x^2 + 6x = -\frac{3}{2}$.
  The coefficient of x is now $B = 6$.
- Add $\left(\frac{B}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = 9$ to both sides to get
  $x^2 + 6x + 9 = -\frac{3}{2} + 9$. This becomes
- $(x+3)^2 = \frac{15}{2}$. Therefore
- $x + 3 = \pm\sqrt{\frac{15}{2}}$ and so $\boxed{x = -3 \pm \sqrt{\frac{15}{2}}}$

As a check, use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-12 \pm \sqrt{12^2 - 4 \cdot 2 \cdot 3}}{2 \cdot 2} = \frac{-12 \pm \sqrt{120}}{4}$$
$$= \frac{-12 \pm \sqrt{4 \cdot 30}}{4} = -3 \pm \frac{2\sqrt{30}}{4} = \boxed{-3 \pm \frac{\sqrt{30}}{2}}$$

The boxed answers are equal because

$$\sqrt{\frac{15}{2}} = \frac{\sqrt{15}}{\sqrt{2}} = \frac{\sqrt{15}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{30}}{2}$$

It's easier to use the quadratic formula: complete the square only if that method is requested.

Details: To solve $ax^2 + bx + c = 0$ by completing the square

- Divide by a to get $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.
- Rewrite as $x^2 + Bx = -C$ where $B = \frac{b}{a}$ and $C = \frac{c}{a}$.
- Complete the square: $(x + \frac{B}{2})^2 = -C + \frac{B^2}{4} = \frac{B^2 - 4C}{4}$
- This has the form $(x+k)^2 = K$ with $k = \frac{B}{2}$ so
  $x = -k \pm \sqrt{K} = -\frac{B}{2} \pm \frac{\sqrt{B^2 - 4C}}{\sqrt{4}} = \frac{-B \pm \sqrt{B^2 - 4C}}{2}$

To obtain the Quadratic Formula from the last line above, substitute $\frac{b}{a}$ for B and $\frac{c}{a}$ for C.

1.5.3 Rational equations

Definition: An equation is rational

if it contains at least one fraction.

To solve a rational equation

- Rewrite the equation without parentheses or nested fractions.
- Multiply both sides by the LCD of all denominators.
- Solve the resulting polynomial equation.
- Reject any solution that makes the denominator of any fraction in the original equation equal to 0.

Example 18: Solve $\frac{4}{x} = 2 + \frac{4}{3-x}$

Solution: The LCD is $x(3-x)$

Multiply both sides by the LCD :

$$x(3-x)\left(\frac{4}{x}\right) = x(3-x)\left(2 + \frac{4}{3-x}\right)$$

Distribute, cancel $\quad \cancel{x}(3-x)\frac{4}{\cancel{x}} = x(3-x)(2) + x\cancel{(3-x)}\left(\frac{4}{\cancel{3-x}}\right)$

Multiply out $\quad 12 - 4x = (3x - x^2)2 + 4x$

Rewrite with everything on left side $\quad 12 - 4x = 6x - 2x^2 + 4x$

$2x^2 - 14x + 12 = 0$

Divide by 2, factor $\quad x^2 - 7x + 6 = (x-6)(x-1) = 0$

Set each factor to 0 $\quad x = 6$ and $x = 1$.

Accept both answers since neither makes the original denominators $3 - x$ or x equal to 0. $\quad \boxed{x = 6\ ;\ x = 1}$

To check $x = 1$ in substitute 1 for x

$\frac{4}{x} =?\ 2 + \frac{4}{3-x}$

$\frac{4}{1} =?\ 2 + \frac{4}{3-1}$

Rewrite

$4 =?\ 2 + \frac{4}{2}$

$4 =?\ 2 + 2$ Yes!

.

To check $x = 6$ in substitute 6 for x

$\frac{4}{x} =?\ 2 + \frac{4}{3-x}$

$\frac{4}{6} =?\ 2 + \frac{4}{3-6}$

Rewrite

$\frac{2}{3} =?\ \frac{6}{3} + -\frac{4}{3}$

$\frac{2}{3} =?\ \frac{6-4}{3}$ Yes!

1.5.4 Equations with radicals

To solve an equation with radicals

- Rewrite it so that an n^{th} root is alone on the left side. Remove that radical by raising both sides to the n^{th} power.
- If necessary, repeat until no radical is left.
- Solve the resulting polynomial equation.
- Check your solution(s) in the original equation.
- Reject any solution that produces an even root of a negative number.

Example 19: Solve $\sqrt{2x+1}+1=x$

Solution: Original equation	$\sqrt{2x+1}+1=x$
Isolate radical on left side	$\sqrt{2x+1}=x-1$
Square both sides	$\left(\sqrt{2x+1}\right)^2=(x-1)^2$
Rewrite as	$2x+1=x^2-2x+1$
$0=p(x)$ and	$2x=x^2-2x$
factor $p(x)$	$0=x^2-4x=x(x-4)$
to get possible solutions	$x=0$ and $x=4$.

Check $x=0$:	$\sqrt{2x+1}+1=x$
Substitute 0 for x	$\sqrt{0+1}+1=?\ 0$
and check if	$1+1=?\ 0$. No! Reject $x=0$

Check $x=4$:	$\sqrt{2x+1}+1=x$
Substitute 4 for x	$\sqrt{2(4)+1}+1=?\ 4$
	$\sqrt{9}+1=?\ 4$
	$3+1=?\ 4$.
	Yes! Accept $\boxed{x=4}$

This rejected root was obtained because of a logic error: the third step above, squaring both sides of an equation, may not be reversible! For example, if you start with $x=2$ and square both sides to obtain $x^2=4$ you have introduced an incorrect solution $x=-2$!

Example 20: Solve $\sqrt{x-5}+\sqrt{x}=5$

Solution:

Subtract $\sqrt{x}$	$\sqrt{x-5}=5-\sqrt{x}$
Square both sides	$(\sqrt{x-5})^2=(5-\sqrt{x})^2$
	$x-5=25-10\sqrt{x}+(\sqrt{x})^2$
	$x-5=25-10\sqrt{x}+x$
Subtract $x+25$	$-30=-10\sqrt{x}$
Divide by -10	$3=\sqrt{x}\Rightarrow x=3^2=9$

Check $x=9$:	$\sqrt{x-5}+\sqrt{x}=5$
	$\sqrt{9-5}+\sqrt{9}=?5$
	$\sqrt{4}+\sqrt{9}=2+3=?5$ Yes! $\boxed{x=9}$

1.5.5 Complex numbers and quadratic equations

Does every equation have a solution?

The following list shows that new types of equations require new kinds of numbers as solutions:

- $x+1=3$ has a natural number solution $x=2$.
- $x+1=0$ has a negative integer solution $x=-1$.
- $3x=1$ has a rational number solution $x=\frac{1}{3}$.
- $x^2=5$ has two real number solutions $x=\pm\sqrt{5}$.
- $x^2=-1$ has no real number solutions!

We complete the process by "inventing" a new number i that solves $x^2=-1$. In other words, $i^2=-1$.

The number i is sometimes called imaginary. But imaginary numbers are just as "real" as real numbers.

Definition of complex numbers

A complex number is an expression $a+bi$, where a and b are real numbers and $i^2=-1$.

- $a+bi=a$ is a real number if $b=0$.
- $a+bi=bi$ is a pure imaginary number if $a=0$.

Constructing a complex number $a+bi$ from real numbers a and b is a bit like constructing the rational number $\frac{a}{b}$ from integers a and b.
The need to invent new types of numbers stops here, since every polynomial equation with complex coefficients has complex solutions.

To do arithmetic with complex numbers: Treat i as a letter, replace i^2 by -1.

Suppose $u=a+bi$ and $v=c+di$. Then

- Add: $u+v=a+c+(b+d)i$
- Subtract: $u-v=a-c+(b-d)i$
- Multiply:
  $uv=(a+bi)(c+di)=ac+adi+bic+bdi^2$
  $=ac+(ad+bc)i-bd=(ac-bd)+(ad+bc)i.$

  Special cases:
- if k is real, $k(c+di)=kc+kdi$.
- $(c+di)(c-di)=c^2+dci-cdi-d^2i^2=c^2+d^2$.

Division is more complicated:

- Division: $(a+bi) \div (c+di) = \dfrac{a+bi}{c+di}$.

  Multiply numerator and denominator by $c-di$:
  $$\frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac-bdi^2)+(bc-ad)i}{c^2-d^2i^2}$$
  $i^2=-1$ so this $= \dfrac{ac+bd}{c^2+d^2} + \dfrac{bc-ad}{c^2+d^2}i$

Don't try to memorize this formula.

Complex numbers are crucial in all of advanced mathematics and physics. In this course, they have one purpose: to solve the general quadratic equation $ax^2+bx+c=0$ where a, b, c are real numbers.

Assume a, b, c are real, $a \neq 0$, and $D = b^2-4ac < 0$. Then the quadratic equation $ax^2+bx+c=0$ has two complex solutions

$$x = r_1 = \frac{-b+\sqrt{-D}\ i}{2a} \text{ and } x = r_2 = \frac{-b-\sqrt{-D}\ i}{2a}$$

These are abbreviated as $x = \dfrac{-b \pm \sqrt{-D}\ i}{2a}$

Proof: Substitute r_1 and r_2 for x in $ax^2+bx+c=0$.

Example 21:
Solve $x^2+x+4=0$ and check your answer(s).
Solution: $a=1, b=1, c=4$.
$D = b^2-4ac = 1^2-4\cdot1\cdot4 = 1-16 = -15$.
There are two solutions:
$$x = \frac{-b\pm\sqrt{D}}{2a} = \frac{-1\pm\sqrt{-15}}{2} = \frac{-1\pm\sqrt{15}\ i}{2} = -\frac{1}{2} \pm \frac{\sqrt{15}}{2}\ i$$
$$\boxed{x = -\frac{1}{2}+\frac{\sqrt{15}}{2}\ i;\ x = -\frac{1}{2}-\frac{\sqrt{15}}{2}\ i}$$

Check: Rewrite the answers as $x+\frac{1}{2} = \pm\frac{\sqrt{15}}{2}\ i$

Square both sides: $x^2+x+\frac{1}{4} = \frac{15}{4}i^2 = -\frac{15}{4}$
Add $\frac{15}{4}$ to both sides to get
$x^2+x+\frac{16}{4} = x^2+x+4 = 0$ as desired.

Here, as always, the quadratic formula produces correct answers. Nevertheless, check your work to make sure that you have not made a "careless" algebra error.

1.5.6 Hidden quadratic equations

How to recognize a hidden quadratic equations

In a polynomial equation in x, suppose the only two powers of x are x^n and $x^{2n} = (x^n)^2$. Substitute $W = x^n$ to obtain a quadratic equation in W.

Example 22: Solve $x^4 - 8x^2 + 8 = 0$

Solution: Notice that $x^4 = (x^2)^2$.

Substitute W for x^2 to get $x^4 = W^2$. Then

$W^2 - 8W + 8 = 0$. Now apply the quadratic formula:

$$W = \frac{8 \pm \sqrt{(-8)^2 - 4 \cdot 1 \cdot 8}}{2} = \frac{8 \pm \sqrt{32}}{2} = \frac{8 \pm 4\sqrt{2}}{2} = 4 \pm 2\sqrt{2}$$

Since both $4 + 2\sqrt{2}$ and $4 - 2\sqrt{2}$ are positive, their square roots are defined.

If $W = x^2 = 4 + 2\sqrt{2}$ then $x = \pm\sqrt{4 + 2\sqrt{2}}$

If $W = x^2 = 4 - 2\sqrt{2}$ then $x = \pm\sqrt{4 - 2\sqrt{2}}$

Therefore there are 4 solutions:

$$\boxed{\begin{array}{ll} x = \sqrt{4 + 2\sqrt{2}}; & x = -\sqrt{4 + 2\sqrt{2}}; \\ x = \sqrt{4 - 2\sqrt{2}}; & x = -\sqrt{4 - 2\sqrt{2}} \end{array}}$$

Practice your algebra skills by checking these answers.

Example 23: Solve $x^{\frac{1}{3}} + x^{\frac{1}{6}} = 2$.

Solution: Notice that $x^{\frac{1}{3}} = x^{2 \cdot \frac{1}{6}} = (x^{\frac{1}{6}})^2$

Rewrite the equation as $(x^{\frac{1}{6}})^2 + x^{\frac{1}{6}} - 2 = 0$.

Set $W = x^{\frac{1}{6}}$ to get

$W^2 + W - 2 = (W + 2)(W - 1) = 0$

Setting factors to 0 gives $W = -2$ or $W = 1$.

Since $x = W^6$, $x = (-2)^6 = 64$ or $x = 1^6 = 1$.

Check solutions by substituting in the original equation

$x^{\frac{1}{3}} + x^{\frac{1}{6}} = 2$

$x = 1$ gives $1 + 1 = 2$, Yes!

$x = 64$ gives $4 + 2 = 2$, No!

Therefore there is one solution $\boxed{x = 1}$

The reason for rejecting $x = 64$ is that it arose from a contradiction: $x^{\frac{1}{6}}$ is always positive, so we should have noticed that setting $W = x^{\frac{1}{6}}$ equal to -2 was an error.

In Example 28, however, both values of W were positive, and so there was no real need to check the answer.

1.5.8 Section 1.5 Quiz

Click on ▸ Wolfram Calculator to find an answer checker.

Click on ▸ Wolfram Algebra Examples to see how to check various types of algebra problems.

1. Do the WebAssign exercises.

2. Solve each of the following equations:
a) $x^2 = x$ b) $x(x+1) = 20$ c) $x^3 = 16x$
d) $x^4 = 16x^2$ e) $x^2 = 20$
f) $x(x+1)(x+2) = (x+1)(x^2+5x+6)$
g) $12x^2 + 30x = 5x - 12$

3. Solve each of the following equations:
a) $x^2 + 4 = 0$ b) $x^3 + x^2 - x = 0$ c) $x^4 - 4 = 0$

4. Solve each of the following equations:
a) $x^{4/3} - 5x^{2/3} + 6 = 0$ b) $\frac{4}{x-2} + \frac{12}{x} = \frac{108}{x^2}$
c) $\sqrt{3x+7} = 2x - 7$ d) $\frac{10}{x} - \frac{12}{x-3} + 4 = 0$
e) $\frac{x}{2x+7} - \frac{x+1}{x+3} = 1$ f) $\sqrt{5-x} + 1 = x - 2$

1.5.7 Section 1.5 Quiz

Solve equations for x and check your answers.

Ex. 1.5.1: Solve $x^2 = 4$.

Ex. 1.5.2: Solve $x^2 = -4$.

Ex. 1.5.3: Solve $(x+k)^2 = K$.

Ex. 1.5.4 Use the Zero Product Rule to solve

- $x^2 + 3x + 2 = 0$
- $x = x^3$
- $x^6 - x^4 = 0$
- $4x^6 = 12x^5 - 12x^4$
- $x^3 = 7x^2$
- $x^4 + 3x^2 = 4x^3$

Ex. 1.5.5: Solve $x^2 = x$.

Ex. 1.5.6: Solve $x^4 - 20 = 0$.

Ex. 1.5.7: Solve $x^4 - 25 = 0$.

Ex. 1.5.8: Solve for x:

- $2x^2 + 5x + 1 = 0$
- $2x^2 + 2x + 1 = 0$

Ex. 1.5.9: Factor $x^2 + bx + c = 6x^2 + 13x + 6$.

Ex. 1.5.10: Factor $12x^2 + 44x + 35$.

Ex. 1.5.11: What is the degree of equation $x^2 = x$

Ex. 1.5.12: Find the degree of $E = F$ if $E = x^3 + 3x^2$ and $F = 2x^3 - 5x^2 + x^3 + 12$.

Ex. 1.5.13: Find the prime power factorization of 360.

Ex. 1.5.14: Find the GCF of $72, 180$, and 1200.

Ex. 1.5.15: Factor each polynomial

- $5x^3 + x^4 + 6x^2$
- $x^5 + 3x^4 - x^3$
- $-8x^3 + 6x^4 - x^5$

Ex. 1.5.16: Factor each polynomial by using a special factorization.

- $x^2 - 9$
- $4x^2 - 9$
- $9 - x^2$
- $25 - 4x^2$
- $x^2 + 6x + 9$
- $4x^2 - 12x + 9$

Ex. 1.5.17: Solve $2x^2 + 12x + 3 = 0$ by completing the square. Check your answer by using the quadratic formula.

Ex. 1.5.18: Solve $\frac{4}{x} = 2 + \frac{4}{3-x}$

Ex. 1.5.19: Solve $\sqrt{2x+1} + 1 = x$

Ex. 1.5.20: Solve $\sqrt{x} + \sqrt{x-5} = 5$

Ex. 1.5.21: Solve $x^2 + x + 4 = 0$.

Ex. 1.5.22: Solve $x^4 - 8x^2 + 8 = 0$

Ex. 1.5.23: Solve $x^{\frac{1}{3}} + x^{\frac{1}{6}} = 2$.

Section 1.5 Review: Equations

Solve each equation. In 1.5.3, assume $K \geq 0$ and $b \geq 0$.

Ex. 1.5.1:

- $x^2 = 4 \Rightarrow$
- $x^2 = 7 \Rightarrow$
- $3x^2 = -4 \Rightarrow$
- $x^2 - 9 = 0 \Rightarrow$

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- $x^2 = -4 \Rightarrow$
- $5x^2 = 45 \Rightarrow$
- $x^2 = 80 \Rightarrow$
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Ex. 1.5.3:
- $(x+k)^2 = K \Rightarrow$
- $(x+a)^2 = b \Rightarrow$
- $(x-8)^2 = 12 \Rightarrow$
- $(2x-k)^2 = K \Rightarrow$

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- $x^2 + 3x + 2 = 0 \Rightarrow$
- $x = x^3 \Rightarrow$
- $x^6 - x^4 = 0 \ \Rightarrow$
- $4x^6 = 12x^5 - 12x^4 \qquad \Rightarrow$
- $x^3 = 7x^2 \Rightarrow$
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- $x^2 - 10x + 25 = 0 \Rightarrow x = 5$
- $7x^2 = -9x^3 \Rightarrow x = 0; -\frac{7}{9}$
- $x^6 - 100x^4 = 0 \Rightarrow x = 0; \pm 10$
- $4x^4 = 12x^5 - 12x^4 \Rightarrow x = 0; \frac{4}{3}$
- $x^3 = 7x^2 \Rightarrow x = 0; 7$
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- $(2x-k)^2 = K \Rightarrow x = \frac{\pm\sqrt{K}+k}{2}$

Ex. 1.5.4 Use the Zero Product Rule to find real roots of
- $x^2 + 3x + 2 = 0 \Rightarrow x = -1; -2$
- $x = x^3 \Rightarrow x = 0; \pm 1$
- $x^6 - x^4 = 0 \;\; \Rightarrow x = 0; \pm 1$
- $4x^6 = 12x^5 - 12x^4 \qquad \Rightarrow x = 0$ and solutions of $x^2 - 3x - 3 = 0$
- $x^3 = 7x^2 \Rightarrow x = 0; \frac{1}{7}$
- $x^4 + 3x^2 = 4x^3 \Rightarrow x = 0; 1; 3$
- $x^2 - 3x + 2 = 0 \Rightarrow x = 1; 2$
- $x^2 = x^3 \Rightarrow x = 0; 1$
- $x^6 - x^2 = 0 \;\; \Rightarrow x = 0; \pm 1$
- $x^6 = 6x^5 - 5x^4 \;\; \Rightarrow x = 0; 1; 5$
- $x^2 + 7x = 8; \;\; \Rightarrow x = 1, -8$
- $x^4 + 2x^2 = 3x^3 \Rightarrow x = 0; 1; 2$
- $x^2 + x - 12 = 0 \Rightarrow x = 3; -4$
- $x^2 = 4x^4 \Rightarrow x = 0; \pm\frac{1}{2}$
- $x^6 - 4x^4 = 0 \;\; \Rightarrow x = 0; \pm 2$
- $4x^6 = 12x^5 - 8x^4 \;\; \Rightarrow x = 0; 1; 2$
- $x^3 = -3x^2 \Rightarrow x = 0; -3$
- $x^4 + 12x^2 = -8x^3 \Rightarrow x = 0; -2, -6$
- $x^2 - 10x + 25 = 0 \Rightarrow x = 5$
- $7x^2 = -9x^3 \Rightarrow x = 0; -\frac{7}{9}$
- $x^6 - 100x^4 = 0 \;\; \Rightarrow x = 0; \pm 10$
- $4x^4 = 12x^5 - 12x^4 \;\; \Rightarrow x = 0; \frac{4}{3}$
- $x^3 = 7x^2 \Rightarrow x = 0; 7$
- $x^4 + 3x^2 = 4x^3 \Rightarrow x = 0; 1; 3$

Find all real solutions of the following equations

Ex. 1.5.5:
- $x^2 = x \Rightarrow$
- $8x^2 = x \Rightarrow$
- $x^2 = 8x \Rightarrow$
- $x^2 = -9x \Rightarrow$

Section 1.5 Review: Equations

Solve each equation. In 1.5.3, assume $K \geq 0$ and $b \geq 0$.

Ex. 1.5.1:
- $x^2 = 4 \Rightarrow x = \pm 2$
- $x^2 = 7 \Rightarrow x = \pm\sqrt{7}$
- $3x^2 = -4 \Rightarrow$ No solution
- $x^2 - 9 = 0 \Rightarrow x = \pm 3$

Ex. 1.5.2:
- $x^2 = -4 \Rightarrow$ No solution
- $5x^2 = 45 \Rightarrow x = \pm 3$
- $x^2 = 80 \Rightarrow x = \pm 4\sqrt{5}$
- $x^2 = 121 \Rightarrow x = \pm 11$

Ex. 1.5.3:
- $(x+k)^2 = K \Rightarrow x = \pm\sqrt{K} - k$
- $(x+a)^2 = b \Rightarrow x = \pm\sqrt{b} - a$
- $(x-8)^2 = 12 \Rightarrow x = \pm 2\sqrt{3} + 8$
- $(2x-k)^2 = K \Rightarrow x = \frac{\pm\sqrt{K}+k}{2}$

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- $x^2 + 3x + 2 = 0 \Rightarrow x = -1; -2$
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- $x^3 = 7x^2 \Rightarrow x = 0; \frac{1}{7}$
- $x^4 + 3x^2 = 4x^3 \Rightarrow x = 0; 1; 3$
- $x^2 - 3x + 2 = 0 \Rightarrow x = 1; 2$
- $x^2 = x^3 \Rightarrow x = 0; 1$
- $x^6 - x^2 = 0 \Rightarrow x = 0; \pm 1$
- $x^6 = 6x^5 - 5x^4 \Rightarrow x = 0; 1; 5$
- $x^2 + 7x = 8; \Rightarrow x = 1, -8$
- $x^4 + 2x^2 = 3x^3 \Rightarrow x = 0; 1; 2$
- $x^2 + x - 12 = 0 \Rightarrow x = 3; -4$
- $x^2 = 4x^4 \Rightarrow x = 0; \pm\frac{1}{2}$
- $x^6 - 4x^4 = 0 \Rightarrow x = 0; \pm 2$
- $4x^6 = 12x^5 - 8x^4 \Rightarrow x = 0; 1; 2$
- $x^3 = -3x^2 \Rightarrow x = 0; -3$
- $x^4 + 12x^2 = -8x^3 \Rightarrow x = 0; -2, -6$
- $x^2 - 10x + 25 = 0 \Rightarrow x = 5$
- $7x^2 = -9x^3 \Rightarrow x = 0; -\frac{7}{9}$
- $x^6 - 100x^4 = 0 \Rightarrow x = 0; \pm 10$
- $4x^4 = 12x^5 - 12x^4 \Rightarrow x = 0; \frac{4}{3}$
- $x^3 = 7x^2 \Rightarrow x = 0; 7$
- $x^4 + 3x^2 = 4x^3 \Rightarrow x = 0; 1; 3$

Find all real solutions of the following equations

Ex. 1.5.5:
- $x^2 = x \Rightarrow x = 0; 1$
- $8x^2 = x \Rightarrow x = 0; \frac{1}{8}$
- $x^2 = 8x \Rightarrow x = 0; 8$
- $x^2 = -9x \Rightarrow x = 0; -9$

Section 1.5 Review: Equations

Solve each equation. In 1.5.3, assume $K \geq 0$ and $b \geq 0$.

Ex. 1.5.1:
- $x^2 = 4 \Rightarrow \; x = \pm 2$
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- $x^2 = -4 \Rightarrow$ No solution
- $5x^2 = 45 \Rightarrow x = \pm 3$
- $x^2 = 80 \Rightarrow x = \pm 4\sqrt{5}$
- $x^2 = 121 \Rightarrow \; x = \pm 11$

Ex. 1.5.3:
- $(x+k)^2 = K \Rightarrow x = \pm\sqrt{K} - k$
- $(x+a)^2 = b \Rightarrow x = \pm\sqrt{b} - a$
- $(x-8)^2 = 12 \Rightarrow x = \pm 2\sqrt{3} + 8$
- $(2x-k)^2 = K \Rightarrow x = \frac{\pm\sqrt{K}+k}{2}$

Ex. 1.5.4 Use the Zero Product Rule to find real roots of
- $x^2 + 3x + 2 = 0 \Rightarrow x = -1; -2$
- $x = x^3 \Rightarrow x = 0; \pm 1$
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- $x^3 = 7x^2 \Rightarrow x = 0; \frac{1}{7}$
- $x^4 + 3x^2 = 4x^3 \Rightarrow x = 0; 1; 3$
- $x^2 - 3x + 2 = 0 \Rightarrow x = 1; 2$
- $x^2 = x^3 \Rightarrow x = 0; 1$
- $x^6 - x^2 = 0 \; \Rightarrow x = 0; \pm 1$
- $x^6 = 6x^5 - 5x^4 \; \Rightarrow x = 0; 1; 5$
- $x^2 + 7x = 8; \; \Rightarrow x = 1, -8$
- $x^4 + 2x^2 = 3x^3 \Rightarrow x = 0; 1; 2$
- $x^2 + x - 12 = 0 \Rightarrow x = 3; -4$
- $x^2 = 4x^4 \Rightarrow x = 0; \pm\frac{1}{2}$
- $x^6 - 4x^4 = 0 \; \Rightarrow x = 0; \pm 2$
- $4x^6 = 12x^5 - 8x^4 \; \Rightarrow x = 0; 1; 2$
- $x^3 = -3x^2 \Rightarrow x = 0; -3$
- $x^4 + 12x^2 = -8x^3 \Rightarrow x = 0; -2, -6$
- $x^2 - 10x + 25 = 0 \Rightarrow x = 5$
- $7x^2 = -9x^3 \Rightarrow x = 0; -\frac{7}{9}$
- $x^6 - 100x^4 = 0 \; \Rightarrow x = 0; \pm 10$
- $4x^4 = 12x^5 - 12x^4 \; \Rightarrow x = 0; \frac{4}{3}$
- $x^3 = 7x^2 \Rightarrow x = 0; 7$
- $x^4 + 3x^2 = 4x^3 \Rightarrow x = 0; 1; 3$

Find all real solutions of the following equations

Ex. 1.5.5:
- $x^2 = x \Rightarrow x = 0; 1$
- $8x^2 = x \Rightarrow x = 0; \frac{1}{8}$
- $x^2 = 8x \Rightarrow x = 0; 8$
- $x^2 = -9x \Rightarrow x = 0; -9$

Ex. 1.5.6:
- $x^4 - 20 = 0 \Rightarrow$
- $x^4 - 64 = 0 \Rightarrow$
- $2x^4 - 20 = 0 \Rightarrow$
- $x^4 + 20 = 0 \Rightarrow$

Section 1.5 Review: Equations

Solve each equation. In 1.5.3, assume $K \geq 0$ and $b \geq 0$.

Ex. 1.5.1:
- $x^2 = 4 \Rightarrow \; x = \pm 2$
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- $(x-8)^2 = 12 \Rightarrow x = \pm 2\sqrt{3} + 8$
- $(2x-k)^2 = K \Rightarrow x = \frac{\pm\sqrt{K}+k}{2}$

Ex. 1.5.4 Use the Zero Product Rule to find real roots of
- $x^2 + 3x + 2 = 0 \Rightarrow x = -1; -2$
- $x = x^3 \Rightarrow x = 0; \pm 1$
- $x^6 - x^4 = 0 \;\Rightarrow x = 0; \pm 1$
- $4x^6 = 12x^5 - 12x^4 \;\Rightarrow x = 0$ and solutions of $x^2 - 3x - 3 = 0$
- $x^3 = 7x^2 \Rightarrow x = 0; \frac{1}{7}$
- $x^4 + 3x^2 = 4x^3 \Rightarrow x = 0; 1; 3$
- $x^2 - 3x + 2 = 0 \Rightarrow x = 1; 2$
- $x^2 = x^3 \Rightarrow x = 0; 1$
- $x^6 - x^2 = 0 \;\Rightarrow x = 0; \pm 1$
- $x^6 = 6x^5 - 5x^4 \;\Rightarrow x = 0; 1; 5$
- $x^2 + 7x = 8; \;\Rightarrow x = 1, -8$
- $x^4 + 2x^2 = 3x^3 \Rightarrow x = 0; 1; 2$
- $x^2 + x - 12 = 0 \Rightarrow x = 3; -4$
- $x^2 = 4x^4 \Rightarrow x = 0; \pm\frac{1}{2}$
- $x^6 - 4x^4 = 0 \;\Rightarrow x = 0; \pm 2$
- $4x^6 = 12x^5 - 8x^4 \;\Rightarrow x = 0; 1; 2$
- $x^3 = -3x^2 \Rightarrow x = 0; -3$
- $x^4 + 12x^2 = -8x^3 \Rightarrow x = 0; -2, -6$
- $x^2 - 10x + 25 = 0 \Rightarrow x = 5$
- $7x^2 = -9x^3 \Rightarrow x = 0; -\frac{7}{9}$
- $x^6 - 100x^4 = 0 \;\Rightarrow x = 0; \pm 10$
- $4x^4 = 12x^5 - 12x^4 \;\Rightarrow x = 0; \frac{4}{3}$
- $x^3 = 7x^2 \Rightarrow x = 0; 7$
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Find all real solutions of the following equations

Ex. 1.5.5:
- $x^2 = x \Rightarrow x = 0; 1$
- $8x^2 = x \Rightarrow x = 0; \frac{1}{8}$
- $x^2 = 8x \Rightarrow x = 0; 8$
- $x^2 = -9x \Rightarrow x = 0; -9$

Ex. 1.5.6:
- $x^4 - 20 = 0 \Rightarrow x = \pm\sqrt[4]{20}$
- $x^4 - 64 = 0 \Rightarrow x = \pm 2\sqrt[4]{4}$
- $2x^4 - 20 = 0 \Rightarrow x = \pm\sqrt[4]{10}$
- $x^4 + 20 = 0 \Rightarrow$ No real solutions

Ex. 1.5.7: Find all real roots of:

- $x^4 - 25 = 0 \Rightarrow$
- $x^4 - 49 = 0 \Rightarrow$
- $x^4 - 100 = 0 \Rightarrow$
- $x^4 - 144 = 0 \Rightarrow$

Ex. 1.5.7: Find all real roots of:

- $x^4 - 25 = 0 \Rightarrow x = \pm\sqrt{5}$
- $x^4 - 49 = 0 \Rightarrow x = \pm\sqrt{7}$
- $x^4 - 100 = 0 \Rightarrow x = \pm\sqrt{10}$
- $x^4 - 144 = 0 \Rightarrow x = \pm 12$

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- $x^4 - 100 = 0 \Rightarrow x = \pm\sqrt{10}$
- $x^4 - 144 = 0 \Rightarrow x = \pm 12$

Ex. 1.5.8: Find all real roots of:

- $2x^2 + 5x + 1 = 0 \Rightarrow$
- $-2x^2 + 5x + 1 = 0 \Rightarrow$
- $x^2 + 5x + 1 = 0 \Rightarrow$
- $2x^2 - 5x + 10 = 0 \Rightarrow$
- $2x^2 + 2x - 1 = 0 \Rightarrow$
- $2x^2 - 12x + 1 = 0 \Rightarrow$
- $3x^2 + 4x + 1 = 0 \Rightarrow$
- $-2x^2 - 2x - 1 = 0 \Rightarrow$

Ex. 1.5.7: Find all real roots of:

- $x^4 - 25 = 0 \Rightarrow x = \pm\sqrt{5}$
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- $x^4 - 144 = 0 \Rightarrow x = \pm 12$

Ex. 1.5.8: Find all real roots of:

- $2x^2 + 5x + 1 = 0 \Rightarrow x = \frac{-5\pm\sqrt{17}}{4}$
- $2x^2 + 2x - 1 = 0 \Rightarrow x = \frac{-1\pm\sqrt{3}}{2}$
- $-2x^2 + 5x + 1 = 0 \Rightarrow x = \frac{5\pm\sqrt{33}}{4}$
- $2x^2 - 12x + 1 = 0 \Rightarrow x = 3 \pm \sqrt{\frac{17}{2}}$
- $x^2 + 5x + 1 = 0 \Rightarrow x = \frac{-5\pm\sqrt{21}}{2}$
- $3x^2 + 4x + 1 = 0 \Rightarrow x = -1; -\frac{1}{3}$
- $2x^2 - 5x + 10 = 0 \Rightarrow$ No real solution
- $-2x^2 - 2x - 1 = 0 \Rightarrow$ No real solution

Ex. 1.5.7: Find all real roots of:

- $x^4 - 25 = 0 \Rightarrow x = \pm\sqrt{5}$
- $x^4 - 49 = 0 \Rightarrow x = \pm\sqrt{7}$
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- $x^2 + 5x + 1 = 0 \Rightarrow x = \frac{-5\pm\sqrt{21}}{2}$
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- $2x^2 - 5x + 10 = 0 \Rightarrow$ No real solution
- $-2x^2 - 2x - 1 = 0 \Rightarrow$ No real solution

Ex. 1.5.9: Factor $ax^2 + bx + c =$

- $6x^2 + 13x + 6 \Rightarrow$
- $12x^2 - 7x - 12 \Rightarrow$
- $12x^2 + 38x + 20 \Rightarrow$
- $24x^2 + 79x + 40 \Rightarrow$

Ex. 1.5.7: Find all real roots of:

- $x^4 - 25 = 0 \Rightarrow x = \pm\sqrt{5}$
- $x^4 - 49 = 0 \Rightarrow x = \pm\sqrt{7}$
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- $2x^2 - 5x + 10 = 0 \Rightarrow$ No real solution
- $-2x^2 - 2x - 1 = 0 \Rightarrow$ No real solution

Ex. 1.5.9: Factor $ax^2 + bx + c =$

- $6x^2 + 13x + 6 \Rightarrow = (3x + 2)(2x + 3)$
- $12x^2 - 7x - 12 \Rightarrow = (4x + 3)(3x - 4)$
- $12x^2 + 38x + 20 \Rightarrow = 2(2x + 5)(3x + 2)$
- $24x^2 + 79x + 40 \Rightarrow = (8x + 5)(3x + 8)$

Ex. 1.5.7: Find all real roots of:

- $x^4 - 25 = 0 \Rightarrow x = \pm\sqrt{5}$
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Ex. 1.5.10: Factor

- $12x^2 + 44x + 35 \Rightarrow$
- $27x^2 + 48x + 5 \Rightarrow$
- $-32x^2 - 20x + 25 \Rightarrow$
- $20x^2 + 41x + 20 \Rightarrow$

Ex. 1.5.7: Find all real roots of:
- $x^4 - 25 = 0 \Rightarrow x = \pm\sqrt{5}$
- $x^4 - 49 = 0 \Rightarrow x = \pm\sqrt{7}$
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Ex. 1.5.8: Find all real roots of:
- $2x^2 + 5x + 1 = 0 \Rightarrow x = \frac{-5\pm\sqrt{17}}{4}$
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- $24x^2 + 79x + 40 \Rightarrow = (8x + 5)(3x + 8)$

Ex. 1.5.10: Factor
- $12x^2 + 44x + 35 \Rightarrow = (6x + 7)(2x + 5)$
- $27x^2 + 48x + 5 \Rightarrow = (3x + 5)(9x + 1)$
- $-32x^2 - 20x + 25 \Rightarrow = (4x + 5)(-8x + 5)$
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- $x^4 - 25 = 0 \Rightarrow x = \pm\sqrt{5}$
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- $20x^2 + 41x + 20 \Rightarrow = (4x + 5)(5x + 4)$

Ex. 1.5.11: What is the degree of equation
- $x^2 = x \Rightarrow$
- $x^2 + 4 = (x + 2)^2 \Rightarrow$
- $x^2 - 4 = (x + 7)^2 \Rightarrow$
- $(x + 2)^2 = x(x + 4) \Rightarrow$

Ex. 1.5.7: Find all real roots of:
- $x^4 - 25 = 0 \Rightarrow x = \pm\sqrt{5}$
- $x^4 - 49 = 0 \Rightarrow x = \pm\sqrt{7}$
- $x^4 - 100 = 0 \Rightarrow x = \pm\sqrt{10}$
- $x^4 - 144 = 0 \Rightarrow x = \pm 12$

Ex. 1.5.8: Find all real roots of:
- $2x^2 + 5x + 1 = 0 \Rightarrow x = \frac{-5\pm\sqrt{17}}{4}$
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- $-2x^2 + 5x + 1 = 0 \Rightarrow x = \frac{5\pm\sqrt{33}}{4}$
- $2x^2 - 12x + 1 = 0 \Rightarrow x = 3 \pm \sqrt{\frac{17}{2}}$
- $x^2 + 5x + 1 = 0 \Rightarrow x = \frac{-5\pm\sqrt{21}}{2}$
- $3x^2 + 4x + 1 = 0 \Rightarrow x = -1; -\frac{1}{3}$
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- $-32x^2 - 20x + 25 \Rightarrow = (4x + 5)(-8x + 5)$
- $20x^2 + 41x + 20 \Rightarrow = (4x + 5)(5x + 4)$

Ex. 1.5.11: What is the degree of equation
- $x^2 = x \Rightarrow 2$
- $x^2 + 4 = (x + 2)^2 \Rightarrow 1$
- $x^2 - 4 = (x + 7)^2 \Rightarrow 1$
- $(x + 2)^2 = x(x + 4) \Rightarrow$ undefined

Ex. 1.5.7: Find all real roots of:
- $x^4 - 25 = 0 \Rightarrow x = \pm\sqrt{5}$
- $x^4 - 49 = 0 \Rightarrow x = \pm\sqrt{7}$
- $x^4 - 100 = 0 \Rightarrow x = \pm\sqrt{10}$
- $x^4 - 144 = 0 \Rightarrow x = \pm 12$

Ex. 1.5.8: Find all real roots of:
- $2x^2 + 5x + 1 = 0 \Rightarrow x = \frac{-5\pm\sqrt{17}}{4}$
- $2x^2 + 2x - 1 = 0 \Rightarrow x = \frac{-1\pm\sqrt{3}}{2}$
- $-2x^2 + 5x + 1 = 0 \Rightarrow x = \frac{5\pm\sqrt{33}}{4}$
- $2x^2 - 12x + 1 = 0 \Rightarrow x = 3 \pm \sqrt{\frac{17}{2}}$
- $x^2 + 5x + 1 = 0 \Rightarrow x = \frac{-5\pm\sqrt{21}}{2}$
- $3x^2 + 4x + 1 = 0 \Rightarrow x = -1; -\frac{1}{3}$
- $2x^2 - 5x + 10 = 0 \Rightarrow$ No real solution
- $-2x^2 - 2x - 1 = 0 \Rightarrow$ No real solution

Ex. 1.5.9: Factor $ax^2 + bx + c =$
- $6x^2 + 13x + 6 \Rightarrow = (3x+2)(2x+3)$
- $12x^2 - 7x - 12 \Rightarrow = (4x+3)(3x-4)$
- $12x^2 + 38x + 20 \Rightarrow = 2(2x+5)(3x+2)$
- $24x^2 + 79x + 40 \Rightarrow = (8x+5)(3x+8)$

Ex. 1.5.10: Factor
- $12x^2 + 44x + 35 \Rightarrow = (6x+7)(2x+5)$
- $27x^2 + 48x + 5 \Rightarrow = (3x+5)(9x+1)$
- $-32x^2 - 20x + 25 \Rightarrow = (4x+5)(-8x+5)$
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Ex. 1.5.11: What is the degree of equation
- $x^2 = x \Rightarrow 2$
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Ex. 1.5.12: Find the degree of the equation $E = F$ if
- $E = x^3 + 3x^2$ and $F = 2x^3 - 5x^2 + x^3 + 12 \Rightarrow$
- $E = x^3 + 3x^2$ and $F = x^3 - 5x^2 \Rightarrow$
- $E = 2(x^3 + 3x^2)$ and $F = 2x^3 + 6x^2 + 12x \Rightarrow$
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- $E = 2(x^3 + 3x^2)$ and $F = 2x^3 + 6x^2 + 12x \Rightarrow 1$
- $E = x^3 + 3x^2$ and $F = (x + 1)^3 \Rightarrow 1$

Ex. 1.5.13: Find the prime power factorization

- of 360
- of 5000
- of 8100
- of 4096

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- of 360 $= 2^3 3^2 5$
- of 5000 $= 2^3 5^4$
- of 8100 $= 2^2 3^4 5^2$
- of 4096 $= 2^{12}$

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- $50, 100, 175 \Rightarrow$
- $40, 80, 16 \Rightarrow$
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- $5x^3 + x^4 + 6x^2 =$
- $x^5 + 3x^4 - x^3 =$
- $-8x^3 + 6x^4 - x^5 =$
- $5x^3 - x^4 + 6x^2 =$

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- $x^4 - 4x^3 - 5x^2 =$
- $x^4 + 5x^3 + 6x^2 =$
- $2x^4 - x^3 - 6x^2 =$
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Ex. 1.5.16: Factor each polynomial by using a special factorization.
- $x^2 - 9 =$
- $9 - x^2 =$
- $x^2 + 6x + 9 =$
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- $25 - 4x^2 =$
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- $4x^2 - 36 =$
- $9 - 4x^2 =$
- $100 - 49x^2 =$
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- $4x^2 - 1 =$
- $100 - x^2 =$
- $121 - 9x^2 =$
- $x^2 - 20x + 100 =$
- $4x^2 - 28x + 49 =$

Ex. 1.5.14: Find the Greatest Common Factor of
- $72, 180, 1200 \Rightarrow$ GCF 12
- $50, 100, 175 \Rightarrow$ GCF 25
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- $2x^2 + 12x + 3 = 0 \Rightarrow$
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- $\frac{4}{x} = 2 + \frac{4}{3-x} \Rightarrow$
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- $x^2 + x + 4 = 0 \Rightarrow$
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- $x^2 - x - 9 = 0 \Rightarrow$
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Ex. 1.5.22:

- $x^4 - 8x^2 + 8 = 0 \quad \Rightarrow$
- $4x^4 + 8x^2 = 12 \quad \Rightarrow$
- $u^4 - 4u^2 + 3 = 0 \quad \Rightarrow$
- $5 - u^2 - u^4 = 0 \quad \Rightarrow$

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Ex. 1.5.18:

- $\frac{4}{x}=2+\frac{4}{3-x} \Rightarrow x=1;6$
- $\frac{4}{x}=3+\frac{2}{3-x} \Rightarrow x=1;4$
- $\frac{-5}{x}=4+\frac{4}{3-x} \Rightarrow x=-1;\ \frac{15}{4}$
- $\frac{4}{x-3}=2+\frac{4}{3-x} \Rightarrow x=7$

Ex. 1.5.19:

- $\sqrt{2x+1}+1=x \Rightarrow x=4$
- $2\sqrt{2x+1}=3+x \Rightarrow x=1\pm 2i$
- $\sqrt{x+1}+1=2x \Rightarrow x=\frac{5}{4}$
- $\sqrt{9-5x}+1=3x \Rightarrow x=1$

Ex. 1.5.20:

- $\sqrt{x}+\sqrt{x-5}=5 \Rightarrow x=9$
- $\sqrt{x+9}-\sqrt{x}=2 \Rightarrow x=\frac{25}{16}$
- $\sqrt{x}+\sqrt{5-x}=5 \Rightarrow x=\frac{5}{2}(1\pm\sqrt{15}i)$
- $x-\sqrt{x+9}+1=0 \Rightarrow x=\frac{\sqrt{33}-1}{2}$

Ex. 1.5.21:

- $x^2+x+4=0 \Rightarrow x=\frac{-1\pm\sqrt{15}i}{2}$
- $x^2-x-9=0 \Rightarrow x=\frac{1\pm\sqrt{37}}{2}$
- $4-3x-2x^2=0 \Rightarrow x=\frac{-3\pm\sqrt{41}i}{4}$
- $-x^2-x-4=0 \Rightarrow x=\frac{-1\pm\sqrt{15}i}{2}$

Ex. 1.5.22:

- $x^4-8x^2+8=0$ $\Rightarrow x=\pm\sqrt{4+2\sqrt{2}};\ x=\pm\sqrt{4-2\sqrt{2}}$
- $4x^4+8x^2=12$ $\Rightarrow x=\pm 1;\ x=\pm\sqrt{3}i$
- $u^4-4u^2+3=0$ $\Rightarrow u=\pm 1;\ u=\pm\sqrt{3}$
- $5-u^2-u^4=0$ $\Rightarrow u=\pm\sqrt{(\sqrt{21}-1)/2};\ u=\pm\sqrt{(\sqrt{21}+1)/2}\ i$

Ex. 1.5.17: Solve by completing the square and check your answer by using the quadratic formula.

- $2x^2 + 12x + 3 = 0 \Rightarrow x = -3 \pm \sqrt{\frac{15}{2}}$
- $4x + 3 - x^2 = 0 \Rightarrow x = 2 \pm \sqrt{7}$
- $2x^2 + 3x + 3 = 0 \Rightarrow x = \frac{-3 \pm \sqrt{15}i}{4}$
- $6x^2 + 13x + 6 = 0 \Rightarrow x = -\frac{3}{2}; -\frac{2}{3}$

Ex. 1.5.18:
- $\frac{4}{x} = 2 + \frac{4}{3-x} \Rightarrow x = 1; 6$
- $\frac{4}{x} = 3 + \frac{2}{3-x} \Rightarrow x = 1; 4$
- $\frac{-5}{x} = 4 + \frac{4}{3-x} \Rightarrow x = -1; \frac{15}{4}$
- $\frac{4}{x-3} = 2 + \frac{4}{3-x} \Rightarrow x = 7$

Ex. 1.5.19:
- $\sqrt{2x+1} + 1 = x \Rightarrow x = 4$
- $2\sqrt{2x+1} = 3 + x \Rightarrow x = 1 \pm 2i$
- $\sqrt{x+1} + 1 = 2x \Rightarrow x = \frac{5}{4}$
- $\sqrt{9-5x} + 1 = 3x \Rightarrow x = 1$

Ex. 1.5.20:
- $\sqrt{x} + \sqrt{x-5} = 5 \Rightarrow x = 9$
- $\sqrt{x+9} - \sqrt{x} = 2 \Rightarrow x = \frac{25}{16}$
- $\sqrt{x} + \sqrt{5-x} = 5 \Rightarrow x = \frac{5}{2}(1 \pm \sqrt{15}i)$
- $x - \sqrt{x+9} + 1 = 0 \Rightarrow x = \frac{\sqrt{33}-1}{2}$

Ex. 1.5.21:
- $x^2 + x + 4 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{15}i}{2}$
- $x^2 - x - 9 = 0 \Rightarrow x = \frac{1 \pm \sqrt{37}}{2}$
- $4 - 3x - 2x^2 = 0 \Rightarrow x = \frac{-3 \pm \sqrt{41}i}{4}$
- $-x^2 - x - 4 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{15}i}{2}$

Ex. 1.5.22:
- $x^4 - 8x^2 + 8 = 0$ $\Rightarrow x = \pm\sqrt{4 + 2\sqrt{2}}; x = \pm\sqrt{4 - 2\sqrt{2}}$
- $4x^4 + 8x^2 = 12$ $\Rightarrow x = \pm 1; x = \pm\sqrt{3}i$
- $u^4 - 4u^2 + 3 = 0$ $\Rightarrow u = \pm 1; u = \pm\sqrt{3}$
- $5 - u^2 - u^4 = 0$ $\Rightarrow u = \pm\sqrt{(\sqrt{21}-1)/2}; u = \pm\sqrt{(\sqrt{21}+1)/2}\ i$

Ex. 1.5.23:
- $x^{\frac{1}{3}} + x^{\frac{1}{6}} = 2$ $\Rightarrow$
- $x^{\frac{1}{2}} - 4x^{\frac{1}{4}} + 2 = 0$ $\Rightarrow$
- $x^{\frac{1}{2}} - 4x^{\frac{1}{4}} + 3 = 0$ $\Rightarrow$
- $x^{\frac{1}{2}} - 4x^{\frac{1}{4}} + 4 = 0$ $\Rightarrow$

Ex. 1.5.17: Solve by completing the square and check your answer by using the quadratic formula.

- $2x^2+12x+3=0 \Rightarrow x=-3\pm\sqrt{\frac{15}{2}}$
- $4x+3-x^2=0 \Rightarrow x=2\pm\sqrt{7}$
- $2x^2+3x+3=0 \Rightarrow x=\frac{-3\pm\sqrt{15}i}{4}$
- $6x^2+13x+6=0 \Rightarrow x=-\frac{3}{2};-\frac{2}{3}$

Ex. 1.5.18:

- $\frac{4}{x}=2+\frac{4}{3-x} \Rightarrow x=1;6$
- $\frac{4}{x}=3+\frac{2}{3-x} \Rightarrow x=1;4$
- $\frac{-5}{x}=4+\frac{4}{3-x} \Rightarrow x=-1;\frac{15}{4}$
- $\frac{4}{x-3}=2+\frac{4}{3-x} \Rightarrow x=7$

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- $x^2+x+4=0 \Rightarrow x=\frac{-1\pm\sqrt{15}i}{2}$
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- $-x^2-x-4=0 \Rightarrow x=\frac{-1\pm\sqrt{15}i}{2}$

Ex. 1.5.22:

- $x^4-8x^2+8=0$ $\Rightarrow x=\pm\sqrt{4+2\sqrt{2}};\ x=\pm\sqrt{4-2\sqrt{2}}$
- $4x^4+8x^2=12$ $\Rightarrow x=\pm 1;\ x=\pm\sqrt{3}i$
- $u^4-4u^2+3=0$ $\Rightarrow u=\pm 1;\ u=\pm\sqrt{3}$
- $5-u^2-u^4=0$ $\Rightarrow u=\pm\sqrt{(\sqrt{21}-1)/2};\ u=\pm\sqrt{(\sqrt{21}+1)/2}\ i$

Ex. 1.5.23:

- $x^{\frac{1}{3}}+x^{\frac{1}{6}}=2$ $\Rightarrow x=1$
- $x^{\frac{1}{2}}-4x^{\frac{1}{4}}+2=0$ $\Rightarrow x=68\pm 48\sqrt{2}$
- $x^{\frac{1}{2}}-4x^{\frac{1}{4}}+3=0$ $\Rightarrow x=1;81$
- $x^{\frac{1}{2}}-4x^{\frac{1}{4}}+4=0$ $\Rightarrow x=16$

Section 1.6:Intervals and Inequalities

- 1.6.1: Intervals on the real number line
- 1.6.2: Absolute value inequalities
- Section 1.6 Review

Section 1.6 Preview: Definitions and Theorems

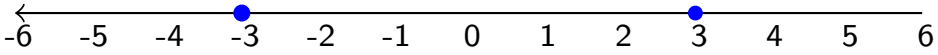
- Definition 1.6.1: Given expressions E and F, an *inequality in x* is
- Definition 1.6.2: On the real number line, inequality signs are
- Definition 1.6.3: Combining inequalities into intervals
- Definition 1.6.4: A finite interval is
- Definition 1.6.5: An infinite interval is
- Definition 1.6.6: The length of an interval between a and b is
- Definition 1.6.7: Closed interval: $[-4, 3]$: all x with $-4 \leq x \leq 3$
- Definition 1.6.8: Open interval: $(-4, 3)$: all x with $-4 < x < 3$
- Definition 1.6.9: Half-open interval: $[-4, 3)$: all x with $-4 \leq x < 3$
- Definition 1.6.10: Half-open interval: $(-4, 3]$: all x with $-4 < x \leq 3$
- Definition 1.6.11: $(4, \infty)$: all x with $4 < x$
- Definition 1.6.12: $[4, \infty)$: all x with $4 \leq x$
- Definition 1.6.13: $(-\infty, 4)$: all x with $x < 4$
- Definition 1.6.14: $(-\infty, 4]$: all x with $x \leq 4$
- Definition 1.6.15: $(-\infty, \infty)$: all real numbers
- Definition 1.6.16: The absolute value $|x|$ of a real number x is

Section 1.6 Preview: Procedures

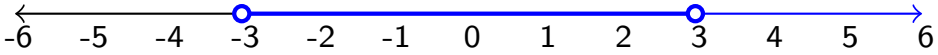
- Procedure 1.6.1: To solve inequality $P < Q$ in x, rewrite it as $P - Q < 0$.
- Procedure 1.6.2: To rewrite the inequality $x < y < z$
- Procedure 1.6.3: To draw an interval's endpoint(s)
- Procedure 1.6.4: To solve absolute value inequalities:

1.6.1 Intervals on the real number line

In the following diagram, -3 is to the left of 3. Write this as $-3 < 3$ and say "-3 is less than 3."



The combined inequality $-3 < x < 3$ means: $-3 < x$ *and* $x < 3$. All such x values form the *interval* $(-3, 3)$, shown on the diagram below, where the hollow dots show that -3 and 3 are not included.

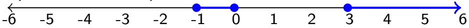


Definition: Given expressions E and F, an *inequality in x:* is $E \leq F$ $E < F$; $E \geq F$, or $E > F$

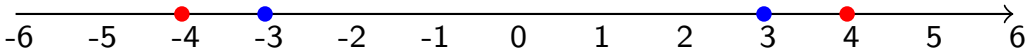
- A real number a is a *solution* of an inequality if substituting a for x yields a correct statement .
- To *solve an inequality*, find all solutions.

Example: The solutions of the equation $x(x+1)(x-3) = 0$ are $\{x = 0; x = -1; x = 3\}$. We will soon see that the solutions of $x(x+1)(x-3) \geq 0$ form two intervals that can be written in three ways:

- (interval form) x in $[-1, 0]$ or x in $[3, \infty)$.
- (inequality form) $-1 \leq x \leq 0$ or $3 \leq x$
- (diagram form):



We have solved equations by rewriting them as simpler equations with the same solutions. Rewriting inequalities is similar, but there is one major change. Multiplying two numbers by -1 reverses their order on the number line: 3 is to the left of 4 but -3 is to the right of -4:



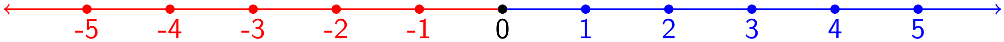
In the following, $<$ may be replaced by $\leq$ or by $>$ or by $\geq$.

***To solve inequality $P < Q$ in x*, rewrite it to say that x is in one or more intervals on the number line. You may**

- add (subtract) any expression E to (from) both sides to get $P + E < Q + E$ or $P - E < Q - E$.
- multiply or divide both sides by a *number* $k > 0$ to get $kP < kQ$ or $\frac{P}{k} < \frac{Q}{k}$.
- multiply or divide both sides by a negative number $k < 0$ provided you do one of the following:
  - Exchange the sides of the inequality to get $-kQ < -kP$ and $\frac{Q}{-k} < \frac{P}{-k}$ or
  - Reverse the inequality sign to get $-kP > -kQ$ and $\frac{P}{-k} > \frac{Q}{-k}$. This answer is nicer: the number to the left on the number line is also on the left side of the inequality.

Be careful: Never multiply or divide an inequality by a letter or expression whose sign is not known.

More examples of interval notation



On the real number line

- $-3 < 5$ means: -3 is to the left of 5.
- $5 > -3$ means: 5 is to the right of -3.
- $x \leq 5$ means: $x < 5$ or $x = 5$
- $x \geq -3$ means: $x > -3$ or $x = -3$

Combining inequalities into intervals

- $3 \leq x \leq 5$ means $3 \leq x$ and $x \leq 5$
- $3 < x < 5$ means $3 < x$ and $x < 5$
- $3 \leq x < 5$ means $3 \leq x$ and $x < 5$
- $3 < x \leq 5$ means $3 < x$ and $x \leq 5$

As stated above, it's best to write inequalities that match the order of numbers on the number line: For example, rewrite $5 \geq x > 3$ as $3 < x \leq 5$.

Warning: Mixed inequalities ,such as $3 < x > 4$ (involving both $>$ and $<$) are illegal! Never use them.

Also, $a < b < c$ implies $a < c$. Therefore statements such as $4 < x < 3$ (which would imply $4 < 3$) may not be used as an abbreviation for $4 < x$ and $x < 3$, both of which can't be true.

When you have a legitimate double inequality such as $x < y < z$, you can do all the things to x, y, z simultaneously that are permitted steps in rewriting a single inequality.

To rewrite the inequality $x < y < z$ you may

- Add (subtract) any number or expression to (from) x, y, and z;
- Multiply or divide x, y, and z by the same real number $k > 0$.
- Multiply or divide x, y, z by the same negative real number $k < 0$ and reverse the inequalities. For example if $x < y < z$ then $3x < 3y < 3z$ but $-3x > -3y > -3z$, best rewritten as $-3z < -3y < -3x$.

Intervals are parts of the number line described by double inequalities.

A double inequality such as $-4 \leq x \leq 3$ says that x is any number between and including the two specified numbers, in this case -4 and 3, on the number line.

To draw a picture of all x values that satisfy an inequality, we describe the piece or pieces of the number line where x lies. These pieces are called intervals.

A finite interval is the set of real numbers x satisfying one of the following

- $a < x < b$: the open interval (a, b)
- $a \leq x \leq b$: the closed interval $[a, b]$
- $a \leq x < b$: the half-open interval $[a, b)$
- $a < x \leq b$: the half-open interval $(a, b]$

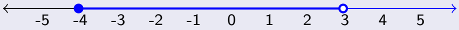
An infinite interval is the set of real numbers x satisfying one of the following

- $a < x$: the interval (a, ∞)
- $x < b$: the interval $(-\infty, b)$
- $a \leq$: the interval $[a, \infty)$
- $x \leq b$: the interval $(-\infty, b]$

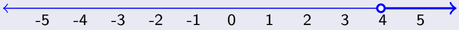
a and b are the interval's endpoints.

Here are two basic examples.

$[-4, 3)$: all x with $-4 \leq x < 3$



$(4, \infty)$: all x with $4 < x$



Meaning of brackets and parentheses

- $(4$ means: omit 4 at the interval's left end.
- $[4$ means: include 4 at the interval's left end.
- $3)$ means: omit 3 at the interval's right end.
- $3]$ means: include 3 at the interval's right end.

Endpoints must be drawn carefully:

How to draw an interval's endpoint(s)

- To include the endpoint, draw a solid dot.
- To omit the endpoint, draw a hollow circle.

Illustrations of intervals

Definition: The *length of an interval* between a and b is

- $b - a$ if the interval is finite, with left endpoint a and right endpoint b
- *undefined* if the interval is infinite.

Closed interval: $[-4, 3]$: all x with $-4 \leq x \leq 3$

Open interval: $(-4, 3)$: all x with $-4 < x < 3$

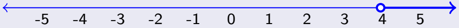
Half-closed interval: $[-4, 3)$: all x with $-4 \leq x < 3$

Half-open interval: $(-4, 3]$: all x with $-4 < x \leq 3$

The above 4 intervals have length $3 - (-4) = 7$.

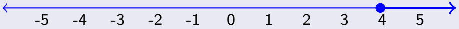
The length of the 5 intervals below is undefined.

$(4, \infty)$: all x with $4 < x$

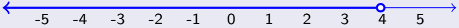


Don't write $(4, \infty)$ as $4 < x < \infty$. Reason: inequality signs compare numbers, but ∞ is not a number!

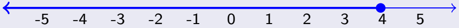
$[4, \infty)$: all x with $4 \leq x$



$(-\infty, 4)$: all x with $x < 4$

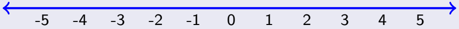


$(-\infty, 4]$: all x with $x \leq 4$



To describe the entire real number line as an interval:

$(-\infty, \infty)$: all real numbers x

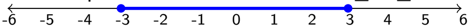


1.6.2 Absolute value inequalities.

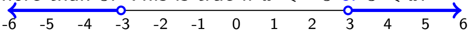
The absolute value $|x|$ of a real number x

- is the distance from x to 0 on the number line.
- $|x| = x$ if $x \geq 0$ but $|x| = -x$ if $x \leq 0$.

• $|x| \leq 3$ says that the distance between x and 0 is less than or equal to 3. This is true if $-3 \leq x \leq 3$:



• $|x| > 3$ says that the distance between x and 0 is more than 3. This is true if $x < -3$ or $3 < x$.



Be careful: Never rewrite $3 < x$ *or* $x < -3$ as $3 < x < -3$, since no such x exists.

How to solve absolute value inequalities:

Let $K > 0$ be a real number.

- To solve $|x| < K$, solve $-K < x < K$.
- To solve $|x| > K$, solve $x < -K$ *or* $K < x$.
  Let $K \geq 0$ be a real number and E any expression in x.
- To solve $|E| \leq K$ solve $-K \leq E \leq K$ for x.
- To solve $|E| \geq K$ solve $E \leq -K$ *or* $K \leq E$ for x.

In this section, boxed answers are written in two ways: as one or more intervals, or as one or more inequalities.

Example 1: Solve $|2x + 4| \leq 3$. Here $E = 2x + 4$; $K = 3$

Rewrite as $-K \leq E \leq K$

$-3 \leq 2x + 4 \leq 3$

Subtract 4 $-7 \leq 2x \leq -1$

Divide by 2 $\boxed{-\frac{7}{2} \leq x \leq -\frac{1}{2}}$ (Inequality form)

$\boxed{\left[-\frac{7}{2}, -\frac{1}{2}\right]}$ (Interval form)

Now let $E = -2x + 4$ and $K = 3$:

Example 2: Solve $|-2x + 4| > 3$

Rewrite as $E \leq -K$ *or* $K \leq E$

$-2x + 4 < -3$ *or* $3 < -2x + 4$

Subtract 4 $-2x < -7$ *or* $-1 < -2x$

Divide by -2 and *reverse the inequality signs*

to obtain $x > \frac{7}{2}$ *or* $\frac{1}{2} > x$

Rewrite as $\boxed{x < \frac{1}{2} \text{ or } \frac{7}{2} < x}$ (Inequality)

$\boxed{x \text{ in } \left(-\infty, \frac{1}{2}\right) \cup \left(\frac{7}{2}, \infty\right)}$ (Interval)

Reminder: Write $(-\infty, \frac{1}{2})$ in inequality form as $x < \frac{1}{2}$, not as $-\infty < x < \frac{1}{2}$.

1.6.3 Section 1.6 Quiz

Ex. 1.6.1: Solve $|2x + 4| < 3$.

Ex. 1.6.2: Solve $|2x + 4| \geq 3$.

Section 1.6 Review: Intervals and Inequalities

Ex. 1.6.1: Solve the following inequalities. Write your answer with both interval and inequality notation.

- $|2x + 4| < 3$
- $|2x - 4| < 3$
- $|2x + 4| < 0$
- $|2x + 4| > 0$

Ex. 1.6.1: Solve the following inequalities. Write your answer with both interval and inequality notation.

- $|2x+4|<3 \quad \Rightarrow -\frac{7}{2}<x<-\frac{1}{2} \quad x$ in $(-\frac{7}{2},-\frac{1}{2})$
- $|2x-4|<3 \quad \Rightarrow \frac{1}{2}<x<\frac{7}{2} \quad x$ in $(\frac{1}{2},\frac{7}{2})$
- $|2x+4|<0 \quad \Rightarrow$ No solution
- $|2x+4|>0 \quad \Rightarrow x<-2; x>-2 \quad x$ in $(-\infty,-2)\cup(-2,\infty)$

Section 1.6 Review: Intervals and Inequalities

Ex. 1.6.1: Solve the following inequalities. Write your answer with both interval and inequality notation.

- $|2x+4|<3 \quad \Rightarrow -\frac{7}{2}<x<-\frac{1}{2} \quad x$ in $(-\frac{7}{2},-\frac{1}{2})$
- $|2x-4|<3 \quad \Rightarrow \frac{1}{2}<x<\frac{7}{2} \quad x$ in $(\frac{1}{2},\frac{7}{2})$
- $|2x+4|<0 \quad \Rightarrow$ No solution
- $|2x+4|>0 \quad \Rightarrow x<-2; x>-2 \quad x$ in $(-\infty,-2)\cup(-2,\infty)$

Ex. 1.6.2: Solve the following inequalities. Write your answer with both interval and inequality notation.

- $|2x+4|\geq 3$
- $|2x-4|\geq 3$
- $|2x+4|\leq 0$
- $|2x+4|\geq 0$

Ex. 1.6.1: Solve the following inequalities. Write your answer with both interval and inequality notation.

- $|2x+4|<3 \quad \Rightarrow -\frac{7}{2}<x<-\frac{1}{2} \quad$ x in $(-\frac{7}{2},-\frac{1}{2})$
- $|2x-4|<3 \quad \Rightarrow \frac{1}{2}<x<\frac{7}{2} \quad$ x in $(\frac{1}{2},\frac{7}{2})$
- $|2x+4|<0 \quad \Rightarrow$ No solution
- $|2x+4|>0 \quad \Rightarrow x<-2; x>-2 \quad$ x in $(-\infty,-2)\cup(-2,\infty)$

Ex. 1.6.2: Solve the following inequalities. Write your answer with both interval and inequality notation.

- $|2x+4|\geq 3 \quad \Rightarrow x\leq -\frac{7}{2}; x\geq -\frac{1}{2} \quad$ x in $(-\infty,-\frac{7}{2}]\cup[-\frac{1}{2},\infty)$
- $|2x-4|\geq 3 \quad \Rightarrow x\leq \frac{1}{2}; x\geq \frac{7}{2} \quad$ x in $(-\infty,\frac{1}{2}]\cup[\frac{7}{2},\infty)$
- $|2x+4|\leq 0 \quad \Rightarrow x=-2 \quad$ x in $[-2,-2]$
- $|2x+4|\geq 0 \quad \Rightarrow x$ in $(-\infty,\infty)$ This inequality is true for all real numbers.
  However, the solution cannot be written as an inequality.
  It doesn't make sense to write $-\infty<x<\infty$.
  Reason: since ∞ is not a number,
  it can't be compared to any number x.

Section 1.7: Formulas and functions

- 1.7.1: Substituting in formulas
- 1.7.2: Defining and using functions
- 1.7.3: Function evaluation
- 1.7.4: Using parentheses when you work with functions
- 1.7.5: Difference quotients
- 1.7.6: Combining functions using algebra
- 1.7.7: Function composition
- 1.7.8: Inverse functions
- Section 1.7 Review

Section 1.7 Preview: Definitions and Theorems

- Definition 1.7.1: Function definitions and notation
- Definition 1.7.2: The natural domain of function $f(x)$ consists of
- Definition 1.7.3: The domain of $f+g, f-g$, and the product fg is
- Definition 1.7.4: The *domain of a quotient function* f/g is
- Definition 1.7.5: Composing functions
- Definition 1.7.6: Functions f and g are *inverse functions* if
- Definition 1.7.7: Square and square root are not inverse functions unless their domains are chosen carefully.

Section 1.7 Preview: Procedures

- Procedure 1.7.1: How and when to use parentheses
- Procedure 1.7.2: To find the value of a function
- Procedure 1.7.3: To simplify any expression involving a function value $f(E)$

1.7.1: Substituting in formulas requires parentheses

How and when to use parentheses

- Review Section 1.1.2: When you do PEMDAS, put parentheses around the result of each E, M, or D operation.
- Review Section 1.1.3: Use parentheses when you substitute an expression for a letter, unless the substituted expression is also a letter.
- Use parentheses when you substitute a letter for an expression. See Example 3 below.
- Use parentheses when you work with functions. Continue reading.

Be careful: If you leave out required parentheses, the work from then on will be a waste of time.

The following were done earlier, in Section 1.3.

Example 1: Substitute $a-b$ for x in expression ax^2+bx+c.

Answer: $\boxed{a(a-b)^2+b(a-b)+c}$

Example 2: If $E=AB-B^2$, find E if $A=x+3$ and $B=y-3$. Answer: $\boxed{(x+3)(y-3)-(y-3)^2}$

Example 3: Suppose
$E=(x+2)^3(y-3)^4-3(x+2)^4(y-3)^3$.
Rewrite E by setting $U=x+2$ and $V=y-3$.

Answer: $\boxed{E=U^3V^4-3U^4V^3}$

Example 4: State the commutative law $AB=BA$ in the case $A=x+3$ and $B=y-2$.

Answer: $\boxed{(x+3)(y-2)=(y-2)(x+3)}$

Example 5: If $y=x^2-3x$ and $x=z+2$, express y in terms of z. Answer: $\boxed{y=(z+2)^2-3(z+2)}$

Example 6:
Rewrite $x^2-x(x+1)-(x+3)^2$ as a polynomial.

Solution: To simplify $\quad x^2-x(x+1)-(x+3)^2$

Parentheses the results of operations M and E: $\quad x^2-(x^2+x)-(x^2+6x+9)$

Distribute minus signs $\quad =x^2-x^2-x-x^2-6x-9$

Answer: $=\boxed{-x^2-7x-9}$

1.7.2: Defining and using functions

A crucial ideas in physics is:
The position of a moving particle depends on time.
This means: at any particular time, a particle will be in one and only one place.

Let's say, for our problem, that Time is the number of seconds after 3:00 PM. At Time 0, a ball is dropped from a cliff 1600 feet high. Let Height be how many feet the ball is above the ground.

Isaac Newton figured out the following:
Starting from Time $= 0$ until the ball hits the ground,
Height equals $1600 - 16 \cdot \text{Time}^2$

Abbreviate this by writing h for Height and t for Time:
Let $h = 1600 - 16t^2$. Then h is the ball's height (in feet) at time t seconds after 3:00 PM .

The following table shows the ball's height at various times. When $t = 10$, the height is $h = 0$, so the ball hits the ground after 10 seconds.

Be careful: Time t *does not have to be, and usually is not, a whole number.* It can be any real number in the interval [0,10].
A seemingly minor change in notation puts h and t together: $h(t) = 1600 - 16t^2$
This expresses h as a *function* of t

Function definition: $h(t) := 1600 - 16t^2$
Read this as: "h of t *is defined to be* $1600 - 16t^2$."

The words "Function definition" tell you two things:
- h is the name of the function being defined.
- t is the value of the input to the function.

Since (very unusually) $h(t)$ could mean $h \cdot t$, math texts use the word "Let" to state that a function is being defined, as in Let $h(t) = 1600 - t^2$.

Time $t =$	Height at time t is $h(t) = 1600 - 16t^2$ ft	
0	$h(0) = 1600 - 16(0)^2$	$= 1600$ ft
0.5	$h(0.5) = 1600 - 16(0.5)^2$	$= 1596$ ft
1	$h(1) = 1600 - 16(1)^2$	$= 1584$ ft
2.5	$h(2.5) = 1600 - 16(2.5)^2$	$= 1500$ ft
3	$h(3) = 1600 - 16(3)^2$	$= 1456$ ft
4	$h(4) = 1600 - 16(4)^2$	$= 1384$ ft
6	$h(6) = 1600 - 16(6)^2$	$= 1024$ ft
7	$h(7) = 1600 - 16(7)^2$	$= 816$ ft
8	$h(8) = 1600 - 16(8)^2$	$= 576$ ft
9	$h(9) = 1600 - 16(9)^2$	$= 304$ ft
10	$h(10) = 1600 - 16(10)^2$	$= 0$ ft

Functions describe how one quantity depends on another

A quantity is a number together with a unit of measurement, such as feet or seconds. One quantity of interest often depends on another. Classical physics assumes that the position of a moving object depends on time. Other examples include:

- The area of a square depends on the length of its side.
- The price of a shopping bag full of apples depends on how many pounds of apples are in it.
- The position of a missile depends on the time that has elapsed since it was launched.

Functions are often used to describe how one quantity depends on another.

I drop a ball off the top of a 1600-foot building. From the time the ball is dropped until it hits the ground, the ball's height (measured in feet) above the ground at time t seconds is $1600 - 16t^2$. The ball's height above the ground depends on how long it has been falling.

Use the symbol $h(t)$ to *abbreviate* "the ball's height in feet at time t seconds after it is dropped."

Then $h(t) = 1600 - 16t^2$ from time $t = 0$ until the ball hits the ground.

Example 7 : How high above the ground is the ball after 2 seconds?

Solution: To find $h(2)$, substitute 2 for t in $h(t) = 1600 - 16t^2$. Then
$h(2) = 100 - 16(2)^2 = 100 - 16(4) = 1600 - 64 = 1536$

Answer:
After 2 seconds, the ball is 1536 feet above the ground.

Example 8: When does the ball hit the ground?

Solution: Since $h(t)$ is the ball's height at time t, and the ball's height when it hits the ground is 0, solve $h(t) = 0$ for t.

$h(t) = 1600 - 16t^2 = 0$
$16t^2 = 1600$
$t^2 = 100$
$t = \pm\sqrt{100} = \pm 10$ and so $t = 10$ or $t = -10$.

Since we require $t \geq 0$, the only valid answer is $t = 10$.

Answer: The ball hits the ground 10 seconds after it is dropped.

1.7.3: Function evaluation: use parentheses when you substitute for the independent variable.

Function definitions and notation

In the *function definition* $h(t) := 1600 - 16t^2$

- The function name is h;
- The input (or independent) variable is t;
- The value of h at t is $h(t) = 1600 - 16t^2$. This a function value, not a definition, and so the colon is omitted.

How to find the value of a function

Let t be any letter and F any expression.
Let $h(t) = F$ be a function definition.
Let E be any number or expression.

- To find $h(E)$, replace every t in expression F by (E). But if E is a letter, omit the parentheses.
- To evaluate or to find the value of h at E, simplify $h(E)$.

Example: If $h(x) := x^2$, then $h(a+2) = (a+2)^2$. Here we *substituted* (plugged in) $a+2$ for the independent variable x. We replaced x by $(a+2)$.
Be careful: The parentheses are crucial! Omitting them will usually yield an incorrect answer!

Example 9: Let $h(t) = 3 - t - t^2$. Find
a) $h(-7)$ b) $h(t+3)$ c) $h(t+k)$ d) $h(2t+3)$

Solution:
To find $h(-7)$, substitute -7 for t in $3 - t - t^2$. This means: replace every letter t in $3 - t - t^2$ by (-7).

a): $h(-7) = 3 - (-7) - (-7)^2$. Similarly:
b): $h(t+3) = 3 - (t+3) - (t+3)^2$
c): $h(t+k) = 3 - (t+k) - (t+k)^2$
d): $h(2t+3) = 3 - (2t+3) - (2t+3)^2$

Example 10: Let $f(x) = ax^2 + bx + c$. Find
a) $f(-7)$ b) $f(t+3)$ c) $f(ax+b)$ d) $f(x+h)$

Solution:
To find $f(-7)$, substitute -7 for x in $ax^2 + bx + c$ This means: replace every letter x in $ax^2 + bx + c$ by (-7).

a): $f(-7) = a(-7)^2 + b(-7) + c$
b): $f(t+3) = a(t+3)^2 + b(t+3) + c$
c): $f(ax+b) = a(ax+b)^2 + b(ax+b) + c$
d): $f(x+h) = a(x+h)^2 + b(x+h) + c$

Exercise: Rewrite each answer above as a polynomial by multiplying out and collecting terms.

1.7.4: Using parentheses when you work with functions

We will often say "If $f(x) = ...$" instead of "Let $f(x) = ...$" to state a function definition.

Example 11: If $f(x) = x^2 - 3x$, find $f(b)$.

Solution: Replace x by b to get $\boxed{f(b) = b^2 - 3b.}$

Parentheses around b are not needed. However, finding $f(2)$ requires parentheses around 2. Why?

Example 12: If $f(a) = 1 - 2a - 3a^2$, find $f(-2)$.

Solution: Replace every a in $1 - 2a - 3a^2$ by $(-2\ )$

$$\boxed{f(-2) = 1 - 2(-2) - 3(-2)^2 = -7}$$

The parentheses are colored red only for emphasis.

Example 13: If $f(x) = a^2 + b^2$, find $f(x + h)$.

Solution: Since there is no x in $= a^2 + b^2$, do nothing!

$$\boxed{f(x + h) = a^2 + b^2}$$

In this example, there aren't any $x's$ on the right side of the function definition. As a result, the function value $f(E) = a^2 + b^2$ doesn't depend on E. We say that f is a constant function.

To simplify any expression involving a function value $f(E)$,

first replace $f(E)$ by $(f(E))$.

Example 14: If $f(x) = x^2 - 3x$, rewrite $f(x + h) - f(x)$ as a polynomial.

Solution: First find $f(x + h)$: substitute $x + h$ for x in $f(x) = x^2 - 3x$ to get

$f(x + h) = (x + h)^2 - 3(x + h)$. Then finding $f(x + h) - f(x)$ requires inserting parentheses:

$$= (f(x + h)) - (f(x))$$
$$= ((x + h)^2 - 3(x + h)) - (x^2 - 3x)$$
$$= (x + h)^2 - 3(x + h) - x^2 + 3x$$
$$= x^2 + 2xh + h^2 - (3x + 3h) - x^2 + 3x$$
$$= x^2 + 2xh + h^2 - 3x - 3h - x^2 + 3x =$$
$$\boxed{2xh + h^2 - 3h}$$

Example 15: If $f(x) = x^2 - 3x$, find $f(a)f(b) - f(c)$

Solution: Since a, b, c are letters, no parentheses are needed to find $f(a) = a^2 - 3a$; $f(b) = b^2 - 3b$; and $f(c) = c^2 - 3c$. To find $f(a)f(b) - f(c)$, parenthesize every function value to obtain $(f(a))\ (f(b)) - (f(c)) =$

$$\boxed{(a^2 - 3a)(b^2 - 3b) - (c^2 - 3c\ )}$$

This is the answer, since the question said only to *find* $f(a)f(b) - f(c)$. If the question had said " find the value of" or "find and simplify", the answer would be $a^2b^2 - 3a^2b - 3ab^2 + 9ab - c^2 + 3c$

Example 15 continued: Let $f(x) = 1 - x^2$. Rewrite each of the bulleted expressions as a polynomial.

- $2x^2 - f(x) = 2x^2 - (1 - x^2) = 2x^2 - 1 + x^2 = \boxed{3x^2 - 1}$
- $x^2 f(x) = x^2(1 - x^2) = x^2 - x^4 = \boxed{-x^4 + x^2}$. Reminder: polynomial exponents decrease from left to right.
- $2f(x+h) - f(x)$

$= 2(1 - (x+h)^2) - (1 - x^2)$

$= (2 - 2(x+h)^2) - (1 - x^2)$ Here we inserted parentheses around the result of the multiply operation

$= 2 - 2(x+h)^2) - (1 - x^2)$ but in this example they aren't needed.

$= 2 - 2(x^2 + 2xh + h^2) - 1 + x^2$

$= 2 - 2x^2 - 4xh - 2h^2 - 1 + x^2 = \boxed{-x^2 - 4xh - 2h^2 + 1}$

- $2 - f(x)f(y)$

$= 2 - (1 - x^2)(1 - y^2)$

When you multiply out, put the product in parentheses:

$= 2 - (1 - y^2 - x^2 + x^2y^2)$

$= 2 - 1 + y^2 + x^2 - x^2y^2 = 1 + y^2 + x^2 - x^2y^2 = \boxed{-x^2y^2 + x^2 + y^2 + 1}$

1.7.5: Difference quotients

Example 16: Given $f(x) = 3x - x^2$, find and simplify $\dfrac{f(x+h) - f(x)}{h}$.

Start with the function definition $\quad f(x) = 3x - x^2$

Substitute $x + h$ for x: $\quad f(x+h) = 3(x+h) - (x+h)^2$

Here is the problem: $\quad \dfrac{f(x+h) - f(x)}{h}$

Substitute for $f(x)$ and $f(x+h)$: $\quad = \dfrac{(3(x+h) - (x+h)^2) - (3x - x^2)}{h}$

Expand $(x+h)^2$: $\quad = \dfrac{3x + 3h - (x^2 + 2hx + h^2) - (3x - x^2)}{h}$.

Distribute minus signs: $\quad = \dfrac{3x + 3h - x^2 - 2hx - h^2 - 3x + x^2}{h}$

Collect terms $\quad = \dfrac{3h - 2hx - h^2}{h}$

Factor and cancel $\quad = \dfrac{\not{h}(3 - 2x - h)}{\not{h}} = \boxed{-2x - h + 3}$

Section 1.7.2 studied the function $f(t) = 96t - 16t^2$, the height above ground of a rock thrown upward from the ground at speed 96 feet per second. For various small values of h, we calculated the average rate of change of $f(t)$ from $t = 2$ to $t = 2 + h$ as $\frac{f(2+h) - f(2)}{h}$. Let's do the same thing here, but for arbitrary h.

Example 17: Given $f(t) = 96t - 16t^2$, find and simplify $\dfrac{f(2+h)-f(2)}{h}$.

Solution:

Start with the function definition $f(t) = 96t - 16t^2$

Substitute 2 for t: $f(2) = 96(2) - 16(2)^2 = 128$

Substitute $2+h$ for t: $f(2+h) = 96(2+h) - 16(2+h)^2$

Average rate of change is: $\dfrac{f(2+h)-f(2)}{h}$

Substitute for $f(2+h)$: $= \dfrac{(96(2+h) - 16(2+h)^2) - 128}{h}$.

Expand $(x+h)^2$: $= \dfrac{96(2) + 96h - 16(4 + 4h + h^2) - 128}{h}$

Distribute -16: $= \dfrac{192 + 96h - 64 - 64h - 16h^2 - 128}{h}$

Collect, factor, cancel $= \dfrac{32h - 16h^2}{h} = \dfrac{\cancel{h}(32 - 16h)}{\cancel{h}} = \boxed{32 - 16h}$

Conclusion: the average velocity of the rock from time $t = 2$ to time $t = 2 + h$ is $32 - 16h$ feet/second.

Combining functions using algebra.

Functions studied so far have been defined by a formula $f(x) := \ldots$. where x is a real number. Such a formula might not make sense for all values of x. For example $f(x) = \sqrt{x}$ makes sense only for $x \geq 0$.

The *natural domain* of function $f(x)$ consists of

all real numbers x for which $f(x)$ is a real number.
Example: the domain of $f(x) = \sqrt{x}$ is $[0, \infty)$.

We'll omit the word *natural* from now on: other types of domains are discussed in some later math courses.

Functions can be added, subtracted, multiplied, divided, and raised to a power, just like numbers or algebra expressions.
If $f(x) = x^2$ and $g(x) = x + 2$, let $h = f + g$ be the sum of the functions f and g, defined by
$h(x) = (f + g)(x) = f(x) + g(x) = x^2 + x + 2$.

Suppose f and g have different domains. Since $h(x) = f(x) + g(x)$, both $f(x)$ and $g(x)$ must be defined if $h(x)$ is to make sense. The same idea applies to function subtraction and multiplication.

The *domain of $f + g, f - g$, and the product fg*

consists of all x in the domains of *both* f and g.

Example 18: Let $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{x}$.
Find the formula for $h = f + g$ and state its domain.
Solution: The domain of f is all x with $x \geq 0$ while the domain of g is $x \neq 0$. Therefore x is in the domain provided $x \geq 0$ and $x \neq 0$. These are both satisfied precisely when $x > 0$.
Answer: $h(x) = \sqrt{x} + \frac{1}{x}$ with domain $(0, \infty)$.

Quotients of functions work differently:

The *domain of a quotient function f/g*

consists of numbers x that are in the domains of both f and g and also satisfy $g(x) \neq 0$.

Example 19: Find the domain of $h(x) = \frac{x+2}{\sqrt{x}} = \frac{f(x)}{g(x)}$

Solution $f(x) = x + 2$ is defined for all x while $g(x) = \sqrt{x}$ is defined for $x \geq 0$. So they are both defined for $x \geq 0$. However, the division requires the denominator to be non-zero,so the requested domain is $x > 0$, or $(0, \infty)$.

Composing functions

Function composition is a completely different method of combining functions. It uses the output of one function as the input for the other.

Composing functions

Given functions f and g

- f composed with g is the function $h = f \circ g$ defined by $h(x) = f(g(x))$
- g composed with f is the function $h = g \circ f$ defined by $h(x) = g(f(x))$

Some people like to draw a diagram:

Input $x \Rightarrow \boxed{g} \Rightarrow g(x) \Rightarrow \boxed{f} \Rightarrow f(g(x)) =$ output

Function composition is not commutative:
$f \circ g$ and $g \circ f$ are usually different functions.

Example 20: Let $f(x) = x + 2$ and let $g(x) = x^2$. Find and simplify $f(g(x))$ and $g(f(x))$.

Solution:
$f(g(x)) = f(x^2) = (x^2) + 2 = x^2 + 2$

$g(f(x)) = g(x+2) = (x+2)^2 = x^2 + 4x + 4.$

Example 21: Let $s(t) = 10 + 3t$ be a square's side length at time t, and let $A(s) = s^2$ be the area of a square with side length s. Find a composite function that gives the square's area at time t.

Solution: Area $= A(s(t)) = A(10 + 3t) = \boxed{(10 + 3t)^2}$

Example 22:
A spherical balloon is being filled with air. If its radius at time t is $2t + 1$, find its volume at time t.

Solution:
The volume of a spherical balloon with radius r is $\frac{4}{3}\pi r^3$. Since r is a function of time, Volume $=$
$V(r(t)) = V(3 + 4t) = \boxed{\frac{4}{3}\pi(2t+1)^3}$

Example 23: Given $f(x) = 3x - x^2$ and $g(x) = 4x + 7$, find $f(g(x))$ and $g(f(x))$ and rewrite each as a completely factored product.

Solution:

- $f(g(x))$
$= f(4x+7) = 3(4x+7) - (4x+7)^2$
$= (4x+7)(3 - (4x+7))$
$= (4x+7)(3 - 4x - 7)$
$=(4x+7)(-4x-4) = \boxed{-4(x+1)(4x+7)}$
- $g(f(x)) = 4(3x - x^2) + 7 = -4x^2 + 12x + 7.$

Inverse functions undo each other

Let's try to factor $= -4x^2 + 12x + 7$. The discriminant is $D = b^2 - 4ac = 144 - 4(-4)(7) = 144 + 112 = 256 = 16^2$, a perfect square and so the result factors. Use the AC method: find r, s with $rs = ac = -28$ and $r + s = b = -2$. Then $r = 14$ $s = -2$ so
$-4x^2 + 12x + 7 = -4x^2 + 14x - 2x + 7$
$= -2x(2x + 7) - 1(2x - 7) = \boxed{-(2x + 1)(2x - 7)}$

The factoring method: the roots of $f(x) = -4x^2 + 12x + 7 = ax^2 + bx + c$ are $r_1, r_2 = \frac{-12 \pm \sqrt{256}}{-8} = \frac{3}{2} \pm 2$. Then
$f(x) = a(x - r_1)(x - r_2) = -4(x - \frac{3}{2} + 2)(x - \frac{3}{2} - 2) = -4(x - \frac{1}{2})(x - \frac{7}{2}) = -(2x - 1)(2x - 7)$
In this example, as is usually the case, $g(f(x))$ and $f(g(x))$ are different. Reversing the order of composition changes the result. However, there are important settings in both math and physics where the order doesn't matter: function composition is sometimes commutative.

An important example is when $f(g(x)) = g(f(x)) = x$ for all x. If so, we say that f and g undo each other, because applying each function to the other's output gets back to the original input.

Example 24: Show that $f(x) = \dfrac{3x + 2}{5}$ and $g(x) = \dfrac{5x - 2}{3}$ undo each other.

Solution:
- $f(g(x)) = f\left(\frac{5x-2}{3}\right) = \dfrac{3\left(\frac{5x-2}{3}\right)+2}{5}$
  $= \frac{5x-2+2}{5} = \frac{5x}{5} = \boxed{x}$
- $g(f(x)) = g\left(\frac{3x+2}{5}\right) = \dfrac{5\left(\frac{3x+2}{5}\right)-2}{3}$
  $= \frac{3x+2-2}{3} = \frac{3x}{3} = \boxed{x}$

Functions f and g are *inverse functions*

means: $f(g(x)) = g(f(x)) = x$ for all real numbers x in both of their domains.

Example 25: Find the composite functions $f(g(x))$ and $g(f(x))$ if $f(x) := x + 2$ and $g(x) := x - 2$
Solution:
- $g(f(x)) = g(x + 2) = (x + 2) - 2 = x$.
  Subtract 2 undoes add 2.
- $f(g(x)) = f(x - 2) = (x - 2) + 2 = x$.
  Add 2 undoes subtract 2.
- Subtract 2 and add 2 are inverse functions

Of course, there's nothing special about 2. For *any* real number a, subtract a and add a undo each other.

More briefly, we say

- Subtraction and addition are inverse functions.
- Subtraction and addition undo each other.

Example 26: Find the composite functions $f(g(x))$ and $g(f(x))$ if $f(x) = 2x$ and $g(x) = x/2$

Solution:

- $g(f(x)) = g(2x) = 2x \div 2 = \frac{2x}{2} = x$.
  Divide by 2 undoes multiply by 2.
- $f(g(x)) = f(x/2) = 2 \cdot x/2 = x$.
  Multiply by 2 undoes divide by 2.
- Multiply by 2 and divide by 2 are inverse functions

Of course, there's nothing special about 2. For any real number $a \neq 0$, multiply by a and divide by a undo each other.

- Multiplication and division are inverse functions.
- Multiplication and division undo each other.

Inverse functions will be discussed in detail in Section 2.8. Right now, the important thing to know is that

The functions square and square root are not inverse functions *unless*

their domains are restricted to positive real numbers.

Squares and square roots

- If $x \geq 0$, then $(\sqrt{x})^2 = x$ and $\sqrt{x^2} = x$.
- If $x < 0$, then $\sqrt{x}$ is undefined and $\sqrt{x^2} = -x$, *not* x.
- If x is any real number, then $\sqrt{x^2} = |x|$.

For example, let $x = -2$.

- $\sqrt{(-2)^2} = -2$ is incorrect since $\sqrt{(-2)^2} = \sqrt{4} = 2$.
- However, $\sqrt{(-2)^2} = |2|$.
- $(\sqrt{-2})^2 = -2$ is incorrect since $\sqrt{-2}$ is undefined.

If we let $x = 2$, everything is fine:

- $\sqrt{2^2} = 2$ is correct since $\sqrt{2^2} = \sqrt{4} = 2$.
- $\sqrt{(2)^2} = 2 = |2|$.
- $(\sqrt{2})^2 = 2$ is correct.

1.7.9 Section 1.7 Quiz

Ex. 1.7.1: Substitute $a-b$ for x in expression ax^2+bx+c.

Ex. 1.7.2: If $E=AB-B^2$, find E if $A=x+3$ and $B=y-3$.

Ex. 1.7.3: Suppose $E=(x+2)^3(y-3)^4-3(x+2)^4(y-3)^3$. Rewrite E by setting $U=x+2$ and $V=y-3$.

Ex. 1.7.4: State the commutative law $AB=BA$ in the case $A=x+3$ and $B=y-2$.

Ex. 1.7.5: If $y=x^2-3x$ and $x=z+2$, express y in terms of z.

Ex. 1.7.6: Rewrite $x^2-x(x+1)-(x+3)^2$ as a polynomial.

If $h(t)=1600-16t^2$ is a ball's height at time t.

Ex. 1.7.7: How high is the ball after 2 seconds?

Ex. 1.7.8: When does the ball hit the ground?

Ex. 1.7.9: $h(t):=3-t-t^2$. Find

a) $h(-7)$ b) $h(t+3)$ c) $h(t+k)$ d) $h(2t+3)$

Ex. 1.7.10: $f(x):=ax^2+bx+c$. Find

a) $f(-7)$ b) $f(t+3)$ c) $f(ax+b)$ d) $f(x+h)$

Ex. 1.7.11: If $f(x):=x^2-3x$, find $f(b)$.

Ex. 1.7.12: If $f(x)=x^2-3x$, find $f(x+h)$.

Ex. 1.7.13: If $f(x)=a^2+b^2$, find $f(x+h)$.

Ex. 1.7.14: If $f(x)=x^2-3x$, rewrite $f(x+h)-f(x)$ as a polynomial.

Ex. 1.7.15:

If $f(x)=x^2-3x$, find $f(a)f(b)-f(c)$.

If $f(x)=1-x^2$, find

- $2x^2-f(x)$ • $2f(x+h)-f(x)$ • $2-f(x)f(y)$

Ex. 1.7.16: Given $f(x)=3x-x^2$, simplify $\frac{f(x+h)-f(x)}{h}$.

Ex. 1.7.17: Given $f(t)=96t-16t^2$, simplify $\frac{f(2+h)-f(2)}{h}$.

Ex. 1.7.18: Let $f(x)=\sqrt{x}$ and $g(x)=\frac{1}{x}$. Find the formula for $h=f+g$ and state its domain.

Ex. 1.7.19: Find the domain of $h(x)=\frac{x+2}{\sqrt{x}}=\frac{f(x)}{g(x)}$

Ex. 1.7.20: Let $f(x)=x+2$ and let $g(x)=x^2$. Find $f(g(x))$ and $g(f(x))$.

1.7.9 Section 1.7 Quiz

Ex. 1.7.21: Let $s(t) = 10 + 3t$ be a square's side length at time t, and let $A(s) = s^2$ be the area of a square with side length s. Find a composite function that gives the square's area at time t.

Ex. 1.7.22: The volume of a spherical balloon with radius r is $\frac{4}{3}\pi r^3$.

Suppose the balloon is being filled with air, and its radius at time t is $3+4t$. Find the balloon's volume at time t.

Ex. 1.7.23: Given $f(x) = 3x - x^2$ and $g(x) = 4x + 7$, find $f(g(x))$ and $g(f(x))$ and rewrite each as a completely factored product.

Ex. 1.7.24: Given $f(x) = \frac{3x+2}{5}$ and $g(x) = \frac{5x-2}{3}$, find $f(g(x))$ and $g(f(x))$.

Ex. 1.7.25: Find $f(g(x))$ and $g(f(x))$ if $f(x) = x + 2$ and $g(x) = x - 2$.

Ex. 1.7.26: Find $f(g(x))$ and $g(f(x))$ if $f(x) = 2x$ and $g(x) = x/2$.

1.7.10 Section 1.7 Algebra Practice

Substitute $x+3$ for A and $x-h$ for B in each expression below. Don't multiply out.

Do all the problems at the left before you check your answers below.

a) A^2-B^2
b) $A-BA$
c) $A-B-xA^2$
d) $3A-4B+6C$
e) $(A-B)(C-B)$
f) $-BA^3C$

1.7.10 Section 1.7 Algebra Practice

Substitute $x + 3$ for A and $x - h$ for B in each expression below. Don't multiply out.

a) $A^2 - B^2$
b) $A - BA$
c) $A - B - xA^2$
d) $3A - 4B + 6C$
e) $(A - B)(C - B)$
f) $-BA^3C$

Do all the problems at the left before you check your answers below.

a) $(x + 3)^2 - (x - h)^2$
b) $(x + 3) - (x - h)(x + 3)$
c) $(x + 3) - (x - h) - x(x + 3)^2$
d) $3(x + 3) - 4(x - h) + 6C$
e) $((x + 3) - (x - h))(C - (x - h))$
f) $-(x - h)(x + 3)^3C$

1.7.10 Section 1.7 Algebra Practice

Substitute $x+3$ for A and $x-h$ for B in each expression below. Don't multiply out.

a) $A^2 - B^2$
b) $A - BA$
c) $A - B - xA^2$
d) $3A - 4B + 6C$
e) $(A - B)(C - B)$
f) $-BA^3C$

Do all the problems at the left before you check your answers below.

a) $(x+3)^2 - (x-h)^2$
b) $(x+3) - (x-h)(x+3)$
c) $(x+3) - (x-h) - x(x+3)^2$
d) $3(x+3) - 4(x-h) + 6C$
e) $((x+3) - (x-h))(C - (x-h))$
f) $-(x-h)(x+3)^3C$

Substitute D for $x+3$ and E for $x-h$ in each expression below.

a) $(x+3)^2 - (x-h)^2$
b) $(x+3) - (x-h)(x+3)$
c) $(x+3) - (x-h) - x(x+3)^2$
d) $3(x+3) - 4(x-h) + 6C$
e) $((x+3) - (x-h))(C - (x-h))$
f) $-(x-h)(x+3)^3C$

Do all the problems at the left before you check your answers below.

1.7.10 Section 1.7 Algebra Practice

Substitute $x+3$ for A and $x-h$ for B in each expression below. Don't multiply out.

a) $A^2 - B^2$
b) $A - BA$
c) $A - B - xA^2$
d) $3A - 4B + 6C$
e) $(A - B)(C - B)$
f) $-BA^3C$

Do all the problems at the left before you check your answers below.

a) $(x+3)^2 - (x-h)^2$
b) $(x+3) - (x-h)(x+3)$
c) $(x+3) - (x-h) - x(x+3)^2$
d) $3(x+3) - 4(x-h) + 6C$
e) $((x+3) - (x-h))(C - (x-h))$
f) $-(x-h)(x+3)^3C$

Substitute D for $x+3$ and E for $x-h$ in each expression below.

a) $(x+3)^2 - (x-h)^2$
b) $(x+3) - (x-h)(x+3)$
c) $(x+3) - (x-h) - x(x+3)^2$
d) $3(x+3) - 4(x-h) + 6C$
e) $((x+3) - (x-h))(C - (x-h))$
f) $-(x-h)(x+3)^3C$

Do all the problems at the left before you check your answers below.

a) $D^2 - E^2$
b) $D - ED$
c) $D - E - xD^2$
d) $3D - 4E + 6C$
e) $(D - E)(C - E)$
f) $-ED^3C$

1.7.10 Section 1.7 Algebra Practice

Substitute $x+3$ for A and $x-h$ for B in each expression below. Don't multiply out.

a) $A^2 - B^2$
b) $A - BA$
c) $A - B - xA^2$
d) $3A - 4B + 6C$
e) $(A-B)(C-B)$
f) $-BA^3C$

Do all the problems at the left before you check your answers below.

a) $(x+3)^2 - (x-h)^2$
b) $(x+3) - (x-h)(x+3)$
c) $(x+3) - (x-h) - x(x+3)^2$
d) $3(x+3) - 4(x-h) + 6C$
e) $((x+3) - (x-h))(C - (x-h))$
f) $-(x-h)(x+3)^3C$

Substitute in each of the following by letting $F = 3x^2 + x, f = 6x + 1, G = x^3 + 1, g = 3x^2$. Remove unnecessary parentheses: for example $((x+2)) = (x+2)$, but do not multiply out.

a) $fG + Fg$ b) $fG - Fg$ c) $\dfrac{fG - Fg}{G^2}$

d) f^2F^2 e) $(fF)^2$ f) $f^2G - g^2F$

Please make sure that you have answered all the questions before you check your answers below.

Substitute D for $x+3$ and E for $x-h$ in each expression below.

a) $(x+3)^2 - (x-h)^2$
b) $(x+3) - (x-h)(x+3)$
c) $(x+3) - (x-h) - x(x+3)^2$
d) $3(x+3) - 4(x-h) + 6C$
e) $((x+3) - (x-h))(C - (x-h))$
f) $-(x-h)(x+3)^3C$

Do all the problems at the left before you check your answers below.

a) $D^2 - E^2$
b) $D - ED$
c) $D - E - xD^2$
d) $3D - 4E + 6C$
e) $(D-E)(C-E)$
f) $-ED^3C$

1.7.10 Section 1.7 Algebra Practice

Substitute $x+3$ for A and $x-h$ for B in each expression below. Don't multiply out.

a) $A^2 - B^2$
b) $A - BA$
c) $A - B - xA^2$
d) $3A - 4B + 6C$
e) $(A-B)(C-B)$
f) $-BA^3C$

Do all the problems at the left before you check your answers below.

a) $(x+3)^2 - (x-h)^2$
b) $(x+3) - (x-h)(x+3)$
c) $(x+3) - (x-h) - x(x+3)^2$
d) $3(x+3) - 4(x-h) + 6C$
e) $((x+3)-(x-h))(C-(x-h))$
f) $-(x-h)(x+3)^3C$

Substitute D for $x+3$ and E for $x-h$ in each expression below.

a) $(x+3)^2 - (x-h)^2$
b) $(x+3) - (x-h)(x+3)$
c) $(x+3) - (x-h) - x(x+3)^2$
d) $3(x+3) - 4(x-h) + 6C$
e) $((x+3)-(x-h))(C-(x-h))$
f) $-(x-h)(x+3)^3C$

Do all the problems at the left before you check your answers below.

a) $D^2 - E^2$
b) $D - ED$
c) $D - E - xD^2$
d) $3D - 4E + 6C$
e) $(D-E)(C-E)$
f) $-ED^3C$

Substitute in each of the following by letting $F = 3x^2 + x, f = 6x+1, G = x^3+1, g = 3x^2$. Remove unnecessary parentheses: for example $((x+2)) = (x+2)$, but do not multiply out.

a) $fG + Fg$ b) $fG - Fg$ c) $\dfrac{fG - Fg}{G^2}$

d) f^2F^2 e) $(fF)^2$ f) $f^2G - g^2F$

Please make sure that you have answered all the questions before you check your answers below.

a) $(6x+1)(x^3+1) + (3x^2+x)(3x^2)$
b) $(6x+1)(x^3+1) - (3x^2+x)(3x^2)$
c) $\dfrac{(6x+1)(x^3+1) - (3x^2+x)(3x^2)}{(x^3+1)^2}$
d) $(6x+1)^2(3x^2+x)^2$
e) $((6x+1)(3x^2+x))^2$
f) $(6x+1)^2(x^3+1) - (3x^2)^2(3x^2+x)$

Section 1.7 Review: Formulas and functions

Ex. 1.7.1: Substitute $a - b$ for x in each of the following expressions:

- $ax^2 + bx + c$
- $3 - x$
- $abcdx$
- $3 - xx$

Section 1.7 Review: Formulas and functions

Ex. 1.7.1: Substitute $a - b$ for x in each of the following expressions:

- $ax^2 + bx + c \Rightarrow a(a-b)^2 + b(a-b) + c$
- $abcdx \Rightarrow abcd(a-b)$
- $3 - x \Rightarrow 3 - (a-b)$
- $3 - xx \Rightarrow 3 - (a-b)(a-b)$

Ex. 1.7.1: Substitute $a - b$ for x in each of the following expressions:

- $ax^2 + bx + c \ \Rightarrow a(a-b)^2 + b(a-b) + c$
- $abcdx \ \Rightarrow abcd(a-b)$
- $3 - x \ \Rightarrow 3 - (a-b)$
- $3 - xx \ \Rightarrow 3 - (a-b)(a-b)$

Ex. 1.7.2: If $A = x + 3$ and $B = y - 3$, find

- $E = AB - B^2$
- $F = B^2 - AC$
- $G = AC - AB$
- $H = A + B + C$

Ex. 1.7.1: Substitute $a - b$ for x in each of the following expressions:

- $ax^2 + bx + c \;\Rightarrow\; a(a-b)^2 + b(a-b) + c$
- $abcdx \;\Rightarrow\; abcd(a-b)$
- $3 - x \;\Rightarrow\; 3 - (a-b)$
- $3 - xx \;\Rightarrow\; 3 - (a-b)(a-b)$

Ex. 1.7.2: If $A = x + 3$ and $B = y - 3$, find

- $E = AB - B^2 \;\Rightarrow\; (x+3)(y-3) - (y-3)^2$
- $F = B^2 - AC \;\Rightarrow\; (y-3)^2 - (x+3)C$
- $G = AC - AB \;\Rightarrow\; (x+3)C - (x+3)(y-3)$
- $H = A + B + C \;\Rightarrow\; (x+3) + (y-3) + C$

Section 1.7 Review: Formulas and functions

Ex. 1.7.1: Substitute $a - b$ for x in each of the following expressions:

- $ax^2 + bx + c \Rightarrow a(a-b)^2 + b(a-b) + c$
- $abcdx \Rightarrow abcd(a-b)$
- $3 - x \Rightarrow 3 - (a-b)$
- $3 - xx \Rightarrow 3 - (a-b)(a-b)$

Ex. 1.7.2: If $A = x + 3$ and $B = y - 3$, find

- $E = AB - B^2 \Rightarrow (x+3)(y-3) - (y-3)^2$
- $F = B^2 - AC \Rightarrow (y-3)^2 - (x+3)C$
- $G = AC - AB \Rightarrow (x+3)C - (x+3)(y-3)$
- $H = A + B + C \Rightarrow (x+3) + (y-3) + C$

Ex. 1.7.3: Rewrite E, F, G, H by setting $U = x + 2$ and $V = y - 3$:

- $E = (x+2)^3(y-3)^4 - 3(x+2)^4(y-3)^3$
- $F = -(x+2)^4 - 4(y-3)^3$
- $G = (x+2)^3(y-3)^4 - 3(x+2)^4(y-3)^3$
- $H = (x+2)^3(y-3)^4 - 3(x+2)^4 V^3$

Section 1.7 Review: Formulas and functions

Ex. 1.7.1: Substitute $a-b$ for x in each of the following expressions:

- $ax^2+bx+c \;\Rightarrow\; a(a-b)^2+b(a-b)+c$
- $abcdx \;\Rightarrow\; abcd(a-b)$
- $3-x \;\Rightarrow\; 3-(a-b)$
- $3-xx \;\Rightarrow\; 3-(a-b)(a-b)$

Ex. 1.7.2: If $A=x+3$ and $B=y-3$, find

- $E=AB-B^2 \;\Rightarrow (x+3)(y-3)-(y-3)^2$
- $F=B^2-AC \;\Rightarrow (y-3)^2-(x+3)C$
- $G=AC-AB \;\Rightarrow (x+3)C-(x+3)(y-3)$
- $H=A+B+C \;\Rightarrow (x+3)+(y-3)+C$

Ex. 1.7.3: Rewrite E,F,G,H by setting $U=x+2$ and $V=y-3:$

- $E=(x+2)^3(y-3)^4-3(x+2)^4(y-3)^3 \quad \Rightarrow E=U^3\,V^4-3\,U^4\,V^3.$
- $F=-(x+2)^4-4(y-3)^3 \quad \Rightarrow F=-U^4-4\,V^3.$
- $G=(x+2)^3(y-3)^4-3(x+2)^4(y-3)^3 \quad \Rightarrow G=U^3\,V^4-3\,U^4\,V^3.$
- $H=(x+2)^3(y-3)^4-3(x+2)^4\,V^3 \quad \Rightarrow H=U^3(y-3)^4-3\,U^4\,V^3$

Section 1.7 Review: Formulas and functions

Ex. 1.7.1: Substitute $a-b$ for x in each of the following expressions:

- $ax^2+bx+c \;\Rightarrow\; a(a-b)^2+b(a-b)+c$
- $abcdx \;\Rightarrow\; abcd(a-b)$
- $3-x \;\Rightarrow\; 3-(a-b)$
- $3-xx \;\Rightarrow\; 3-(a-b)(a-b)$

Ex. 1.7.2: If $A=x+3$ and $B=y-3$, find

- $E=AB-B^2 \;\Rightarrow\; (x+3)(y-3)-(y-3)^2$
- $F=B^2-AC \;\Rightarrow\; (y-3)^2-(x+3)C$
- $G=AC-AB \;\Rightarrow\; (x+3)C-(x+3)(y-3)$
- $H=A+B+C \;\Rightarrow\; (x+3)+(y-3)+C$

Ex. 1.7.3: Rewrite E, F, G, H by setting $U=x+2$ and $V=y-3:$

- $E=(x+2)^3(y-3)^4-3(x+2)^4(y-3)^3 \quad\Rightarrow E=U^3V^4-3U^4V^3.$
- $F=-(x+2)^4-4(y-3)^3 \quad\Rightarrow F=-U^4-4V^3.$
- $G=(x+2)^3(y-3)^4-3(x+2)^4(y-3)^3 \quad\Rightarrow G=U^3V^4-3U^4V^3.$
- $H=(x+2)^3(y-3)^4-3(x+2)^4V^3 \quad\Rightarrow H=U^3(y-3)^4-3U^4V^3$

Ex. 1.7.4: Let $A=x+3$ and $B=y-2$. State each of the following

- $AB=BA$
- $A+(B+C)=(A+B)+C$
- $A(B+C)=AB+AC$
- $(B+C)A=BA+CA$

Section 1.7 Review: Formulas and functions

Ex. 1.7.1: Substitute $a-b$ for x in each of the following expressions:

- $ax^2+bx+c \;\Rightarrow\; a(a-b)^2+b(a-b)+c$
- $abcdx \;\Rightarrow\; abcd(a-b)$
- $3-x \;\Rightarrow\; 3-(a-b)$
- $3-xx \;\Rightarrow\; 3-(a-b)(a-b)$

Ex. 1.7.2: If $A=x+3$ and $B=y-3$, find

- $E=AB-B^2 \;\Rightarrow\; (x+3)(y-3)-(y-3)^2$
- $F=B^2-AC \;\Rightarrow\; (y-3)^2-(x+3)C$
- $G=AC-AB \;\Rightarrow\; (x+3)C-(x+3)(y-3)$
- $H=A+B+C \;\Rightarrow\; (x+3)+(y-3)+C$

Ex. 1.7.3: Rewrite E, F, G, H by setting $U=x+2$ and $V=y-3$:

- $E=(x+2)^3(y-3)^4-3(x+2)^4(y-3)^3 \quad\Rightarrow E=U^3V^4-3U^4V^3.$
- $F=-(x+2)^4-4(y-3)^3 \quad\Rightarrow F=-U^4-4V^3.$
- $G=(x+2)^3(y-3)^4-3(x+2)^4(y-3)^3 \quad\Rightarrow G=U^3V^4-3U^4V^3.$
- $H=(x+2)^3(y-3)^4-3(x+2)^4V^3 \quad\Rightarrow H=U^3(y-3)^4-3U^4V^3$

Ex. 1.7.4: Let $A=x+3$ and $B=y-2$. State each of the following

- $AB=BA \quad\Rightarrow (x+3)(y-2)=(y-2)(x+3)$
- $A+(B+C)=(A+B)+C \quad\Rightarrow (x+3)+((y-2)+C)=((x+3)+(y-2))+C$
- $A(B+C)=AB+AC \quad\Rightarrow (x+3)((y-2)+C)=(x+3)(y-2)+(x+3)C$
- $(B+C)A=BA+CA \quad\Rightarrow ((y-2)+C)(x+3)=(y-2)(x+3)+C(x+3)$

Ex. 1.7.1: Substitute $a-b$ for x in each of the following expressions:

- $ax^2+bx+c \Rightarrow a(a-b)^2+b(a-b)+c$
- $abcdx \Rightarrow abcd(a-b)$
- $3-x \Rightarrow 3-(a-b)$
- $3-xx \Rightarrow 3-(a-b)(a-b)$

Ex. 1.7.2: If $A=x+3$ and $B=y-3$, find

- $E=AB-B^2 \Rightarrow (x+3)(y-3)-(y-3)^2$
- $F=B^2-AC \Rightarrow (y-3)^2-(x+3)C$
- $G=AC-AB \Rightarrow (x+3)C-(x+3)(y-3)$
- $H=A+B+C \Rightarrow (x+3)+(y-3)+C$

Ex. 1.7.3: Rewrite E,F,G,H by setting $U=x+2$ and $V=y-3$:

- $E=(x+2)^3(y-3)^4-3(x+2)^4(y-3)^3 \Rightarrow E=U^3V^4-3U^4V^3.$
- $F=-(x+2)^4-4(y-3)^3 \Rightarrow F=-U^4-4V^3.$
- $G=(x+2)^3(y-3)^4-3(x+2)^4(y-3)^3 \Rightarrow G=U^3V^4-3U^4V^3.$
- $H=(x+2)^3(y-3)^4-3(x+2)^4V^3 \Rightarrow H=U^3(y-3)^4-3U^4V^3$

Ex. 1.7.4: Let $A=x+3$ and $B=y-2$. State each of the following

- $AB=BA \Rightarrow (x+3)(y-2)=(y-2)(x+3)$
- $A+(B+C)=(A+B)+C \Rightarrow (x+3)+((y-2)+C)=((x+3)+(y-2))+C$
- $A(B+C)=AB+AC \Rightarrow (x+3)((y-2)+C)=(x+3)(y-2)+(x+3)C$
- $(B+C)A=BA+CA \Rightarrow ((y-2)+C)(x+3)=(y-2)(x+3)+C(x+3)$

Ex. 1.7.5: Express y in terms of z if

- $y=x^2-3x$ and $x=z+2$
- $y=z^2-3x$ and $x=z^2+2$
- $y=x-ab$ and $a=z+2,\ b=z-2$
- $y=x^x$ and $x=z+2$

Section 1.7 Review: Formulas and functions

Ex. 1.7.1: Substitute $a-b$ for x in each of the following expressions:

- $ax^2+bx+c \Rightarrow a(a-b)^2+b(a-b)+c$
- $abcdx \Rightarrow abcd(a-b)$
- $3-x \Rightarrow 3-(a-b)$
- $3-xx \Rightarrow 3-(a-b)(a-b)$

Ex. 1.7.2: If $A=x+3$ and $B=y-3$, find

- $E=AB-B^2 \Rightarrow (x+3)(y-3)-(y-3)^2$
- $F=B^2-AC \Rightarrow (y-3)^2-(x+3)C$
- $G=AC-AB \Rightarrow (x+3)C-(x+3)(y-3)$
- $H=A+B+C \Rightarrow (x+3)+(y-3)+C$

Ex. 1.7.3: Rewrite E, F, G, H by setting $U=x+2$ and $V=y-3$:

- $E=(x+2)^3(y-3)^4-3(x+2)^4(y-3)^3 \Rightarrow E=U^3V^4-3U^4V^3.$
- $F=-(x+2)^4-4(y-3)^3 \Rightarrow F=-U^4-4V^3.$
- $G=(x+2)^3(y-3)^4-3(x+2)^4(y-3)^3 \Rightarrow G=U^3V^4-3U^4V^3.$
- $H=(x+2)^3(y-3)^4-3(x+2)^4V^3 \Rightarrow H=U^3(y-3)^4-3U^4V^3$

Ex. 1.7.4: Let $A=x+3$ and $B=y-2$. State each of the following

- $AB=BA \Rightarrow (x+3)(y-2)=(y-2)(x+3)$
- $A+(B+C)=(A+B)+C \Rightarrow (x+3)+((y-2)+C)=((x+3)+(y-2))+C$
- $A(B+C)=AB+AC \Rightarrow (x+3)((y-2)+C)=(x+3)(y-2)+(x+3)C$
- $(B+C)A=BA+CA \Rightarrow ((y-2)+C)(x+3)=(y-2)(x+3)+C(x+3)$

Ex. 1.7.5: Express y in terms of z if

- $y=x^2-3x$ and $x=z+2 \Rightarrow y=(z+2)^3-3(z+2)$
- $y=z^2-3x$ and $x=z^2+2 \Rightarrow y=z^2-3(z^2+2)$
- $y=x-ab$ and $a=z+2, b=z-2 \Rightarrow y=x-(z+2)(z-2)$
- $y=x^x$ and $x=z+2 \Rightarrow y=(z+2)^{(z+2)}$

Ex. 1.7.6: Rewrite as a polynomial:

- $x^2 - x(x+1) - (x+3)^2$
- $a^2 - (a+3)(3-a)$
- $x^3 - x(2-x)^2 - (-x+3)^2$
- $(uv+1)^2 - 2uv$

Ex. 1.7.6: Rewrite as a polynomial:

- $x^2 - x(x+1) - (x+3)^2 \ \Rightarrow -x^2 - 7x - 9$
- $x^3 - x(2-x)^2 - (-x+3)^2 \ \Rightarrow 3x^2 + 2x - 9$
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Ex. 1.7.7: How high is a ball after 2 seconds if its height in feet at time $t > 0$ is

- $h(t) = 1600 - 16t^2$
- $g(t) = 32t^2 - 64t$
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- $h(t) = 1600 - 16t^2 \Rightarrow 1536$ feet
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- $u(t) = 16t \Rightarrow 32$ feet

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- $h(t) = 1600 - 16t^2$
- $g(t) = 32t^2 - 64t$
- $h(t) = 1600t - t^2$
- $u(t) = 16t$:

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Ex. 1.7.8: When does the ball hit the ground if its height in feet at time $t > 0$ is
- $h(t) = 1600 - 16t^2 \Rightarrow t = 10$ seconds
- $g(t) = 32t^2 - 64t \Rightarrow t = 2$ seconds
- $h(t) = 1600t - t^2 \Rightarrow t = 1600$ seconds
- $u(t) = 16t$: $\Rightarrow$ Never!

Ex. 1.7.6: Rewrite as a polynomial:

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- $u(t) = 16t$: $\Rightarrow$ Never!

Ex. 1.7.9: If $h(t) := 3 - t - t^2$ find and simplify

- $h(-7)$
- $h(t+3)$
- $h(t+k)$
- $h(2t+3)$

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- $x^2 - x(x+1) - (x+3)^2 \Rightarrow -x^2 - 7x - 9$
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- $u(t) = 16t$: $\Rightarrow$ Never!

Ex. 1.7.9: If $h(t) := 3 - t - t^2$ find and simplify
- $h(-7) = 3 - (-7) - (-7)^2 = -39$
- $h(t+3) = 3 - (t+3) - (t+3)^2 = -t^2 - 7t - 9$
- $h(t+k) = 3 - (t+k) - (t+k)^2 = -k^2 - 2kt - k - t^2 - t + 3$
- $h(2t+3) = 3 - (2t+3) - (2t+3)^2 = -4t^2 - 14t - 9$

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Ex. 1.7.10: If $f(x) := ax^2 + bx + c$ find and simplify

- $f(-7)$
- $f(t+3)$
- $f(-a)$
- $f(1-a)$

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Ex. 1.7.10: If $f(x) := ax^2 + bx + c$ find and simplify
- $f(-7) = 49a - 7b + c$
- $f(t+3) = at^2 + 6at + bt + 9a + 3b + c$
- $f(-a) = a^3 - ab + c$
- $f(1-a) = a^3 - 2a^2 - ab + a + b + c$

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Ex. 1.7.11: If $f(x) := x^2 - 3x$ find and simplify

- $f(b)$
- $f(a+2b)$
- $f(-87+83)$
- $2f(b) - f(2b)$

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- $f(b) = b^2 - 3b$
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- $2f(b) - f(2b) = -2b^2$

Ex. 1.7.12: If $f(a) = 1 - 2a - 3a^2$ find and simplify
- $f(-2)$
- $f(a+2)$
- $f(a+h)$
- $f(a+h) - f(a)$

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- $x^2 - x(x+1) - (x+3)^2 \ \Rightarrow -x^2 - 7x - 9$
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- $f(-87+83) = 28$
- $2f(b) - f(2b) = -2b^2$

Ex. 1.7.12: If $f(a) = 1 - 2a - 3a^2$ find and simplify
- $f(-2) = 1 - 2(-2) - 3(-2)^2 = -7$
- $f(a+2) = -3a^2 - 14a - 15$
- $f(a+h) = -3a^2 - 6ah - 2a - 3h^2 - 2h + 1$
- $f(a+h) - f(a) = -6ah - 3h^2 - 2h$

Ex. 1.7.13: If

- $f(x) = a^2 + b^2$, find $f(x+h)$
- $f(x) = x + b^2$, find $f(x+h)$
- $f(x) = a^2 + b^2$, find $f(x+h)$
- $f(x) = abcdx$, find $f(x-h)$

Ex. 1.7.13: If

- $f(x) = a^2 + b^2$, find $f(x+h) \ = a^2 + b^2$
- $f(x) = x + b^2$, find $f(x+h) \ = x + h + b^2$
- $f(x) = a^2 + b^2$, find $f(x+h) \ = a^2 + b^2$
- $f(x) = abcdx$, find $f(x-h) \ = abcd(x-h)$

Ex. 1.7.13: If

- $f(x) = a^2 + b^2$, find $f(x+h) \; = a^2 + b^2$
- $f(x) = x + b^2$, find $f(x+h) \; = x + h + b^2$
- $f(x) = a^2 + b^2$, find $f(x+h) \; = a^2 + b^2$
- $f(x) = abcdx$, find $f(x-h) \; = abcd(x-h)$

Ex. 1.7.14: Rewrite $f(x+h) - f(x)$ as a polynomial if

- $f(x) = x^2 - 3x$
- $f(x) = x^3 - 3x$
- $f(x) = x^3$
- $f(x) = 1 - 8x - 3x^2$

Ex. 1.7.13: If

- $f(x) = a^2 + b^2$, find $f(x+h)$ $= a^2 + b^2$
- $f(x) = a^2 + b^2$, find $f(x+h)$ $= a^2 + b^2$
- $f(x) = x + b^2$, find $f(x+h)$ $= x + h + b^2$
- $f(x) = abcdx$, find $f(x-h)$ $= abcd(x-h)$

Ex. 1.7.14: Rewrite $f(x+h) - f(x)$ as a polynomial if

- $f(x) = x^2 - 3x \Rightarrow h^2 + 2hx - 3h$
- $f(x) = x^3 \Rightarrow h^3 + 3h^2x + 3hx^2$
- $f(x) = x^3 - 3x \Rightarrow h^3 + 3h^2x + 3hx^2 - 3h$
- $f(x) = 1 - 8x - 3x^2 \Rightarrow -3h^2 - 6hx - 8h$

Ex. 1.7.13: If
- $f(x) = a^2 + b^2$, find $f(x+h) \;= a^2 + b^2$
- $f(x) = x + b^2$, find $f(x+h) \;= x + h + b^2$
- $f(x) = a^2 + b^2$, find $f(x+h) \;= a^2 + b^2$
- $f(x) = abcdx$, find $f(x-h) \;= abcd(x-h)$

Ex. 1.7.14: Rewrite $f(x+h) - f(x)$ as a polynomial if
- $f(x) = x^2 - 3x \;\Rightarrow h^2 + 2hx - 3h$
- $f(x) = x^3 - 3x \;\Rightarrow h^3 + 3h^2x + 3hx^2 - 3h$
- $f(x) = x^3 \;\Rightarrow h^3 + 3h^2x + 3hx^2$
- $f(x) = 1 - 8x - 3x^2 \;\Rightarrow -3h^2 - 6hx - 8h$

Ex. 1.7.15: If $f(x) = x^2 - 3x$, find
- $f(a)f(b) - f(c) \quad =$
- $f(a)^3 f(b) - f(c)^3 \quad =$
- $f(a)(f(b) - f(c)) \quad =$
- $f(abc) \quad =$

Ex. 1.7.13: If
- $f(x) = a^2 + b^2$, find $f(x+h) \;= a^2 + b^2$
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- $f(x) = abcdx$, find $f(x-h) \;= abcd(x-h)$

Ex. 1.7.14: Rewrite $f(x+h) - f(x)$ as a polynomial if
- $f(x) = x^2 - 3x \;\Rightarrow h^2 + 2hx - 3h$
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- $f(x) = x^3 - 3x \;\Rightarrow h^3 + 3h^2x + 3hx^2 - 3h$
- $f(x) = 1 - 8x - 3x^2 \;\Rightarrow -3h^2 - 6hx - 8h$

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- $f(a)f(b) - f(c) \;= (a^2 - 3a)(b^2 - 3b) - (c^2 - 3c\ )$
- $f(a)^3 f(b) - f(c)^3 \;= (a^2 - 3a)^3(b^2 - 3b) - (c^2 - 3c)^3$
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If $f(x) = 1 - x^2$, rewrite as polynomials
- $2x^2 - f(x)$
- $x^2 f(x)$.
- $2f(x+h) - f(x)$
- $2 - f(x)f(y)$

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If $f(x) = 1 - x^2$, rewrite as polynomials

- $2x^2 - f(x) = 3x^2 - 1$
- $2f(x+h) - f(x) = -x^2 - 4xh - 2h^2 + 1$
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- $2 - f(x)f(y) = -x^2y^2 + x^2 + y^2 + 1$
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- $f(x^2)$.
- $x^3 f(x)$
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- $2f(x+h) - f(x) = -x^2 - 4xh - 2h^2 + 1$
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- $f(x+h) - f(x) = -h^2 - 2hx$
- $-f(x+1)$
- $f(\frac{1}{3x+h}) - f(\frac{1}{3x})$
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- $f(\frac{1}{(3x+3h)}) - f(\frac{1}{3x})$

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Section 1.7 Review

Ex. 1.7.16: Find and simplify $\frac{f(x+h)-f(x)}{h}$ if

- $f(x) = 3x - x^2$
- $f(x) = x^3 - 3x$
- $f(x) = x^3$
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Section 1.7 Review

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Ex. 1.7.17: Given $f(t) = 96t - 16t^2$, find and simplify

- $\frac{f(2+h)-f(2)}{h}$
- $\frac{f(3+h)-f(3)}{h}$
- $\frac{f(2+h)-f(2-h)}{h}$
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Section 1.7 Review

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Ex. 1.7.17: Given $f(t) = 96t - 16t^2$, find and simplify

- $\frac{f(2+h)-f(2)}{h} = -16(h + 2 \cdot 2 - 6) = -16h + 32$
- $\frac{f(3+h)-f(3)}{h} = -16h$
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Section 1.7 Review

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Ex. 1.7.18: Using interval notation, state the domain of $h = f + g$ if

- $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{x}$.
- $f(x) = \sqrt{x-2}$ and $g(x) = \frac{1}{x-3}$.
- $f(x) = \sqrt[3]{x}$ and $g(x) = \frac{1}{x}$.
- $f(x) = \sqrt[3]{x + \frac{2}{4-x}}$ and $g(x) = \sqrt{\frac{1}{x}}$

Section 1.7 Review

Ex. 1.7.16: Find and simplify $\frac{f(x+h)-f(x)}{h}$ if

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Ex. 1.7.18: Using interval notation, state the domain of $h = f + g$ if

- $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{x}$. $\Rightarrow D : (0, \infty)$
- $f(x) = \sqrt{x-2}$ and $g(x) = \frac{1}{x-3}$. $\Rightarrow D : [2, 3) \cup (3, \infty)$
- $f(x) = \sqrt[3]{x}$ and $g(x) = \frac{1}{x}$. $\Rightarrow D : (\infty, 0) \cup (0, \infty)$
- $f(x) = \sqrt[3]{x + \frac{2}{4-x}}$ and $g(x) = \sqrt{\frac{1}{x}}$ $\Rightarrow D : (-\infty, 0) \cup (0, 4) \cup (4, \infty)$

Section 1.7 Review

Ex. 1.7.16: Find and simplify $\frac{f(x+h)-f(x)}{h}$ if

- $f(x) = 3x - x^2 \Rightarrow -h - 2x + 3$
- $f(x) = x^3 - 3x \Rightarrow h^2 + 3hx + 3x^2 - 3$
- $f(x) = x^3 \Rightarrow h^2 + 3hx + 3x^2$
- $f(x) = 1 - 8x - 3x^2 \Rightarrow -3h - 6x - 8$

Ex. 1.7.17: Given $f(t) = 96t - 16t^2$, find and simplify

- $\frac{f(2+h)-f(2)}{h} = -16(h + 2 \cdot 2 - 6) = -16h + 32$
- $\frac{f(2+h)-f(2-h)}{h} = 64$
- $\frac{f(3+h)-f(3)}{h} = -16h$
- $\frac{f(3+h)-f(3-h)}{h} = 0$

Ex. 1.7.18: Using interval notation, state the domain of $h = f + g$ if

- $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{x}$. $\Rightarrow D : (0, \infty)$
- $f(x) = \sqrt{x-2}$ and $g(x) = \frac{1}{x-3}$. $\Rightarrow D : [2, 3) \cup (3, \infty)$
- $f(x) = \sqrt[3]{x}$ and $g(x) = \frac{1}{x}$. $\Rightarrow D : (\infty, 0) \cup (0, \infty)$
- $f(x) = \sqrt[3]{x + \frac{2}{4-x}}$ and $g(x) = \sqrt{\frac{1}{x}}$ $\Rightarrow D : (-\infty, 0) \cup (0, 4) \cup (4, \infty)$

Ex. 1.7.19: Using inequalities, state the domain of

- $h(x) = \frac{x+2}{\sqrt{x}}$
- $h(x) = \frac{\sqrt{x^2}}{x}$
- $h(x) = \frac{x+2}{\sqrt{x+2}}$
- $h(x) = \frac{\sqrt[3]{x}}{\sqrt{x^2+1}}$

Section 1.7 Review

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Ex. 1.7.19: Using inequalities, state the domain of

- $h(x) = \frac{x+2}{\sqrt{x}} \Rightarrow D : x > 0$
- $h(x) = \frac{x+2}{\sqrt{x+2}} \Rightarrow D : x > -2$
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Section 1.7 Review

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- $f(x) = x^3 \Rightarrow h^2 + 3hx + 3x^2$
- $f(x) = 1 - 8x - 3x^2 \Rightarrow -3h - 6x - 8$

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- $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{x}$. $\Rightarrow D : (0, \infty)$
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- $h(x) = \frac{\sqrt[3]{x}}{\sqrt{x^2+1}} \Rightarrow D : (-\infty, \infty)$

Ex. 1.7.20: Find $f(g(x))$ and $g(f(x))$ if
- $f(x) = x + 2$ and $g(x) = x^2$:
- $f(x) = \frac{1}{x+1}$ and $g(x) = x^2$:
- $f(x) = \sqrt{x}$ and $g(x) = x^2$:
- $f(x) = x^2 - 2$ and $g(x) = 2 - x^2$:

Section 1.7 Review

Ex. 1.7.16: Find and simplify $\frac{f(x+h)-f(x)}{h}$ if

- $f(x) = 3x - x^2 \Rightarrow -h - 2x + 3$
- $f(x) = x^3 - 3x \Rightarrow h^2 + 3hx + 3x^2 - 3$
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- $h(x) = \frac{\sqrt[3]{x}}{\sqrt{x^2+1}} \Rightarrow D : (-\infty, \infty)$

Ex. 1.7.20: Find $f(g(x))$ and $g(f(x))$ if

- $f(x) = x + 2$ and $g(x) = x^2$: $\Rightarrow f(g(x)) = x^2 + 2$ and $g(f(x)) = (x+2)^2$
- $f(x) = \frac{1}{x+1}$ and $g(x) = x^2$: $\Rightarrow f(g(x)) = \frac{1}{x^2+1}$ and $g(f(x)) = \frac{1}{(x+1)^2}$
- $f(x) = \sqrt{x}$ and $g(x) = x^2$: $\Rightarrow f(g(x)) = \sqrt{x^2} = |x|$; $g(f(x)) = (\sqrt{x})^2 = x$ if $x \geq 0$
- $f(x) = x^2 - 2$ and $g(x) = 2 - x^2$: $\Rightarrow f(g(x)) = (2 - x^2)^2 - 2$ and $g(f(x)) = 2 - (x^2 - 2)^2$

Some geometry formulas that are composite functions of time

A sphere of radius r has volume πr^3, surface area $4\pi r^2$ and diameter $2r$.
A radius r disc has area πr^2 , circumference $2\pi r$, and diameter $2r$

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Ex. 1.7.21: Let $s(t) = 10 + 3t$ be a square's side length at time t. Find and simplify its

- Area at time t
- Perimeter at time t

Let $r(t) = 3t^3$ be the radius of a disc at time t. Find and simplify its

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Ex. 1.7.21: Let $s(t) = 10 + 3t$ be a square's side length at time t. Find and simplify its

- Area at time t $= A(s(t)) = s(t)^2 = (10 + 3t)^2 = 9t^2 + 60t + 100.$
- Perimeter at time t $= P(s(t)) = 4s(t) = 4(10 + 3t) = 12t + 40.$

Let $r(t) = 3t^3$ be the radius of a disc at time t. Find and simplify its

- Area at time t $= A(r(t)) = \pi r(t)^3 = \pi(3t^3)^2 = 9\pi t^6.$
- Circumference at time t $= C(r(t)) = 2\pi r(t) = 6\pi t^3$

Some geometry formulas that are composite functions of time

A sphere of radius r has volume πr^3, surface area $4\pi r^2$ and diameter $2r$.
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Ex. 1.7.22: Let $r(t) = 2t + 1$ be a sphere's radius at time t. Find its

- Volume at time t
- Surface Area at time t

Let $h(t) = 3t$ be the side length of a cube at time t. Find and simplify its

- Surface Area at time t
- Volume at time t

Some geometry formulas that are composite functions of time

A sphere of radius r has volume πr^3, surface area $4\pi r^2$ and diameter $2r$.
A radius r disc has area πr^2 , circumference $2\pi r$, and diameter $2r$
A square with side length s has area s^2 and perimeter $4s$.
A cube with side length s has surface area $6s^2$ and volume s^3.

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- Area at time t $= A(s(t)) = s(t)^2 = (10 + 3t)^2 = 9t^2 + 60t + 100.$
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Let $r(t) = 3t^3$ be the radius of a disc at time t. Find and simplify its

- Area at time t $= A(r(t)) = \pi r(t)^3 = \pi(3t^3)^2 = 9\pi t^6.$
- Circumference at time t $= C(r(t)) = 2\pi r(t) = 6\pi t^3$

Ex. 1.7.22: Let $r(t) = 2t + 1$ be a sphere's radius at time t. Find its

- Volume at time t $= V(r(t)) = \frac{4}{3}\pi r(t)^3 = \frac{4}{3}\pi(2t + 1)^3 = \frac{4}{3}\pi(8t^3 + 12t^2 + 6t + 1).$
- Surface Area at time t $= S(r(t)) = 4\pi r(t)^2 = 4\pi(2t + 1)^2 = 4\pi(4t^2 + 4t + 1).$

Let $h(t) = 3t$ be the side length of a cube at time t. Find and simplify its

- Surface Area at time t $= A(h(t)) = 6h(t)^2 = 6(3t)^2 = 54t^2.$
- Volume at time t $= V(h(t)) = h(t)^3 = (3t)^3 = 27t^3.$

Ex. 1.7.23: Find $f(g(x))$ and $g(f(x))$ and simplify each result if

- $f(x) = 3x - x^2$ and $g(x) = 4x + 7$
- $f(t) = t^2 - 1$ and $g(t) = t - 4$
- $f(x) = 2x - 1$ and $g(x) = 3x + 1$
- $f(t) = \frac{1}{t+1}$ and $g(t) = \frac{1}{t-1}$

Ex. 1.7.23: Find $f(g(x))$ and $g(f(x))$ and simplify each result if

- $f(x) = 3x - x^2$ and $g(x) = 4x + 7$ $\Rightarrow f(g(x)) = -4(x+1)(4x+7); g(f(x)) = -4x^2 + 12x + 7$
- $f(t) = t^2 - 1$ and $g(t) = t - 4$ $\Rightarrow f(g(t)) = (t-3)(t-5); \quad g(f(t)) = t^2 - 5$
- $f(x) = 2x - 1$ and $g(x) = 3x + 1$ $\Rightarrow f(g(x)) = 6x + 1; \quad g(f(x)) = 6x - 2$
- $f(t) = \frac{1}{t+1}$ and $g(t) = \frac{1}{t-1}$ $\Rightarrow f(g(t)) = \frac{t-1}{t} \quad g(f(t)) = -\frac{t+1}{t}$

Ex. 1.7.23: Find $f(g(x))$ and $g(f(x))$ and simplify each result if

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- $f(t) = t^2 - 1$ and $g(t) = t - 4$ $\Rightarrow f(g(t)) = (t-3)(t-5);\quad g(f(t)) = t^2 - 5$
- $f(x) = 2x - 1$ and $g(x) = 3x + 1$ $\Rightarrow f(g(x)) = 6x + 1;\quad g(f(x)) = 6x - 2$
- $f(t) = \frac{1}{t+1}$ and $g(t) = \frac{1}{t-1}$ $\Rightarrow f(g(t)) = \frac{t-1}{t}\quad g(f(t)) = -\frac{t+1}{t}$

Ex. 1.7.24: Given $f(x) = \frac{3x+2}{5}$ and $g(x) = \frac{5x-2}{3}$, find

- $f(g(x))$ and $g(f(x))$
- $f(g(2x))$ and $g(f(2x))$
- $f(g(x-3))$ and $g(f(x-3))$
- $f(g(x^2))$ and $g(f(x^2))$

Ex. 1.7.23: Find $f(g(x))$ and $g(f(x))$ and simplify each result if

- $f(x) = 3x - x^2$ and $g(x) = 4x + 7$ $\Rightarrow f(g(x)) = -4(x+1)(4x+7); g(f(x)) = -4x^2 + 12x + 7$
- $f(t) = t^2 - 1$ and $g(t) = t - 4$ $\Rightarrow f(g(t)) = (t-3)(t-5); \quad g(f(t)) = t^2 - 5$
- $f(x) = 2x - 1$ and $g(x) = 3x + 1$ $\Rightarrow f(g(x)) = 6x + 1; \quad g(f(x)) = 6x - 2$
- $f(t) = \frac{1}{t+1}$ and $g(t) = \frac{1}{t-1}$ $\Rightarrow f(g(t)) = \frac{t-1}{t} \quad g(f(t)) = -\frac{t+1}{t}$

Ex. 1.7.24: Given $f(x) = \frac{3x+2}{5}$ and $g(x) = \frac{5x-2}{3}$, find

- $f(g(x))$ and $g(f(x))$ $\Rightarrow f(g(x)) = g(f(x)) = x$
- $f(g(2x))$ and $g(f(2x))$ $\Rightarrow f(g(x)) = g(f(x)) = 2x$
- $f(g(x-3))$ and $g(f(x-3))$ $\Rightarrow f(g(x)) = g(f(x)) = x - 3$
- $f(g(x^2))$ and $g(f(x^2))$ $\Rightarrow f(g(x)) = g(f(x)) = x^2$

Ex. 1.7.23: Find $f(g(x))$ and $g(f(x))$ and simplify each result if

- $f(x) = 3x - x^2$ and $g(x) = 4x + 7$ $\Rightarrow f(g(x)) = -4(x+1)(4x+7); g(f(x)) = -4x^2 + 12x + 7$
- $f(t) = t^2 - 1$ and $g(t) = t - 4$ $\Rightarrow f(g(t)) = (t-3)(t-5);\quad g(f(t)) = t^2 - 5$
- $f(x) = 2x - 1$ and $g(x) = 3x + 1$ $\Rightarrow f(g(x)) = 6x + 1;\quad g(f(x)) = 6x - 2$
- $f(t) = \frac{1}{t+1}$ and $g(t) = \frac{1}{t-1}$ $\Rightarrow f(g(t)) = \frac{t-1}{t}\quad g(f(t)) = -\frac{t+1}{t}$

Ex. 1.7.24: Given $f(x) = \frac{3x+2}{5}$ and $g(x) = \frac{5x-2}{3}$, find

- $f(g(x))$ and $g(f(x))$ $\Rightarrow f(g(x)) = g(f(x)) = x$
- $f(g(2x))$ and $g(f(2x))$ $\Rightarrow f(g(x)) = g(f(x)) = 2x$
- $f(g(x-3))$ and $g(f(x-3))$ $\Rightarrow f(g(x)) = g(f(x)) = x - 3$
- $f(g(x^2))$ and $g(f(x^2))$ $\Rightarrow f(g(x)) = g(f(x)) = x^2$

Ex. 1.7.25: Find $f(g(x))$ and $g(f(x))$ if

- $f(x) = x + 2$ and $g(x) = x - 2$
- $f(x) = 4x$ and $g(x) = x/4$
- $f(x) = x - 5$ and $g(x) = x + 6$
- $f(x) = \frac{4x}{3}$ and $g(x) = 7x$

Ex. 1.7.23: Find $f(g(x))$ and $g(f(x))$ and simplify each result if

- $f(x) = 3x - x^2$ and $g(x) = 4x + 7$ $\Rightarrow f(g(x)) = -4(x+1)(4x+7); g(f(x)) = -4x^2 + 12x + 7$
- $f(t) = t^2 - 1$ and $g(t) = t - 4$ $\Rightarrow f(g(t)) = (t-3)(t-5); \quad g(f(t)) = t^2 - 5$
- $f(x) = 2x - 1$ and $g(x) = 3x + 1$ $\Rightarrow f(g(x)) = 6x + 1; \quad g(f(x)) = 6x - 2$
- $f(t) = \frac{1}{t+1}$ and $g(t) = \frac{1}{t-1}$ $\Rightarrow f(g(t)) = \frac{t-1}{t} \quad g(f(t)) = -\frac{t+1}{t}$

Ex. 1.7.24: Given $f(x) = \frac{3x+2}{5}$ and $g(x) = \frac{5x-2}{3}$, find

- $f(g(x))$ and $g(f(x))$ $\Rightarrow f(g(x)) = g(f(x)) = x$
- $f(g(2x))$ and $g(f(2x))$ $\Rightarrow f(g(x)) = g(f(x)) = 2x$
- $f(g(x-3))$ and $g(f(x-3))$ $\Rightarrow f(g(x)) = g(f(x)) = x - 3$
- $f(g(x^2))$ and $g(f(x^2))$ $\Rightarrow f(g(x)) = g(f(x)) = x^2$

Ex. 1.7.25: Find $f(g(x))$ and $g(f(x))$ if

- $f(x) = x + 2$ and $g(x) = x - 2$ $\Rightarrow f(g(x)) = x; \quad g(f(x)) = x$
- $f(x) = 4x$ and $g(x) = x/4$ $\Rightarrow f(g(x)) = x; \quad g(f(x)) = x$
- $f(x) = x - 5$ and $g(x) = x + 6$ $\Rightarrow f(g(x)) = x + 1; \quad g(f(x)) = x + 1$
- $f(x) = \frac{4x}{3}$ and $g(x) = 7x$ $\Rightarrow f(g(x)) = \frac{28x}{3}; \quad g(f(x)) = \frac{28x}{3}$

Ex. 1.7.23: Find $f(g(x))$ and $g(f(x))$ and simplify each result if

- $f(x) = 3x - x^2$ and $g(x) = 4x + 7$ $\Rightarrow f(g(x)) = -4(x+1)(4x+7); g(f(x)) = -4x^2 + 12x + 7$
- $f(t) = t^2 - 1$ and $g(t) = t - 4$ $\Rightarrow f(g(t)) = (t-3)(t-5);\quad g(f(t)) = t^2 - 5$
- $f(x) = 2x - 1$ and $g(x) = 3x + 1$ $\Rightarrow f(g(x)) = 6x + 1;\quad g(f(x)) = 6x - 2$
- $f(t) = \frac{1}{t+1}$ and $g(t) = \frac{1}{t-1}$ $\Rightarrow f(g(t)) = \frac{t-1}{t}\quad g(f(t)) = -\frac{t+1}{t}$

Ex. 1.7.24: Given $f(x) = \frac{3x+2}{5}$ and $g(x) = \frac{5x-2}{3}$, find

- $f(g(x))$ and $g(f(x))$ $\Rightarrow f(g(x)) = g(f(x)) = x$
- $f(g(2x))$ and $g(f(2x))$ $\Rightarrow f(g(x)) = g(f(x)) = 2x$
- $f(g(x-3))$ and $g(f(x-3))$ $\Rightarrow f(g(x)) = g(f(x)) = x - 3$
- $f(g(x^2))$ and $g(f(x^2))$ $\Rightarrow f(g(x)) = g(f(x)) = x^2$

Ex. 1.7.25: Find $f(g(x))$ and $g(f(x))$ if

- $f(x) = x + 2$ and $g(x) = x - 2$ $\Rightarrow f(g(x)) = x;\quad g(f(x)) = x$
- $f(x) = 4x$ and $g(x) = x/4$ $\Rightarrow f(g(x)) = x;\quad g(f(x)) = x$
- $f(x) = x - 5$ and $g(x) = x + 6$ $\Rightarrow f(g(x)) = x + 1;\quad g(f(x)) = x + 1$
- $f(x) = \frac{4x}{3}$ and $g(x) = 7x$ $\Rightarrow f(g(x)) = \frac{28x}{3};\quad g(f(x)) = \frac{28x}{3}$

Ex. 1.7.26: Find $f(g(x))$ and $g(f(x))$ if

- $f(x) = 2x$ and $g(x) = \frac{x}{2}$
- $f(x) = 2 + x$ and $g(x) = 2 - x$
- $f(x) = 1 + 2x$ and $g(x) = 1 + \frac{x}{2}$
- $f(x) = 1 + 2x$ and $g(x) = 3 + \frac{2}{x}$

Ex. 1.7.23: Find $f(g(x))$ and $g(f(x))$ and simplify each result if

- $f(x) = 3x - x^2$ and $g(x) = 4x + 7$ $\Rightarrow f(g(x)) = -4(x+1)(4x+7); g(f(x)) = -4x^2 + 12x + 7$
- $f(t) = t^2 - 1$ and $g(t) = t - 4$ $\Rightarrow f(g(t)) = (t-3)(t-5); \quad g(f(t)) = t^2 - 5$
- $f(x) = 2x - 1$ and $g(x) = 3x + 1$ $\Rightarrow f(g(x)) = 6x + 1; \quad g(f(x)) = 6x - 2$
- $f(t) = \frac{1}{t+1}$ and $g(t) = \frac{1}{t-1}$ $\Rightarrow f(g(t)) = \frac{t-1}{t} \quad g(f(t)) = -\frac{t+1}{t}$

Ex. 1.7.24: Given $f(x) = \frac{3x+2}{5}$ and $g(x) = \frac{5x-2}{3}$, find

- $f(g(x))$ and $g(f(x))$ $\Rightarrow f(g(x)) = g(f(x)) = x$
- $f(g(2x))$ and $g(f(2x))$ $\Rightarrow f(g(x)) = g(f(x)) = 2x$
- $f(g(x-3))$ and $g(f(x-3))$ $\Rightarrow f(g(x)) = g(f(x)) = x - 3$
- $f(g(x^2))$ and $g(f(x^2))$ $\Rightarrow f(g(x)) = g(f(x)) = x^2$

Ex. 1.7.25: Find $f(g(x))$ and $g(f(x))$ if

- $f(x) = x + 2$ and $g(x) = x - 2$ $\Rightarrow f(g(x)) = x; \quad g(f(x)) = x$
- $f(x) = 4x$ and $g(x) = x/4$ $\Rightarrow f(g(x)) = x; \quad g(f(x)) = x$
- $f(x) = x - 5$ and $g(x) = x + 6$ $\Rightarrow f(g(x)) = x + 1; \quad g(f(x)) = x + 1$
- $f(x) = \frac{4x}{3}$ and $g(x) = 7x$ $\Rightarrow f(g(x)) = \frac{28x}{3}; \quad g(f(x)) = \frac{28x}{3}$

Ex. 1.7.26: Find $f(g(x))$ and $g(f(x))$ if

- $f(x) = 2x$ and $g(x) = \frac{x}{2}$ $\Rightarrow f(g(x)) = g(f(x)) = x$
- $f(x) = 2 + x$ and $g(x) = 2 - x$ $\Rightarrow f(g(x)) = 4 - x; \quad g(f(x)) = -x$
- $f(x) = 1 + 2x$ and $g(x) = 1 + \frac{x}{2}$ $\Rightarrow f(g(x)) = 3 + x; \quad g(f(x)) = \frac{3}{2} + x$
- $f(x) = 1 + 2x$ and $g(x) = 3 + \frac{2}{x}$ $\Rightarrow f(g(x)) = \frac{7x+4}{x}; \quad g(f(x)) = \frac{6x+5}{2x+1}$

- 1.8.1: The coordinate grid
- 1.8.2: X,Y-coordinates
- 1.8.3: Plotting points
- 1.8.4: Graphs of equations
- 1.8.5: Locating points
- 1.8.6: Completing the square
- 1.8.7: Graph symmetry
- Section 1.8 Quiz

Section 1.8 Preview: Definitions/Theorems

- Definition 1.8.1: A point $P(a, b)$ in the x, y-plane lies on the graph of an equation with variables x, y if
- Definition 1.8.2: Equations of vertical and horizontal lines
- Definition 1.8.3: The intercepts of a graph are the points
- Definition 1.8.4: The distance between points (a, b) and (c, d) is
- Definition 1.8.5: The standard form equation of a circle is
- Definition 1.8.6: The reflection of point (x, y) across an axis or the origin is
- Definition 1.8.7: A graph is symmetric about an axis or the origin if
- Definition 1.8.9: Two equations in x and y are equivalent if
- Definition 1.8.10: Let P and Q be polynomial expressions in one variable x. Then the equations $y = P$ and $y = Q$ are equivalent if

Section 1.8 Preview: Procedures

- Procedure 1.8.1: To find the coordinates of a point P in the x, y-plane
- Procedure 1.8.2: To plot a point $(x, y) = (a, b)$ by starting at the origin
- Procedure 1.8.3: To plot a point with given coordinates (a, b)
- Procedure 1.8.4: When you draw a graph, pay attention to the following details:
- Procedure 1.8.5: To find the intercepts of the graph of an equation
- Procedure 1.8.6: To complete the square in $x^2 + Bx$
- Procedure 1.8.7: To complete the square in an equation with $x^2 + Bx$ on one side
- Procedure 1.8.8: To complete the square in an equation involving $Ax^2 + Bx$
- Procedure 1.8.9: Applications of completing the square include
- Procedure 1.8.10: To determine whether the graph of an equation is symmetric
- Procedure 1.8.11: To reflect the graph of an equation across the origin or an axis

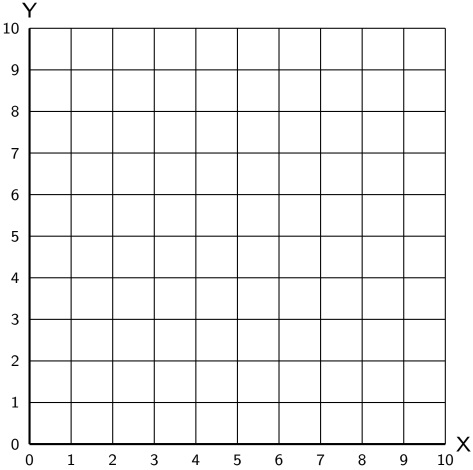
1.8.1 The coordinate grid

Applied mathematics and science rely on analyzing and understanding graphs of equations in the x, y-co-ordinate plane.

Section 1.1 showed how to locate points on the number line. Choose a point 0. The point 3 units to the right of 0 is called 3, while the point 3 units to the left of 0 is called -3. Every point on the number line is labeled with a single number, positive or negative.

It's harder to locate points in the plane.
It helps to think of the map of Manhattan: avenues are vertical lines; streets are horizontal lines. You might agree to meet a friend at the corner of 3^{rd} Avenue and 4^{th} Street.

The street map of Manhattan is a model for the co-ordinate plane.



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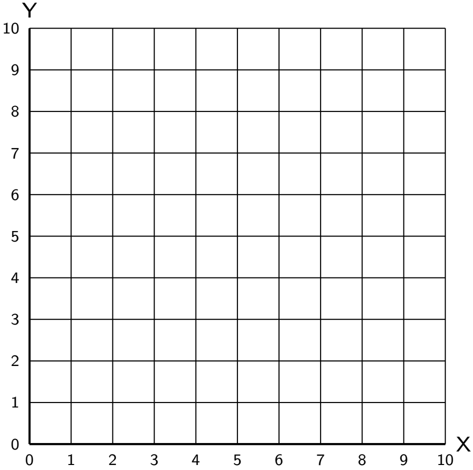

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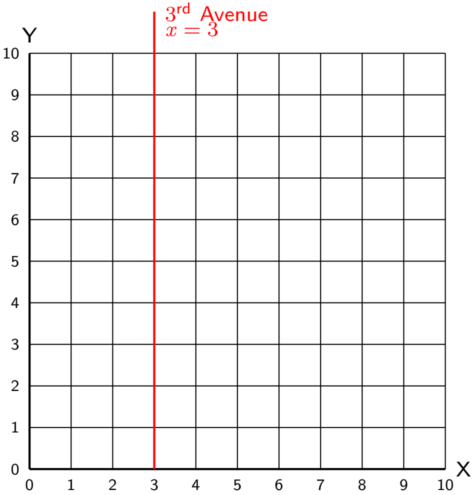
- The vertical line's Avenue name is $x = 3$.



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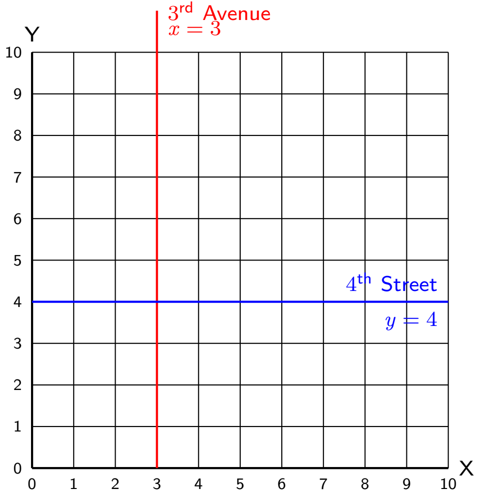
- The vertical line's Avenue name is $x = 3$.
- The horizontal line's Street name is $y = 4$.



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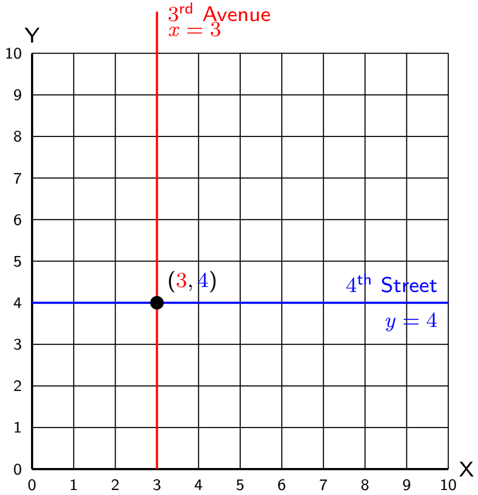
It helps to think of the map of Manhattan: avenues are vertical lines; streets are horizontal lines. You might agree to meet a friend at the corner of 3^{rd} Avenue and 4^{th} Street.

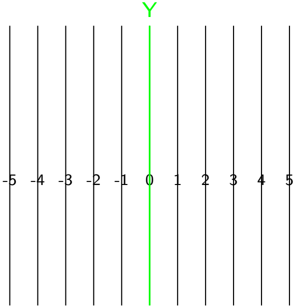
The street map of Manhattan is a model for the co-ordinate plane.

- The vertical line's Avenue name is $x = 3$.
- The horizontal line's Street name is $y = 4$.
- The corner where Avenue $x = 3$ meets Street $y = 4$ is **point** $(3, 4)$.

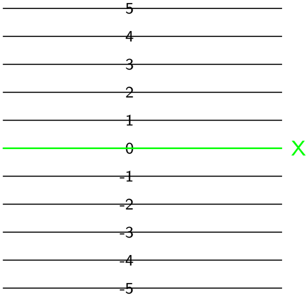


Since the number line contains negative numbers, so must the coordinate grid. We will construct the region in the plane containing all points (x, y) with $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$. The green horizontal line will become the x-axis, the green vertical line the y-axis. Streets and avenues are shown below, labeled with positive and negative avenue and street numbers.

Avenues (South to North)



Streets (East to West)



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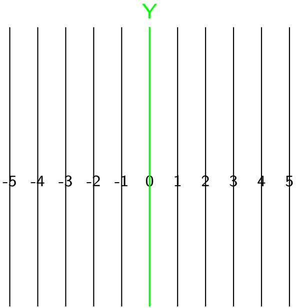
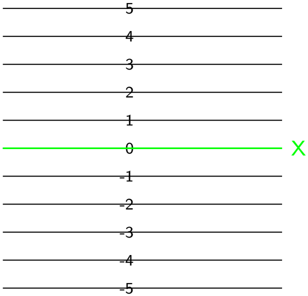


Streets (East to West)



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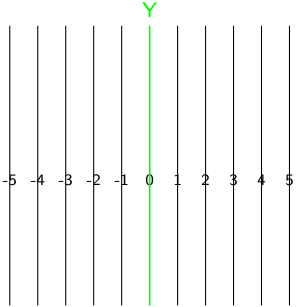
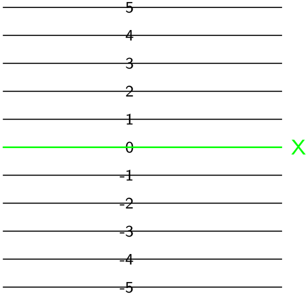
Now combine streets and avenues on a single grid, the x, y-plane, at the right.

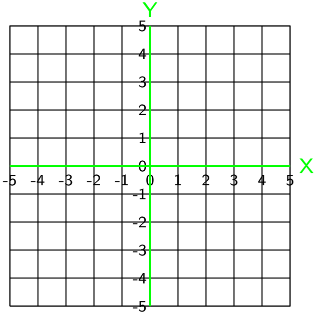
Avenues (South to North)



Streets (East to West)



Combine to form x, y-plane.



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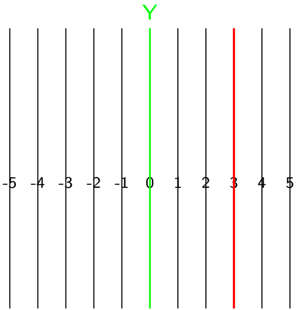
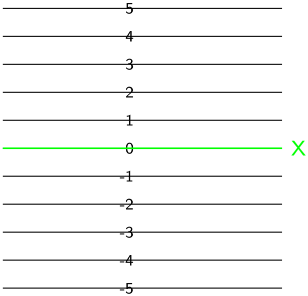
Now combine streets and avenues on a single grid, the x, y-plane, at the right.

Now highlight 3rd Avenue in red

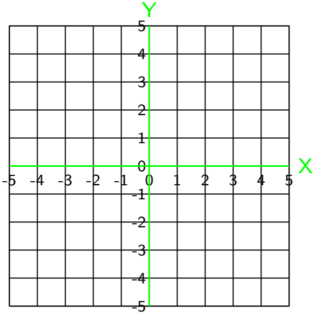
Avenues (South to North)



Streets (East to West)



Combine to form x, y-plane.



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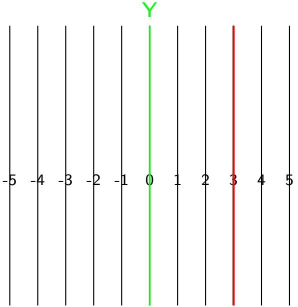
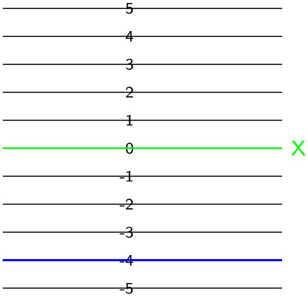
Now combine streets and avenues on a single grid, the x, y-plane, at the right.

Now highlight 3rd Avenue in red and -4th Street in blue.

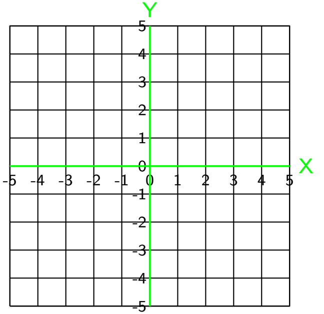
Avenues (South to North)



Streets (East to West)



Combine to form x, y-plane.



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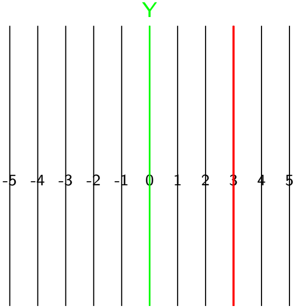
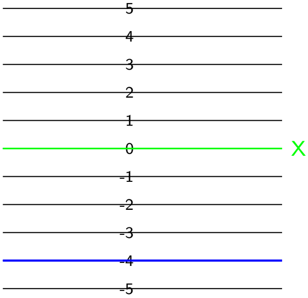
Now combine streets and avenues on a single grid, the x, y-plane, at the right.

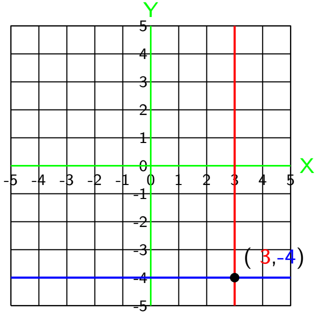
Now highlight 3rd Avenue in red and -4th Street in blue.

3rd Avenue and -4th Street meet at the corner of 3rd and -4th

Avenues (South to North) Streets (East to West) Combine to form x, y-plane.







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Now combine streets and avenues on a single grid, the x, y-plane, at the right.

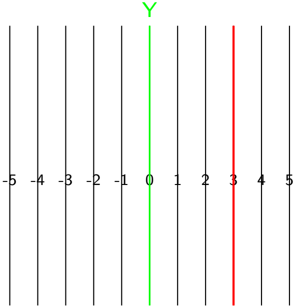
Now highlight 3rd Avenue in red and -4th Street in blue.

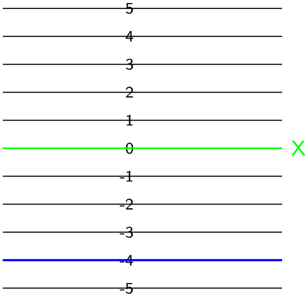
3rd Avenue and -4th Street meet at the corner of 3rd and -4th

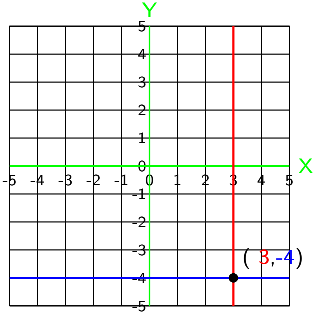
This translates to mathematical language as

The vertical line $x = 3$ and the horizontal line y = -4 meet at Point(x,y) =(3,-4).

Avenues (South to North) Streets (East to West) Combine to form x, y-plane.

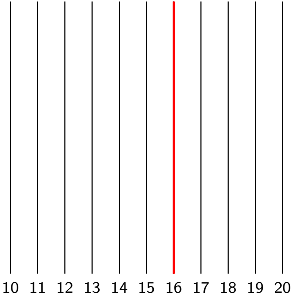


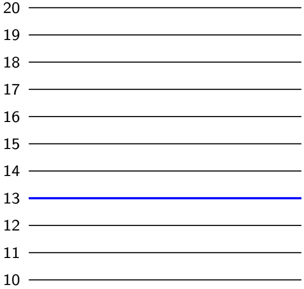




The grid below shows the rectangle containing all points (x, y) with $10 \leq x \leq 20$ and $10 \leq y \leq 20$.
Since the x-axis ($y = 0$) and the y-axis ($x = 0$) are not visible,
write avenue numbers(x-labels) below the grid, street numbers (y-labels) at the left.

Avenues (South-North) Streets (East-West)





Street Map: 16th Avenue and 13th Street meet at the

The grid below shows the rectangle containing all points (x, y) with $10 \le x \le 20$ and $10 \le y \le 20$.
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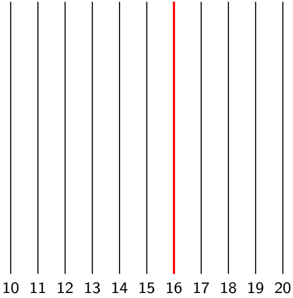
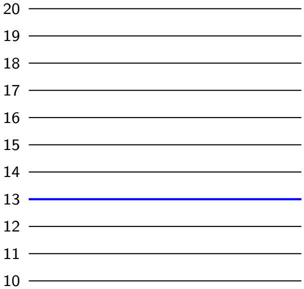
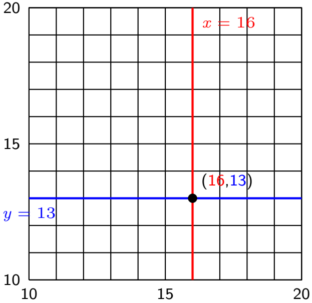
Avenues (South-North)



Streets (East-West)



combine to form grid:



Street Map: 16th Avenue and 13th Street meet at the corner of 16th and 13th

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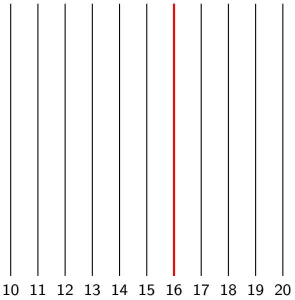
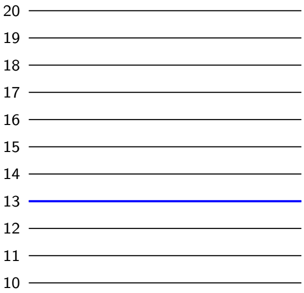
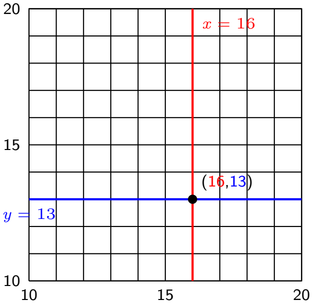
Avenues (South-North)



Streets (East-West)



combine to form grid:



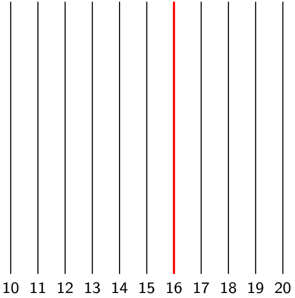
Street Map: 16th Avenue and 13th Street meet at the corner of 16th and 13th

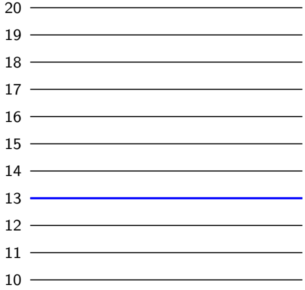
This translates to mathematical language as

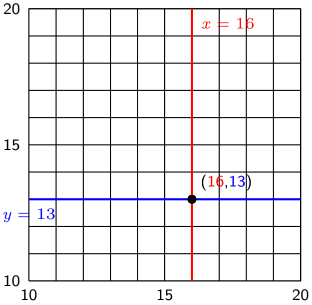
The vertical line x = 16 and the horizontal line y = 13 meet at Point $(x, y) = (16, 13)$

The grid below shows the rectangle containing all points (x, y) with $10 \leq x \leq 20$ and $10 \leq y \leq 20$.
Since the x-axis ($y = 0$) and the y-axis ($x = 0$) are not visible,
write avenue numbers(x-labels) below the grid, street numbers (y-labels) at the left.

Avenues (South-North) Streets (East-West) combine to form grid:







Street Map: 16th Avenue and 13th Street meet at the corner of 16th and 13th

This translates to mathematical language as

The vertical line x = 16 and the horizontal line y = 13 meet at Point $(x, y) = (16, 13)$

Placing x- and y-labels at the left of and below the grid rather than next to the x- and y-axes makes them much easier to read. Grids will be drawn in this manner from now on, whether or not they contain the origin (0,0) ;

If you are drawing the grid by hand, don't waste time including labels for every gridline! In the above grid, its fine to include just the x-labels and y-labels $10, 15$, and 20.

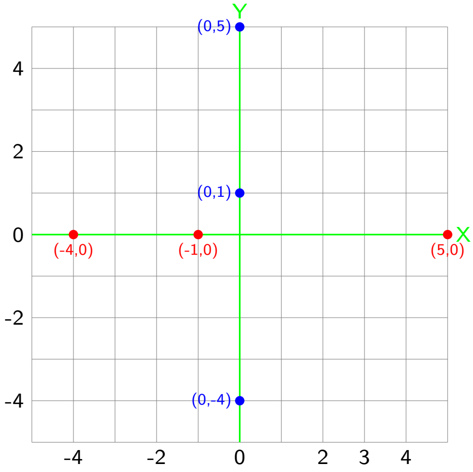
1.8.2 Finding the coordinates of a point in the plane

How to find the coordinates of a point P in the x, y-plane

- The x-coordinate of P is the x-label at the bottom of the vertical line through P.
- The y-coordinate of P is the y-label to the left of the horizontal line through P.
- The (x,y)-coordinates of point P are (a, b), where a is its x-coordinate and b is its y-coordinate.

Based on the above, it's easy to see that

- Every point on the x-axis has y-coordinate 0. See the red points at the right.
- Every point on the y-axis has x- coordinate 0. See the blue points at the right.



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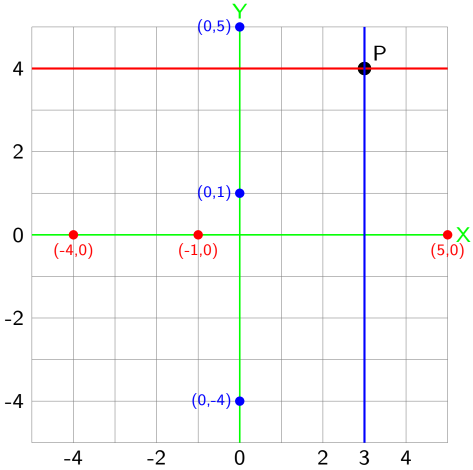
Example: Find the coordinates of point P at the right

- The vertical line through P meets the x-label 3.
- The horizontal line through P meets the y-label 4.

Answer: Therefore the coordinates of P are $(3, 4)$.



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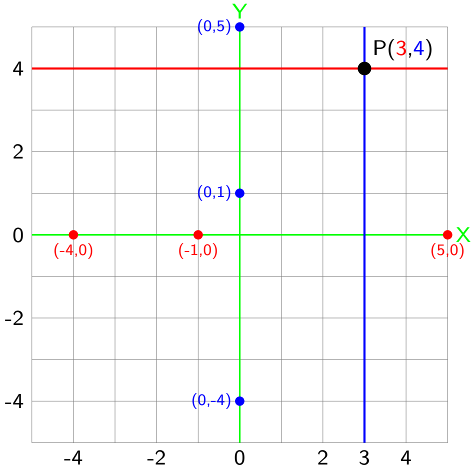
All points on a vertical line have the same x-coordinate.

- Every point on the blue line has x-coordinate 3.
- The blue line's equation is $x = 3$.
- The green y-axis is vertical: its equation is $x = 0$.



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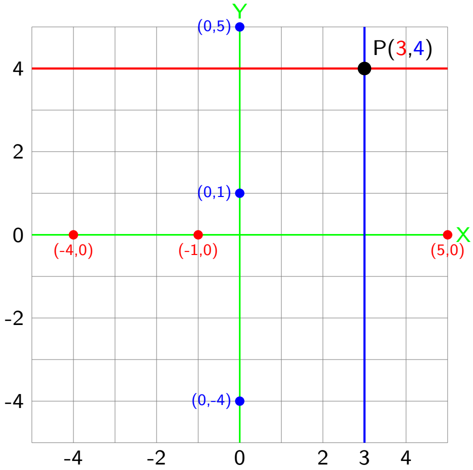
All points on a horizontal line have the same y-coordinate.

- Every point on the red line has y-coordinate 4.
- The red line's equation is $y = 4$.
- The green x-axis is horizontal: its equation is $y = 0$.



1.8.3 Plotting a point whose coordinates are given

The following method for plotting a point is the one that most students know from high school. However, it works well only if the grid contains the origin $(0, 0)$.

How to plot a point $(x, y) = (a, b)$ by starting at the origin

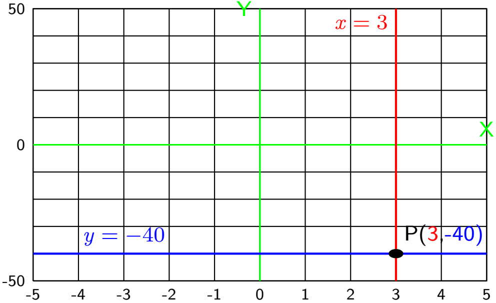
- If $a \geq 0$, move a units right. If $a \leq 0$, move $|a|$ units left. You are now at point $(a, 0)$. Then:
- If $b \geq 0$, move b units up. If $b \leq 0$, move $|b|$ units down. This brings you to point (a, b).

We prefer the following method, which relies on having drawn x-labels below and y-labels to the left of the grid.

How to plot a point with given coordinates (a, b)

- Draw the vertical line through the x-label a
- Draw the horizontal line through the y-label b.
- Draw a dot where those lines meet, and label it (a, b).
- If you want to name the dot P, label it $P(a, b)$.

Use this method to plot point $P(3, -40)$:



The blue line's equation is $y = -40$ because:

- 5 boxes are 50 units high on the y-scale;
- therefore each box is $50/5 = 10$ units high, and so
- the y-label to the left of the blue line is $-50 + 10 = -40$.

A grid is usually not a checkerboard:

- Grid boxes don't need to be squares with side 1.
- Not every gridline needs a scale number next to it.
- Vertical and horizontal scales don't need to match. In the grid above, a distance of 1 on the x-axis equals a distance of 15 on the y-axis!.

Every pair of numbers (x, y) corresponds to a point in the x, y-plane. The order matters: $P(3, 4)$ and $Q(4, 3)$ are different points.

Fundamental principle of analytic geometry:

A point $P(a, b)$ in the x, y-plane lies on the graph of an equation with variables x, y if and only if the point's coordinates satisfy the equation. This means: substituting a for x and b for y yields a true numerical statement.

Example 1: $x^2 + y^2 = 25$ is an equation.

- $(3, 4)$ satisfies the equation because substituting 3 for x and 4 for y yields the true statement $3^3 + 4^2 = 25$, but
- $(4, 2)$ doesn't satisfy the equation because substituting 4 for x and 2 for y yields the false statement $4^2 + 2^2 = 25$.

The graph of the equation $x^2 + y^2 = 25$ is a circle with center at $(0, 0)$ and radius 5. The graph consists of all points (x, y) in the x, y-plane with $x^2 + y^2 = 25$.

The points $(3, 4), (-3, 4), (-3, -4)$, and $(3, -4)$ all lie on the graph but $(-4, -2)$ does not, since $(-4)^2 + (-2)^2 = 20$ rather than 25.

Example 2: The equation $x^2 + y^2 + 25 = 0$ can be rewritten as $x^2 + y^2 = -25$. Since the square of any real number is non-negative, the graph is empty: it contains no points at all.

Equations of vertical and horizontal lines

- For any point $P(c, d)$
  - the equation of the vertical line through P is $x = c$.
  - The equation of the horizontal line through P is $y = d$.
- If any point on a vertical line has x-coordinate c, every point on that line has x-coordinate c,
- If any point on a horizontal line has y-coordinate d, every point on that line has y-coordinate d.

Example 3: Find the equations of the horizontal and vertical lines through point $(3, 4)$.

Solution:

- The horizontal line through $(3, 4)$ has equation $y = 4$.
- The vertical line through $(3, 4)$ has equation $x = 3$.

The statements above illustrate an important point: If a variable is missing from an equation, the variable is arbitrary.

Be careful: If y is missing from an equation in x and y, every solution is of the form (x, y) where x solves the equation and y is arbitrary. *Don't make the mistake of thinking that the missing variable is zero.*

For example, the equation $x^2 = 1$ is missing letter y.

That equation has two solutions: $x = 1$ and $x = -1$, while y is arbitrary. Therefore the graph of the equation consists of the two vertical lines $x = 1$ and $x = -1$. On each of these lines, y is arbitrary, not just 0.

The following was discussed earlier, but it's so important that we repeat it here. A cause for concern is that New York State Regents Exams seem to assume that almost all coordinate grids consists of boxes measuring 1 unit square.

When you draw a graph:

- Depending on the problem, you need not choose only x-values that are whole numbers centered around $x = 0$.
- In a science course, it would be unusual if each box measured 1 unit by 1 unit.
- The sides of a box drawn as a square need not match: The x-scale numbers could be $1, 2, 3, 4$ while the y-scale numbers could be $10000, 11000, 12000, 13000$.
- If you are drawing a graph with scales that run from -10 to 10, do not waste time writing scale numbers $-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.
  It's fine to use just $-10, -5, 0, 5, 10$.

x-intercepts and y-intercepts.

If you know that the graph of an equation is near the origin, a good way to start drawing it is to find its intercepts.

Definition: the intercepts of a graph

- The x-intercepts are the x-coordinates of all points where the graph meets the x-axis. Those points are called x-intercept points
- The y-intercepts are the y-coordinates of all points where the graph meets the y-axis. Those points are called y-intercept points.

How to find the intercepts of the graph of an equation

- To find the y-intercepts, set $x = 0$ and solve the equation for y.
- To find the x-intercepts, set $y = 0$ and solve the equation for x.

Example 4: Find all x- and y-intercepts of the graph of the equation $x^2 + y^2 = 4$. At what points does the graph meet the x-axis? the y-axis?

We will see in the next section that the graph of $x^2 + y^2 = 4$ is a circle with center $(0, 0)$ and radius 2.

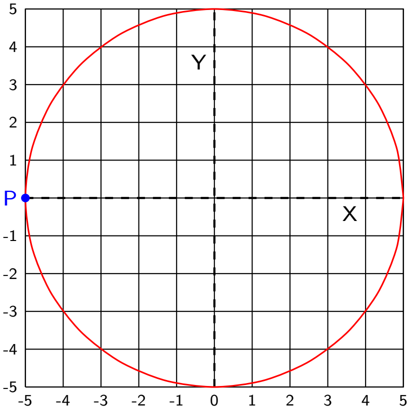
Solution:
Setting $y = 0$ gives $x^2 = 4$.
This equation has two solutions: $x = 2$ and $x = -2$.
The x-intercepts are 2 and -2.
The graph meets the x-axis at points $(2, 0)$ and $(-2, 0)$.

Setting $x = 0$ gives $y^2 = 4$.
This equation has two solutions: $y = 2$ and $y = -2$.
The y-intercepts are 2 and -2
The graph meets the y-axis at points $(0, 2)$ and $(0, -2)$.

1.8.5 Locating points on graphs

Below is the graph of the radius 5 circle with equation $x^2 + y^2 = 25$. Roll the mouse wheel slowly, or click an appropriate arrow key, and figure out the coordinates of each new point before the answer appears on the screen.

Example 5: Points on the top semicircle



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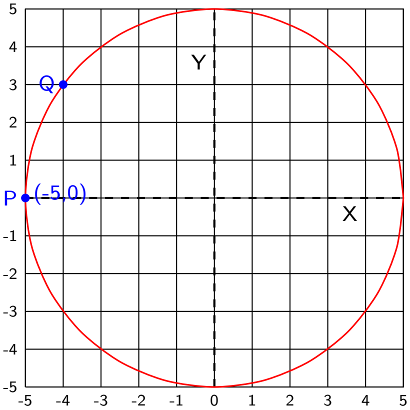

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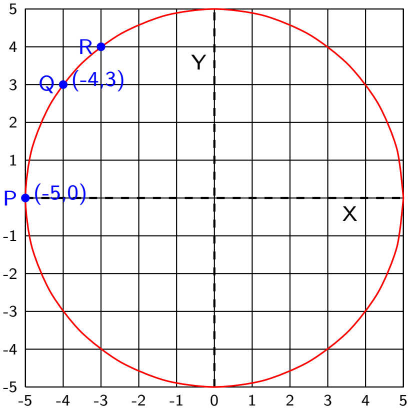

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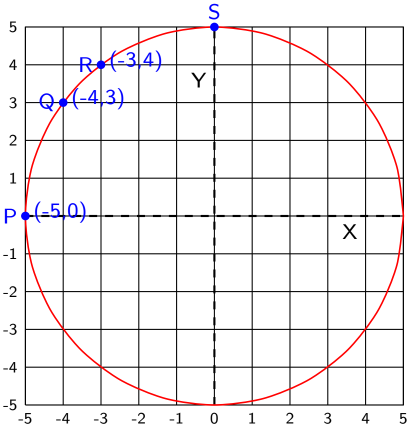

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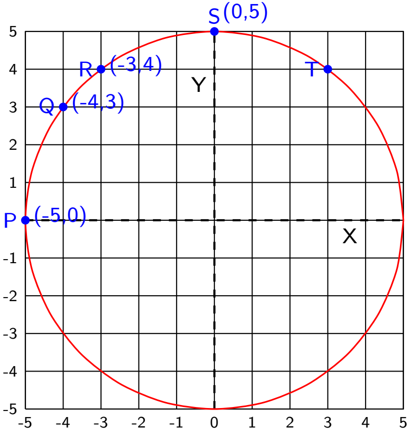

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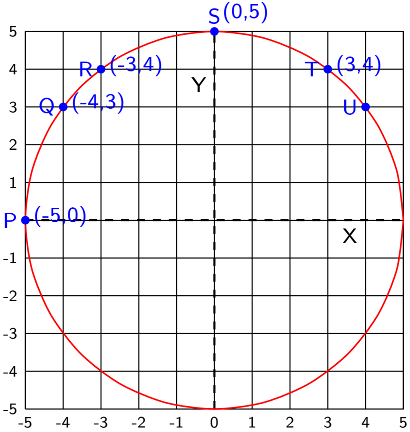

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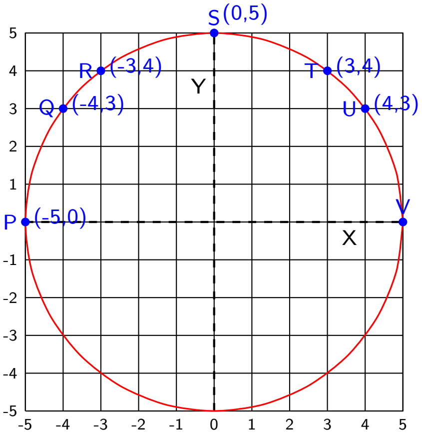

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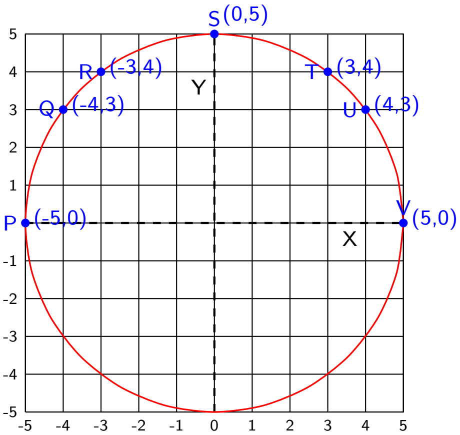

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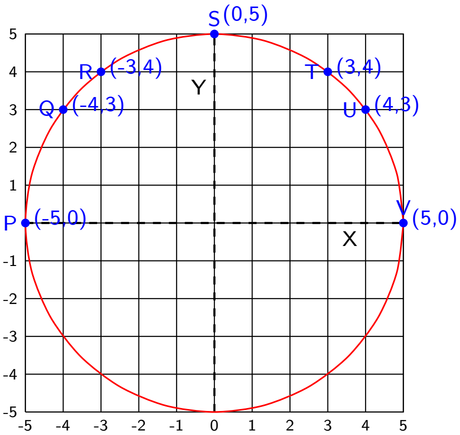
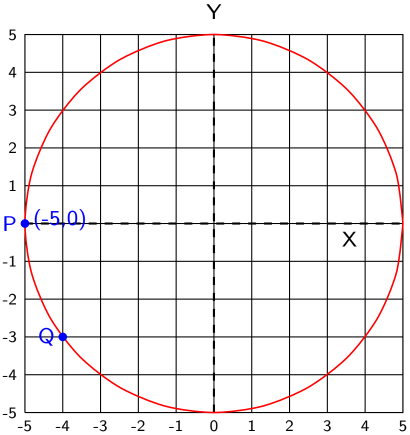


Example 6: Points on the bottom semicircle



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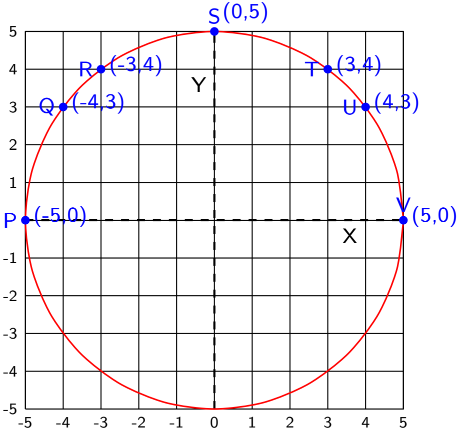

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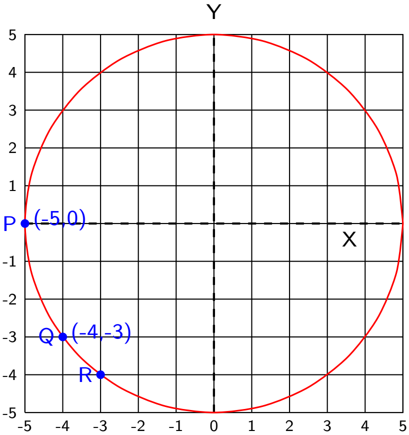

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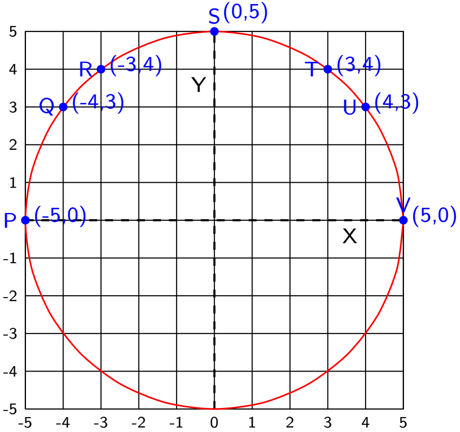

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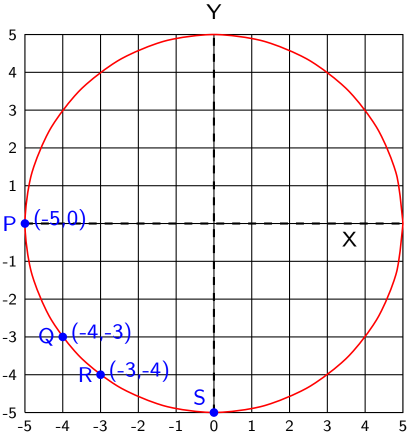

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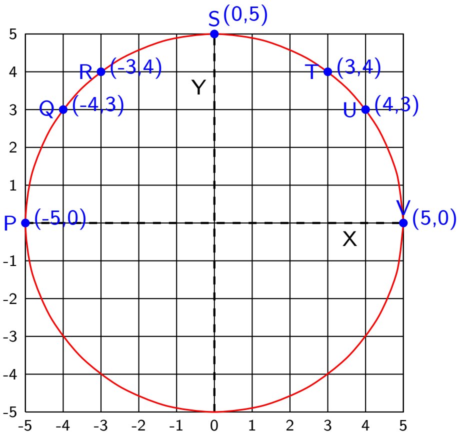

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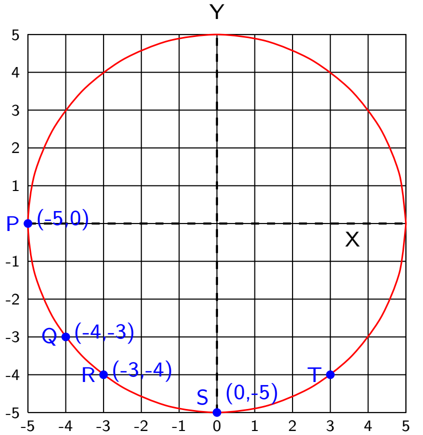

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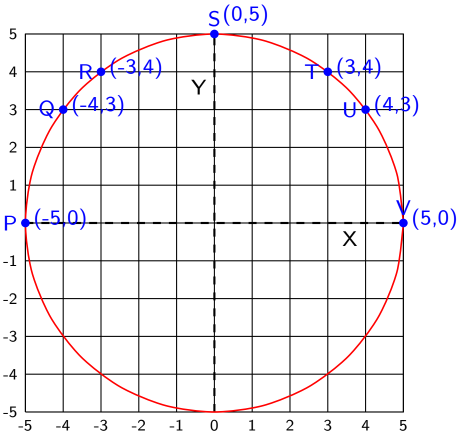

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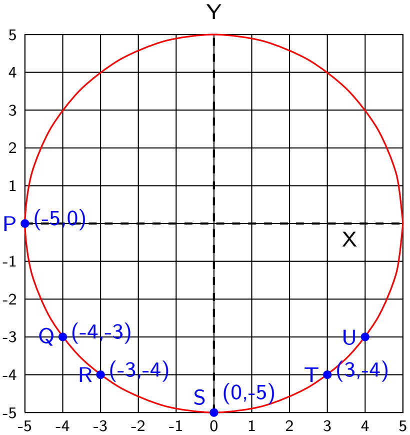

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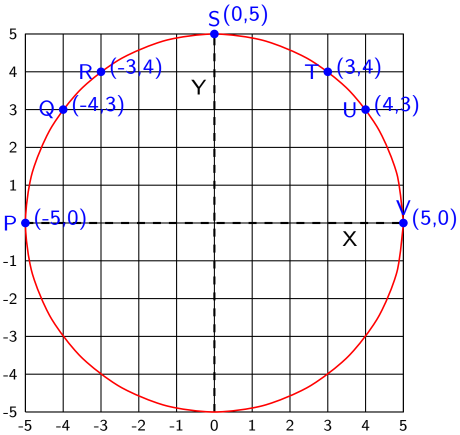

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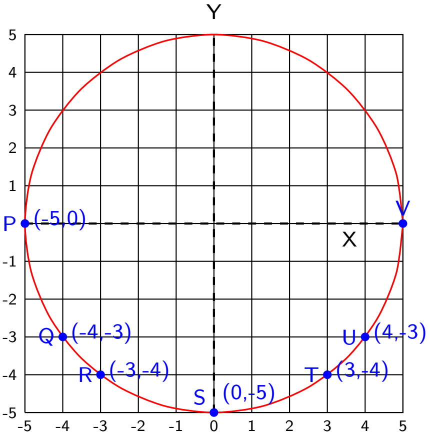

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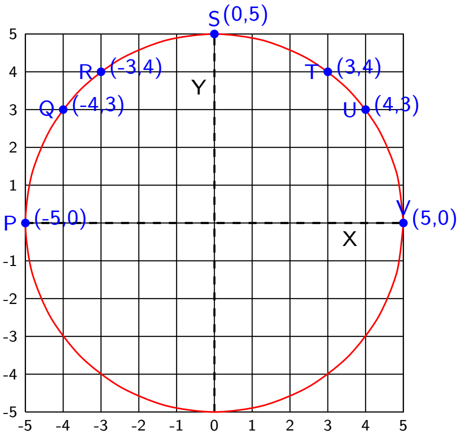

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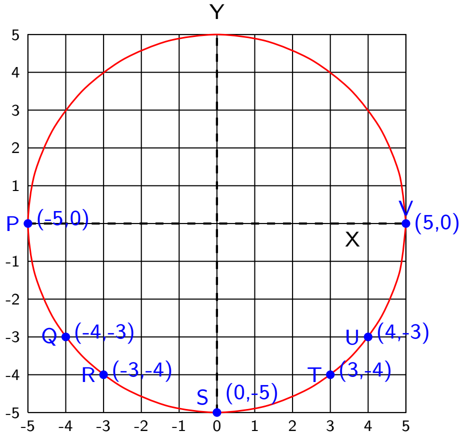
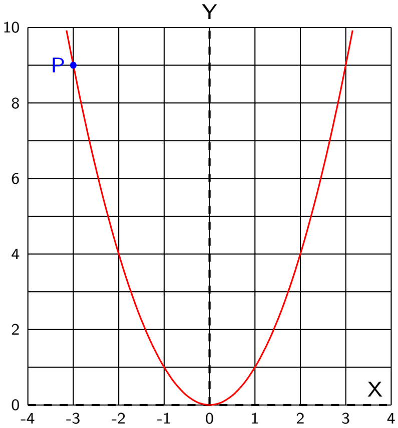


Example 7: Below is the graph of $y = x^2$.
Roll the mouse wheel slowly or click an appropriate arrow key, then figure out the coordinates of each new point before the answer appears on the screen.



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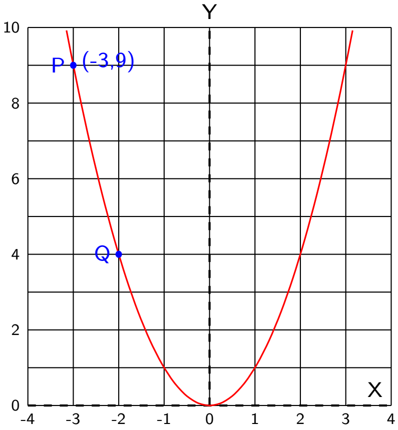

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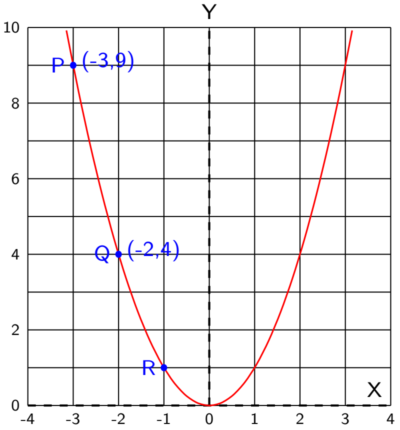

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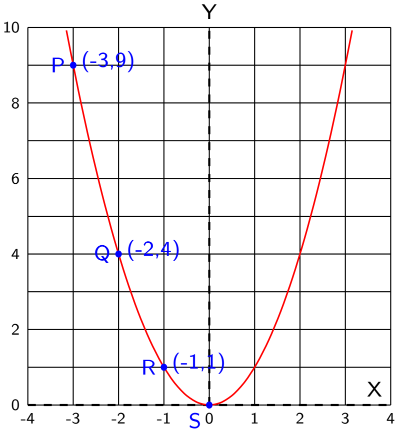

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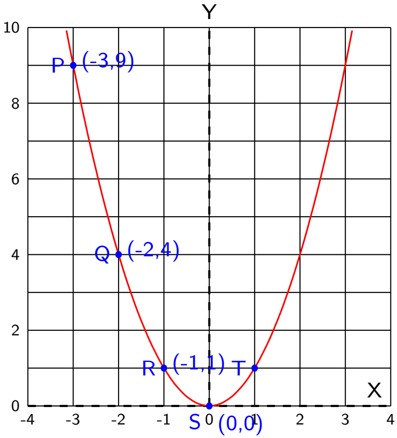

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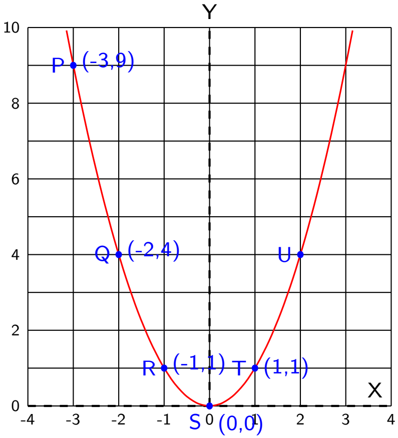

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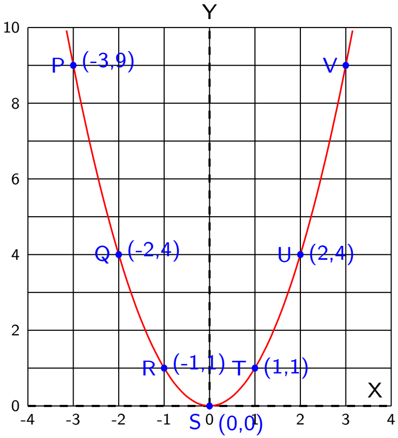

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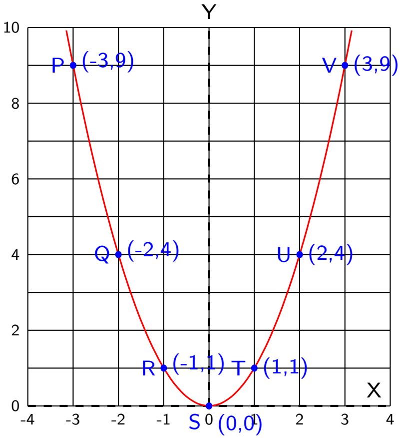

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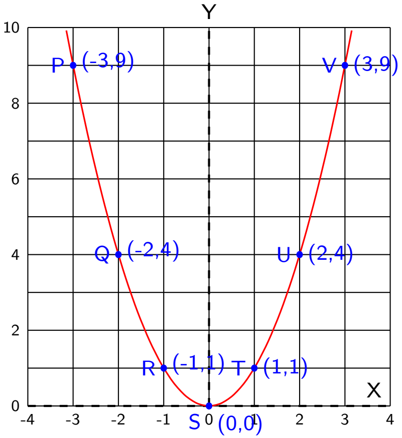

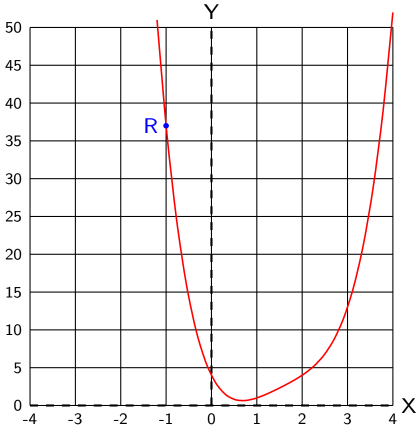

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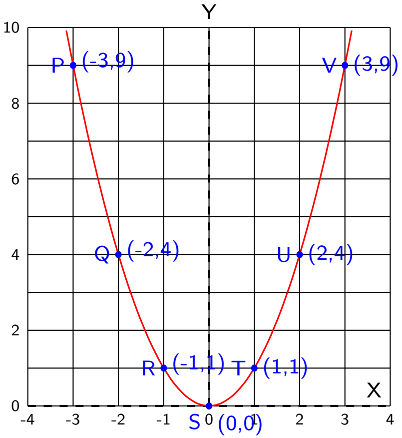

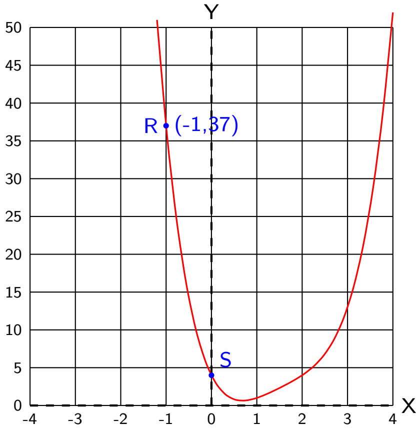

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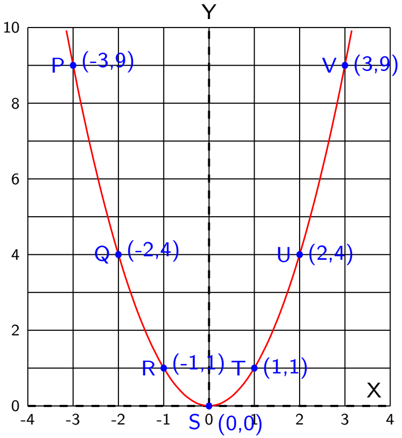

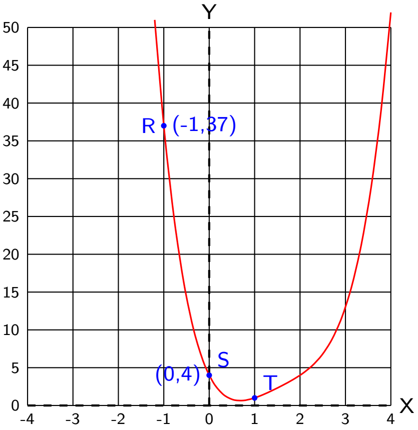

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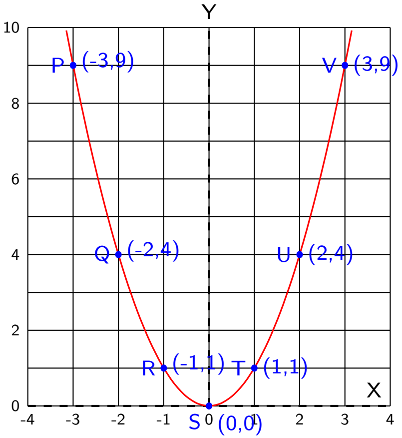

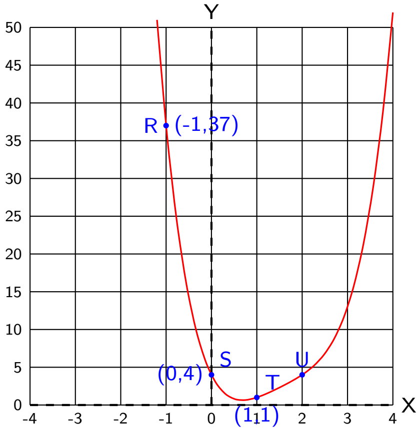

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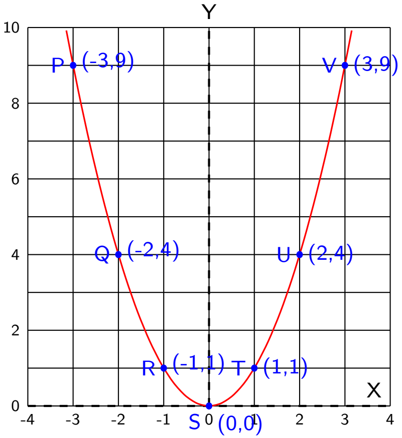

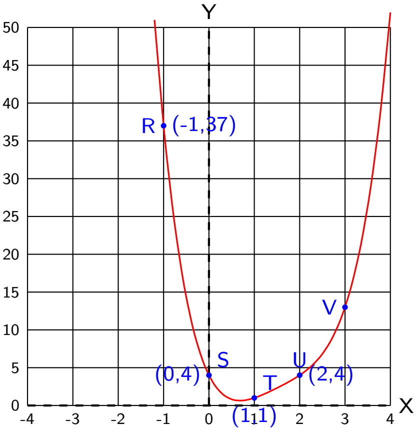

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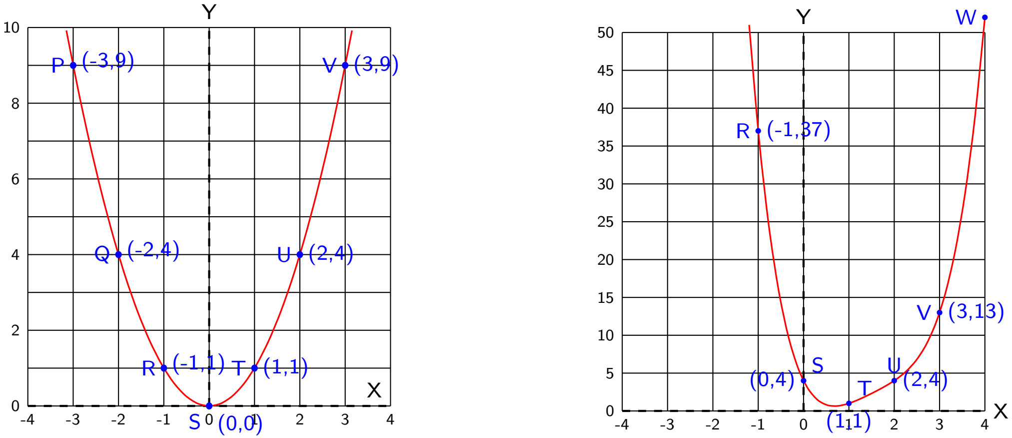

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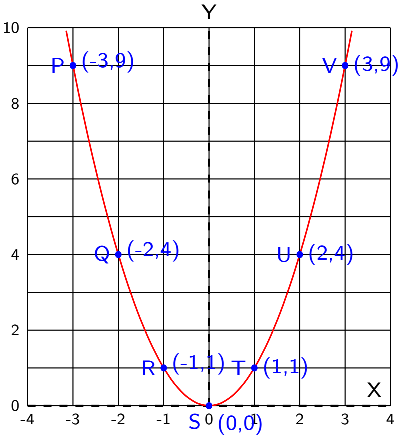

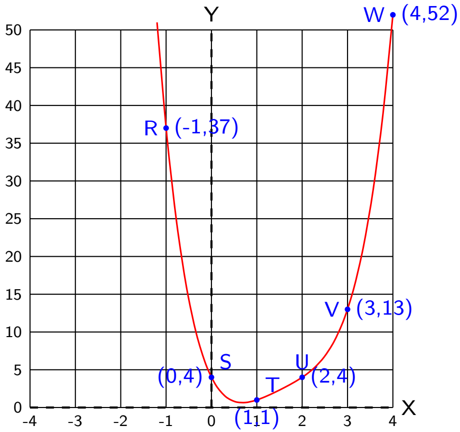

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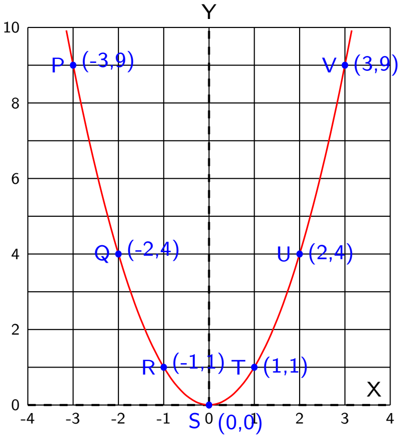
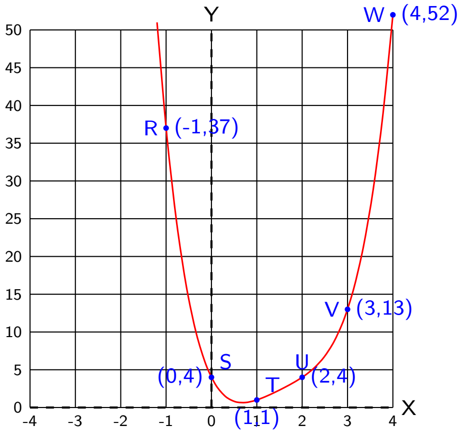




Further work requires studying distances between points in the plane.

Distance Formula: The distance between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the x, y-plane is

$\overline{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ This equals $\overline{QP} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

1.8.6 Completing the square

The circle with radius 2 and center $(4, 5)$ consists of all points (x, y) whose distance from $(4, 5)$ is 2. Using the distance formula, this says $\sqrt{(x-4)^2 + (y-5)^2} = 2$. Squaring both sides yields $(x-4)^2 + (y-5)^2 = 4$.

Definition: The standard form equation of a circle

with center (h, k) and radius r is
$(x-h)^2 + (y-k)^2 = r^2$.

Example 8 : Find the center and radius of the circle with equation $(x-4)^2 + (y+5)^2 = 49$.

Solution: We want the given equation $(x-4)^2 + (y+5)^2 = 49$ to match the general formula $(x-h)^2 + (y-k)^2 = r^2$. To make the equations match, we take $h = 4$ and $k = -5$ and $r^2 = 49 = 7^2$.

Answer: Center is $(h, k) = (4, -5)$ and radius is 7.

Error Warnings:

- The coordinates h and k are the *negatives* of the numbers following x and y.
- Any graph has many equations. There is no such thing as *the* equation of a graph.
- The language "standard form equation" singles out a particular equation that is useful, since it displays in a convenient way the center and radius of the circle.

But there are many other forms that are also of interest. For example, if you don't like parentheses, multiplying out the standard form circle equation $(x-4)^2 + (y+5)^2 = 49$ gives $x^2 - 8x + 16 + y^2 + 10y + 25 = 49$. This gives $x^2 - 8x + y^2 + 10y = 8$, an equation that conceals the circle's radius and center. To recover the missing information, we *complete the square in* $x^2 - 8x$.

To complete the square in $x^2 + Bx$, substitute

$$\left(x + \frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^2 \text{ for } x^2 + Bx$$

Example 9:

Complete the square in the equation $x^2 + 4x = 12$

Here $x^2 + Bx = x^2 + 4x$ so $B = 4$ and $B/2 = 2$

Substitute: $(x+2)^2 - 2^2 = 12$ and so $\boxed{(x+2)^2 = 16}$

Example 10: Find the standard form equation, and the center and radius, of the circle with equation $3x^2 + 3y^2 = 24 + 24x - 30y$ by completing the square.

Solution: First divide by 3 to get

$x^2 + y^2 = 8 + 8x - 10y$.

$x^2 - 8x + y^2 + 10y = 8$, which is the equation discussed above.

To complete the square in $x^2-8x+y^2+10y=8$,

- Let $x^2-8x=x^2+Bx$: Use $B=-8$; $B/2=-4$.
and substitute $(x-4)^2-(-4)^2=(x-4)^2-16$ for x^2-8x.
- Let $y^2+10y=y^2+By$: use $B=10$, $B/2=5$
and substitute $(y+5)^2-5^2=(y+5)^2-25$ for y^2+10y.

These substitution change $x^2-8x+y^2+10y=8$ to:
$(x-4)^2-16+(y-5)^2-25=8$.
Add $16+25$ to both sides to get
$(x-4)^2+(y-5)^2=8+16+25=49=7^2$.

$(x-4)^2+(y+5)^2=7^2$: Center: $(4,-5)$ Radius: 7

Example 11: Complete the square in $y=2x^2+x+1$.
Solution:

Original equation:	$y=2x^2+x+1$
Factor out coeff. of x^2:	$y=2(x^2+\frac{1}{2}x)+1$
Here $B=\frac{1}{2}$; $\frac{B}{2}=\frac{1}{4}$	$y=2((x+\frac{1}{4})^2-\frac{1}{4}^2)+1$
Remember parentheses	when you substitute!
Distributive Law:	$y=2(x+\frac{1}{4})^2-2\cdot\frac{1}{16}+1$
$y=2(x+\frac{1}{4})^2-\frac{1}{8}+1$	$y=2(x+\frac{1}{4})^2+\frac{7}{8}$

Method 2: To complete the square in an equation with x^2+Bx on one side:

Replace x^2+Bx by $x^2+Bx+\left(\frac{B}{2}\right)^2=(x+\frac{B}{2})^2$ and make corresponding changes involving $\left(\frac{B}{2}\right)^2$ to the other side of the equation.

Example 11, Method 2: Solve: $21=2x^2+x$

Solution: Factor	coefficient 2 of x^2 from
just the x terms:	$21=2(x^2+\frac{1}{2}x)$
Complete square	of $x^2+\frac{1}{2}x$ using $B=\frac{1}{2}$; so
	$\frac{B}{2}=\frac{1}{4}$ and $(\frac{B}{2})^2=\frac{1}{16}$:
Replace	$x^2+\frac{1}{2}x$ on the right
side (RS) by	$x^2+\frac{1}{2}x+\frac{1}{16}$
so RS becomes	$2(x^2+\frac{1}{2}x+\frac{1}{16})$
This added	$2\cdot\frac{1}{16}$ (*not* $\frac{1}{16}$) to the right side
so we must	also add $2\cdot\frac{1}{16}$ to the left side!
to get equation	$21+2\cdot\frac{1}{16}=2(x^2+\frac{1}{2}x+\frac{1}{16})$
Rewrite	$\frac{21+\frac{1}{8}}{2}=(x+\frac{1}{4})^2$
as	$\frac{169}{16}=(x+\frac{1}{4})^2$
and so	$\pm\sqrt{\frac{169}{16}}=x+\frac{1}{4}$
$x=-\frac{1}{4}\pm\frac{13}{4}$	$x=3$ or $x=-\frac{7}{2}$

We can also solve Example 10 by using Method 2:
To put the circle equation $x^2 + y^2 = 8 + 8x - 10y$ in standard form, rewrite it as
$x^2 - 8x + (\quad) + y^2 + 10y + (\quad) = 8 + (\quad) + (\quad)$
To fill in the first blank, complete the square in $x^2 - 8x$ to get $x^2 - 8x + (-8/2)^2 = x^2 - 8x + 16$.
To fill in the second blank, complete the square in $y^2 + 10$ to get $y^2 + 10y + (10/2)^2 = y^2 + 10y + 25$.
Fill in the blanks on both left and right with 16 and 25
$x^2 - 8x + (16) + y^2 + 10y + (25) = 8 + (16) + (25) = 49$.
Rewrite with squares to get the standard form equation:

$(x-4)^2 + (y+5)^2 = 7^2$: Center: $(4, -5)$ Radius: 7

Finally, here is the general procedure:

To complete the square in an equation involving $Ax^2 + Bx$

Divide by A and complete the square in $x^2 + \frac{B}{A}x$

Original equation:	$y = 2x^2 + x + 1$
Divide by 2	$\frac{y}{2} = x^2 + \frac{1}{2}x + \frac{1}{2}$
Complete the square with $B = \frac{1}{2}$:	$\frac{y}{2} + (\frac{1}{4})^2 = (x^2 + \frac{1}{2}x + (\frac{1}{4})^2) + \frac{1}{2}$
Extract the square	$\frac{y}{2} + \frac{1}{16} = (x + \frac{1}{4})^2 + \frac{1}{2}$
Multiply by 2 :	$y + \frac{1}{8} = 2(x + \frac{1}{4})^2 + 1$
Solve for y:	$y = 2(x + \frac{1}{4})^2 + \frac{7}{8}$

Applications of completing the square include

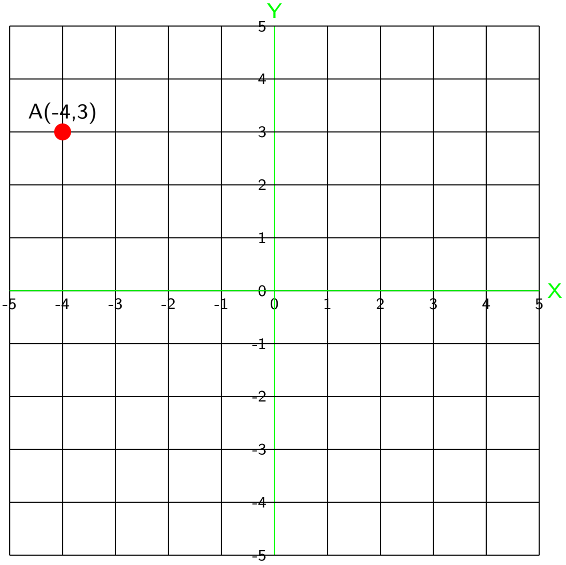
- finding the standard form equation of circles, parabolas, ellipses, hyperbolas;
- deriving the Quadratic Equation.

1.8.7 Graph symmetry

The reflection of point (x, y)

- across the y-axis is the point $(-x, y)$
- through the origin is the point $(-x, -y)$
- across the x-axis is the point $(x, -y)$

- Start at point $A : (x, y) = (-4, 3)$. It is 4 units from the y-axis, 3 from the x-axis, and 5 from the origin. Now click slowly through this slide.



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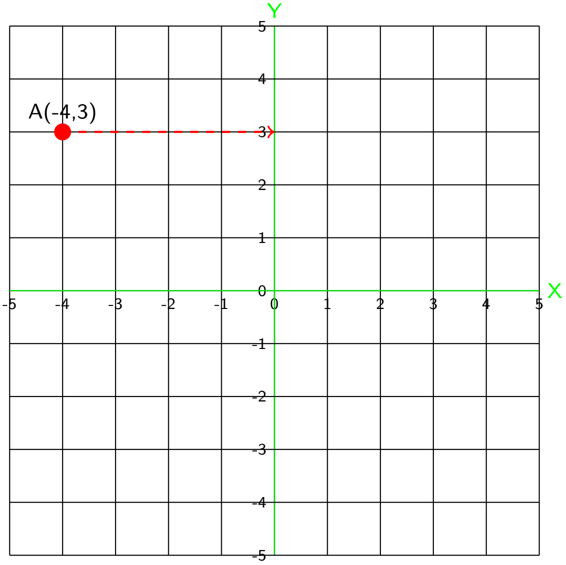

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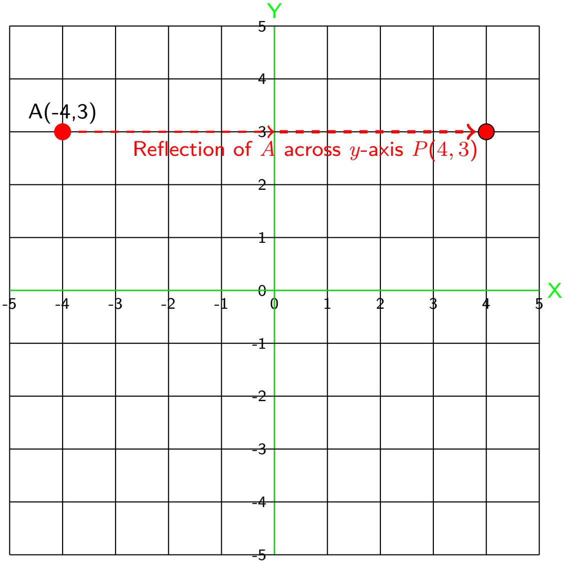
- If you draw a horizontal line to the y-axis, then continue that line an equal distance past the y-axis, you arrive at the reflection of A across the y-axis, point $P(4, 3)$.



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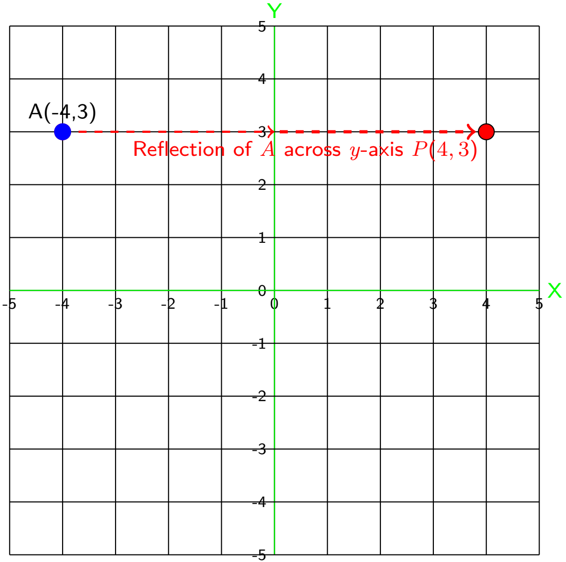
- Go back to $A(-4, 3)$



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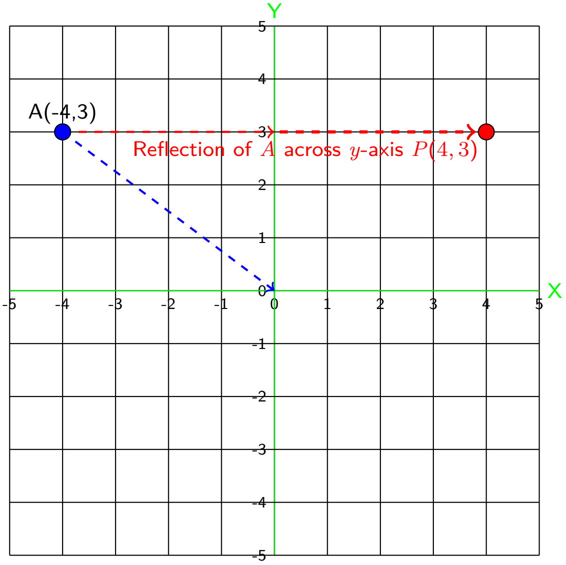
- Go back to $A(-4, 3)$ If you draw a line to the origin,



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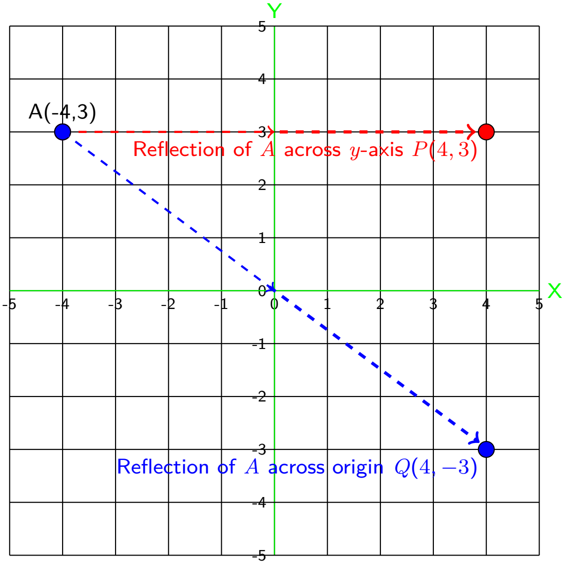
- Go back to $A(-4, 3)$ If you draw a line to the origin, then continue that line an equal distance past the origin, you arrive at the reflection of A across the origin, point $Q(4, -3)$.



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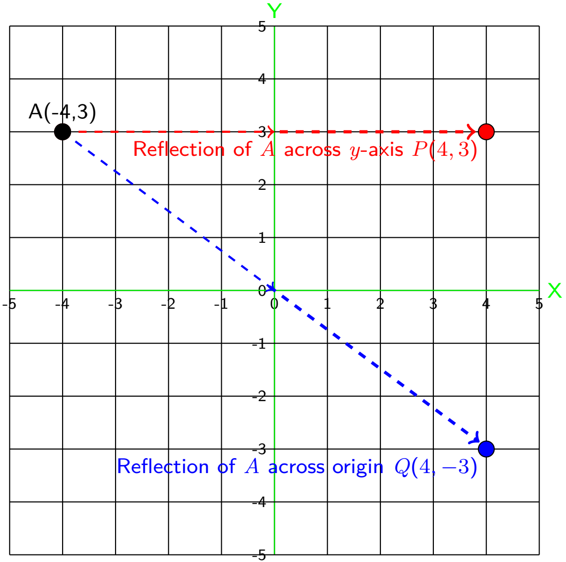

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- Go back to $A(-4, 3)$ If you draw a line to the origin, then continue that line an equal distance past the origin, you arrive at the reflection of A across the origin, point $Q(4, -3)$.
- Go back to $A(-4, 3)$

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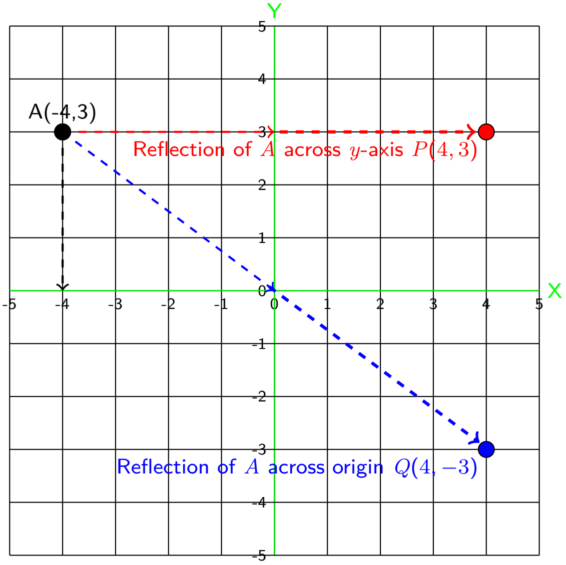
- Go back to $A(-4, 3)$ If you draw a line to the x-axis



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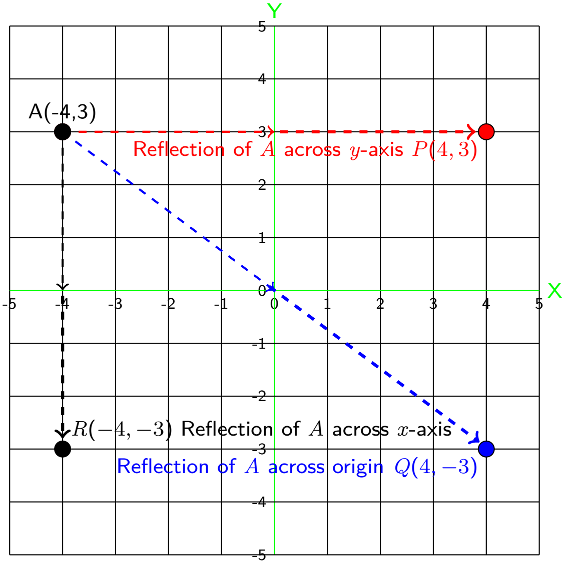
- If you draw a horizontal line to the y-axis, then continue that line an equal distance past the y-axis, you arrive at the reflection of A across the y-axis, point $P(4, 3)$.
- Go back to $A(-4, 3)$ If you draw a line to the origin, then continue that line an equal distance past the origin, you arrive at the reflection of A across the origin, point $Q(4, -3)$.
- Go back to $A(-4, 3)$ If you draw a line to the x-axis then continue that line an equal distance past the x-axis, you arrive at the reflection of A across the x-axis, point $R(4, -3)$.



Definition: ***A graph is symmetric***

- about the y-axis if the reflection of any point on the graph across the y-axis is also on the graph.
- about the x-axis if the reflection of any point on the graph across the x-axis is also on the graph.
- about the origin if the reflection of any point on the graph through the origin is also on the graph.

Reminder: $x = 4; y = 3$ is a solution of the equation $x^2 + y^2 = 25$ because substituting 4 for x and 3 for y gives the true statement $4^2 + 3^2 = 25$.

Two equations in x and y are equivalent means:

- Both equations have the same solutions.
- Both equations have the same graphs

To show that two equations are equivalent, just show that one can be obtained from the other by the usual rules for solving equations:

Multiplying both sides of an equation by a nonzero number, or adding any number to both sides, yields an equivalent equation.

Let P and Q be polynomial expressions in one variable x. Then the equations $y = P$ and $y = Q$ are equivalent

if and only if $P = Q$ is an identity, i.e. if $P - Q$ simplifies to the zero polynomial.

To determine whether the graph of an equation is

- x-axis symmetric, decide if substituting $-y$ for y in the equation yields an equivalent equation.
- y-axis symmetric, decide if substituting $-x$ for x in the equation yields an equivalent equation.
- origin symmetric, decide if substituting $-x$ for x and $-y$ for y in the equation yields an equivalent equation.

Exercise: Show that a graph is origin symmetric if and only if it is both x-axis symmetric and y-axis symmetric.

Example 12:

Is the graph of equation $y = 8x^5 + 7x^3 + 8x$ x-axis symmetric?

Solution: Let $P = 8x^5 + 7x^3 + 8x$. We want to know if $y = P$ and $-y = P$ are equivalent. The second equation says $y = -P$, so let $Q = -P$. Then the eqauations are equivalent if $P - Q$ reduces to the zero polynomial. But

$P - Q = P - (-P) = 2P = -2(+8x^5 + 7x^3 + 8x)$is simplified but not zero.

The graph is not x-axis symmetric.

Exercise : Modify the above argument to show why the graph of equation $y = P$ is not x-axis equivalent unless $P = 0$.

Example 13:

Is the graph of $y = 7x^4 + 8x^2$ y-axis symmetric?

Solution: Substitute $-x$ for x in

- $y = 7x^4 + 8x^2$ to get $y = 7(-x)^4 + 8(-x)^2$ which is the same as
- $y = 7x^4 + 8x^2$. Since the original and new equations are identical, they have the same solutions and so

The graph is y-axis symmetric.

Example 14:

Is the graph of $y = x^3 + x$ origin symmetric?

Solution: Substitute $-x$ for x and $-y$ for y in

- $y = x^3 + x$ to get $-y = (-x)^3 + (-x) = -x^3 - x$ Multiplying by -1 gives an equivalent equation
- $y = x^3 + x$, the one we started with.

Therefore The graph is origin-symmetric.

When we study functions in detail, we will see that reflecting graphs across the x-axis, y-axis, or across the origin will be a crucial tool for sketching complicated graphs. For now, we only state the language that will be used.

To reflect the graph of an equation across the

- y-axis: Substitute $-x$ for x in the equation to obtain a new equation and draw its graph.
- x-axis: substitute $-y$ for y in the equation to obtain a new equation and draw its graph.
- origin: substitute $-x$ for x and $-y$ for y in the equation to obtain a new equation and draw its graph.

Section 1.8 Quiz

Ex. 1.8.1: Which of the pairs (3,4) and (4,2) satisfies the equation $x^2 + y^2 = 25$?

Ex. 1.8.2: Explain why the graph of $x^2 + y^2 + 25 = 0$ is empty.

Ex. 1.8.3: Find the equations of the vertical and horizontal lines through $(3, 4)$.

Ex. 1.8.4: Find all x- and y- intercepts of the graph of the equation $x^2 + y^2 = 4$.

At what points does the graph meet the x-axis? the y-axis?

Ex. 1.8.5: Go back in these notes and do these examples interactively.

Ex. 1.8.6:: Go back in these notes and do these examples interactively.

Ex. 1.8.7: Go back in these notes and do these examples interactively.

Ex. 1.8.8: Find the center and radius of the circle with equation $(x - 4)^2 + (y + 5)^2 = 49$.

Ex. 1.8.9: Complete the square in the equation $x^2 + 4x = 12$.

Ex. 1.8.10: Find the standard form equation, center, and radius of the circle with equation $3x^2 + 3y^2 = 24 + 24x - 30y$.

Ex. 1.8.11: Complete the square in $y = 2x^2 + x + 1$

Ex. 1.8.12: Is the graph of $y = x + 7x^3 + 8x^5$ x-axis symmetric?

Ex. 1.8.13: Is the graph of $y = 7x^4 + 8x^2$ y-axis symmetric?

Ex. 1.8.14: Is the graph of $y = x^3 + x$ origin symmetric?

Section 1.8 Review: The x, y-coordinate plane

Ex. 1.8.1: Which of the pairs $(3, 4)$ and $(4, 2)$ satisfies the equation

- $x^2 + y^2 = 25$?
- $x^2 - y^2 = 25$?
- $x^2 + \frac{x}{y} = 18$?
- $y + 2x = 10$?

Section 1.8 Review: The x, y-coordinate plane

Ex. 1.8.1: Which of the pairs $(3, 4)$ and $(4, 2)$ satisfies the equation • $x^2 + y^2 = 25$? $\Rightarrow (3, 4)$
• $x^2 - y^2 = 25$? $\Rightarrow$ Neither • $x^2 + \frac{x}{y} = 18$? $\Rightarrow (4, 2)$ • $y + 2x = 10$? $\Rightarrow$ Both

Section 1.8 Review: The x, y-coordinate plane

Ex. 1.8.1: Which of the pairs $(3, 4)$ and $(4, 2)$ satisfies the equation • $x^2 + y^2 = 25? \Rightarrow (3, 4)$
• $x^2 - y^2 = 25? \Rightarrow$ Neither • $x^2 + \frac{x}{y} = 18? \Rightarrow (4, 2)$ • $y + 2x = 10? \Rightarrow$ Both

Ex. 1.8.2: Which of the following graphs are empty? Explain why.

- $x^2 + y^2 + 25 = 0$
- $\frac{\sqrt{x}+\sqrt{-x}}{x} = 0$
- $\sqrt{x} + \sqrt{-x} = 0$
- $x^2 + y^3 + 25 = 0$

Section 1.8 Review: The x, y-coordinate plane

Ex. 1.8.1: Which of the pairs $(3,4)$ and $(4,2)$ satisfies the equation • $x^2+y^2=25$? $\Rightarrow (3,4)$
• $x^2-y^2=25$? $\Rightarrow$ Neither • $x^2+\frac{x}{y}=18$? $\Rightarrow (4,2)$ • $y+2x=10$? $\Rightarrow$ Both

Ex. 1.8.2: Which of the following graphs are empty? Explain why.

- $x^2+y^2+25=0$ $\Rightarrow$ Empty, because sum of squares x^2+y^2 must be positive.
- $\frac{\sqrt{x}+\sqrt{-x}}{x}=0$ $\Rightarrow$ Empty: Unless $x=0$, either $x<0$ or $-x<0$ and so either $\sqrt{x}$ or $\sqrt{-x}$ is undefined. But if $x=0$, then $\frac{0+0}{0}$ is undefined.
- $\sqrt{x}+\sqrt{-x}=0$ $\Rightarrow$ Not empty, satisfied by $x=0$
- $x^2+y^3+25=0$ $\Rightarrow$ Not empty since $y=\sqrt[3]{25-x^2}$ is defined for all x.

Ex. 1.8.1: Which of the pairs $(3,4)$ and $(4,2)$ satisfies the equation
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- $x^2+y^3+25=0$ $\Rightarrow$ Not empty since $y=\sqrt[3]{25-x^2}$ is defined for all x.

Ex. 1.8.3: Find the equations of the vertical and horizontal lines through
- $(3,4)$
- $(0,0)$
- $(-9,-3)$
- $(.001, .0001)$

Section 1.8 Review: The x, y-coordinate plane

Ex. 1.8.1: Which of the pairs $(3,4)$ and $(4,2)$ satisfies the equation • $x^2 + y^2 = 25$? $\Rightarrow (3,4)$
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- $x^2 + y^3 + 25 = 0$ $\Rightarrow$ Not empty since $y = \sqrt[3]{25 - x^2}$ is defined for all x.

Ex. 1.8.3: Find the equations of the vertical and horizontal lines through

- $(3,4) \Rightarrow x = 3; y = 4$
- $(0,0) \Rightarrow x = 0; y = 0$
- $(-9,-3) \Rightarrow x = -9; y = -3$
- $(.001, .0001) \Rightarrow x = .001; y = .0001$

Section 1.8 Review: The x, y-coordinate plane

Ex. 1.8.1: Which of the pairs $(3,4)$ and $(4,2)$ satisfies the equation • $x^2+y^2=25? \Rightarrow (3,4)$
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- $(0,0) \Rightarrow x=0; y=0$
- $(-9,-3) \Rightarrow x=-9; y=-3$
- $(.001,.0001) \Rightarrow x=.001; y=.0001$

Ex. 1.8.4: Find all x- and y- intercepts of the graph of the equation

- $x^2+y^2=4$
- $x^2y^2=4$
- $y=x^2+4x+3$
- $y=x+\frac{1}{x}$

At what points does the graph meet the x-axis? the y-axis?

- $x^2+y^2=4$
- $x^2y^2=4$
- $y=x^2+4x+3$
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Section 1.8 Review: The x, y-coordinate plane

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- $\frac{\sqrt{x}+\sqrt{-x}}{x} = 0$ $\Rightarrow$ Empty: Unless $x = 0$, either $x < 0$ or $-x < 0$ and so either $\sqrt{x}$ or $\sqrt{-x}$ is undefined. But if $x = 0$, then $\frac{0+0}{0}$ is undefined.
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- $(3, 4) \Rightarrow x = 3; y = 4$
- $(0, 0) \Rightarrow x = 0; y = 0$
- $(-9, -3) \Rightarrow x = -9; y = -3$
- $(.001, .0001) \Rightarrow x = .001; y = .0001$

Ex. 1.8.4: Find all x- and y- intercepts of the graph of the equation

- $x^2 + y^2 = 4 \Rightarrow x-$ints are -2 and 2 $y-$ints are -2 and 2
- $x^2y^2 = 4 \Rightarrow x-$int is 0 $y-$int is 0
- $y = x^2 + 4x + 3 \Rightarrow x$-ints are $-3, -1$ $y-$int is 3
- $y = x + \frac{1}{x} \Rightarrow$ None

At what points does the graph meet the x-axis? the y-axis?

- $x^2 + y^2 = 4 \Rightarrow x-$axis $(-2, 0); (2, 0)$ $y-$axis $(0, -2); (0, 2)$
- $x^2y^2 = 4 \Rightarrow x-$axis $(0, 0)$ $y-$axis $(0, 0)$
- $y = x^2 + 4x + 3 \Rightarrow x-$axis $(-3, 0); (-1, 0)$ $y-$intercept$(0, 3)$
- $y = x + \frac{1}{x} \Rightarrow$ None

Section 1.8 Review: The x, y-coordinate plane

Ex. 1.8.1: Which of the pairs $(3, 4)$ and $(4, 2)$ satisfies the equation • $x^2 + y^2 = 25? \Rightarrow (3, 4)$
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- $\frac{\sqrt{x}+\sqrt{-x}}{x} = 0$ $\Rightarrow$ Empty: Unless $x = 0$, either $x < 0$ or $-x < 0$ and so either $\sqrt{x}$ or $\sqrt{-x}$ is undefined. But if $x = 0$, then $\frac{0+0}{0}$ is undefined.
- $\sqrt{x} + \sqrt{-x} = 0$ $\Rightarrow$ Not empty, satisfied by $x = 0$
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- $x^2 + y^2 = 4 \Rightarrow x-$ints are -2 and 2 $y-$ints are-2 and 2
- $x^2y^2 = 4 \Rightarrow x-$int is 0 $y-$int is 0
- $y = x^2 + 4x + 3 \Rightarrow x$-ints are $-3, -1$ $y-$int is 3
- $y = x + \frac{1}{x} \Rightarrow$ None

At what points does the graph meet the x-axis? the y-axis?

- $x^2 + y^2 = 4 \Rightarrow x-$axis $(-2, 0); (2, 0)$ $y-$axis $(0, -2); (0, 2)$
- $x^2y^2 = 4 \Rightarrow x-$axis $(0, 0)$ $y-$axis $(0, 0)$
- $y = x^2 + 4x + 3 \Rightarrow x-$axis $(-3, 0); (-1, 0)$ $y-$intercept$(0, 3)$
- $y = x + \frac{1}{x} \Rightarrow$ None

Ex. 1.8.5: Ex. 1.8.6: Ex. 1.8.7: Do these examples interactively.

Ex. 1.8.8: Find the center and radius of the circle with equation

- $(x-4)^2+(y+5)^2=49$
- $(x+4)^2+(y+5)^2-16=0$
- $(x-4)^2+(y-5)^2=3$
- $(x+4)^2+(y-5)^2=7$

Ex. 1.8.8: Find the center and radius of the circle with equation

- $(x-4)^2+(y+5)^2=49 \Rightarrow C(4,-5);\, R=7$
- $(x+4)^2+(y+5)^2-16=0 \Rightarrow C(-4,-5);\, R=4$
- $(x-4)^2+(y-5)^2=3 \Rightarrow C(4,5);\, R=\sqrt{3}$
- $(x+4)^2+(y-5)^2=7 \Rightarrow C(-4,5);\, R=\sqrt{7}$

Ex. 1.8.8: Find the center and radius of the circle with equation

- $(x-4)^2+(y+5)^2=49 \Rightarrow C(4,-5);\ R=7$
- $(x+4)^2+(y+5)^2-16=0 \Rightarrow C(-4,-5);\ R=4$
- $(x-4)^2+(y-5)^2=3 \Rightarrow C(4,5);\ R=\sqrt{3}$
- $(x+4)^2+(y-5)^2=7 \Rightarrow C(-4,5);\ R=\sqrt{7}$

Ex. 1.8.9: Complete the square in

- $x^2+4x=12$
- $x^2-5x=12$
- $-x^2+8x=12$
- $x^2+4x=-4$

Ex. 1.8.8: Find the center and radius of the circle with equation

- $(x-4)^2+(y+5)^2=49 \Rightarrow C(4,-5);\ R=7$
- $(x+4)^2+(y+5)^2-16=0 \Rightarrow C(-4,-5);\ R=4$
- $(x-4)^2+(y-5)^2=3 \Rightarrow C(4,5);\ R=\sqrt{3}$
- $(x+4)^2+(y-5)^2=7 \Rightarrow C(-4,5);\ R=\sqrt{7}$

Ex. 1.8.9: Complete the square in

- $x^2+4x=12 \Rightarrow (x+2)^2=12+4=16$
- $x^2-5x=12 \Rightarrow (x-\frac{5}{2})^2=12+\frac{25}{4}=\frac{73}{4}$
- $-x^2+8x=12 \Rightarrow (x-4)^2=-12+16=4$
- $x^2+4x=-4 \Rightarrow (x+2)^2=-4+4=0$

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- $-x^2+8x=12 \Rightarrow (x-4)^2=-12+16=4$
- $x^2+4x=-4 \Rightarrow (x+2)^2=-4+4=0$

Ex. 1.8.10: Find the standard form equation, center, and radius of the circle with equation

- $3x^2+3y^2=24+24x-30y$
- $x^2+y^2-8=3x-10y$
- $x=x^2+y^2$
- $2x^2+5x+2y^2+6y=0$

Ex. 1.8.8: Find the center and radius of the circle with equation

- $(x-4)^2+(y+5)^2=49 \Rightarrow C(4,-5);\, R=7$
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Ex. 1.8.10: Find the standard form equation, center, and radius of the circle with equation

- $3x^2+3y^2=24+24x-30y \quad \Rightarrow (x-4)^2+(y+5)^2=49: \quad C(4,-5);\, R=7$
- $x^2+y^2-8=3x-10y \quad \Rightarrow (x-\frac{3}{2})^2+(y+5)^2=\frac{141}{4} \quad C(\frac{3}{2}-5);\, R=\frac{1}{2}\sqrt{141}$
- $x=x^2+y^2 \quad \Rightarrow (x-\frac{1}{2})^2+y^2=\frac{1}{4}: \quad C(\frac{1}{2},0);\, R=\frac{1}{2}$
- $2x^2+5x+2y^2+6y=0 \quad \Rightarrow (x+\frac{5}{4})^2+(y+\frac{3}{2})^2=\frac{61}{4}: \quad C(-\frac{5}{4},-\frac{3}{2});\, R=\frac{61\sqrt{2}}{2}$

Ex. 1.8.8: Find the center and radius of the circle with equation

- $(x-4)^2+(y+5)^2=49 \Rightarrow C(4,-5);\, R=7$
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- $2x^2+5x+2y^2+6y=0 \quad \Rightarrow (x+\frac{5}{4})^2+(y+\frac{3}{2})^2=\frac{61}{4}: \quad C(-\frac{5}{4},-\frac{3}{2});\, R=\frac{61\sqrt{2}}{2}$

Ex. 1.8.11: Complete the square in

- $y=2x^2+x+1$
- $y=2x^2-x$
- $y=-4x^2+2x+15/4$
- $y=-48x^2+24x+61$

Ex. 1.8.8: Find the center and radius of the circle with equation

- $(x-4)^2+(y+5)^2=49 \Rightarrow C(4,-5);\, R=7$
- $(x+4)^2+(y+5)^2-16=0 \Rightarrow C(-4,-5);\, R=4$
- $(x-4)^2+(y-5)^2=3 \Rightarrow C(4,5);\, R=\sqrt{3}$
- $(x+4)^2+(y-5)^2=7 \Rightarrow C(-4,5);\, R=\sqrt{7}$

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- $x^2+y^2-8=3x-10y \quad \Rightarrow (x-\frac{3}{2})^2+(y+5)^2=\frac{141}{4} \quad C(\frac{3}{2}-5);\, R=\frac{1}{2}\sqrt{141}$
- $x=x^2+y^2 \quad \Rightarrow (x-\frac{1}{2})^2+y^2=\frac{1}{4}: \quad C(\frac{1}{2},0);\, R=\frac{1}{2}$
- $2x^2+5x+2y^2+6y=0 \quad \Rightarrow (x+\frac{5}{4})^2+(y+\frac{3}{2})^2=\frac{61}{4}: \quad C(-\frac{5}{4},-\frac{3}{2});\, R=\frac{61\sqrt{2}}{2}$

Ex. 1.8.11: Complete the square in

- $y=2x^2+x+1 \Rightarrow y=2(x+\frac{1}{4})^2+\frac{7}{8}$
- $y=2x^2-x \Rightarrow y=2(x-\frac{1}{2})^2-\frac{1}{4}$
- $y=-4x^2+2x+15/4 \Rightarrow -4(x-\frac{1}{4})^2+4$
- $y=-48x^2+24x+61 \Rightarrow -48(x-\frac{1}{4})^2+64$

Ex. 1.8.8: Find the center and radius of the circle with equation
- $(x-4)^2+(y+5)^2=49 \Rightarrow C(4,-5);\, R=7$
- $(x+4)^2+(y+5)^2-16=0 \Rightarrow C(-4,-5);\, R=4$
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- $x=x^2+y^2 \quad \Rightarrow (x-\frac{1}{2})^2+y^2=\frac{1}{4}: \quad C(\frac{1}{2},0);\, R=\frac{1}{2}$
- $2x^2+5x+2y^2+6y=0 \quad \Rightarrow (x+\frac{5}{4})^2+(y+\frac{3}{2})^2=\frac{61}{4}: \quad C(-\frac{5}{4},-\frac{3}{2});\, R=\frac{61\sqrt{2}}{2}$

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- $y=2x^2-x \Rightarrow y=2(x-\frac{1}{2})^2-\frac{1}{4}$
- $y=-4x^2+2x+15/4 \Rightarrow -4(x-\frac{1}{4})^2+4$
- $y=-48x^2+24x+61 \Rightarrow -48(x-\frac{1}{4})^2+64$

Ex. 1.8.12: Is the graph of the following equation x-axis symmetric?
- $y=8x^5+7x^3+x$
- $x^2+2y^2=5$
- $y^2+y^4+y^6=1$
- $y=x$

Ex. 1.8.8: Find the center and radius of the circle with equation

- $(x-4)^2+(y+5)^2=49 \Rightarrow C(4,-5); R=7$
- $(x+4)^2+(y+5)^2-16=0 \Rightarrow C(-4,-5); R=4$
- $(x-4)^2+(y-5)^2=3 \Rightarrow C(4,5); R=\sqrt{3}$
- $(x+4)^2+(y-5)^2=7 \Rightarrow C(-4,5); R=\sqrt{7}$

Ex. 1.8.9: Complete the square in

- $x^2+4x=12 \Rightarrow (x+2)^2=12+4=16$
- $x^2-5x=12 \Rightarrow (x-\frac{5}{2})^2=12+\frac{25}{4}=\frac{73}{4}$
- $-x^2+8x=12 \Rightarrow (x-4)^2=-12+16=4$
- $x^2+4x=-4 \Rightarrow (x+2)^2=-4+4=0$

Ex. 1.8.10: Find the standard form equation, center, and radius of the circle with equation

- $3x^2+3y^2=24+24x-30y \quad \Rightarrow (x-4)^2+(y+5)^2=49: \quad C(4,-5); R=7$
- $x^2+y^2-8=3x-10y \quad \Rightarrow (x-\frac{3}{2})^2+(y+5)^2=\frac{141}{4} \quad C(\frac{3}{2}-5); R=\frac{1}{2}\sqrt{141}$
- $x=x^2+y^2 \quad \Rightarrow (x-\frac{1}{2})^2+y^2=\frac{1}{4}: \quad C(\frac{1}{2},0); R=\frac{1}{2}$
- $2x^2+5x+2y^2+6y=0 \quad \Rightarrow (x+\frac{5}{4})^2+(y+\frac{3}{2})^2=\frac{61}{4}: \quad C(-\frac{5}{4},-\frac{3}{2}); R=\frac{61\sqrt{2}}{2}$

Ex. 1.8.11: Complete the square in

- $y=2x^2+x+1 \Rightarrow y=2(x+\frac{1}{4})^2+\frac{7}{8}$
- $y=2x^2-x \Rightarrow y=2(x-\frac{1}{2})^2-\frac{1}{4}$
- $y=-4x^2+2x+15/4 \Rightarrow -4(x-\frac{1}{4})^2+4$
- $y=-48x^2+24x+61 \Rightarrow -48(x-\frac{1}{4})^2+64$

Ex. 1.8.12: Is the graph of the following equation x-axis symmetric?

- $y=8x^5+7x^3+x \Rightarrow$ No • $x^2+2y^2=5 \Rightarrow$ Yes • $y^2+y^4+y^6=1 \Rightarrow$ Yes • $y=x \Rightarrow$ No

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- $y=7x^4+8x^2$
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Ex. 1.8.14: Is the graph of he following equation origin symmetric?

- $y=x^3+x$
- $y=x^4+x$
- $y+x^2+x=0$
- $(x-1)^2+(y-3)^2=12$

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Ex. 1.8.14: Is the graph of he following equation origin symmetric?

- $y=x^3+x \Rightarrow$Yes
- $y=x^4+x \Rightarrow$ No
- $y+x^2+x=0 \Rightarrow$ No
- $(x-1)^2+(y-3)^2=12 \Rightarrow$ No

- 1.9.1: Straight lines and their graphs
- 1.9.2: Lines and their slopes
- 1.9.3: Parallel and perpendicular lines
- 1.9.4: Equations of lines
- Section 1.9 Review

Section 1.9 Preview: Definitions/Theorems

- Definition 1.9.1: Lines and line segments in the x, y-coordinate plane
- Definition 1.9.2: Horizontal and vertical lines
- Definition 1.9.3: The line segment joining points $P(x_1, y_1)$ and $Q(x_2, y_2)$
- Definition 1.9.4: The distance between points
- Definition 1.9.5: The midpoint of the line segment joining $P(x_1, y_1)$ and $Q(x_2, y_2)$
- Definition 1.9.6: The *slope of a horizontal line* is 0.
- Definition 1.9.7: The *slope of a vertical line* is undefined.
- Definition 1.9.8: Lines are parallel if they never meet.
- Definition 1.9.9: Lines are perpendicular if

Section 1.9 Preview: Procedures

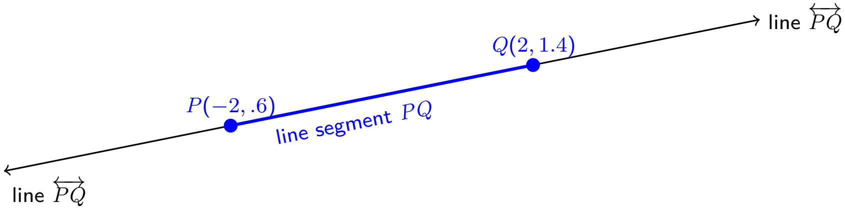
- Procedure 1.9.1: To find the length of a vertical line segment
- Procedure 1.9.2: To find the length of a horizontal line segment
- Procedure 1.9.3: To find the slope of a line segment
- Procedure 1.9.4: To find the slope of a straight line
- Procedure 1.9.5: To tell if two lines are parallel
- Procedure 1.9.6: To find a line's slope and y-intercept
- Procedure 1.9.7: To find *an* equation of a line

1.9.1 Straight lines and their graphs.

In this course, the word "line" means "straight line." See the diagram below.

If P and Q are two points in the x, y-coordinate plane

- Line segment PQ is finite, and joins points P and Q,
- Line segments PQ and QP are identical.
- $\overline{PQ} = \overline{QP}$ is the length of line segments $PQ = QP$.
- line $\overleftrightarrow{PQ}$ is infinite, and extends line segment PQ in both directions.



Horizontal and vertical lines

Definition: Horizontal and vertical lines

- If two points have the same y-coordinate, the line through them is horizontal.
- If a line is horizontal, all of its points have the same y-coordinate.
- If two points have the same x-coordinate , the line through them is vertical.
- If a line is vertical, all of its points have the same x-coordinate.

Definition: The *line segment* joining points P and Q

consists of P, Q, and all points on the infinite straight line through P and Q that lie between them.

To find the length of a vertical line segment

Subtract its bottom y-coordinate from its top y-coordinate.

To find the length of a horizontal line segment

Subtract its left x-coordinate from its right x-coordinate.

For any two points P and Q, the line segment joining them is written PQ. The length of this line segment is $\overline{PQ}$, the distance between P and Q.

Definition: The distance between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the x, y-plane is

$$\overline{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1: Find the distance between points $A(-3, 2)$ and $D(3, 3)$.

Solution: Here $(x_1, y_1) = (-3, 2)$ and $(x_2, y_2) = (3, 3)$.
The distance is

$$\sqrt{(3-(-3))^2 + (3-2)^2} = \sqrt{6^2 + 1^2} = \boxed{\sqrt{37}}$$

The midpoint of the line segment joining $P(x_1, y_1)$ and $Q(x_2, y_2)$

is the point $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ that is halfway between P and Q: thus $\overline{PM} = \overline{MQ} = \frac{1}{2}\overline{PQ}$.

Example 2: Find the midpoint M of the line segment PQ joining the points $P(-3,2)$ and $Q(3,3)$. Check your answer by showing that $\overline{PM} = \overline{MQ} = \frac{1}{2}\overline{PQ}$.

Solution: The midpoint M of the line segment PQ is $\left(\frac{-3+3}{2}, \frac{3+2}{2}\right) = (0, \frac{5}{2})$. Now compute the lengths of the requested line segments:

- $\overline{PQ}$ = distance from P to Q

  $= \sqrt{(3-(-3))^2 + (3-2)^2} = \sqrt{36+1} = \boxed{\sqrt{37}}$
- $\overline{MQ}$ = distance from M to Q

  $= \sqrt{(3-0)^2 + (3-\frac{5}{2})^2}$

  $= \sqrt{9+\frac{1}{4}} = \sqrt{\frac{37}{4}} = \frac{\sqrt{37}}{\sqrt{4}} = \boxed{\frac{1}{2}\sqrt{37}}$
- $= \overline{MP}$ = distance from M to P

  $= \sqrt{(-3-0)^2 + (2-\frac{5}{2})^2}$

  $= \sqrt{9+\frac{1}{4}} = \sqrt{\frac{37}{4}} = \boxed{\frac{1}{2}\sqrt{37}}$
- Q is the midpoint since $\overline{MQ} = \overline{MP} = \frac{1}{2}\sqrt{37} = \frac{1}{2}\overline{PQ}$.

1.9.2 The slope of a line

Slope is often described as $\frac{rise}{run}$. A better description:

Slope = $\frac{\text{change in Y}}{\text{change in X}}$.

The word *change* means new value minus original value.

How to find the slope of a line segment

The slope of line segment PQ joining points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

- $m_{PQ} = \frac{\text{change in Y}}{\text{change in X}} = \frac{y_2 - y_1}{x_2 - x_1}$ if $x_1 \neq x_2$.
- undefined if $x_1 = x_2$. In this case the line is vertical.

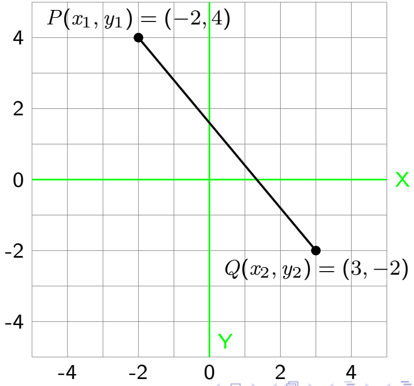
Example 3: Find the slope of line segment PQ in the diagram at the right.

Solution: First find the coordinates of $P(x_1, y_1)$:

- The vertical line through P hits the x-scale number -2 at the bottom of the grid.
  Therefore P has x-coordinate $x_1 = -2$.
- The horizontal line through P hits the y-scale number 4 at the left of the grid.
  Therefore P has y-coordinate $y_1 = 4$.
- Therefore the coordinates of P are $(x_1, y_1) = (-2, 4)$.
- Similarly, the coordinates of Q are $(x_2, y_2) = (3, -2)$.

Answer: The slope of line segment PQ is

$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-4}{3-(-2)} = \frac{-6}{5} = \boxed{-\frac{6}{5}}$$



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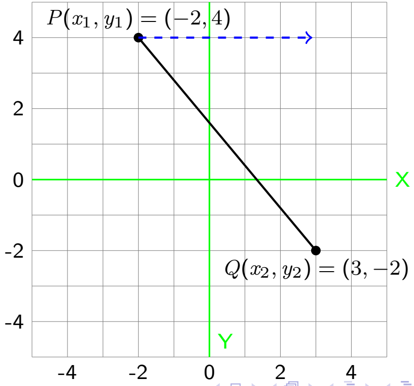
$m_{PQ} = \frac{y_2-y_1}{x_2-x_1} = \frac{-2-4}{3-(-2)} = \frac{-6}{5} = \boxed{-\frac{6}{5}}$

The run is the change $3 - -2 = 5$ of the x-coordinate as you go from P to Q.



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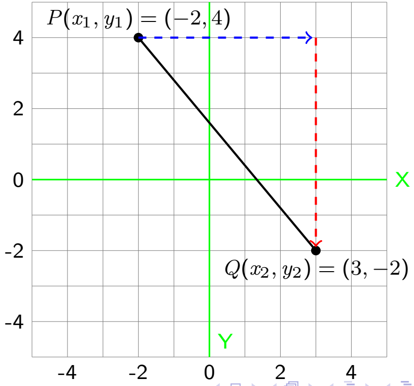
$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-4}{3-(-2)} = \frac{-6}{5} = \boxed{-\frac{6}{5}}$

The run is the change $3 - -2 = 5$ of the x-coordinate as you go from P to Q.
The rise is the change $-2 - 4 = -6$ of the y-coordinate as you go from P to Q.
The run is positive but the “rise” is negative!



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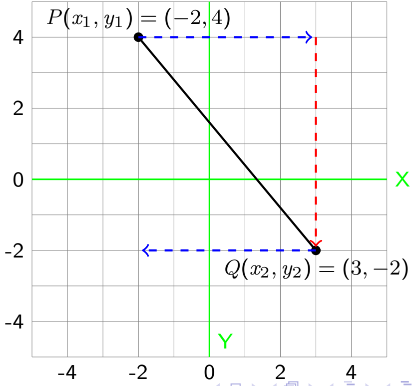
  Therefore P has y-coordinate $y_1 = 4$.
- Therefore the coordinates of P are $(x_1, y_1) = (-2, 4)$.
- Similarly, the coordinates of Q are $(x_2, y_2) = (3, -2)$.

Answer: The slope of line segment PQ is

$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-4}{3-(-2)} = \frac{-6}{5} = \boxed{-\frac{6}{5}}$

The run is the change $3 - -2 = 5$ of the x-coordinate as you go from P to Q.
The rise is the change $-2 - 4 = -6$ of the y-coordinate as you go from P to Q.
The run is positive but the "rise" is negative!
You get the same answer if you calculate changes in y and x by starting at Q and finishing at P.



1.9.2 The slope of a line

Slope is often described as $\frac{rise}{run}$. A better description:

Slope = $\frac{\text{change in Y}}{\text{change in X}}$.

The word *change* means new value minus original value.

How to find the slope of a line segment

The slope of line segment PQ joining points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

- $m_{PQ} = \frac{\text{change in Y}}{\text{change in X}} = \frac{y_2 - y_1}{x_2 - x_1}$ if $x_1 \neq x_2$.
- undefined if $x_1 = x_2$. In this case the line is vertical.

Example 3: Find the slope of line segment PQ in the diagram at the right.

Solution: First find the coordinates of $P(x_1, y_1)$:

- The vertical line through P hits the x-scale number -2 at the bottom of the grid.
  Therefore P has x-coordinate $x_1 = -2$.
- The horizontal line through P hits the y-scale number 4 at the left of the grid.
  Therefore P has y-coordinate $y_1 = 4$.
- Therefore the coordinates of P are $(x_1, y_1) = (-2, 4)$.
- Similarly, the coordinates of Q are $(x_2, y_2) = (3, -2)$.

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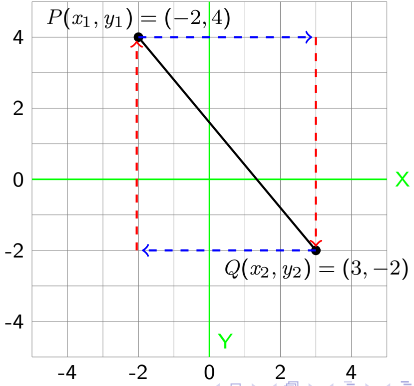
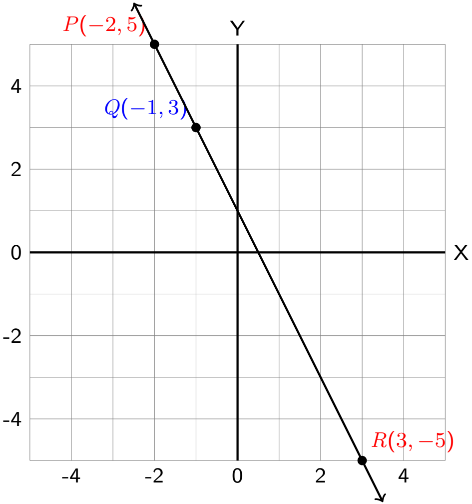
You get the same answer if you calculate changes in y and x by starting at Q and finishing at P. Why?





So far we have defined the slope of a *line segment*.

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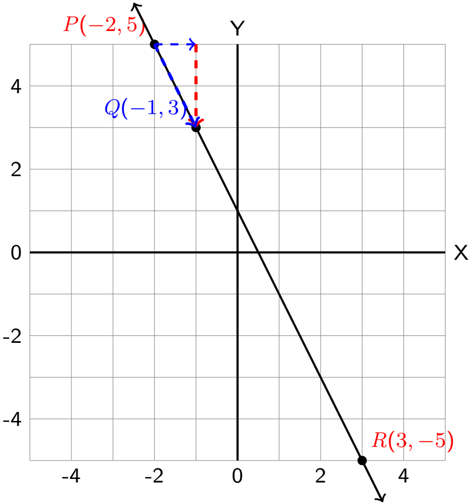

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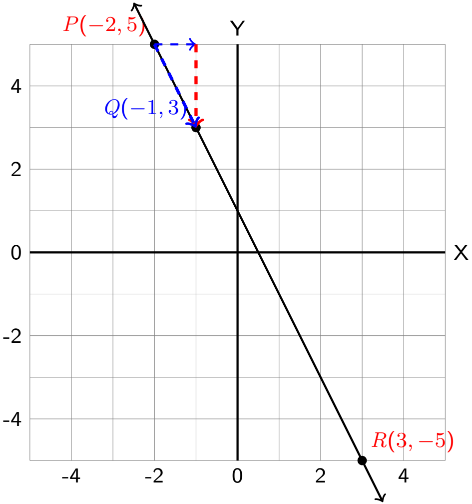

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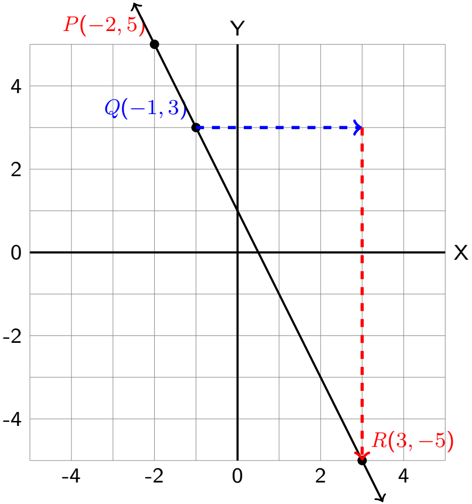

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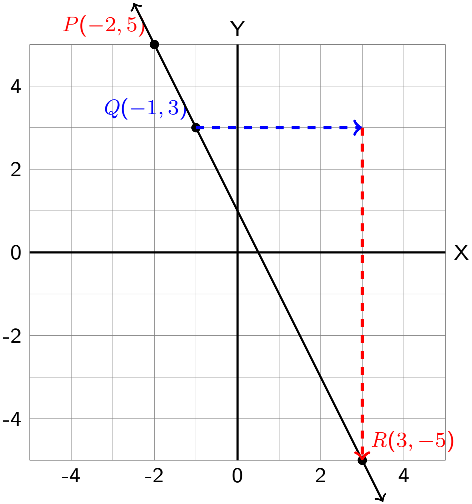

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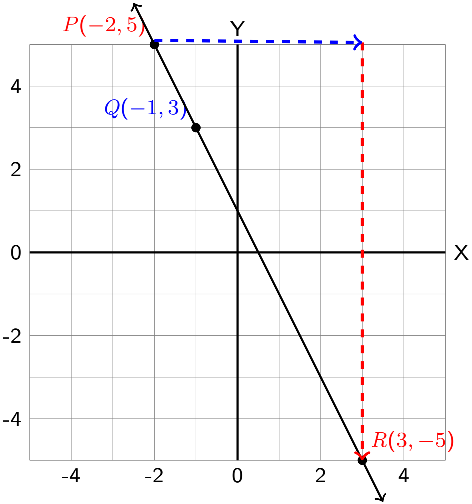

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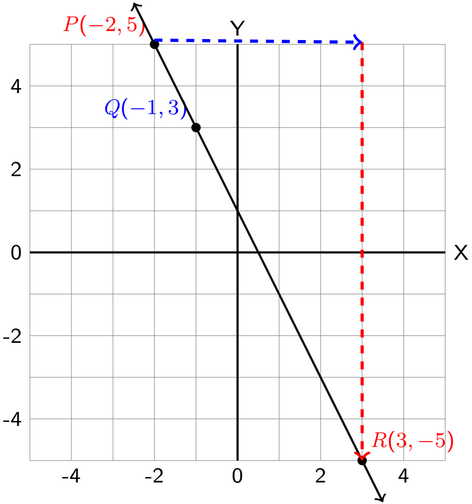

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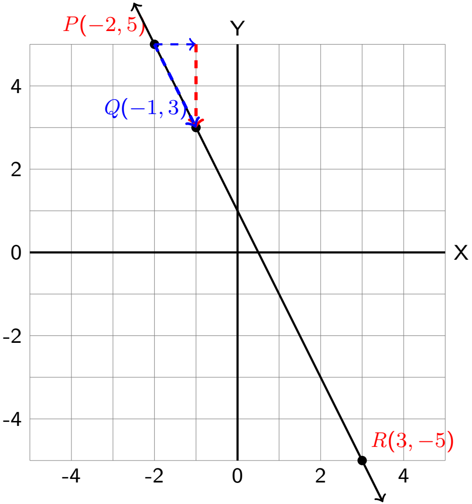

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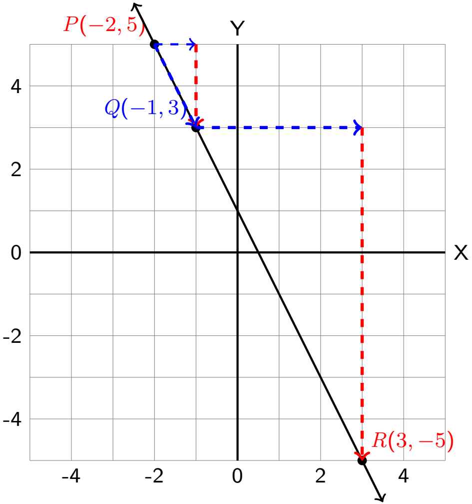
We get the same slope no matter which points we pick. That's because the three triangles are similar.



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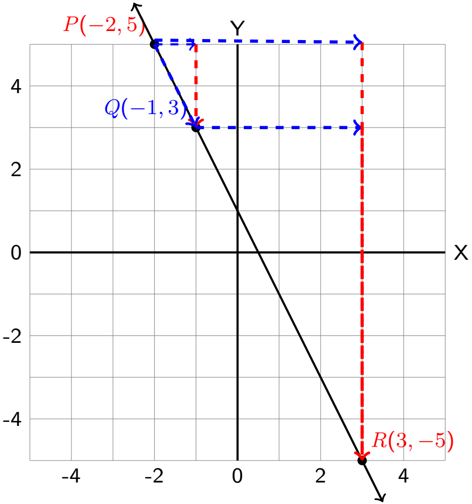

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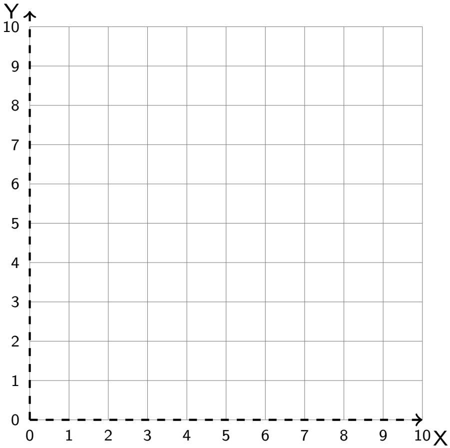
Lines with positive slope rise from left to right.



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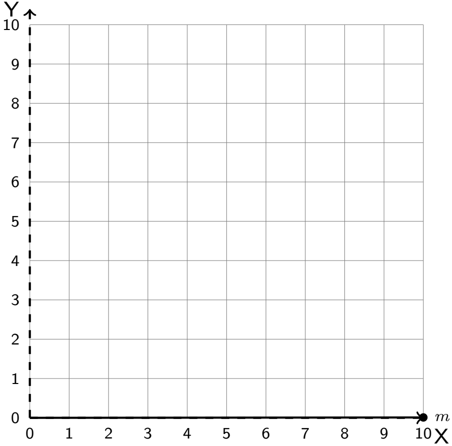

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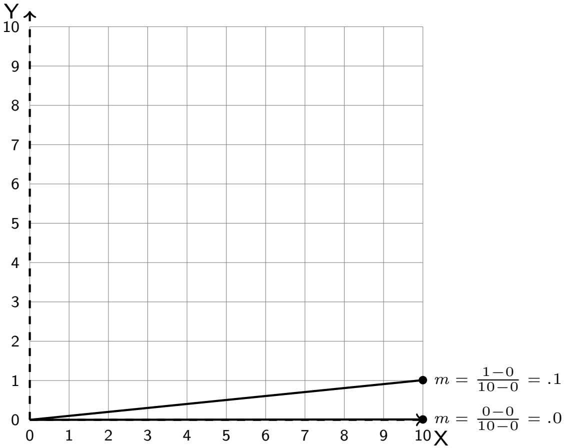

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All these slanted lines have positive slope, since a point moving along the line *from left to right* moves upward.

As the black lines get steeper, their slopes increase from 0.1 to 0.9.

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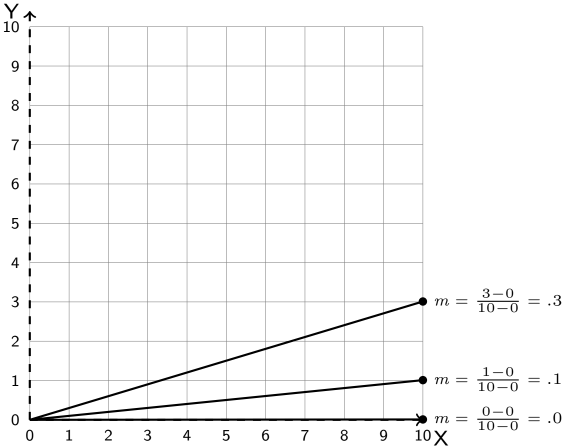

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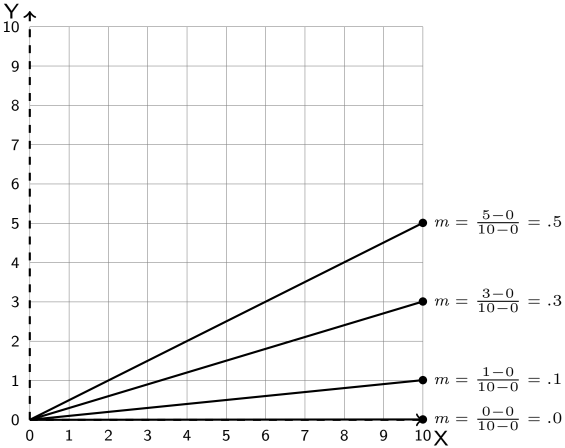

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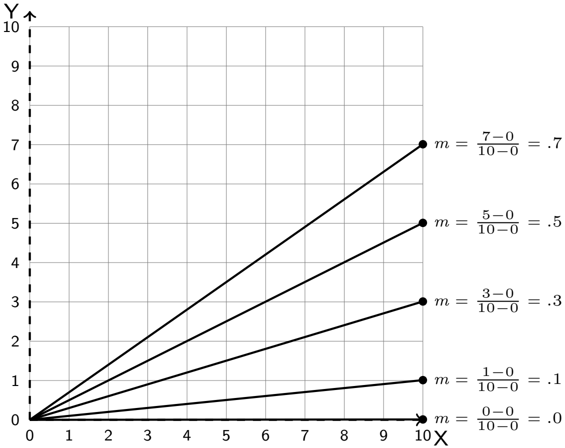

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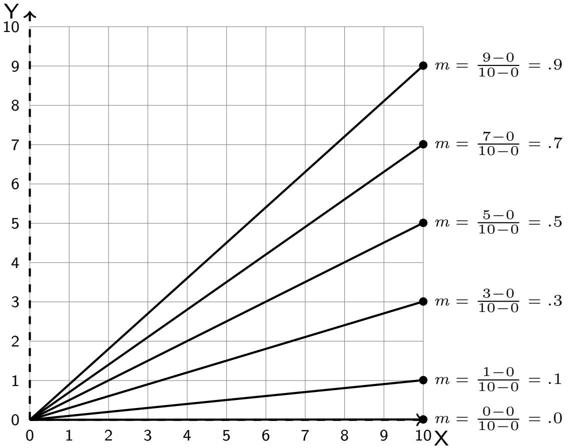

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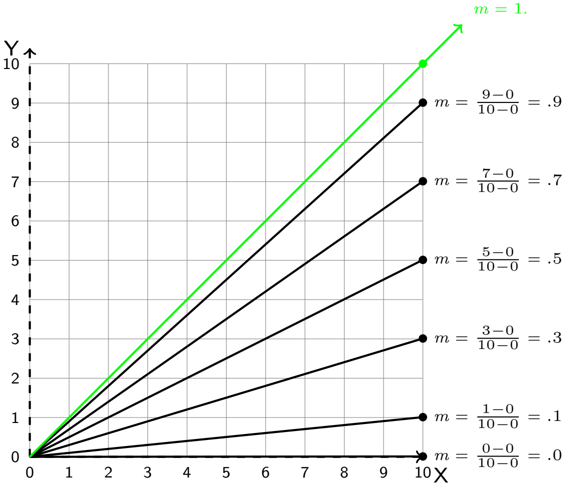

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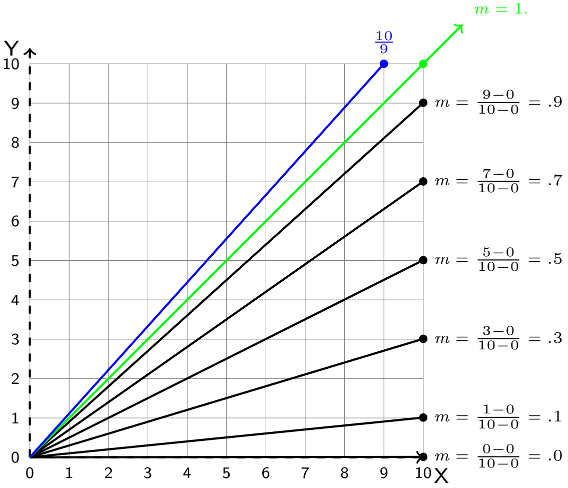

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Now start at the top right of the picture and move left. As the blue lines get steeper, their slopes go from $10/9 = 1.111...$ to 10.

Make sure you can compute the slope of each line segment joining the origin $(0, 0)$ to a blue or black dot.

For example, the highest black line joins the origin $(0, 0)$ to point $(10, 9)$ and so its slope is $\frac{9-0}{10-0} = \frac{9}{10} = 0.9$.

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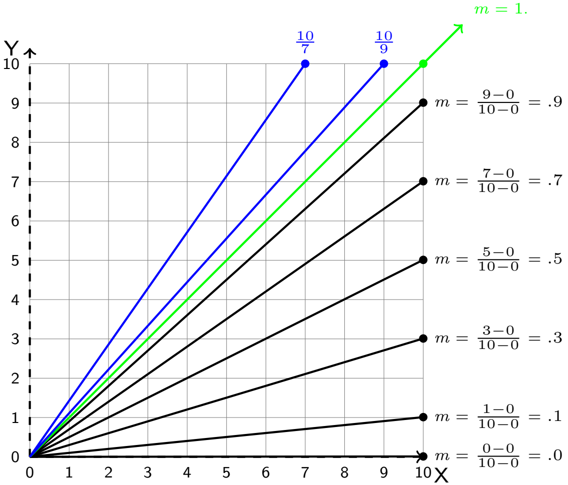

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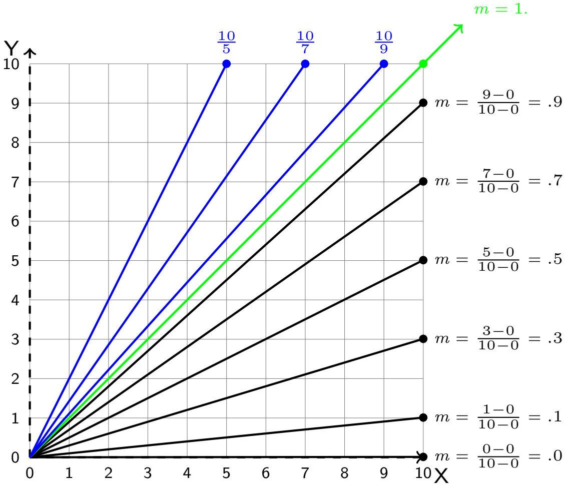

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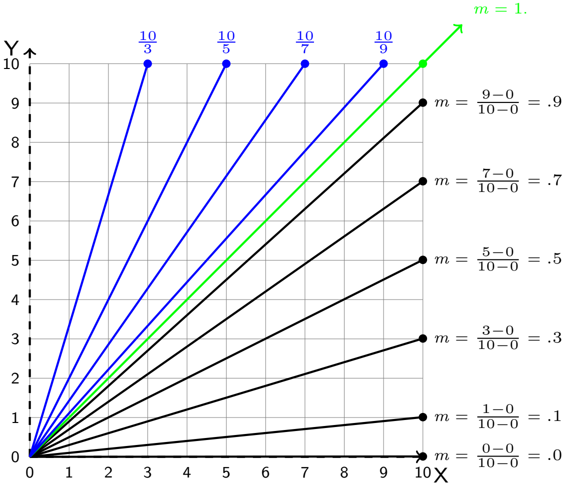

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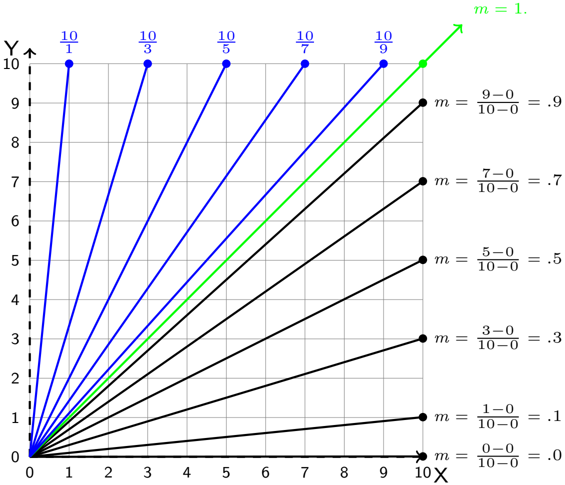

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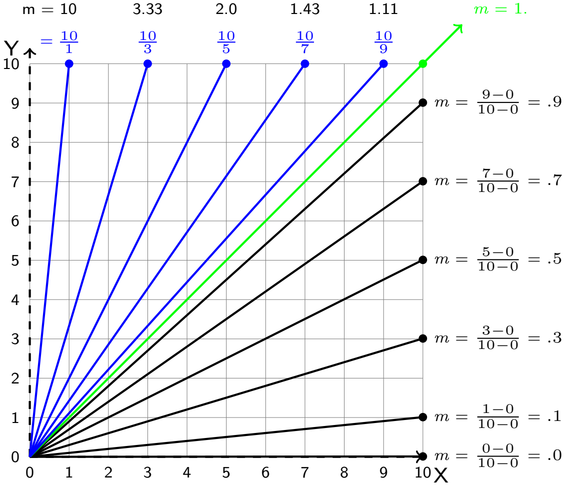

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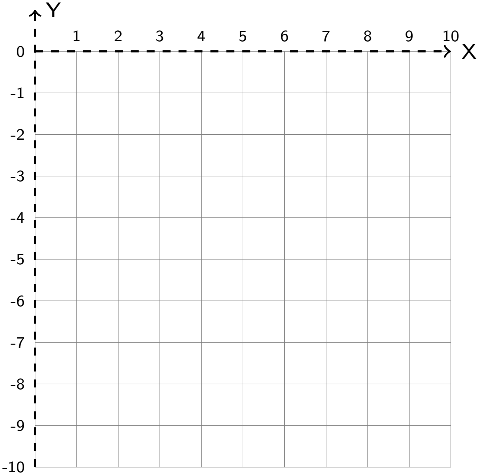

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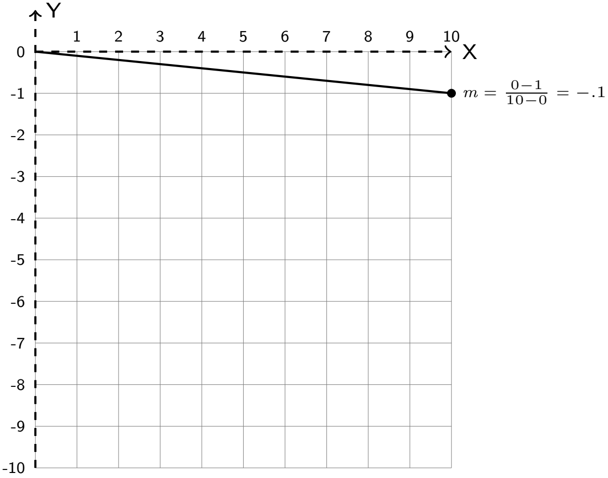

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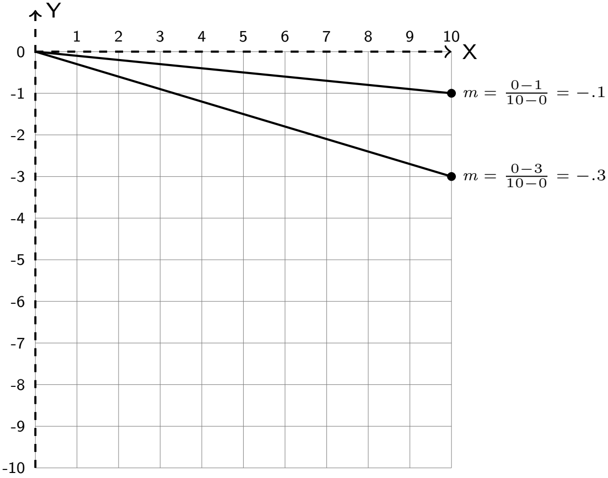

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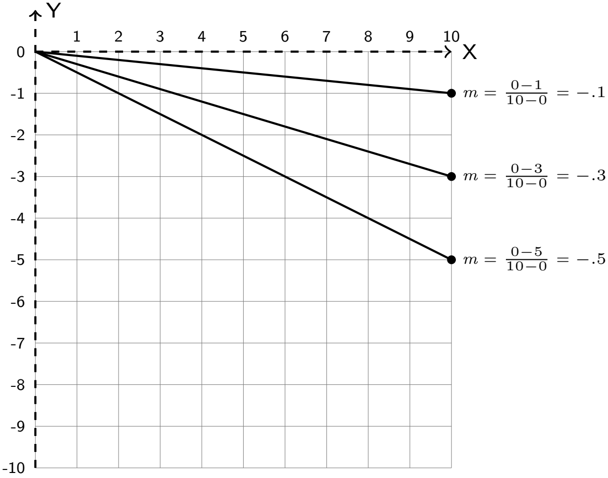

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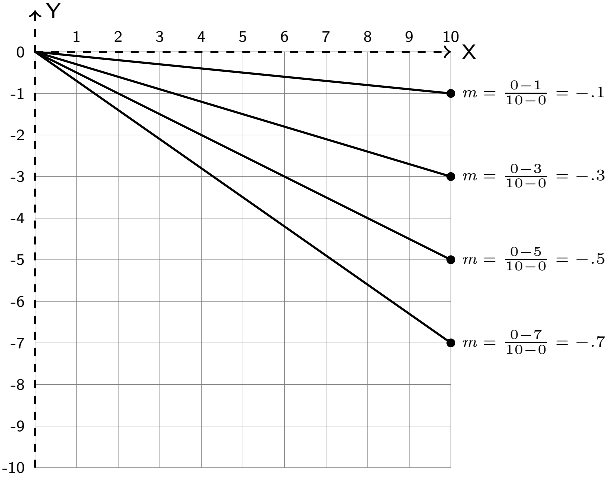
Lines with negative slope fall from left to right.



All these slanted lines have negative slope, since a point moving along the line from left to right moves downward.

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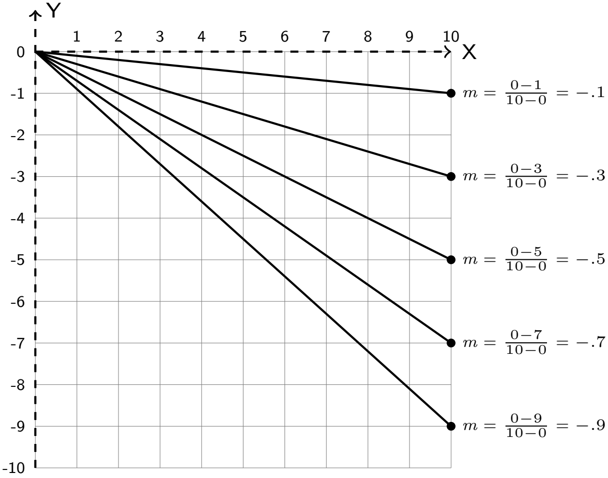

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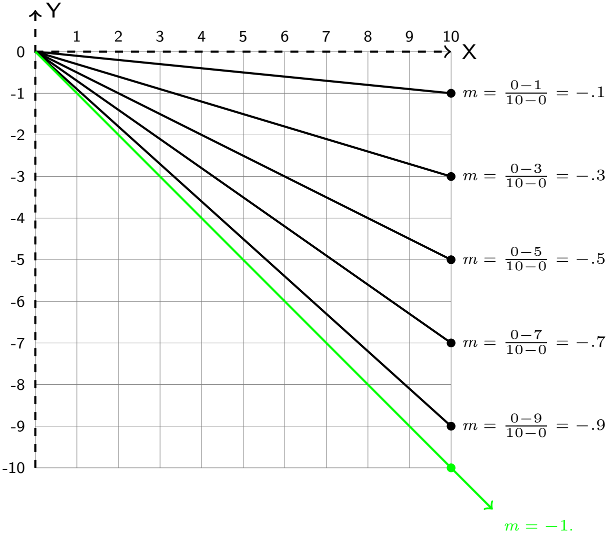

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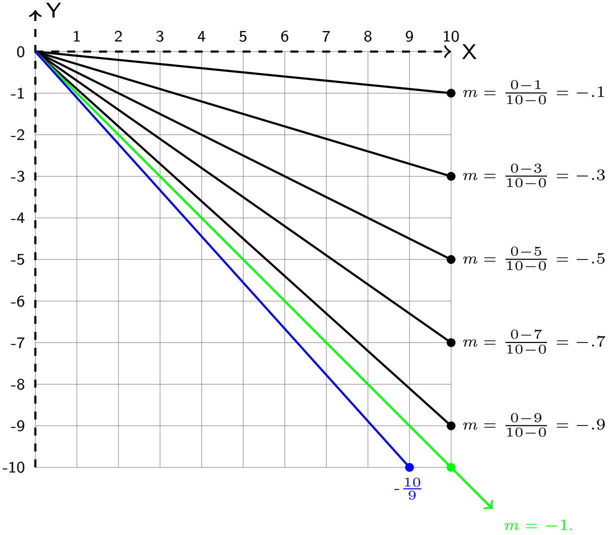
Start at the top right, and move down. The x-axis (horizontal line $y = 0$) has slope 0. As the black lines get steeper, their slopes decrease from $-.1$ to $-.9$.

The downward diagonal green line has slope $\frac{-10-0}{10-0} = -1$.

Lines with negative slope fall from left to right.



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Start at the bottom right and move left.

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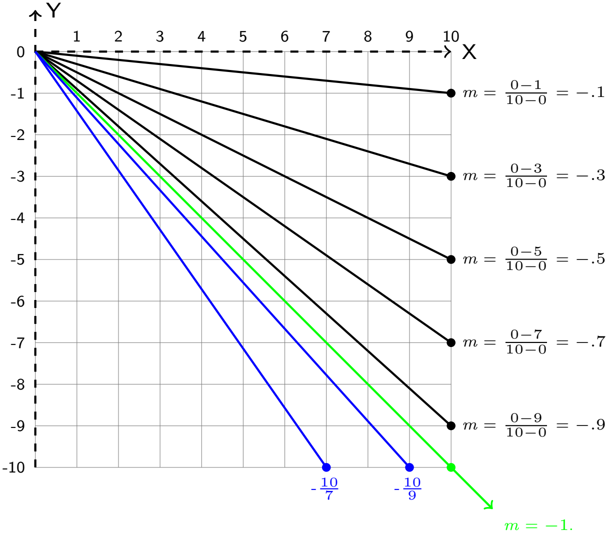

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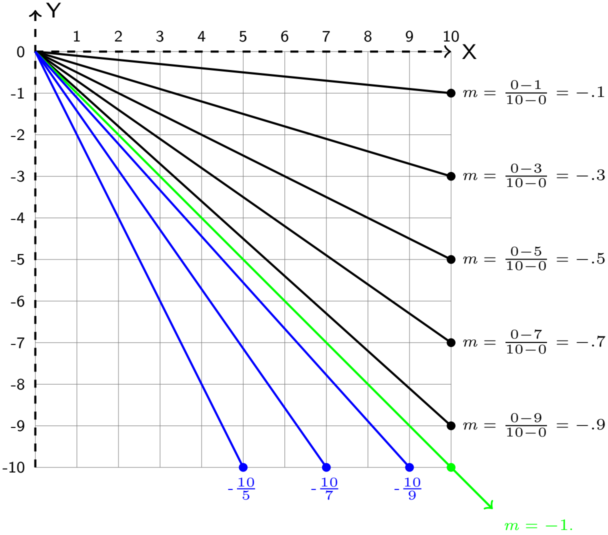
Start at the top right, and move down. The x-axis (horizontal line $y = 0$) has slope 0. As the black lines get steeper, their slopes decrease from $-.1$ to $-.9$.

The downward diagonal green line has slope $\frac{-10-0}{10-0} = -1$.

Start at the bottom right and move left. As the blue lines get steeper, their slopes go from $-10/9 = -1.111...$ to $m = -10$.

Lines with negative slope fall from left to right.



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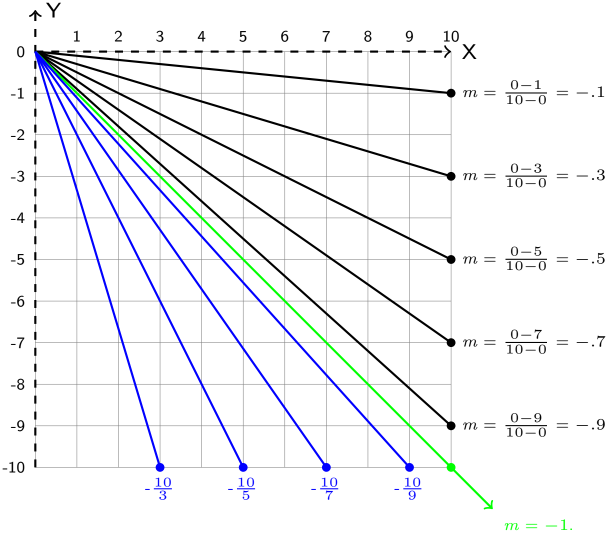

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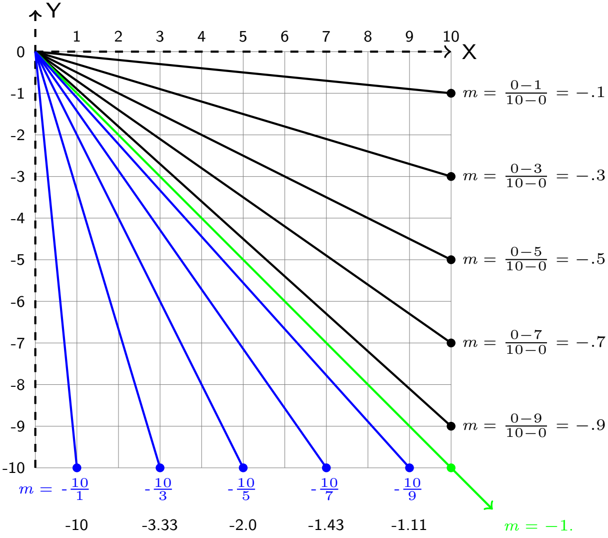

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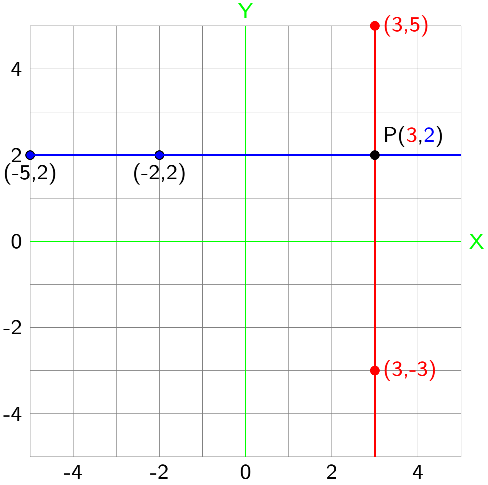
The downward diagonal green line has slope $\frac{-10-0}{10-0} = -1.$

Start at the bottom right and move left. As the blue lines get steeper, their slopes go from $-10/9 = -1.111\ldots$ to $m = -10$.

Equations of vertical and horizontal lines

On the previous pages, we calculated the slopes of slanted lines. What about non-slanted lines, which are vertical or horizontal?



The blue horizontal line through point (c, d) has equation $y = d$. See above, where $(c, d) = (3, 2)$.

For any two (different) points $P(x_1, d)$ and $Q(x_2, d)$ on the horizontal line $y = d$, the slope of segment PQ is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{d - d}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0$ since $x_2 - x_1 \neq 0$.

The *slope of a horizontal line* is 0 because

the rise (change in y) between any two of its points is 0.

The red vertical line through point (c, d) has equation $x = c$.

For any two points $P(c, y_1)$ and $Q(c, y_2)$ on the vertical line $x = c$, the slope of line segment PQ is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{c - c}$ which is undefined since $c - c = 0$.

The *slope of a vertical line* is undefined because

the run (change in x) between any two its points is 0.
Don't say "a vertical line has no slope."
That sounds like "a vertical line has slope zero", which is not the case.

Example 4: Find the equations of the vertical and horizontal lines through point $(5, 6)$.
Solution: Vertical: $x = 5$ Horizontal: $y = 6$.

1.9.3 Parallel and perpendicular lines

Definition of parallel lines

Two lines are parallel if they never meet.

How to tell if two lines are parallel

Two lines are parallel if

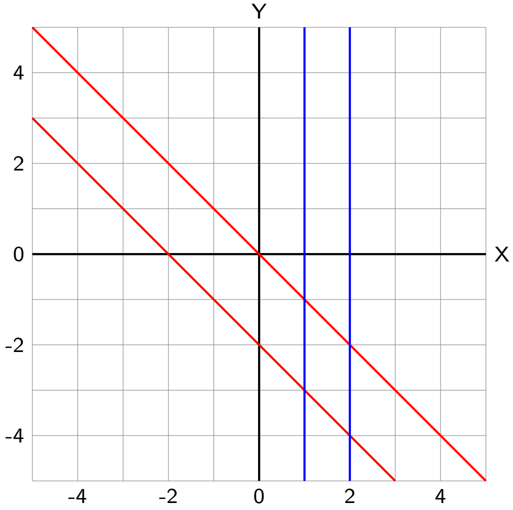
- they are both vertical *or*
- if they have the same slope.

The red lines are parallel: they both have slope -1.

The vertical blue lines are also parallel, of course. However, they don't have the same slope for a simple reason: a vertical line's slope is undefined.

The correct statement is: the slope of both blue lines is undefined.

The blue lines have equations $x = 1$ and $x = 2$.



Definition of perpendicular lines

Two lines are *perpendicular* provided

- one line is vertical and the other is horizontal, *or*
- you can rotate the plane around the origin so that one line becomes vertical and the other becomes horizontal.

How to tell if lines are perpendicular

Two lines are perpendicular if

- one is vertical and one is horizontal, or
- their slopes m_1 and m_2 satisfy $m_1 m_2 = -1$: each slope is minus the reciprocal of the other.

The diagram at the right shows why the product of slopes of perpendicular lines is -1.

Start with the (perpendicular) x- and y- axes. Hold the origin fixed and rotate the plane clockwise so that the x-axis becomes the red line, containing both the origin $(0,0)$ and point $(5,-2)$.

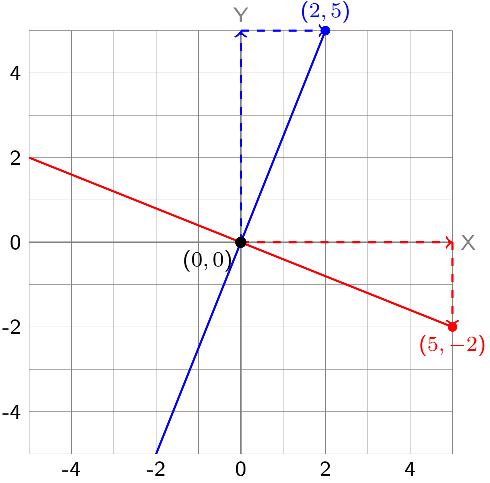
At the same time, the y-axis becomes the blue line, which contains points $(0,0)$ and $(2,5)$.

It follows from the definition above that the red and blue lines are perpendicular.

The red line's slope is $m_1 = \frac{-2-0}{5-0} = -\frac{2}{5}$.

The blue line's slope is $m_2 = \frac{5-0}{2-0} = \frac{5}{2}$.

The product of the slopes is $m_1 m_2 = -\frac{2}{5} \cdot \frac{5}{2} = -1$.



1.9.4 Equations of lines

How to find a line's slope and y-intercept

- If a line is not vertical, its y-intercept is the y-coordinate of the point where the line meets the y-axis. If that meeting point is $(0, b)$, then the line's y-intercept is b.
- If a line is not vertical, its slope is $m = \frac{y_1 - y_0}{x_1 - x_0}$ where (x_0, y_0) and (x_1, y_1) are any two points on the line.
- If a line is vertical, its slope is undefined.
- The y-intercept of the y-axis is indeterminate.
- Any other vertical line doesn't have a y-intercept.

How to find *an* equation of a line

- If the line is vertical and passes through point (x_0, y_0), an equation is $x = x_0$.
- If the line is not vertical, with slope m and y-intercept b:
  - The slope-intercept equation of the line is $y = mx + b$.
  - A point-slope equation is $y - y_0 = m(x - x_0)$. Different points on the line give different point-slope equations.
- Any line's equation can be rewritten (in many ways) in standard form $Ax + By = C$.

For any number d, the slope of the horizontal line $y = d$ is $m = 0$. The slope-intercept form of that line's equation would be $y = 0x + d$, but most people just write $y = d$.

Example 5: Find the point-slope and slope-intercept forms of the equation of the line with slope 2 that passes through point $(5, 7)$. Also, find the line's y-intercept.

Solution : The point is $(x_0, y_0) = (5, 7)$ and so $y - y_0 = m(x - x_0)$ gives the point-slope form

$$\boxed{y - 7 = 2(x - 5).}$$

To get the slope-intercept form, solve for y:

$y = 2(x - 5) + 7 = 2x - 10 + 7$: $\boxed{y = 2x - 3}$

Each of the above can be rewritten in standard form as $\boxed{2x - y = 3}$

Set $x = 0$ in any of the above 3 equations and solve for y to get: $\boxed{\text{the } y\text{-intercept is } -3.}$

Example 6: Find the slope-intercept form of the line through points $P(3,4)$ and $Q(5,7)$.

Solution: First find the slope by using points $P(x_0, y_0)$ and $Q(x_1, y_1)$:

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{7-4}{5-3} = \frac{3}{2}$$

Use the coordinates of either point in the point-slope equation $(y - y_1) = m(x - x_1)$.

Using $(3,4)$ yields $y - 4 = \frac{3}{2}(x-3)$

To find the slope-intercept form of the equation, solve for y:

$$y = \frac{3}{2}(x-3) + 4$$
$$= \frac{3}{2}x - \frac{9}{2} + 4$$
$$= \frac{3}{2}x - \frac{9}{2} + \frac{8}{2} = \frac{3}{2}x - \frac{1}{2} \text{ and so}$$

the equation's slope-intercept form is $\boxed{y = \frac{3}{2}x - \frac{1}{2}}$.

Exercise: Show that $(x,y) = (3,4)$ and $(5,7)$ both satisfy the last equation.

Example 7: Find the slope of the line with standard form equation $2x + 5y = 12$.

Solution: Solving for y gives

$$y = \frac{12-2x}{5} = -\frac{2}{5}x + \frac{12}{5}.$$

This is the slope-intercept form of the line with slope $m = -\frac{2}{5}$ and y-intercept $b = \frac{12}{5}$.

Answer: The slope of the line $2x + 5y = 12$ is $-\frac{2}{5}$.

Example 8: Find an equation of the line L through point $P(4,6)$ that is parallel to the line $2x + 5y = 12$.

Solution: Parallel lines have the same slope.

By Example 7, line L has slope $m = -\frac{2}{5}$.

Line L passes through point $P(4,6)$.

By the point-slope formula its equation is $y - y_0 = m(x - x_0)$ where $(x_0, y_0) = (4,6)$ is the given point on line L.

Answer: The line L has equation $\boxed{y - 6 = -\frac{2}{5}(x-4)}$.

Since the problem asked for *an* equation of L, you don't need to convert to the standard form equation (which would match the form of the given equation).

Example 9: Find an equation of the line L through point $P(4,6)$ that is perpendicular to line $2x+5y=12$.

Solution: By Example 3, line $2x+5y=12$ has slope $m=-\frac{2}{5}$. Since the slopes of perpendicular lines are negative reciprocals, the slope of line L is $\frac{-1}{m}=\frac{-1}{-2/5}=\frac{5}{2}$.

The point-slope formula tells us that line L has equation $y-y_0=\frac{5}{2}(x-x_0)$ where $(x_0,y_0)=(4,6)$ is the given point on line L.

Answer: Line L has equation $\boxed{y-6=\frac{5}{2}(x-4)}$.

Example 10: Find an equation of the line L through point $P(4,6)$ that is perpendicular to the line joining points $(2,4)$ and $(5,7)$.

Solution: The line joining $(2,4)$ and $(5,7)$ has slope $\frac{7-4}{5-2}=1$. The perpendicular line has slope $-1/1=-1$.
Thus L has slope -1 and passes through $(4,6)$
Now apply the point-slope formula.

Answer: Line L has equation $\boxed{y-6=-1(x-4)}$.

Example 11: Find an equation of the line L through point $P(4,6)$ that is perpendicular to the line joining points $Q(2,4)$ and $R(2,7)$.

Solution: Here you need to notice that line QR is vertical, since points Q and R have the same x-coordinate . Since line L is perpendicular to line QR, line L is horizontal. Finally, since line L passes through point $(4,6)$ we have

Answer: Line L has equation $\boxed{y=6}$.

Again: there is no such thing as *"the"* equation of a straight line. A correctly stated question will ask for *an* equation of a straight line.

Be careful:

The following statements are all incorrect. Why?

- Every line has an equation of the form $y=mx+b$.
- Parallel lines have the same slope.
- Perpendicular lines have slopes that are negative reciprocals.
- $\frac{3}{0}=\frac{5}{0}$.

A preview: linear vs. exponential growth.

The slope-intercept equation $y = mx + b$ of a line is important for real-life problems.

Suppose Lita opens an account with a deposit of b dollars, and Lita's bank deposits m dollars (as a gift) once a year thereafter.

After t years, Lita's account will grow to $L(t) = mt + b$ dollars.

Here t (for time) rather than the usual x is the input of the function.

On the same day, Eta opens an account with one dollar and relaxes while the bank gives 1 percent annual interest. That is, once every year the bank multiplies the amount in Eta's account by 1.01.

After t years, Eta's account will grow to $E(t) = (1.01)^t$ dollars

Lita's account grows *Linearly*
Eta's account grows *Exponentially*.

Here's a surprise: No matter how big are Lita's initial deposit b and her bank's annual gift m, Eta's account will eventually have more money than Lita's.
That is, for t big enough, $(1.01)^t > mt + b$.

For example, suppose

- Lita opens her account with a trillion dollars and the bank deposits 10 trillion dollars every second. At the same time,
- Eta deposits just a penny in a bank that gives $1/1000$ of one percent interest each century.
- If they live long enough, Eta's account will be a trillion times larger than Lita's.

Exercise Just how long would they have to live? Use
▸ Wolfram Calculator

1.9.5 Section 1.9 Quiz

Ex 1.9.1 Find the distance between points $A(-3, 2)$ and $D(3, 3)$.

Ex 1.9.2 Find the midpoint M of the line segment PQ joining the points $P(-3, 2)$ and $Q(3, 3)$. Check your answer by showing that $\overline{PM} = \overline{MQ} = \frac{1}{2}\overline{PQ}$.

Ex 1.9.3 Find the slope of the line segment joining points $(-2, 4)$ and $(3, -2)$.

Ex 1.9.4 Find the equations of the vertical and horizontal lines through point $(5, 6)$.

Ex 1.9.5 Find point-slope and slope-intercept forms of the equation of the line with slope 2 that passes through point $(5, 7)$. Also, find the line's y-intercept.

Ex 1.9.6 Find the slope-intercept form of the line through points $P(3, 4)$ and Q$(5, 7)$.

Ex 1.9.7 Find the slope of the line with equation $2x + 5y = 12$.

Ex 1.9.8 Find an equation of the line L through point $P(4, 6)$ that is parallel to the line $2x + 5y = 12$.

Ex 1.9.9 Find an equation of the line through point $P(4, 6)$ that is perpendicular to the line $2x + 5y = 12$.

Ex 1.9.10 Find an equation of the line L through point $P(4, 6)$ that is perpendicular to the line joining points $(2, 4)$ and $(5, 7)$.

Ex 1.9.11 Find an equation of the line L through point $P(4, 6)$ that is perpendicular to the line joining points $Q(2, 4)$ and $R(2, 7)$.

Ex. 1.9.1: Find the distance d between points

- $A(-3,2)$ and $D(3,3)$
- $A(-3,2)$ and $D(-3,3)$
- $A(-3,2)$ and $D(3,2)$
- $A(5,1)$ and $D(8,5)$

Ex. 1.9.1: Find the distance d between points

- $A(-3, 2)$ and $D(3, 3) \Rightarrow d = \sqrt{37}$
- $A(-3, 2)$ and $D(3, 2) \Rightarrow d = 6$
- $A(-3, 2)$ and $D(-3, 3) \Rightarrow d = 1$
- $A(5, 1)$ and $D(8, 5) \Rightarrow d = 5$

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- $A(5,1)$ and $D(8,5) \Rightarrow d = 5$

Ex 1.9.2 Find the midpoint M of the line segment PQ joining points

- $P(-3,2)$ and $Q(3,3)$
- $P(-3,2)$ and $Q(3,2)$
- $P(-3,2)$ and $Q(-3,3)$
- $P(5,1)$ and $Q(8,5)$

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- $A(5,1)$ and $D(8,5) \Rightarrow d = 5$

Ex 1.9.2 Find the midpoint M of the line segment PQ joining points

- $P(-3,2)$ and $Q(3,3) \Rightarrow M(0, \frac{5}{2})$
- $P(-3,2)$ and $Q(3,2) \Rightarrow M(0,2)$
- $P(-3,2)$ and $Q(-3,3) \Rightarrow M(-3, \frac{5}{2})$
- $P(5,1)$ and $Q(8,5) \Rightarrow M(\frac{13}{2}, 3)$
- Check your answers by showing that $\overline{PM} = \overline{MQ} = \frac{1}{2}\overline{PQ}$.

Section 1.9 Review: Lines in the plane

Ex. 1.9.1: Find the distance d between points

- $A(-3,2)$ and $D(3,3) \Rightarrow d = \sqrt{37}$
- $A(-3,2)$ and $D(3,2) \Rightarrow d = 6$
- $A(-3,2)$ and $D(-3,3) \Rightarrow d = 1$
- $A(5,1)$ and $D(8,5) \Rightarrow d = 5$

Ex 1.9.2 Find the midpoint M of the line segment PQ joining points

- $P(-3,2)$ and $Q(3,3) \Rightarrow M(0, \frac{5}{2})$
- $P(-3,2)$ and $Q(3,2) \Rightarrow M(0,2)$
- $P(-3,2)$ and $Q(-3,3) \Rightarrow M(-3, \frac{5}{2})$
- $P(5,1)$ and $Q(8,5) \Rightarrow M(\frac{13}{2}, 3)$
- Check your answers by showing that $\overline{PM} = \overline{MQ} = \frac{1}{2}\overline{PQ}$.

Ex 1.9.3 Find the slope m of the line segment joining points

- $P(-2,4)$ and $P(3,-2)$
- $P(-3,2)$ and $Q(3,2)$
- $P(-3,2)$ and $Q(-3,3)$
- $P(5,1)$ and $Q(8,5)$

Ex. 1.9.1: Find the distance d between points

- $A(-3,2)$ and $D(3,3) \Rightarrow d=\sqrt{37}$
- $A(-3,2)$ and $D(3,2) \Rightarrow d=6$
- $A(-3,2)$ and $D(-3,3) \Rightarrow d=1$
- $A(5,1)$ and $D(8,5) \Rightarrow d=5$

Ex 1.9.2 Find the midpoint M of the line segment PQ joining points

- $P(-3,2)$ and $Q(3,3) \Rightarrow M(0,\frac{5}{2})$
- $P(-3,2)$ and $Q(3,2) \Rightarrow M(0,2)$
- $P(-3,2)$ and $Q(-3,3) \Rightarrow M(-3,\frac{5}{2})$
- $P(5,1)$ and $Q(8,5) \Rightarrow M(\frac{13}{2},3)$
- Check your answers by showing that $\overline{PM}=\overline{MQ}=\frac{1}{2}\overline{PQ}$.

Ex 1.9.3 Find the slope m of the line segment joining points

- $P(-2,4)$ and $P(3,-2) \Rightarrow m=-\frac{6}{5}$
- $P(-3,2)$ and $Q(3,2) \Rightarrow m=0$
- $P(-3,2)$ and $Q(-3,3) \Rightarrow m=$ undefined
- $P(5,1)$ and $Q(8,5) \Rightarrow m=\frac{4}{3}$

Section 1.9 Review: Lines in the plane

Ex. 1.9.1: Find the distance d between points

- $A(-3,2)$ and $D(3,3) \Rightarrow d = \sqrt{37}$
- $A(-3,2)$ and $D(3,2) \Rightarrow d = 6$
- $A(-3,2)$ and $D(-3,3) \Rightarrow d = 1$
- $A(5,1)$ and $D(8,5) \Rightarrow d = 5$

Ex 1.9.2 Find the midpoint M of the line segment PQ joining points

- $P(-3,2)$ and $Q(3,3) \Rightarrow M(0, \frac{5}{2})$
- $P(-3,2)$ and $Q(3,2) \Rightarrow M(0,2)$
- $P(-3,2)$ and $Q(-3,3) \Rightarrow M(-3, \frac{5}{2})$
- $P(5,1)$ and $Q(8,5) \Rightarrow M(\frac{13}{2}, 3)$
- Check your answers by showing that $\overline{PM} = \overline{MQ} = \frac{1}{2}\overline{PQ}$.

Ex 1.9.3 Find the slope m of the line segment joining points

- $P(-2,4)$ and $P(3,-2) \Rightarrow m = -\frac{6}{5}$
- $P(-3,2)$ and $Q(3,2) \Rightarrow m = 0$
- $P(-3,2)$ and $Q(-3,3) \Rightarrow m =$ undefined
- $P(5,1)$ and $Q(8,5) \Rightarrow m = \frac{4}{3}$

Ex 1.9.4 Find the equations of the vertical and horizontal lines through point

- $(5,6)$
- $(0,0)$
- (a,b)
- $(a^2, b+3)$

Ex. 1.9.1: Find the distance d between points

- $A(-3,2)$ and $D(3,3) \Rightarrow d = \sqrt{37}$
- $A(-3,2)$ and $D(3,2) \Rightarrow d = 6$
- $A(-3,2)$ and $D(-3,3) \Rightarrow d = 1$
- $A(5,1)$ and $D(8,5) \Rightarrow d = 5$

Ex 1.9.2 Find the midpoint M of the line segment PQ joining points

- $P(-3,2)$ and $Q(3,3) \Rightarrow M(0, \frac{5}{2})$
- $P(-3,2)$ and $Q(3,2) \Rightarrow M(0,2)$
- $P(-3,2)$ and $Q(-3,3) \Rightarrow M(-3, \frac{5}{2})$
- $P(5,1)$ and $Q(8,5) \Rightarrow M(\frac{13}{2}, 3)$
- Check your answers by showing that $\overline{PM} = \overline{MQ} = \frac{1}{2}\overline{PQ}$.

Ex 1.9.3 Find the slope m of the line segment joining points

- $P(-2,4)$ and $P(3,-2) \Rightarrow m = -\frac{6}{5}$
- $P(-3,2)$ and $Q(3,2) \Rightarrow m = 0$
- $P(-3,2)$ and $Q(-3,3) \Rightarrow m =$ undefined
- $P(5,1)$ and $Q(8,5) \Rightarrow m = \frac{4}{3}$

Ex 1.9.4 Find the equations of the vertical and horizontal lines through point

- $(5,6) \Rightarrow x = 5; y = 6$
- $(0,0) \Rightarrow x = 0; y = 0$
- $(a,b) \Rightarrow x = a; y = b$
- $(a^2, b+3) \Rightarrow x = a^2; y = b+3$

Section 1.9 Review: Lines in the plane

Ex. 1.9.1: Find the distance d between points

- $A(-3,2)$ and $D(3,3) \Rightarrow d = \sqrt{37}$
- $A(-3,2)$ and $D(3,2) \Rightarrow d = 6$
- $A(-3,2)$ and $D(-3,3) \Rightarrow d = 1$
- $A(5,1)$ and $D(8,5) \Rightarrow d = 5$

Ex 1.9.2 Find the midpoint M of the line segment PQ joining points

- $P(-3,2)$ and $Q(3,3) \Rightarrow M(0, \frac{5}{2})$
- $P(-3,2)$ and $Q(3,2) \Rightarrow M(0,2)$
- $P(-3,2)$ and $Q(-3,3) \Rightarrow M(-3, \frac{5}{2})$
- $P(5,1)$ and $Q(8,5) \Rightarrow M(\frac{13}{2}, 3)$
- Check your answers by showing that $\overline{PM} = \overline{MQ} = \frac{1}{2}\overline{PQ}$.

Ex 1.9.3 Find the slope m of the line segment joining points

- $P(-2,4)$ and $P(3,-2) \Rightarrow m = -\frac{6}{5}$
- $P(-3,2)$ and $Q(3,2) \Rightarrow m = 0$
- $P(-3,2)$ and $Q(-3,3) \Rightarrow m =$ undefined
- $P(5,1)$ and $Q(8,5) \Rightarrow m = \frac{4}{3}$

Ex 1.9.4 Find the equations of the vertical and horizontal lines through point

- $(5,6) \Rightarrow x = 5; y = 6$
- $(0,0) \Rightarrow x = 0; y = 0$
- $(a, b) \Rightarrow x = a; y = b$
- $(a^2, b+3) \Rightarrow x = a^2; y = b+3$

Ex 1.9.5 Find the y-intercept and the point-slope and slope-intercept forms of the equation of the line with

- slope 2 that passes through point $(5,7)$
- slope 0 that passes through point $(5,7)$
- slope -3 that passes through point $(0,0)$
- slope 3 that passes through point (a, b)

Ex. 1.9.1: Find the distance d between points

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- $(a^2, b+3) \Rightarrow x = a^2; y = b+3$

Ex 1.9.5 Find the y-intercept and the point-slope and slope-intercept forms of the equation of the line with

- slope 2 that passes through point $(5,7)$ $\Rightarrow y - 7 = 2(x-5); y = 2x - 3;$ y-intercept is -3.
- slope 0 that passes through point $(5,7)$ $\Rightarrow y = 7;$ y-intercept is 7.
- slope -3 that passes through point $(0,0)$ $\Rightarrow y = -3x; y = -3x;$ y-intercept is 0.
- slope 3 that passes through point (a, b) $\Rightarrow y - b = 3(x-a); y = 3x - 3a + b$
  y-intercept is $-3a + b$.

Section 1.9 Review

Ex 1.9.6 Find the slope-intercept form of the line through points

- $P(3,4)$ and Q$(5,7)$
- $P(3,4)$ and Q$(0,7)$
- $P(2,4)$ and Q$(5,4)$
- $P(3,4)$ and Q$(3,7)$

Ex 1.9.6 Find the slope-intercept form of the line through points

- $P(3,4)$ and Q$(5,7) \Rightarrow y = \frac{3}{2}x - \frac{1}{2}$
- $P(3,4)$ and Q$(0,7) \Rightarrow y = -x + 7$
- $P(2,4)$ and Q$(5,4) \Rightarrow y = 0x + 4$ or $y = 4$
- $P(3,4)$ and Q$(3,7)$ $\Rightarrow x = 3$. The line is vertical and doesn't meet the y-axis. It does not have an equation in slope-intercept form.

Section 1.9 Review

Ex 1.9.6 Find the slope-intercept form of the line through points

- $P(3,4)$ and $Q(5,7) \Rightarrow y = \frac{3}{2}x - \frac{1}{2}$
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- $P(3,4)$ and $Q(3,7) \Rightarrow x = 3$. The line is vertical and doesn't meet the y-axis. It does not have an equation in slope-intercept form.

Ex 1.9.7 Find the slope m of the line with equation

- $2x + 5y = 12$
- $2x = 3y$
- $2x = 12$
- $3y = 23$

Section 1.9 Review

Ex 1.9.6 Find the slope-intercept form of the line through points

- $P(3,4)$ and Q$(5,7) \Rightarrow y = \frac{3}{2}x - \frac{1}{2}$
- $P(3,4)$ and Q$(0,7) \Rightarrow y = -x + 7$
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Ex 1.9.7 Find the slope m of the line with equation

- $2x + 5y = 12 \Rightarrow m = -\frac{2}{5}$
- $2x = 3y \Rightarrow m = \frac{2}{3}$
- $2x = 12 \Rightarrow m$ undefined
- $3y = 23 \Rightarrow m = 0$

Section 1.9 Review

Ex 1.9.6 Find the slope-intercept form of the line through points

- $P(3,4)$ and Q$(5,7) \Rightarrow y = \frac{3}{2}x - \frac{1}{2}$
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- $2x = 12 \Rightarrow m$ undefined
- $3y = 23 \Rightarrow m = 0$

Ex 1.9.8 Find an equation of the line L through point $P(4,6)$ that is parallel to the line

- $2x + 5y = 12$
- $2x = 3y$
- $2x = 12$
- $3y = 23$

Section 1.9 Review

Ex 1.9.6 Find the slope-intercept form of the line through points

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Ex 1.9.8 Find an equation of the line L through point $P(4,6)$ that is parallel to the line

- $2x + 5y = 12 \Rightarrow y - 6 = -\frac{2}{5}(x-4)$
- $2x = 3y \Rightarrow y - 6 = \frac{2}{3}(x-4)$
- $2x = 12 \Rightarrow x = 4$
- $3y = 23 \Rightarrow y = 6$

Section 1.9 Review

Ex 1.9.6 Find the slope-intercept form of the line through points

- $P(3,4)$ and $Q(5,7) \Rightarrow y = \frac{3}{2}x - \frac{1}{2}$
- $P(3,4)$ and $Q(0,7) \Rightarrow y = -x + 7$
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- $2x = 12 \Rightarrow x = 4$
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- $2x + 5y = 12$
- $2x = 3y$
- $2x = 12$
- $3y = 23$

Section 1.9 Review

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Section 1.9 Review

Ex 1.9.6 Find the slope-intercept form of the line through points

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- $2x = 12 \Rightarrow y = 6$
- $3y = 23 \Rightarrow x = 4$

Ex 1.9.10 Find an equation of the line L through $P(4,6)$ that is perpendicular to the line joining points

- $(2,4)$ and $(5,7)$
- $(2,0)$ and $(2,7)$
- $(3,7)$ and $(5,7)$
- $(4,4)$ and $(9,5)$

Section 1.9 Review

Ex 1.9.6 Find the slope-intercept form of the line through points

- $P(3,4)$ and Q$(5,7) \Rightarrow y = \frac{3}{2}x - \frac{1}{2}$
- $P(3,4)$ and Q$(0,7) \Rightarrow y = -x + 7$
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Ex 1.9.9 Find an equation of the line through point $P(4,6)$ that is perpendicular to the line

- $2x + 5y = 12 \Rightarrow y - 6 = \frac{5}{2}(x-4)$
- $2x = 3y \Rightarrow y - 6 = -\frac{3}{2}(x-4)$
- $2x = 12 \Rightarrow y = 6$
- $3y = 23 \Rightarrow x = 4$

Ex 1.9.10 Find an equation of the line L through $P(4,6)$ that is perpendicular to the line joining points

- $(2,4)$ and $(5,7) \Rightarrow y = -x + 10$
- $(2,0)$ and $(2,7) \Rightarrow y = 6$
- $(3,7)$ and $(5,7) \Rightarrow x = 4$
- $(4,4)$ and $(9,5) \Rightarrow y - 6 = -5(x-4)$ or $y = -5x + 26$

Section 1.9 Review

Ex 1.9.6 Find the slope-intercept form of the line through points

- $P(3,4)$ and Q$(5,7) \Rightarrow y = \frac{3}{2}x - \frac{1}{2}$
- $P(3,4)$ and Q$(0,7) \Rightarrow y = -x + 7$
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Ex 1.9.8 Find an equation of the line L through point $P(4,6)$ that is parallel to the line

- $2x + 5y = 12 \Rightarrow y - 6 = -\frac{2}{5}(x-4)$
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- $2x = 12 \Rightarrow x = 4$
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- $2x = 12 \Rightarrow y = 6$
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Ex 1.9.10 Find an equation of the line L through $P(4,6)$ that is perpendicular to the line joining points

- $(2,4)$ and $(5,7) \Rightarrow y = -x + 10$
- $(2,0)$ and $(2,7) \Rightarrow y = 6$
- $(3,7)$ and $(5,7) \Rightarrow x = 4$
- $(4,4)$ and $(9,5) \Rightarrow y - 6 = -5(x-4)$ or $y = -5x + 26$

Ex 1.9.11 Find an equation of the line L through point $P(4,6)$ that is

- perpendicular to the line joining points $Q(2,4)$ and $R(2,7)$.
- perpendicular to the line joining points $Q(1,4)$ and $R(3,4)$.
- parallel to the line joining points $Q(2,7)$ and $R(2,9)$.
- parallel to the line joining points $Q(-3,-4)$ and $R(2,7)$.

Section 1.9 Review

Ex 1.9.6 Find the slope-intercept form of the line through points

- $P(3,4)$ and Q$(5,7) \Rightarrow y = \frac{3}{2}x - \frac{1}{2}$
- $P(3,4)$ and Q$(0,7) \Rightarrow y = -x + 7$
- $P(2,4)$ and Q$(5,4) \Rightarrow y = 0x + 4$ or $y = 4$
- $P(3,4)$ and Q$(3,7) \Rightarrow x = 3$. The line is vertical and doesn't meet the y-axis. It does not have an equation in slope-intercept form.

Ex 1.9.7 Find the slope m of the line with equation

- $2x + 5y = 12 \Rightarrow m = -\frac{2}{5}$
- $2x = 3y \Rightarrow m = \frac{2}{3}$
- $2x = 12 \Rightarrow m$ undefined
- $3y = 23 \Rightarrow m = 0$

Ex 1.9.8 Find an equation of the line L through point $P(4,6)$ that is parallel to the line

- $2x + 5y = 12 \Rightarrow y - 6 = -\frac{2}{5}(x-4)$
- $2x = 3y \Rightarrow y - 6 = \frac{2}{3}(x-4)$
- $2x = 12 \Rightarrow x = 4$
- $3y = 23 \Rightarrow y = 6$

Ex 1.9.9 Find an equation of the line through point $P(4,6)$ that is perpendicular to the line

- $2x + 5y = 12 \Rightarrow y - 6 = \frac{5}{2}(x-4)$
- $2x = 3y \Rightarrow y - 6 = -\frac{3}{2}(x-4)$
- $2x = 12 \Rightarrow y = 6$
- $3y = 23 \Rightarrow x = 4$

Ex 1.9.10 Find an equation of the line L through $P(4,6)$ that is perpendicular to the line joining points

- $(2,4)$ and $(5,7) \Rightarrow y = -x + 10$
- $(2,0)$ and $(2,7) \Rightarrow y = 6$
- $(3,7)$ and $(5,7) \Rightarrow x = 4$
- $(4,4)$ and $(9,5) \Rightarrow y - 6 = -5(x-4)$ or $y = -5x + 26$

Ex 1.9.11 Find an equation of the line L through point $P(4,6)$ that is

- perpendicular to the line joining points $Q(2,4)$ and $R(2,7)$. $\Rightarrow y = 6$
- perpendicular to the line joining points $Q(1,4)$ and $R(3,4)$. $\Rightarrow x = 4$
- parallel to the line joining points $Q(2,7)$ and $R(2,9)$. $\Rightarrow y = 6$
- parallel to the line joining points $Q(-3,-4)$ and $R(2,7)$. $\Rightarrow y - 6 = \frac{11}{5}(x-4)$

Chapter 1 Review

- Precalculus Section 1.0 Review: Arithmetic
- Precalculus Section 1.1 Review: Numbers and Expressions
- Precalculus Section 1.2. Review: Fractions
- Precalculus Section 1.3 Review: Powers, roots, and exponents
- Precalculus Section 1.4 Review: Modeling real-life problems
- Precalculus Section 1.5 Review: Equations
- Precalculus Section 1.6 Review: Intervals and Inequalities
- Precalculus Section 1.7 Review: Formulas and functions
- Precalculus Section 1.8 Review: The x, y-coordinate plane
- Precalculus Section 1.9 Review: Lines in the plane

To review a section listed above:

Click on its button to view the first Example in that section as well as three similar questions.
Work out the answers, then click again to see if you are correct. If so, keep on clicking.

If you have trouble answering a question, click on the to its left to access its solution in the text.
Then click on the faint ⥀ Adobe control at the bottom right of the text screen to continue your review.

Precalculus Chapter 2: Functions and their graphs

Section 2.1: Functions

2.1.1: How functions are used
2.1.2: Methods for defining functions
2.1.3: The graph of a function
2.1.4: Domain and range
2.1.5: Relations, functions, and graphs
2.1.6: Examples of graphs of functions
2.1.7: The domain really matters!
2.1 Quiz 2.1 Review

Section 2.2: Sketching graphs

2.2.1: Graphing a real-life function
2.2.2: The effect of the choice of domain
2.2.3: Special features of graphs
2.2.4: Linear functions defined on intervals
2.2.5: Graphing a polynomial on an interval
2.2.6: Piecewise defined functions
2.2.7: A three-part piecewise linear graph
2.2.8: The vertical line test
2.2 Quiz 2.2 Review

Section 2.3: Analyzing graphs

2.3.1: Maximum and minimum points
2.3.2: Increasing, decreasing functions
2.3.3: Given $h(t)$, find t
2.3.4: A degree 3 polynomial
2.3.5: How many solutions of $h(t) = K$?
2.3 Quiz 2.3 Review

Section 2.4: Quadratic functions

2.4.1: Graphs of quadratic functions
2.4.2: Sketching $y = ax^2$
2.4.3: Sketching $y = ax^2 + c$
2.4.4: Completing the square revisited
2.4.5: Rewriting $ax^2 + bx + c$ in standard form
2.4.6: Sketching $y = ax^2 + bx + c$
2.4.7: Transforming to standard form
2.4.8: x-intercepts of a quadratic
2.4 Quiz 2.4 Review

Section 2.5: Graphing polynomials

2.5.1: Polynomial sign analysis
2.5.2: Degree 4 polynomial sign analysis
2.5.3: Accuracy of polynomial sketches
2.5.4: Polynomial inequalities
2.5.5: Rational inequalities
2.5 Quiz 2.5 Review

Section 2.6: Rate of change

2.6.1: Average rate of change
2.6.2: The slope of a line
2.6.3: Speedometers display average velocity
2.6.4: Instantaneous velocity
2.6 Quiz 2.6 Review

Section 2.7: Transformations

2.7.1: Transforming equations
2.7.2: Modifying equation $y = f(x)$
2.7.3: Modifying the argument of $y = f(x)$
2.7.4: Combining transformations
2.7 Quiz 2.7 Review

Section 2.8: Inverse functions

2.8.1: Formulas and functions
2.8.2: Composing f and f^{-1}
2.8.3: The horizontal line test
2.8.4: Restricting the domain
2.8.5: Graphing f and f^{-1} on the same grid
2.8 Quiz 2.8 Review

Chapter 2 Review

- 2.1.1: How functions are used
- 2.1.2: Methods for defining functions
- 2.1.3: The graph of a function
- 2.1.4: Domain and range
- 2.1.5: Relations, functions, graphs
- 2.1.6: Examples of graphs of functions
- 2.1.7: The domain is important
- 2.1.8: Review

Section 2.1 Preview: Definitions

- Definition 2.1.1: *Functions and their graphs*
- Definition 2.1.2: Definition: The symbol $\mathbb{R}$ stands for
- Definition 2.1.3: A function f from set A to set B
- Definition 2.1.4: An *ordered pair* of real numbers is (x, y),
- Definition 2.1.5: A *function* f is any set of ordered pairs of real numbers
- Definition 2.1.6 : Different function formulas define the same function!
- Definition 2.1.7: The *graph of a function* f *in the* x, y*-plane* consists of
- Definition 2.1.8: Relations and graphs
- Definition 2.1.9: The graph of an equation in letters x and y consists of all points (x, y)
- Definition 2.1.10: The graph of a function f defined by formula $y = f(x)$ is the graph of the equation $y = f(x)$.
- Definition 2.1.11: *Vertical Line Test:*

Section 2.1 Preview: Procedures

- Procedure 2.1.1: To define a function f
- Procedure 2.1.2: Examples of functions f and g with domain $\mathbb{R}$
- Procedure 2.1.3: To *draw* the graph of a function,
- Procedure 2.1.4: The *natural domain of a function* $f(x)$
- Procedure 2.1.5: To find the domain and range of function f from its graph
- Procedure 2.1.6: To decide if an equation defines y as a function of x
- Procedure 2.1.7 To substitute an expression for x in $f(x) = E$
- Procedure 2.1.8: To graph the absolute value function

2.1.1 How functions are used

We begin with a quick review of Chapter 1.7, which was concerned with using function notation correctly in algebraic expressions. Now we focus on how to draw and understand graphs of functions.

Functions and their graphs

- are often used to describe how one real-life quantity depends on another.
- To get the big picture of how the quantities are related, try to draw the graph of the function.
- However, the graph may be misleading in certain situations.

I drop a ball off the top of a 100-foot building. From the time the ball is dropped until it hits the ground, the ball's height (measured in feet) above the ground at time t seconds is $100 - 16t^2$. The ball's height above the ground depends on how long it has been falling.

Use the symbol $h(t)$ to *abbreviate* "the ball's height in feet at time t seconds after it is dropped." Then $h(t) = 100 - 16t^2$ from time $t = 0$ until the time t when the ball hits the ground.

Example 1: How high above the ground is the ball after 2 seconds?

Solution: To find $h(2)$, substitute 2 for t in $h(t) = 100 - 16t^2$. Then
$h(2) = 100 - 16(2)^2 = 100 - 16(4) = 100 - 64 = 36$

After 2 seconds, the ball is 36 feet above the ground.

Example 2: When does the ball hit the ground?

Solution: Since $h(t)$ is the ball's height at time t, and the ball's height when it hits the ground is 0,
solve $h(t) = 0$ for t.

$100 - 16t^2 = 0$ and so $100 = 16t^2$. Divide by 16:
$t^2 = \frac{100}{16} = \frac{25}{4}$.
$t = \pm\sqrt{\frac{25}{4}} = \pm\frac{\sqrt{25}}{\sqrt{4}} = \pm\frac{5}{2}$ and so $t = 2.5$ or $t = -2.5$.

Since this problem starts at $t = 0$, the answer will require $t \geq 0$. Therefore

The ball hits the ground 2.5 seconds after it is dropped.

Definition: The symbol $\mathbb{R}$ stands for

the set of all real numbers.

2.1.2 A function is a set of ordered pairs

Here is a textbook definition:

A function f from set A to set B

is a rule that assigns to each input element x in A exactly one output element $f(x)$ in B.

In these notes A will be $\mathbb{R}$ or a subset, while B will always be $\mathbb{R}$.

The statement that a function is a rule doesn't say what a rule is, and it leaves open the possibility that different rules for getting from the input to the output might define different functions.

That's not so. The only thing a function cares about is: for a given input, what is the output?

For example, the rules $f(x) = x^2 - 1$ and $g(x) = (x-1)(x+1)$ require different calculations to find their respective outputs. However, they define the same function, because $f(x) = g(x)$ for every input x.

An *ordered pair* of real numbers is (x, y),

where x and y are real numbers. The ordered pair's input is x and its output is y.

A *function* f is any set of ordered pairs of real numbers in which no input has two different outputs.

- If x is an input, then $f(x)$ is its output.
- The domain of f is all of its inputs.
- The range of f is all of its outputs.
- The function f is the set of all ordered pairs $(x, f(x))$, where x is an input for F.

To describe a function by using a formula, you should state its domain. Even if described by the same rule, functions with different domains are different.

To define a function f

- specify its domain D and
- state a clear method for figuring out $f(x)$ for each input x in D.

Examples of functions f and g with domain $\mathbb{R}$

- $f(x) = x^2 - 1$
- $g(x) = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ 1 & \text{if } x \text{ is not an integer} \end{cases}$

2.1.3 The graph of a function

Different function formulas define the same function!

Here are definitions of two seemingly different functions:

- Let $f(x) = x^2 - 1$ with domain $\mathbb{R}$.
- Let $g(x) = (x+1)(x-1)$ with domain $\mathbb{R}$.
- g is all number pairs $(x, (x+1)(x-1))$.
- f is all number pairs $(x, x^2 - 1)$.
- Although their formulas are different, f and g are the same function because
  - they have the same domain *and*
  - $f(x) = g(x)$ for all x in that domain.

The *graph of a function* f *in the* x, y*-plane* consists of

all points obtained by plotting number pairs $(x, f(x))$, where x is in the domain of the function.

To *draw* the graph of a function,

choose x-values in the domain and draw dots on the coordinate plane, one for each ordered pair $(x, f(x))$.

The word *plot* is both a verb and a noun.
To *plot* a function, draw its graph.
That graph is the *plot* of the function.

Can you think of a third meaning of *plot*?
Answer: The map of a cemetery is a *plot plot* :)

The number of dots that can be plotted, even by a computer, is finite. Section 2.1.7 will show that connecting consecutive dots with line segments may, or may not, produce an accurate drawing.

Most function domains are infinite sets: either all of $\mathbb{R}$ or an interval such as $[3, 7]$.

The *natural domain of a function* $f(x)$

consists of all real number inputs x for which the formula is defined. It does not include $x = a$ if $f(a)$ is

- a fraction with denominator 0 or
- an even root of a negative number.
- or any other undefined expression.

If a function definition does not state its domain, use the function's natural domain.

Example 3: $h(x) = \sqrt{x}$ defines a function with natural domain $[0, \infty)$. That's because the square root of a negative real number is not a real number.

2.1.4 Domain and range

Example 4:
Find the natural domain of the function $f(x) = \frac{1}{\sqrt{2x+5}}$.

Solution:
- The square root of a number less than 0 is not defined. Thus we need $2x + 5 \geq 0$, which means $x \geq -\frac{5}{2}$.
- Division by zero is undefined. Thus we require $2x + 5 \neq 0$ and so $x \neq -\frac{5}{2}$.

Now combine the above requirements.

Domain of f as inequality: $x > -\frac{5}{2}$; as interval: $(-\frac{5}{2}, \infty)$

Example 5: Express the natural domain of $\frac{1}{x^2-3x+2}$ using interval notation.

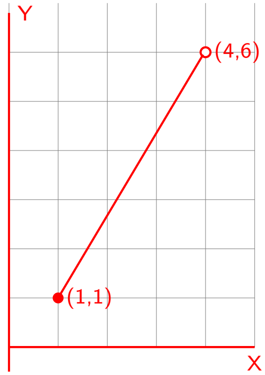
Solution: x is any number that makes the denominator non-zero. To find the exceptions, solve
$x^2 - 3x + 2 = 0 \Rightarrow (x-2)(x-1) = 0 \Rightarrow x = 2; x = 1$.
Thus x can be any number other than 1 or 2.

Answer: The domain of f is $(-\infty, 1) \cup (1, 2) \cup (2, \infty)$.

To find the domain and range of function f from its graph

- The domain of f consists of all x-scale numbers that lie on vertical lines through points on the graph.
- The range of f consists of all y-scale numbers that lie on horizontal lines through points on the graph.

Example 6: Below is a graph given by formula $f(x) = \frac{5x-2}{3}$. Keep on clicking to find the domain and range of the function sketched.



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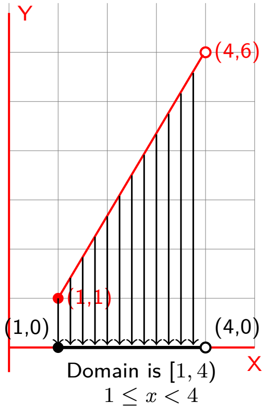
Example 6: Below is a graph given by formula $f(x) = \frac{5x-2}{3}$. Keep on clicking to find the domain and range of the function sketched.

To find the domain, slide all points on the graph down to the x-axis. The domain of f is $[1, 4)$.



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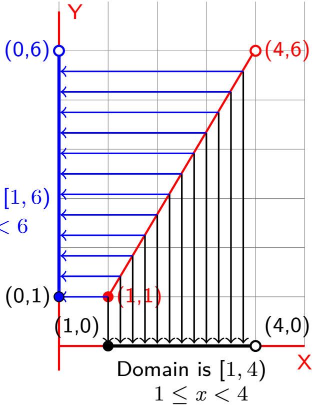
To find the domain, slide all points on the graph down to the x-axis. The domain of f is $[1, 4)$.

To find the range, slide all points on the graph left to the y-axis. The range of f is $[1, 6)$.



2.1.5 Relations, Functions, and Graphs

A function and its graph are pretty much the same: the function is a bunch of number pairs, while a function's graph is all those number pairs plotted as points in the plane. More generally:

Relations and graphs

- A graph is *any* set of points in the x, y-plane.
- A relation between x and y is any set of ordered real number pairs (x, y).
- The set of real number pairs (x, y) that satisfy an equation in letters x and/or y form a relation.
- To graph that relation, plot all of its number pairs as points in the x, y-plane.

For example, the equation $x^2 + 2y = 4$ defines a relation. (a, b) is a point of the relation if and only if substituting a for x and b for y in $x^2 + 2y = 4$ yields a true statement. For example, letting $x = 2$ and $y = 0$ yields the true statement $(2)^2 + 2(0) = 4$. Thus $(2, 0)$ is on the graph of the equation $x^2 + 2y = 4$. However, letting $x = 3$ and $y = 1$ yields a false statement $3^2 + 2(1) = 4$. Therefore point $(3, 1)$ is not on the graph of the equation $x^2 + 2y = 4$.

The graph of an equation in letters x and y

consists of all points (x, y) in the x, y-plane that satisfy the equation.

Example 7: Describe the graph of the equation $x^2 + 2y = 4$ as a set of points in the x, y-plane.
Solution: First solve the equation for $y = \frac{4-x^2}{2}$.
Answer: The graph of $x^2 + 2y = 4$ consists of all points $\left(x, \frac{4-x^2}{2}\right)$ where x is any real number.

The connection between function graphs and function formulas is straightforward.

The graph of a function f defined by formula $y = f(x)$

is the graph of the equation $y = f(x)$.

Example 8: Graphs we will discuss later include:

- The graph of $f(x) = 3x + 4$ is is a slanted line passing through point $(0, 4)$.
- The graph of $f(x) = 16 - x^2$ is a parabola.
- The graph of $f(x) = \sin(x)$ is a wavy curve.

To recognize the graph of a function, use the

Vertical Line Test:

A set of points in the x, y-plane is the graph of some function if and only if every vertical line meets the graph never, or at exactly one point.

To see why, take any vertical line, for example $x = 4$. Then either

- 4 is not in the domain of f, and so there is no point on the graph with input 4. Thus the line $x = 4$ does not meet the graph.
- 4 is in the domain of f. Then x is an input for f, with a single output $f(4)$. Thus there is only one point on line $x = 4$, namely the point $(4, f(4))$.

An equation defines y as a function of x with domain D

if solving the equation for y as a formula in x yields at most one y-value for every x in D.

An equation that does not define a function is $x^2 + y^2 = 1$, the equation of a circle, which certainly doesn't pass the vertical line test. Just draw a picture: clearly the vertical line $x = a$ meets the circle in two points if $-1 < a < 1$.

Indeed solving for $y = \pm\sqrt{1 - x^2}$ gives two different y-values for any x with $-1 < x < 1$.

Expressions as well as numbers can be function inputs.

To substitute an expression for x in $f(x) = E$

Replace every letter x in E by the expression enclosed in parentheses.

Example 9:
Suppose $h(x) = 16 - x^2$. Find and simplify $h(a + b)$.
Solution:
Substitute $a + b$ for x in the function formula.

$$h(a + b) = 16 - (a + b)^2 = 16 - (a^2 + 2ab + b^2)$$
$$= \boxed{16 - a^2 - 2ab - b^2}$$

Make sure to enclose every function value in parentheses before simplifying.

Exercise: If $f(x) = 3 - x - x^2$, find and simplify

a) $\frac{f(x+h)-f(x-h)}{h}$ and b) $\frac{f(3+h)-f(3-h)}{h}$.

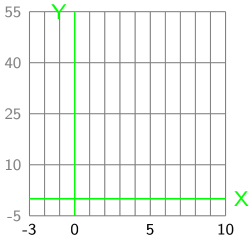
Check that substituting 3 for x in a) and then simplifying gives the answer to b).

2.1.6 Examples of graphs of functions

Example 10: Graph the straight line $y = 4x + 7$ showing domain $-3 \leq x \leq 10$.

Solution:

- Draw a grid with $-3 \leq x \leq 10$.



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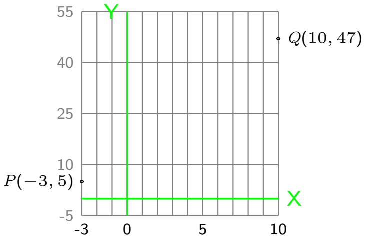
- Draw a grid with $-3 \leq x \leq 10$.
- Plot points $(x, 4x + 7)$ for $x = -3$ and $x = 10$, namely $P(-3, -5)$ and $Q(10, 47)$.



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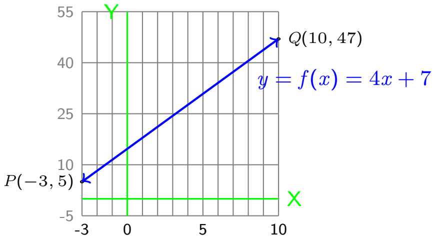

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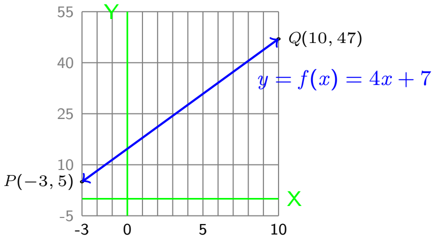

Exercise: Try to draw the graph of $y = g(x) = x^{100}$ with domain $0 \leq x \leq 1$. Drawing line segments between 3 points with equally spaced x-values $0, 0.5$, and 1 gives a totally incorrect picture. Why? Does using a graphing calculator help?

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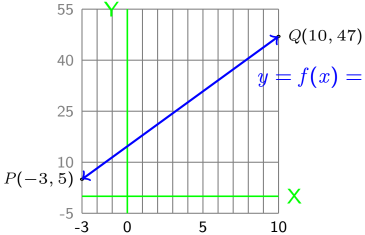

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To graph the absolute value function

$|x| = -x$ if $x \leq 0$ $\quad |x| = x$ if $x \geq 0$.
On the same grid, plot each piece separately.

Example 11: Graph $f(x) = |x|$ showing domain $[-5, 5]$

where $|x| = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } x \geq 0 \end{cases}$

The braces notation above is used very often!

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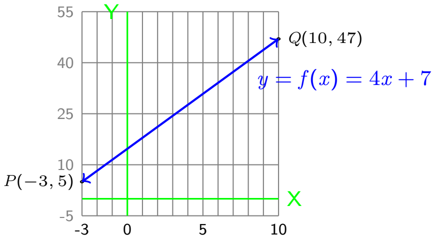

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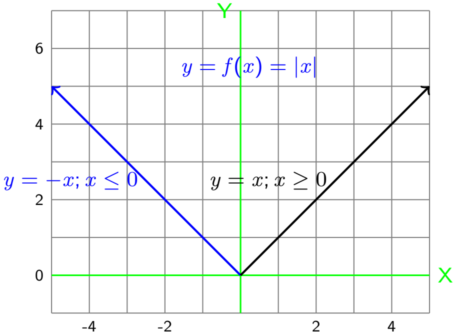

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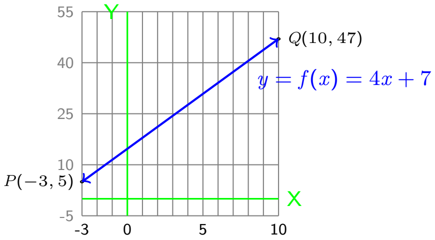

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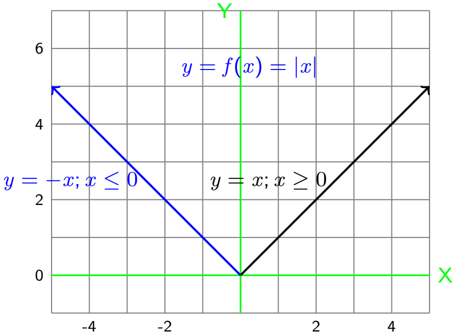
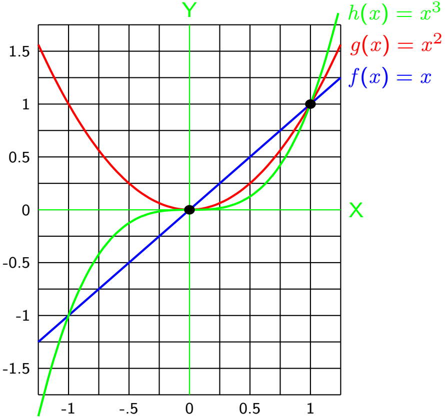


The correct graph was obtained (by luck!) in this case by plotting 3 points with equally spaced x-values.

2.1.7 The domain is important

The three functions $f(x) = x$; $g(x) = x^2$; and $h(x) = x^3$ are all drawn below with domain $[-1.25, 1.25]$. The graphs are very different. However, if you restrict the domain to be the two x-values 0 and 1, the graphs of all three functions are identical: the two black dots.



When you draw the graph of a function, choosing only whole numbers as inputs can produce a totally incorrect graph!

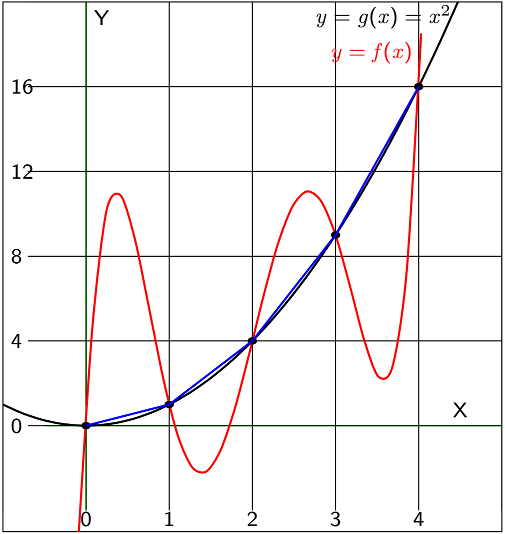
For the simple function $g(x) = x^2$, choosing $x = -2, -1, 0, 1, 2$, connecting the points (x, x^2), and smoothing out the curve gives a reasonable picture. But that's just a matter of luck.

A harder example: use $x = 0, 1, 2, 3, 4$ to sketch $f(x) = 3x^5 - 30x^4 + 105x^3 - 149x^2 + 72x$. As shown in the next frame,

- The tables and graphs of $f(x)$ and $g(x)$, using $x = 0, 1, 2, 3, 4$, are identical.
- However, the graphs of f and g, using domain $[0, 4]$ (*all* real numbers between 0 and 4), are very different.

Exercise: Find $f(0), f(1), f(2), f(3)$, and $f(4)$ without using a calculator. If that's too hard, see Example 11.

A computer could plot 500 points instead of the 5 points that we are working with. Nevertheless, it could still make a serious mistake by connecting the dots to form the graph. Algebra and calculus are needed to verify that the graph is reasonably correct.



Above are the graphs of $g(x) = x^2$ and $f(x) = 3x^5 - 30x^4 + 105x^3 - 149x^2 + 72x$ with domains including the interval $[0, 4]$.

The blue graph, which you might draw by hand, draws straight line segments connecting points $(x, f(x))$ with $x = 0, 1, 2, 3, 4$. It is a decent approximation to the familiar graph of $y = x^2$ but a terrible approximation to the graph of f. That's because f and g have the same graph (5 black dots) for domain $\{0, 1, 2, 3, 4\}$ but very different graphs when the domain $[0, 4]$ includes all in-between points.

x	$f(x)$	$g(x)$
0	0	0
1	1	1
2	4	4
3	9	9
4	16	16

x	$f(x)$	$g(x)$
0.5	0.25	10.09375
1.5	2.25	−1.96875
2.5	6.25	10.46875
3.5	12.25	2.40625
4.5	20.25	108.84375

Example 12: Multiply out and rewrite $x^2 + 3x(x-1)(x-2)(x-3)(x-4)$ as a polynomial.

Answer:
$3x^5 - 30x^4 + 105x^3 - 149x^2 + 72x$, which is $f(x)$!

Exercise: The graph seems to show that $f(x) = g(x) = x^2$ for $x = 0, 1, 2, 3, 4$ but for no other values of x. Prove this.

Hint: First show that $f(x) - x^2 = 3x(x-1)(x-2)(x-3)(x-4)$. Then apply the Zero Product Rule.

2.1.8 Section 2.1 Quiz

Ex. 2.1.1: Suppose the height of a ball above the ground after t seconds is $h(t) = 100 - 16t^2$. How high above the ground is the ball after 2 seconds?

Ex. 2.1.2: When does the ball hit the ground?

Ex. 2.1.3: Find the natural domain of the function $h(x) = \sqrt{x}$.

Ex. 2.1.4: Find the natural domain of the function $f(x) = \frac{1}{\sqrt{2x+5}}$.

Ex. 2.1.5: Using interval notation, find the natural domain of $\frac{1}{x^2-3x+2}$.

Ex. 2.1.6: Draw the graph of $y = f(x) = \frac{5x-2}{3}$ for $1 \le x < 4$. Then find the domain and range of the function f.

Ex. 2.1.7: Describe the graph of the equation $x^2 + 2y = 4$ as a set of ordered pairs.

Ex. 2.1.8: Describe the graphs of the functions • $f(x) = 3x + 4$ and • $g(x) = 16 - x^2$.

Ex. 2.1.9: Suppose $h(x) = 16 - x^2$. Find and simplify $h(a + b)$.

Ex. 2.1.10: Graph the straight line $y = 4x + 7$ showing domain $-3 \le x \le 10$.

Ex. 2.1.11: Draw the graph of $f(x) = |x|$ with domain $[-5, 5]$.

Ex. 2.1.12: Multiply out and rewrite $x^2 + 3x(x-1)(x-2)(x-3)(x-4)$ as a polynomial.

Section 2.1 Review: Functions

Ex. 2.1.1: How high above the ground is the ball after 2 seconds if its height (in feet) above the ground after t seconds is

- $f(t) = 100 - 16t^2 \Rightarrow$
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Ex. 2.1.4 Using both inequality and interval notation, find the natural domain of the functions

- $f(x) = \frac{1}{\sqrt{2x+5}} \Rightarrow$
- $g(x) = \frac{x}{\sqrt{2x-5}} \Rightarrow$
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Ex. 2.1.5 Using interval notation, find the natural domain of

- $f(x) = \frac{1}{x^2-3x+2} \Rightarrow$
- $g(x) = \frac{1}{x^2+5x-6} \Rightarrow$
- $h(x) = \frac{1}{|x|+1} \Rightarrow$
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- $s(x) = \frac{1}{\sqrt{x^2-1}} \Rightarrow (-\infty, -1) \cup (1, \infty)$

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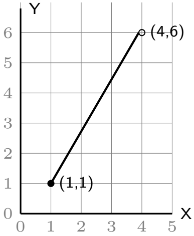

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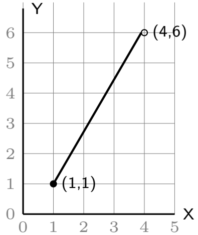

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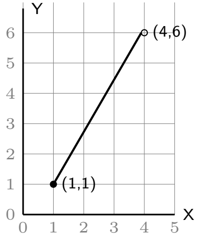

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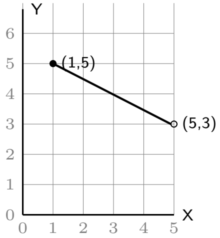

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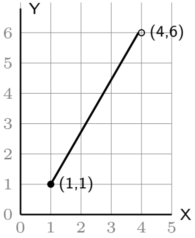

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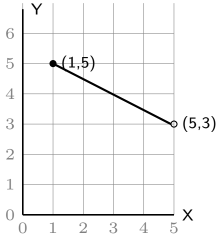

$f(x) = \frac{25-x}{5}$; $0 \le x < 5$
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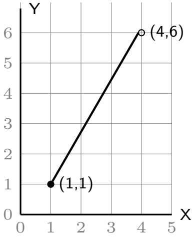

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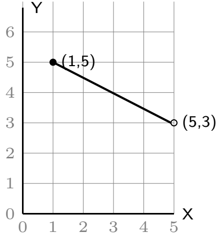

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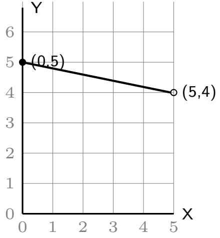

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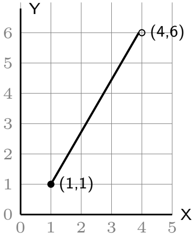

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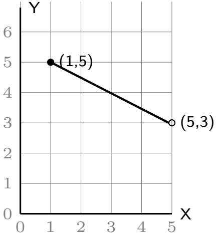

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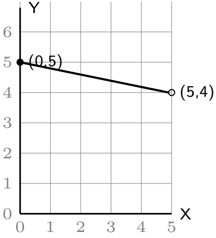

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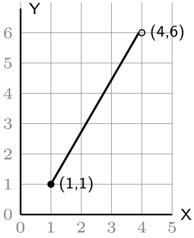

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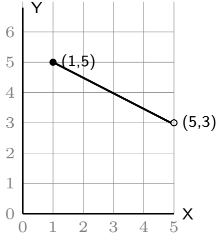

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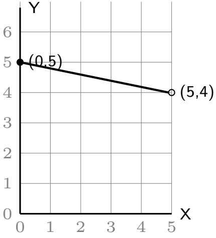

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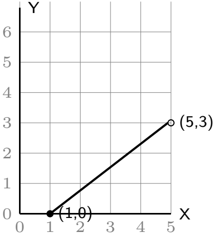

Ex. 2.1.7: Describe the graph of the equation as a set of ordered pairs. Use $\mathbf{R}$ as the symbol for all real numbers.

- $x^2 + 2y = 4 \Rightarrow$
- $y^2 = 4 \Rightarrow$
- $x^2 + y^2 = 4 \Rightarrow$
- $x^2 + 4 = 0 \Rightarrow$

Ex. 2.1.7: Describe the graph of the equation as a set of ordered pairs. Use $\mathbf{R}$ as the symbol for all real numbers.

- $x^2 + 2y = 4 \Rightarrow$ all points $\left(x, \frac{4-x^2}{2}\right)$ where x in $\mathbf{R}$
- $y^2 = 4 \Rightarrow$ all points $(x, \pm 2)$ for x in $\mathbf{R}$
- $x^2 + y^2 = 4 \Rightarrow$ all points $\left(x, \pm\sqrt{4 - x^2}\right)$ for x in $[-2, 2]$
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- $h(a + b) \Rightarrow 16 - a^2 - 2ab - b^2$
- $h(a - b) \Rightarrow 16 - a^2 + 2ab - b^2$
- $h(2x + 1) \Rightarrow -4x^2 - 4x + 15$
- $h(a^2 - b^2) \Rightarrow -a^4 + 2a^2b^2 - b^4 + 16$

Example 10: Draw the graph of the linear function showing the given domain.

$y = 4x + 7$

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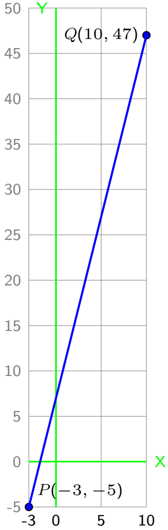

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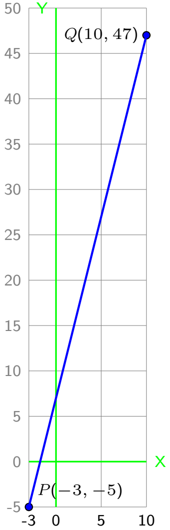

$y = 4x + 7$

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$y = 4x + 7$

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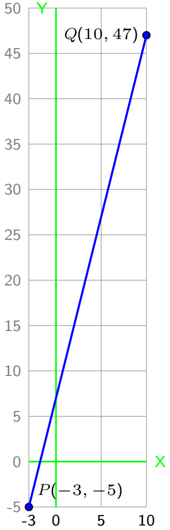
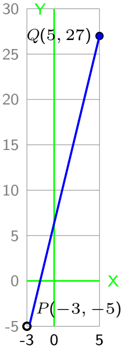


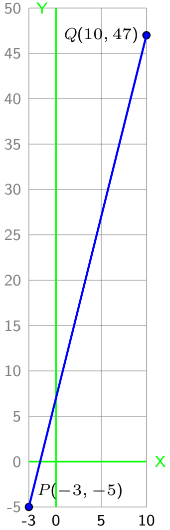
$y = 4x + 7$

$-3 < x \leq 5 \Rightarrow$



Example 10: Draw the graph of the linear function showing the given domain.

$y = 4x + 7$
$-3 \le x \le 10 \Rightarrow$



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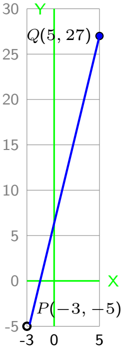

$y = -2x + 7$
$-3 < x \le 10 \Rightarrow$

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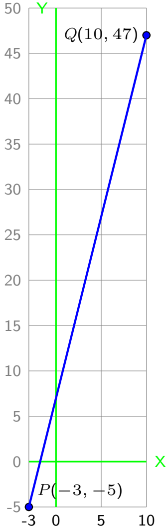

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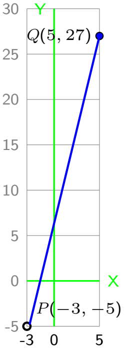

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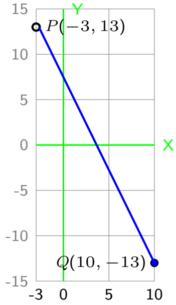

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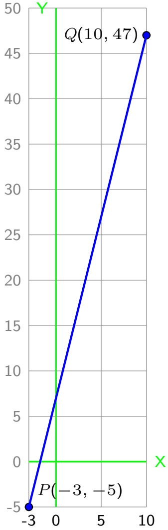

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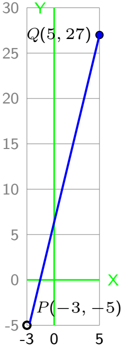

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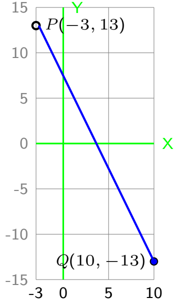

$y = 3x + 5$
$0 < x < 10 \Rightarrow$

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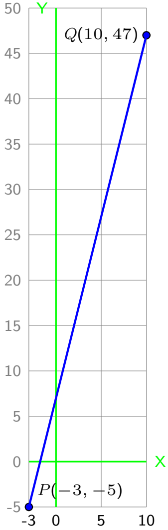
$-3 \leq x \leq 10 \Rightarrow$



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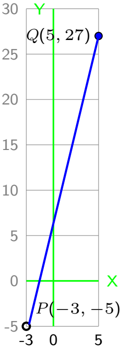

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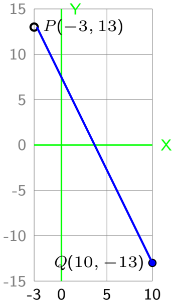

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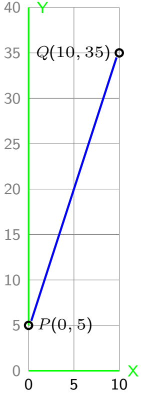

Ex. 2.1.11: Draw the part of the graph of $y = f(x)$ showing only x in $[-5, 5]$.

$y = f(x) = |x|$

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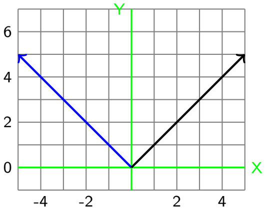
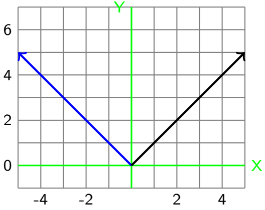


Ex. 2.1.11: Draw the part of the graph of $y = f(x)$ showing only x in $[-5, 5]$.

$y = f(x) = |x|$ $\qquad$ $y = f(x) = |x + 3|$



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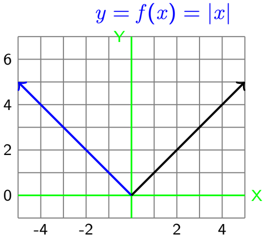

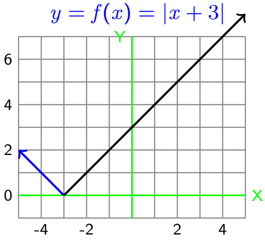

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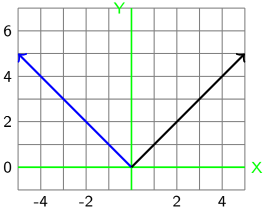

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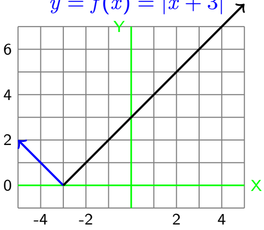

$y = f(x) = 6 - |x|$

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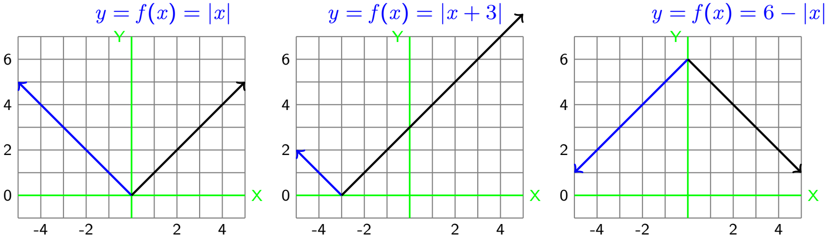

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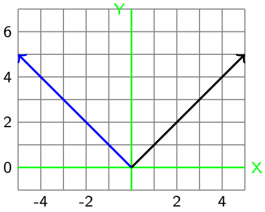

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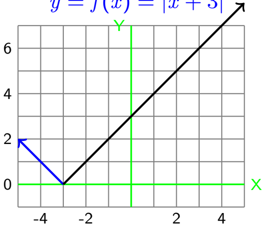

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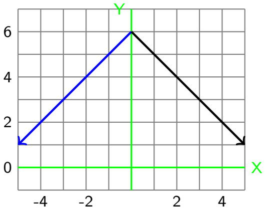

$y = f(x) = |\frac{x}{2}|$

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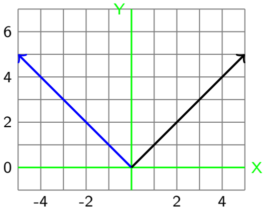

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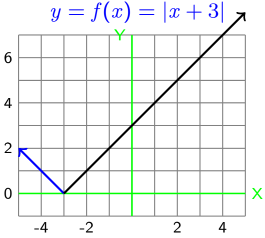

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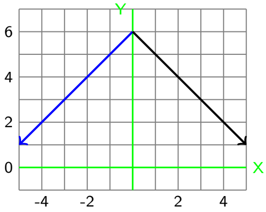

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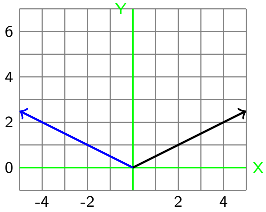

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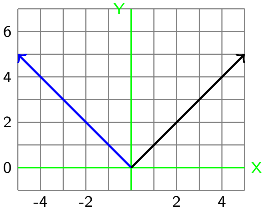

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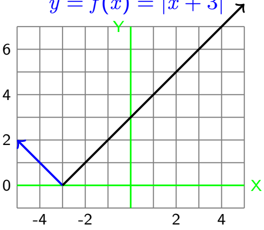

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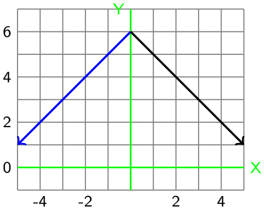

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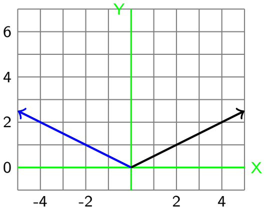


Ex. 2.1.12: Multiply out and rewrite as a polynomial.

- $x^2 + 3x(x-1) \Rightarrow 4x^2 - 3x$
- $x^2 + 3x(x-1)(x-2) \Rightarrow 3x^3 - 8x^2 + 6x$
- $x^2 + 3x(x-1)(x-2)(x-3) \Rightarrow 3x^4 - 18x^3 + 34x^2 - 18x$
- $x^2 + 3x(x-1)(x-2)(x-3)(x-4) \Rightarrow 3x^5 - 30x^4 + 105x^3 - 149x^2 + 72x$

Section 2.2: Sketching graphs

- 2.2.1: Graphing a real-life function
- 2.2.2: The effect of the choice of domain
- 2.2.3: Special features of graphs
- 2.2.4: Linear functions defined on intervals
- 2.2.5: Graphing a polynomial on an interval
- 2.2.6: Piecewise defined functions
- 2.2.7: A three-part piecewise linear graph
- 2.2.8: The vertical line test
- Section 2.2 Review

Section 2.2 Preview: Definitions and Procedures

- Definition 2.2.1:Finite intervals from a to b
- Definition 2.2.2: Vertical line test

- Procedure 2.2.1:To graph $f(x) = mx + b$ on a finite interval

2.2.1 Graphing a real-life function

If a function describes a real-life process, you can understand that process better by studying the function's graph.

Throw a ball up from ground level at time $t = 0$ and release it with velocity 64 feet per second. After t seconds, until the ball hits the ground, its height, measured in feet above ground level, is $y = h(t) = 64t - 16t^2$. The input of function h is time and its output is height.

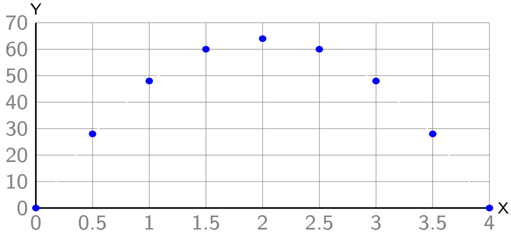
The number pair (input,output) is written in English as (time,height) and in math as $(t, h(t))$.

Since the ball is released at time $t = 0$, the function's domain starts at $t = 0$. When the ball hits the ground, it's height above ground is again 0 feet. To find when that happens, set the height function $h(t) = 64t - 16t^2$ equal to 0 and solve $64t - 16t^2 = 0 \Rightarrow 16t(4 - t) = 0$.

Thus $t = 0$ (when the ball is thrown) and $t = 4$, when it hits the ground. The domain of h is $0 \le t \le 4$.

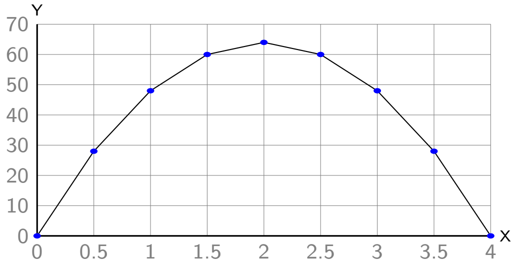
To get an idea of how the ball's height changes, make a table of values for $0 \le t \le 4$, with spacing 0.5 between t-values. Then plot the coordinate pairs $(t, h(t))$ in the right column of the table.

t	$h(t) = 64t - 16t^2$ $= 16 \cdot t(4 - t)$	$(t, h(t))$
0.0	$0 - 0 = 0$	$(0, 0)$
0.5	$32 - 4 = 28$	$(0.5, 28)$
1.0	$64 - 16 = 48$	$(1, 48)$
1.5	$96 - 36 = 60$	$(1.5, 60)$
2.0	$128 - 64 = 64$	$(2, 64)$
2.5	$16 \cdot 2.5(1.5) = 60$	$(2.5, 60)$
3.0	$16 \cdot 3(1) = 48$	$(3, 48)$
3.5	$16 \cdot 3.5(.5) = 28$	$(3.5, 28)$
4.0	$16 \cdot 4(0) = 0$	$(4, 0)$

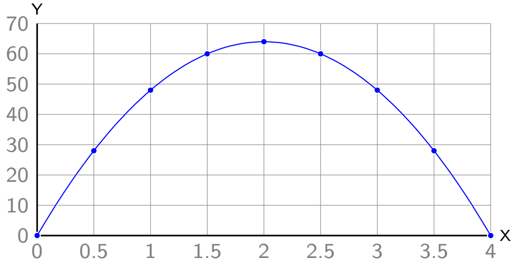


- How should we connect the dots?
- It appears that the ball reaches its maximum height of 64 feet after 2 seconds. How can we be sure?

We can connect dots with line segments by hand



or we can plot more points (using a computer) to avoid sharp corners.

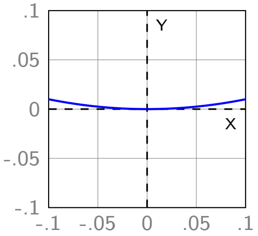


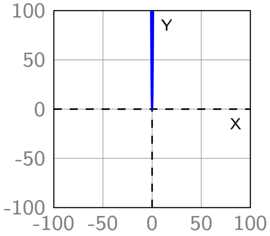
Looking at these tables and graphs suggests that the ball's maximum height is 64 feet, when $t = 2$ seconds. However, appearances can be deceiving.

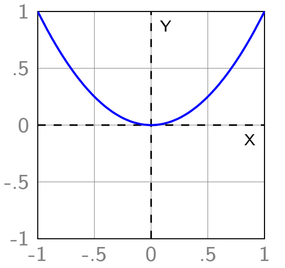
A large part of first semester calculus is devoted to understanding and sketching graphs of complicated functions. The most important features are called local maximum and minimum points of the function. These are hilltops or valley bottoms that appear on the graph of the function.

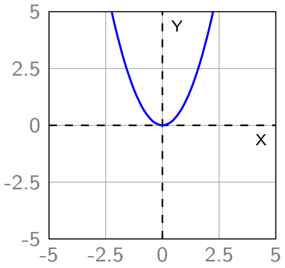
2.2.2 The effect of the choice of domain

Suppose you are asked to graph $y = f(x) = x^2$. You know that the graph should look like a bowl. Specify a viewing window, namely a domain of x-values and a range of y-values. You try some possibilities as shown below.









Neither graph above looks like a high-school sketch of $y = x^2$. The left viewing window is too small; the right window is too big. The viewing windows at the right seem better.

First semester calculus will show how to choose the viewing window intelligently.

2.2.3 Special features of graphs

Exercise. Click on ▸ Wolfram Alpha. Type in the following:
Plot $y = x\hat{}2$ for $-100 <= x <= 100$ and $-100 <= y <= 100$
Then hit Enter to see the graph. Revise the command you typed to find a good viewing window for each of the following functions.

1. $f(x) = x^2$
2. $g(x) = -x^2$
3. $h(x) = x^3$
4. $p(x) = 5000x - x^2 - 10000$
5. $q(x) = x^3 - 300x^2 - 30000x$
6. $r(x) = \sqrt{10000 - x^2 - 9000x}$

On each graph, try to locate the following features:

In Section 3.1, a procedure called "completing the square" will explain how to find the hilltop or valley bottom on the graph of any quadratic function.

For more complicated functions such as $h(x) = x^3$, which has a rising plateau at $(0, 0)$, calculus is required.

2.2.4 Drawing a graph of a linear function whose domain is an interval

Finite intervals from a to b

- $[a, b]$ includes endpoints a and b.
- (a, b) omits endpoints a and b.
- $[a, b)$ includes endpoint a but omits endpoint b.
- $(a, b]$ includes endpoint b but omits endpoint a.

To graph $f(x) = mx + b$ on a finite interval

- For each interval endpoint x, plot point $(x, f(x))$
  - as a filled dot if the endpoint is included in the interval; or
  - as a hollow dot if the endpoint is omitted (not included) in the interval.
- Connect the endpoints with a line segment.

Example 1: Draw the graph of $y = f(x) = x + 1$ with domain $-3 < x \leq 1$.

Solution:
Click through the steps while you watch the graph:

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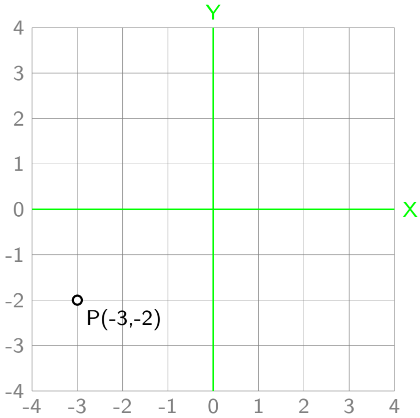
- Since left endpoint $x = -3$ is not in the interval, plot point $(-3, f(-3)) = P(-3, -2)$ as a hollow dot.



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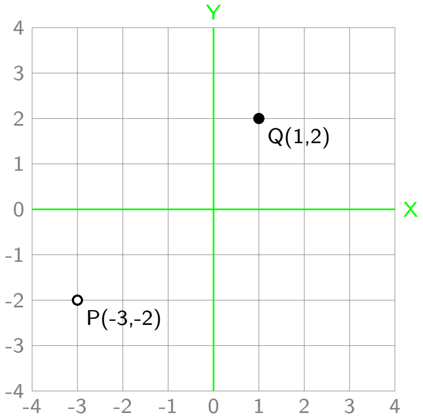
- Since right endpoint $x = 1$ is in the interval, plot point $(1, f(1)) = Q(1, 2)$ as a solid dot.



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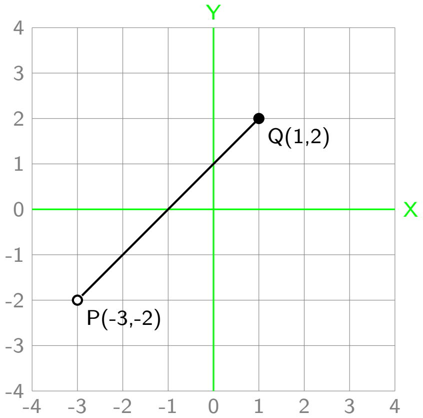
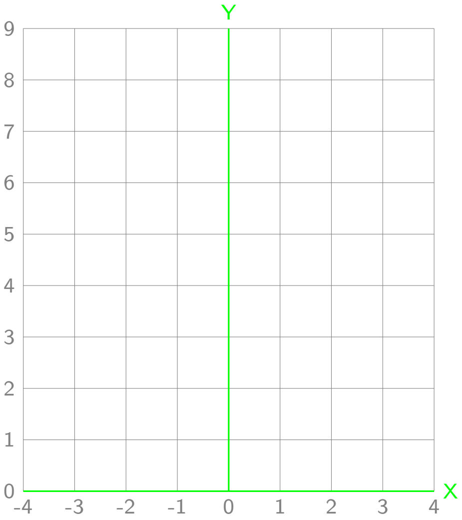
- Since right endpoint $x = 1$ is in the interval, plot point $(1, f(1)) = Q(1, 2)$ as a solid dot.
- Draw the line segment from P to Q.



2.2.5 Graphing of a polynomial function on an interval

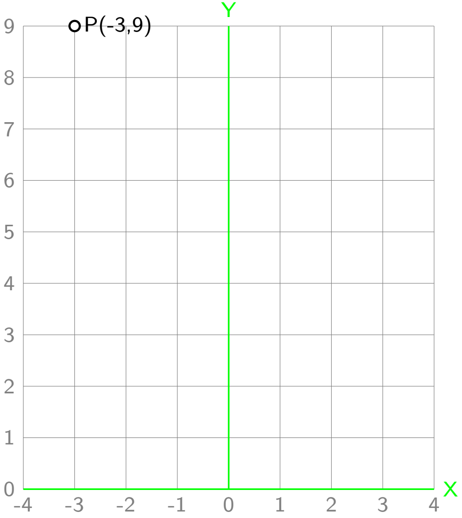
Example 2: Draw the graph of $y = f(x) = x^2$ with domain $-3 < x \leq 1$.



Solution:

2.2.5 Graphing of a polynomial function on an interval

Example 2: Draw the graph of $y = f(x) = x^2$ with domain $-3 < x \leq 1$.



Solution:

- When $x = -3, y = f(x) = x^2 = 9$. The left endpoint of the graph (shown as a hole since $x = -3$ is missing from the domain) is the point $P(-3, 9)$.

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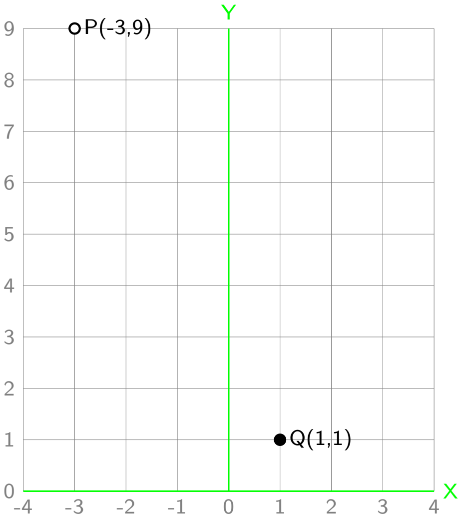

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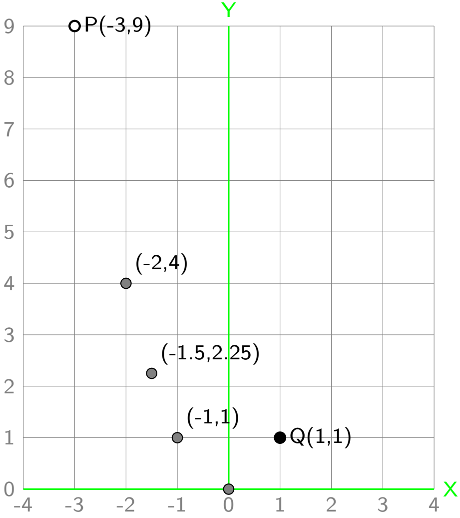

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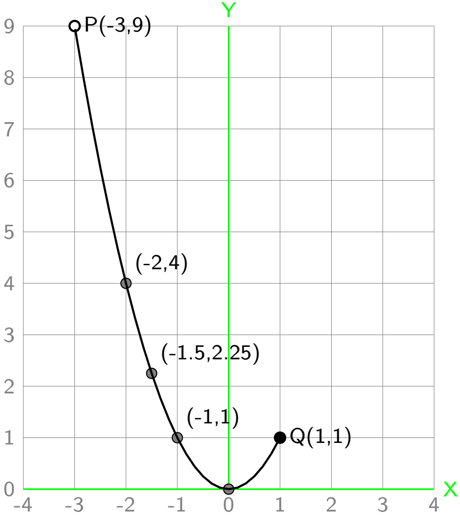

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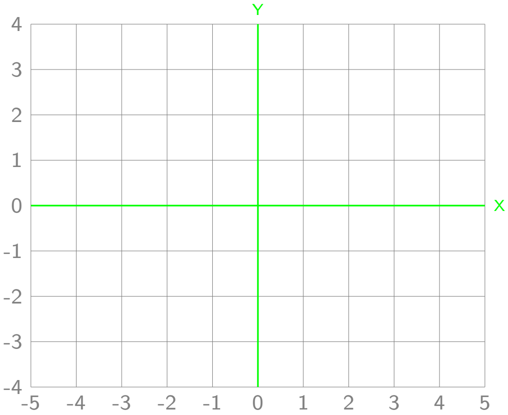
- To finish the graph, fill in other points, some with non-integer x-values: $(0, 0), (-1, 1), (-1.5, 2.25), (-2, 4)$.
- Draw a smooth curve, which you should recognize as the graph of part of a parabola.

2.2.6 Piecewise defined functions

Sometimes a function is defined piecewise, using different formulas on different intervals. The simplest example is the absolute value function, sketched in Section 2.1.6.

To draw the graph of any piecewise defined function, plot all pieces on the same grid.

Example 3: draw the graph of $f(x) = \begin{cases} x+1 & \text{if } -3 < x < 1 \\ 2-x & \text{if } 1 \le x < 4 \end{cases}$



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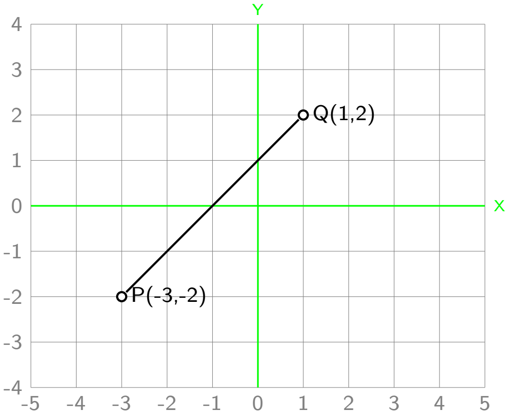

Solution:

- $f(x) = x + 1$ for $-3 < x < 1$ is a line segment from (but omitting) $(-3, f(3)) = P(-3, -2)$ to (but omitting) point $(1, f(1)) = Q(1, 2)$, as shown at the left.

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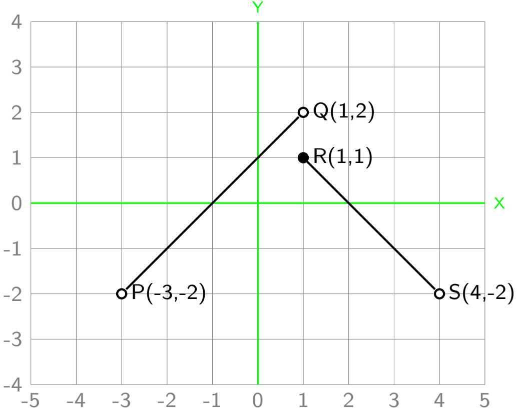

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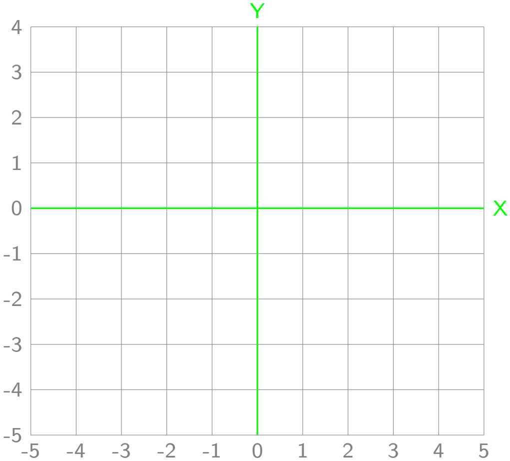
- $f(x) = 2 - x$ for $1 \leq x < 4$ is a line segment from (and including) $(1, f(1)) = R(1, 1)$ to (but omitting) point $(4, f(4)) = S(4, -2)$. This completes the graph.

The next example requires drawing a half-line with domain the infinite interval $(-\infty, 2)$.



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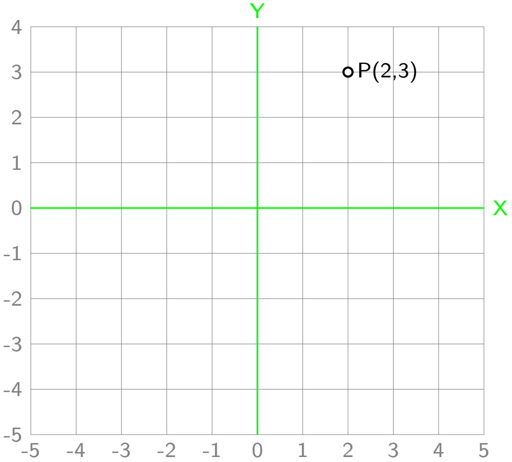

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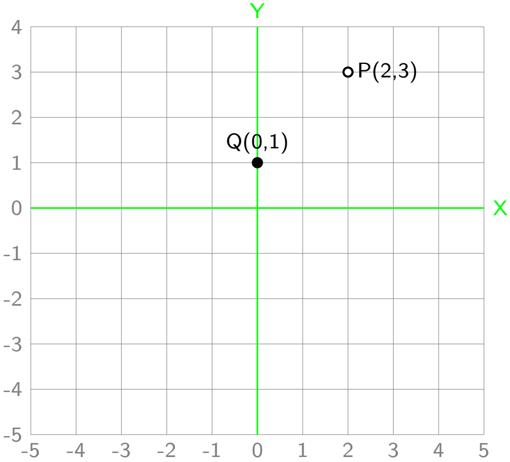

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- Plot any other point on the grid to the left of P. For example, choose $x = 0$ to obtain point $Q(0, 1)$. Plot this point as a filled dot: it is on the graph.

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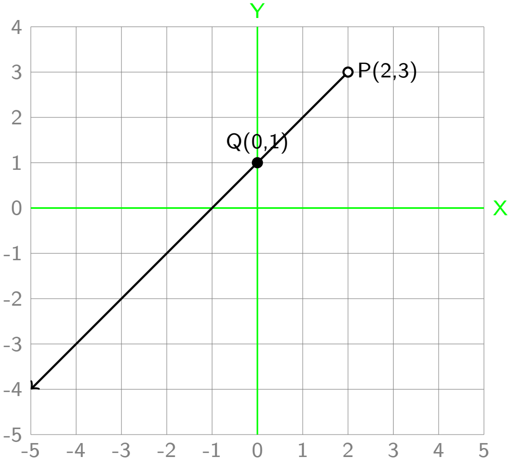

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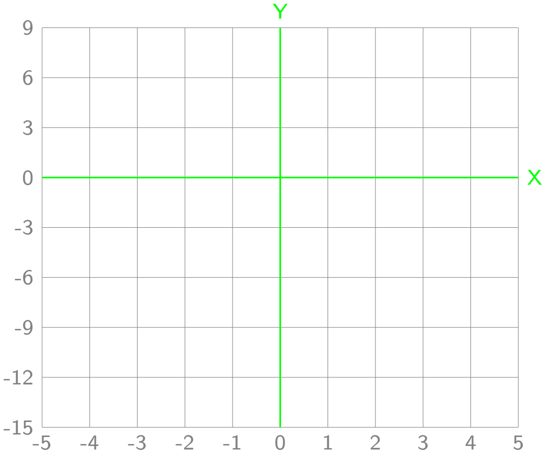
- Plot any other point on the grid to the left of P. For example, choose $x = 0$ to obtain point $Q(0, 1)$. Plot this point as a filled dot: it is on the graph.
- Draw the line from P through Q and extending to the edge of the grid. Put an arrowhead at that edge point to indicate that the line continues forever to the left.

2.2.7 A three-part piecewise linear graph

Example 5: Draw the graph of $f(x) = \begin{cases} 3x & \text{if } x \le -2 \\ x+1 & \text{if } -2 < x \le 1 \\ 7 & \text{if } 1 < x \end{cases}$



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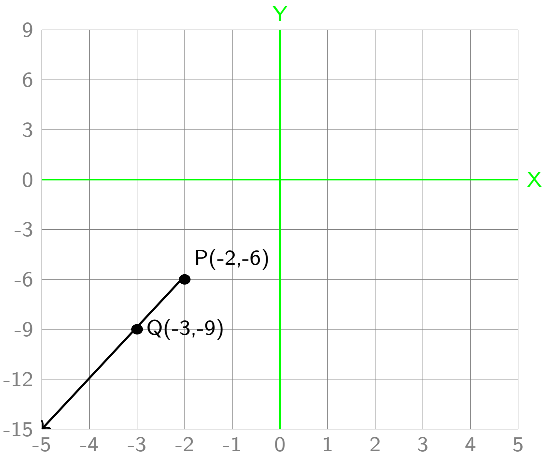

Solution:

- $y = 3x$ for $x \leq -2$ starts at (and includes) $P(-2,-6)$. Then choose another point with x-coordinate satisfying $x \leq -2$. Choose $x = -3$ to get point $Q(-3,-9)$. Draw the line from P to Q to an arrowhead at the edge (in this case, the corner) of the grid.

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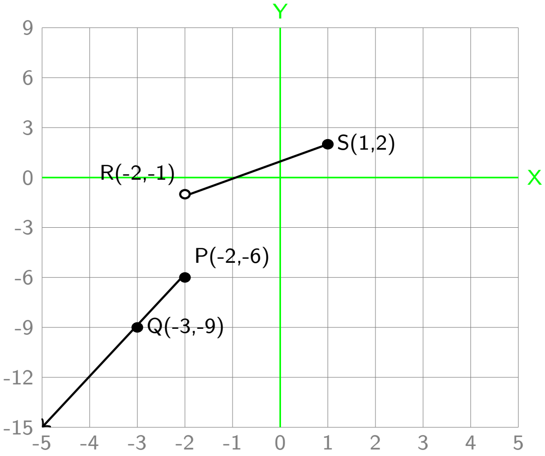

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- $y = x + 1$ for $-2 < x \leq 1$ goes from (but omits) point $R(-2, -1)$ to (and includes) point $S(1, 2)$.

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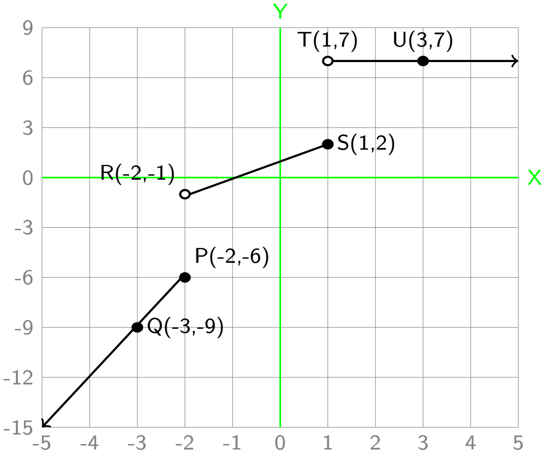

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- $y = x + 1$ for $-2 < x \le 1$ goes from (but omits) point $R(-2, -1)$ to (and includes) point $S(1, 2)$.
- $y = 7$ for $1 < x$ starts at (but omits) point $T(1, 7)$. Then choose another point with x-coordinate satisfying $1 < x$. Choose $x = 3$ to get point $U(3, 7)$. Draw the line from T to U to an arrowhead at the right edge of the grid.

To show details well, we chose a coordinate system in which each square box is 1 x-unit wide and 3 y-units high.

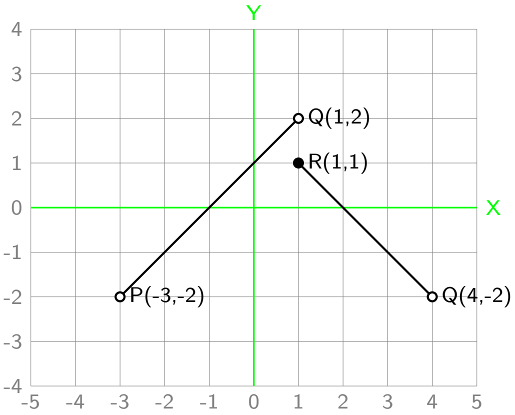
2.2.8 The vertical line test

Vertical line test

A graph is the graph of a function provided every vertical line meets the graph once or never.

Below is the graph of $f(x) = \begin{cases} x+1 & \text{if } -3 < x < 1 \\ 2-x & \text{if } 1 \leq x < 4 \end{cases}$ from Example 3.

Example 6: Use the vertical line test to decide if the graph below is the graph of a function.



Solution:

- For any x other than $x = 1$, it's clear that the vertical line through point $(x, 0)$ meets the graph at one point or not at all.

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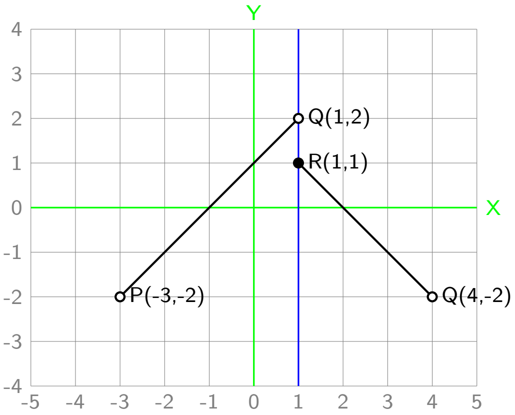

Solution:

- For any x other than $x = 1$, it's clear that the vertical line through point $(x, 0)$ meets the graph at one point or not at all.
- The only tricky part is to check the line $x = 1$, which passes through two locations on the grid. Point $R(1, 1)$ is on the graph, but $Q(1, 2)$ is not. Therefore the vertical blue line $x = 1$ meets the graph at the single point $Q(1, 2)$. That's fine.

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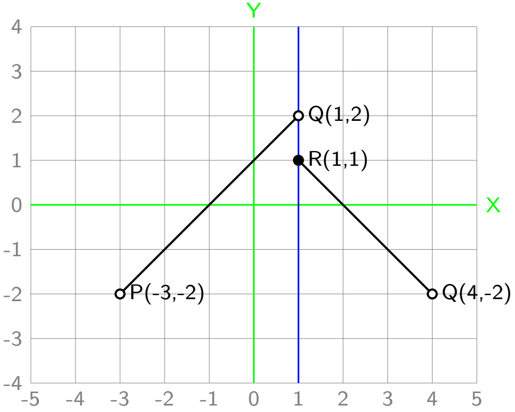

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- Every vertical line meets the graph once or never. Therefore the graph passes the vertical line test.

The graph is the graph of a function.

2.2.9 Precalculus Section 2.2 Quiz

Ex. 2.2.1: Draw the graph of $y = f(x) = x + 1$ with domain $-3 < x \leq 1$.

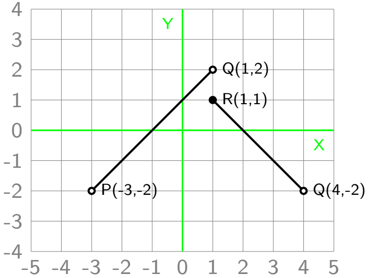
Ex. 2.2.2: Draw the graph of $y = f(x) = x^2$ with domain $-3 < x \leq 1$.

Ex. 2.2.3: Draw the graph of $f(x) = \begin{cases} x+1 & \text{if } -3 < x < 1 \\ 2-x & \text{if } 1 \leq x < 4 \end{cases}$

Ex. 2.2.4: Draw the graph of $y = x + 1$ for $x < 2$.

Ex. 2.2.5: Draw the graph of $f(x) = \begin{cases} 3x & \text{if } x \leq -2 \\ x+1 & \text{if } -2 < x \leq 1 \\ 7 & \text{if } 1 < x \end{cases}$

Ex. 2.2.6: Use the vertical line test to decide if the graph below is the graph of a function.



Section 2.2 Review: Graphing functions

Ex. 2.2.1: Draw the graph f with requested domain.

$y = f(x) = x + 1;$
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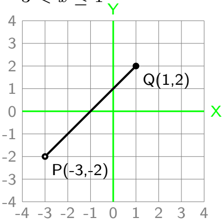

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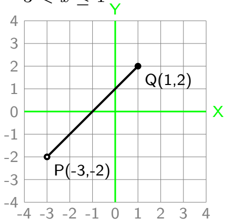

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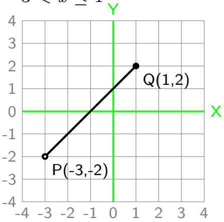

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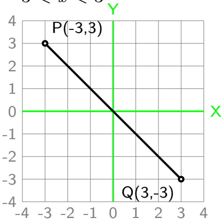

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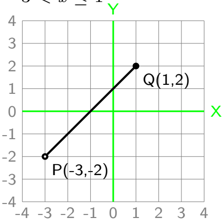

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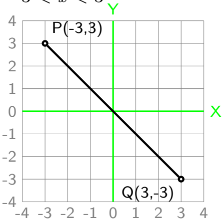

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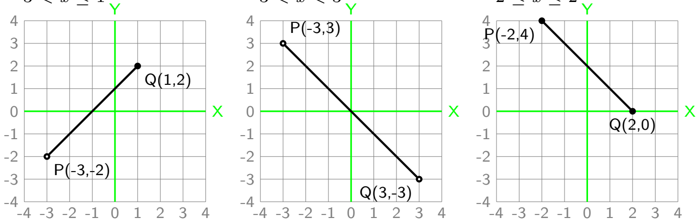

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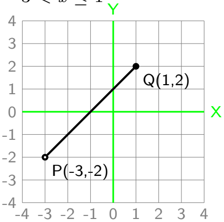

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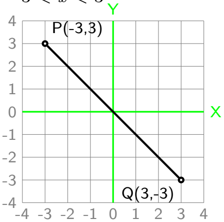

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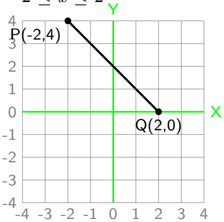

$y = f(x) = \frac{x}{2} + 1$; $-4 < x \leq 4$

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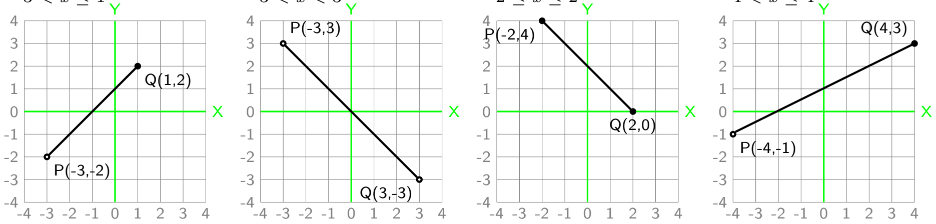

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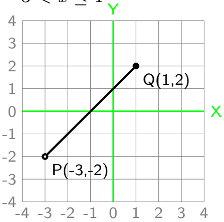

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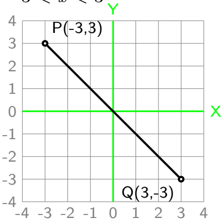

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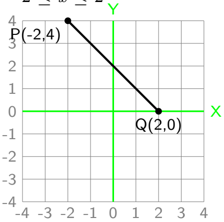

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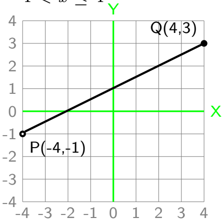

Ex. 2.2.2: Draw the graph of f with specified domain.

$y = f(x) = x^2$
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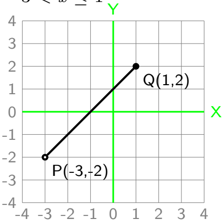

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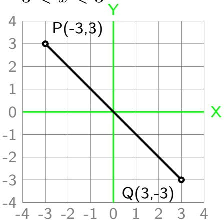
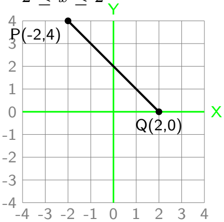
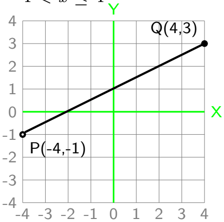


$y = f(x) = 2 - x;$
$-2 \le x \le 2$

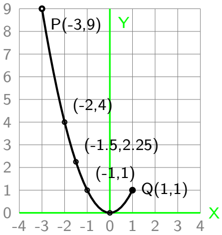


$y = f(x) = \frac{x}{2} + 1;$
$-4 < x \le 4$



Ex. 2.2.2: Draw the graph of f with specified domain.

$y = f(x) = x^2$
$-3 < x \le 1.$



Section 2.2 Review: Graphing functions

Ex. 2.2.1: Draw the graph f with requested domain.

$y = f(x) = x + 1;$
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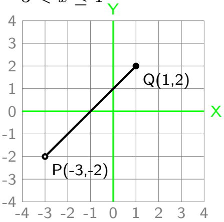

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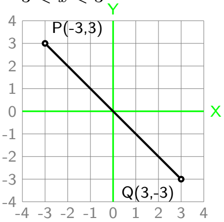
$-3 < x < 3$



$y = f(x) = 2 - x;$
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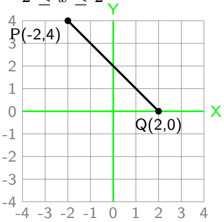

$y = f(x) = \frac{x}{2} + 1;$
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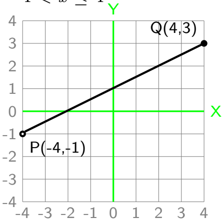

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$y = f(x) = x^2$
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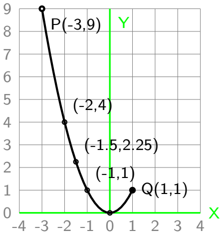
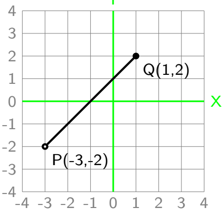


$y = f(x) = 8 - \frac{x^2}{2}$
$-4 < x < 4$

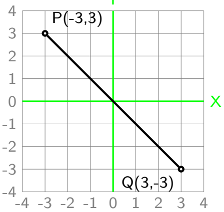
Section 2.2 Review: Graphing functions

Ex. 2.2.1: Draw the graph f with requested domain.

$y = f(x) = x + 1;$
$-3 < x \leq 1$



$y = f(x) = -x;$
$-3 < x < 3$



$y = f(x) = 2 - x;$
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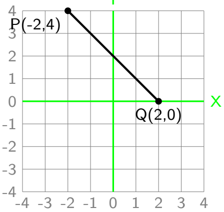

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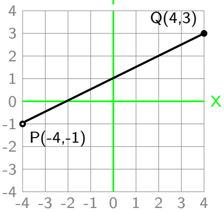
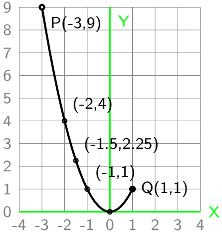
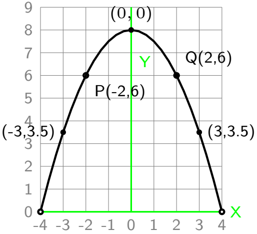


Ex. 2.2.2: Draw the graph of f with specified domain.

$y = f(x) = x^2$
$-3 < x \leq 1.$



$y = f(x) = 8 - \frac{x^2}{2}$
$-4 < x < 4$



Section 2.2 Review: Graphing functions

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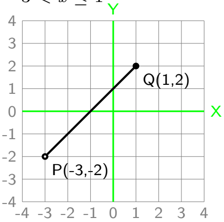

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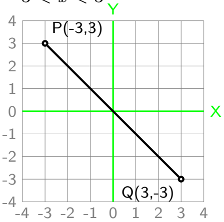

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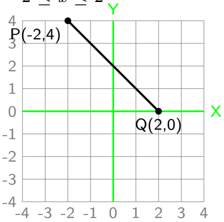

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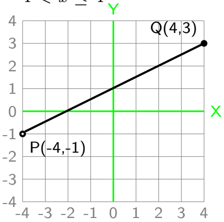

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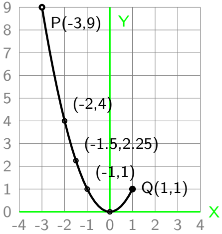
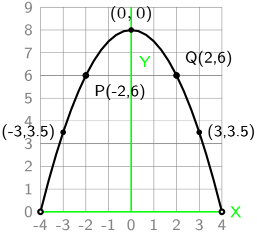


$y = f(x) = 8 - \frac{x^2}{2}$
$-4 < x < 4$

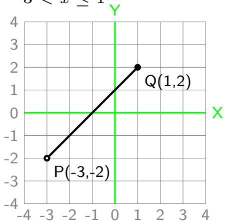


$y = f(x) = \frac{x^2}{9} + x + 3$
$-3 < x \leq 3$

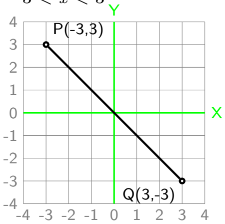
Section 2.2 Review: Graphing functions

Ex. 2.2.1: Draw the graph f with requested domain.

$y = f(x) = x + 1$; $-3 < x \le 1$



$y = f(x) = -x$; $-3 < x < 3$



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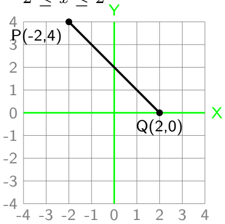
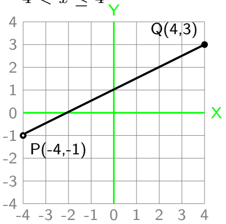
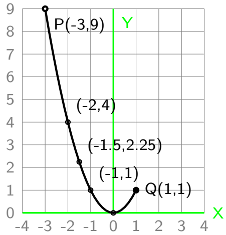


$y = f(x) = \frac{x}{2} + 1$; $-4 < x \le 4$



Ex. 2.2.2: Draw the graph of f with specified domain.

$y = f(x) = x^2$
$-3 < x \le 1.$



$y = f(x) = 8 - \frac{x^2}{2}$
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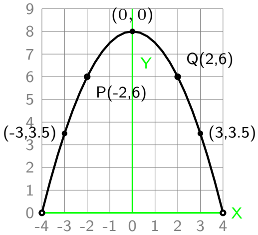

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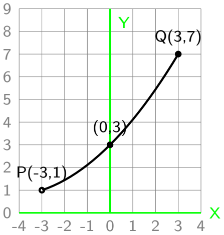

Section 2.2 Review: Graphing functions

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$y = f(x) = x + 1;$
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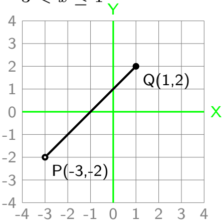

$y = f(x) = -x;$
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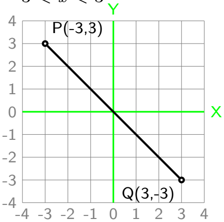

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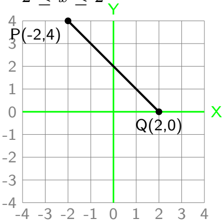

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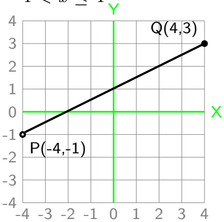

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$y = f(x) = x^2$
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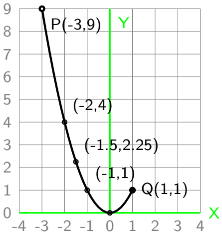
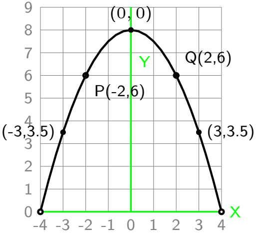
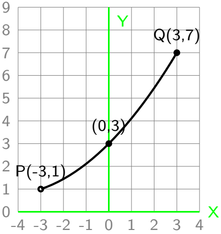


$y = f(x) = 8 - \frac{x^2}{2}$
$-4 < x < 4$



$y = f(x) = \frac{x^2}{9} + x + 3$
$-3 < x \leq 3$



$y = f(x) = 9 - \frac{x^2}{4}$
$-4 \leq x \leq 4$

Section 2.2 Review: Graphing functions

Ex. 2.2.1: Draw the graph f with requested domain.

$y = f(x) = x + 1;$
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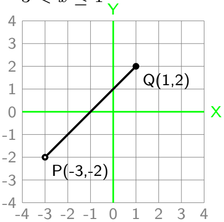

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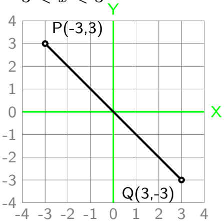

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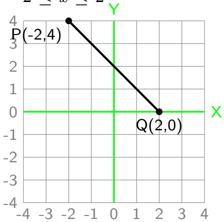

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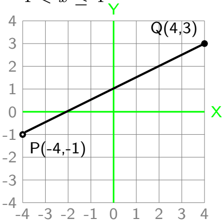

Ex. 2.2.2: Draw the graph of f with specified domain.

$y = f(x) = x^2$
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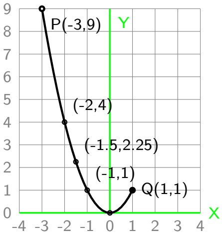

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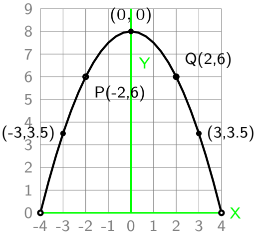

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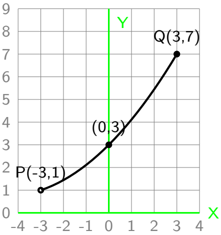

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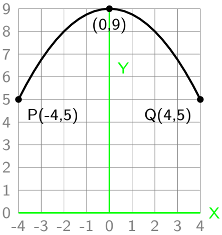

Ex. 2.2.3: Draw the graph of $f(x) =$

$$\begin{cases} x+1 & \text{if } -3 < x < 1 \\ 2-x & \text{if } 1 \le x < 4 \end{cases}$$

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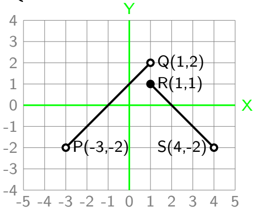
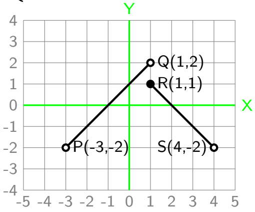


Ex. 2.2.3: Draw the graph of $f(x) =$

$$\begin{cases} x+1 & \text{if } -3 < x < 1 \\ 2-x & \text{if } 1 \le x < 4 \end{cases} \qquad \begin{cases} -4x-12 & \text{if } -4 < x < -2 \\ x & \text{if } -2 \le x < 4 \end{cases}$$



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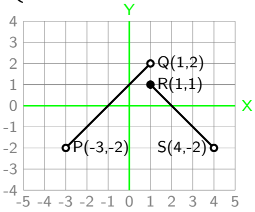

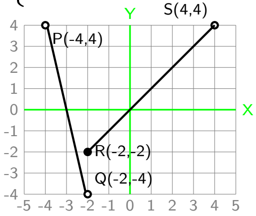

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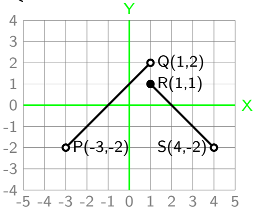
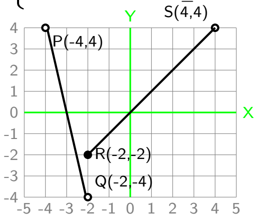
$$\begin{cases} -2x-4 & \text{if } -4 \le x \le 0 \\ -2x+4 & \text{if } 0 < x < 4 \end{cases}$$





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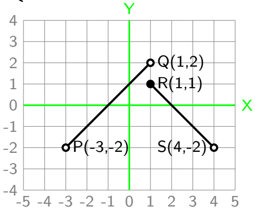

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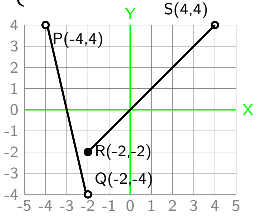
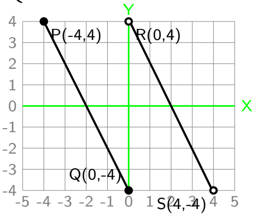


$$\begin{cases} -2x-4 & \text{if } -4 \leq x \leq 0 \\ -2x+4 & \text{if } 0 < x < 4 \end{cases}$$



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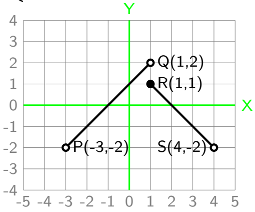
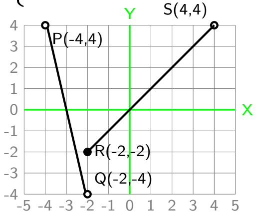
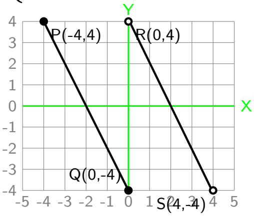


$$\begin{cases} -4x-12 & \text{if } -4 < x < -2 \\ x & \text{if } -2 \le x < 4 \end{cases}$$



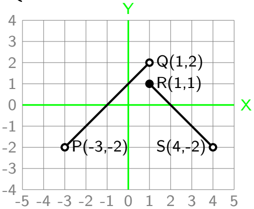
$$\begin{cases} -2x-4 & \text{if } -4 \le x \le 0 \\ -2x+4 & \text{if } 0 < x < 4 \end{cases}$$



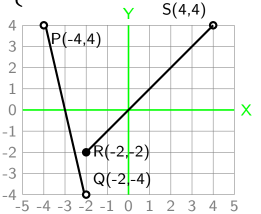
$$\begin{cases} 1 & \text{if } -3 < x < 1 \\ -3 & \text{if } 1 \le x < 4 \end{cases}$$

Ex. 2.2.3: Draw the graph of $f(x) =$

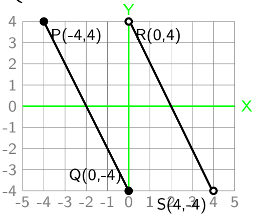
$$\begin{cases} x+1 & \text{if } -3 < x < 1 \\ 2-x & \text{if } 1 \le x < 4 \end{cases}$$



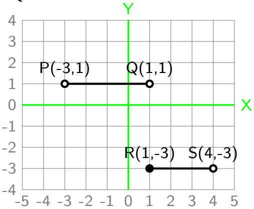
$$\begin{cases} -4x-12 & \text{if } -4 < x < -2 \\ x & \text{if } -2 \le x < 4 \end{cases}$$



$$\begin{cases} -2x-4 & \text{if } -4 \le x \le 0 \\ -2x+4 & \text{if } 0 < x < 4 \end{cases}$$



$$\begin{cases} 1 & \text{if } -3 < x < 1 \\ -3 & \text{if } 1 \le x < 4 \end{cases}$$



Ex. 2.2.3: Draw the graph of $f(x) =$

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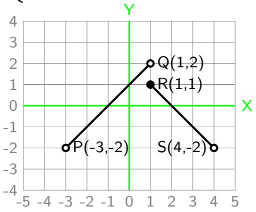
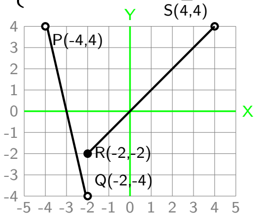
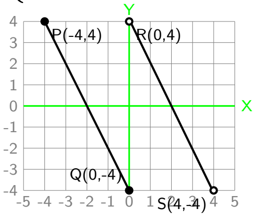


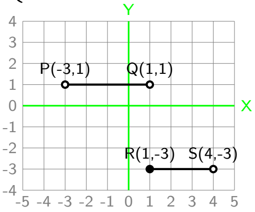
$\begin{cases} -4x-12 & \text{if } -4 < x < -2 \\ x & \text{if } -2 \leq x < 4 \end{cases}$



$\begin{cases} -2x-4 & \text{if } -4 \leq x \leq 0 \\ -2x+4 & \text{if } 0 < x < 4 \end{cases}$



$\begin{cases} 1 & \text{if } -3 < x < 1 \\ -3 & \text{if } 1 \leq x < 4 \end{cases}$

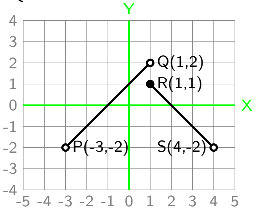


Ex. 2.2.4: Draw the graph of the half-line. Draw a point Q on it.

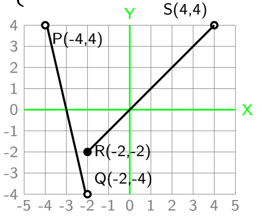
$y = x + 1$ for $x < 2$

Ex. 2.2.3: Draw the graph of $f(x) =$

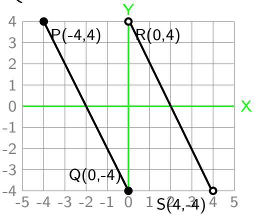
$\begin{cases} x+1 & \text{if } -3 < x < 1 \\ 2-x & \text{if } 1 \leq x < 4 \end{cases}$



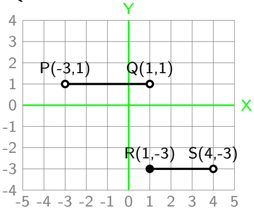
$\begin{cases} -4x-12 & \text{if } -4 < x < -2 \\ x & \text{if } -2 \leq x < 4 \end{cases}$



$\begin{cases} -2x-4 & \text{if } -4 \leq x \leq 0 \\ -2x+4 & \text{if } 0 < x < 4 \end{cases}$

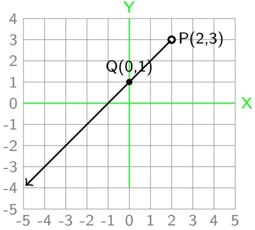


$\begin{cases} 1 & \text{if } -3 < x < 1 \\ -3 & \text{if } 1 \leq x < 4 \end{cases}$



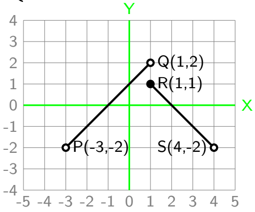
Ex. 2.2.4: Draw the graph of the half-line. Draw a point Q on it.

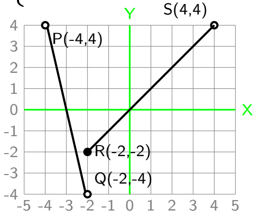
$y = x + 1$ for $x < 2$

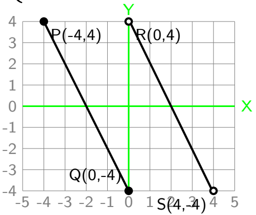


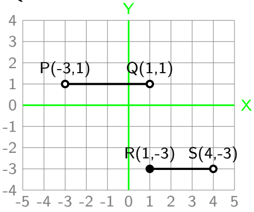
Ex. 2.2.3: Draw the graph of $f(x) =$

$\begin{cases} x+1 & \text{if } -3 < x < 1 \\ 2-x & \text{if } 1 \le x < 4 \end{cases}$ $\begin{cases} -4x-12 & \text{if } -4 < x < -2 \\ x & \text{if } -2 \le x < 4 \end{cases}$ $\begin{cases} -2x-4 & \text{if } -4 \le x \le 0 \\ -2x+4 & \text{if } 0 < x < 4 \end{cases}$ $\begin{cases} 1 & \text{if } -3 < x < 1 \\ -3 & \text{if } 1 \le x < 4 \end{cases}$



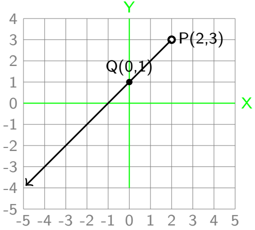






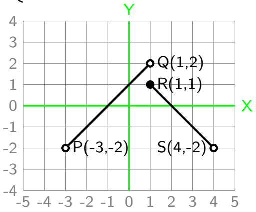
Ex. 2.2.4: Draw the graph of the half-line. Draw a point Q on it.

$y = x + 1$ for $x < 2$ $\qquad$ $y = \frac{1}{2}x + 1$ for $x \ge -3$

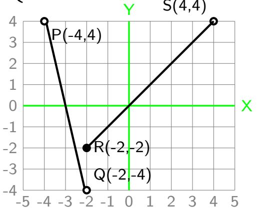


Ex. 2.2.3: Draw the graph of $f(x) =$

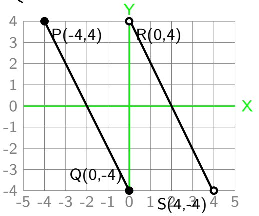
$$\begin{cases} x+1 & \text{if } -3 < x < 1 \\ 2-x & \text{if } 1 \leq x < 4 \end{cases}$$



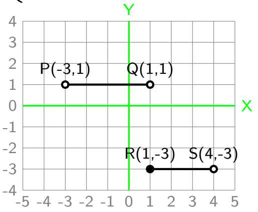
$$\begin{cases} -4x-12 & \text{if } -4 < x < -2 \\ x & \text{if } -2 \leq x < 4 \end{cases}$$



$$\begin{cases} -2x-4 & \text{if } -4 \leq x \leq 0 \\ -2x+4 & \text{if } 0 < x < 4 \end{cases}$$

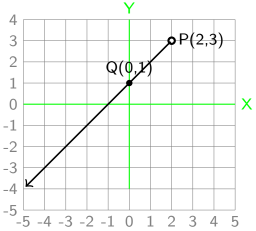


$$\begin{cases} 1 & \text{if } -3 < x < 1 \\ -3 & \text{if } 1 \leq x < 4 \end{cases}$$

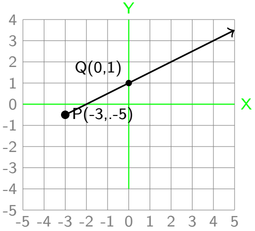


Ex. 2.2.4: Draw the graph of the half-line. Draw a point Q on it.

$y = x + 1$ for $x < 2$

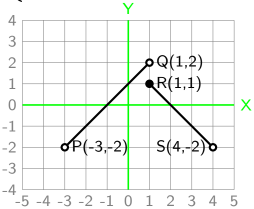


$y = \frac{1}{2}x + 1$ for $x \geq -3$

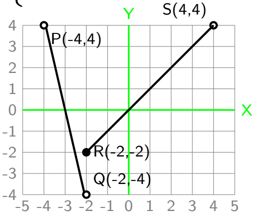


Ex. 2.2.3: Draw the graph of $f(x) =$

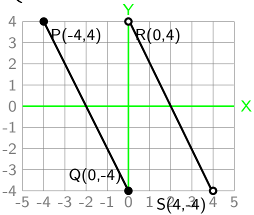
$\begin{cases} x+1 & \text{if } -3 < x < 1 \\ 2-x & \text{if } 1 \le x < 4 \end{cases}$



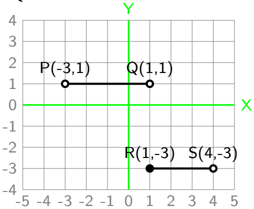
$\begin{cases} -4x-12 & \text{if } -4 < x < -2 \\ x & \text{if } -2 \le x < 4 \end{cases}$



$\begin{cases} -2x-4 & \text{if } -4 \le x \le 0 \\ -2x+4 & \text{if } 0 < x < 4 \end{cases}$



$\begin{cases} 1 & \text{if } -3 < x < 1 \\ -3 & \text{if } 1 \le x < 4 \end{cases}$



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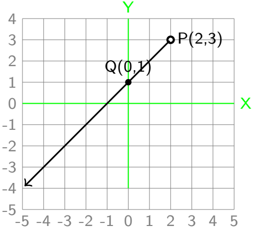
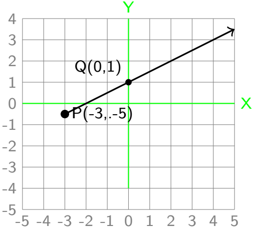
$y = x + 1$ for $x < 2$



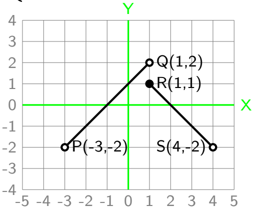
$y = \frac{1}{2}x + 1$ for $x \ge -3$



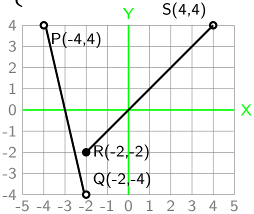
$y = x + 1$ for $x > -5$

Ex. 2.2.3: Draw the graph of $f(x) =$

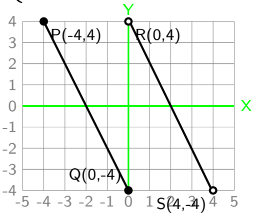
$\begin{cases} x+1 & \text{if } -3 < x < 1 \\ 2-x & \text{if } 1 \le x < 4 \end{cases}$



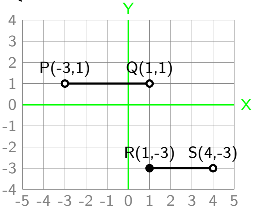
$\begin{cases} -4x-12 & \text{if } -4 < x < -2 \\ x & \text{if } -2 \le x < 4 \end{cases}$



$\begin{cases} -2x-4 & \text{if } -4 \le x \le 0 \\ -2x+4 & \text{if } 0 < x < 4 \end{cases}$

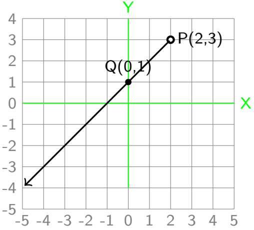


$\begin{cases} 1 & \text{if } -3 < x < 1 \\ -3 & \text{if } 1 \le x < 4 \end{cases}$

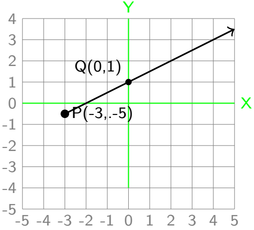


Ex. 2.2.4: Draw the graph of the half-line. Draw a point Q on it.

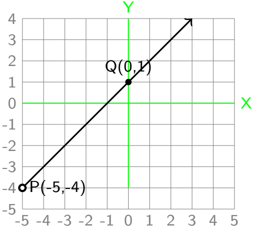
$y = x + 1$ for $x < 2$



$y = \frac{1}{2}x + 1$ for $x \ge -3$

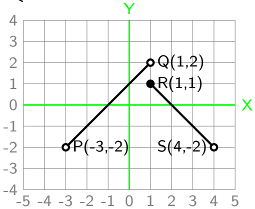


$y = x + 1$ for $x > -5$



Ex. 2.2.3: Draw the graph of $f(x) =$

$\begin{cases} x+1 & \text{if } -3 < x < 1 \\ 2-x & \text{if } 1 \le x < 4 \end{cases}$



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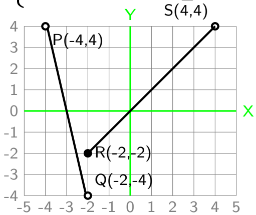

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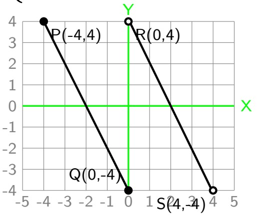
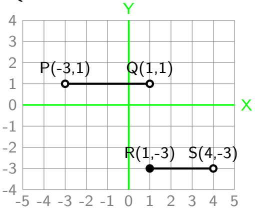
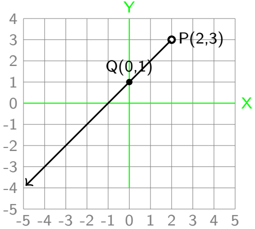


$\begin{cases} 1 & \text{if } -3 < x < 1 \\ -3 & \text{if } 1 \le x < 4 \end{cases}$

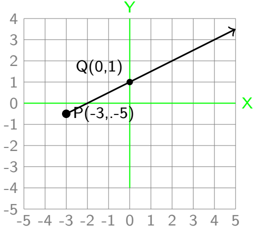


Ex. 2.2.4: Draw the graph of the half-line. Draw a point Q on it.

$y = x + 1$ for $x < 2$



$y = \frac{1}{2}x + 1$ for $x \ge -3$



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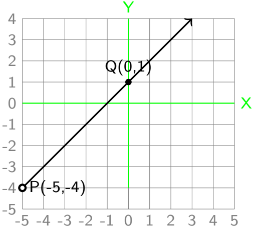
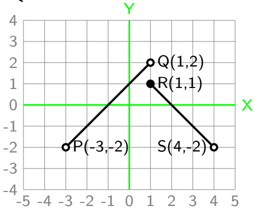


$y = 1$ for $x > -2$

Ex. 2.2.3: Draw the graph of $f(x) =$

$\begin{cases} x+1 & \text{if } -3 < x < 1 \\ 2-x & \text{if } 1 \le x < 4 \end{cases}$



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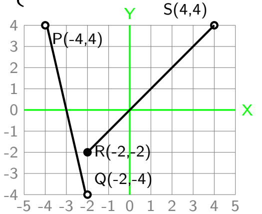
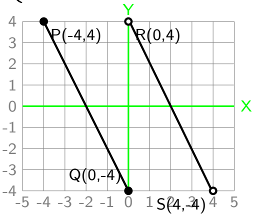
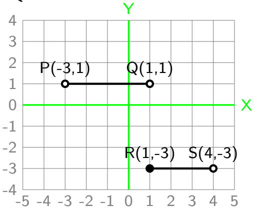


$\begin{cases} -2x-4 & \text{if } -4 \le x \le 0 \\ -2x+4 & \text{if } 0 < x < 4 \end{cases}$



$\begin{cases} 1 & \text{if } -3 < x < 1 \\ -3 & \text{if } 1 \le x < 4 \end{cases}$



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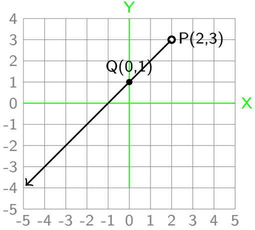
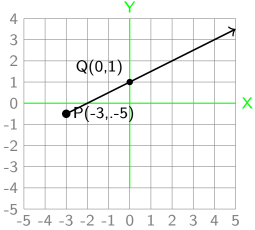
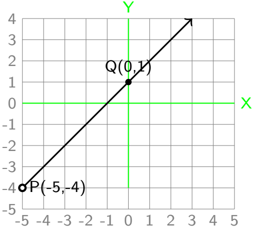
$y = x + 1$ for $x < 2$



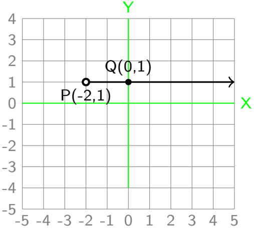
$y = \frac{1}{2}x + 1$ for $x \ge -3$



$y = x + 1$ for $x > -5$

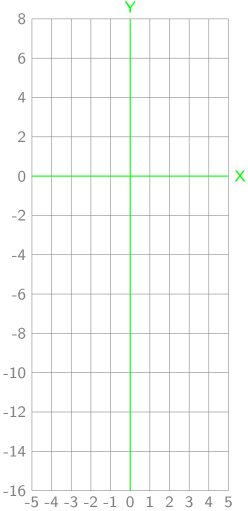


$y = 1$ for $x > -2$



Ex. 2.2.5: Draw the graph of $y = f(x)$ defined below, then click slowly to see how to draw piecewise linear graphs.

$$y = \begin{cases} 3x & \text{if } x \leq -2 \\ x+1 & \text{if } -2 < x \leq 1 \\ 7 & \text{if } 1 < x \end{cases}$$



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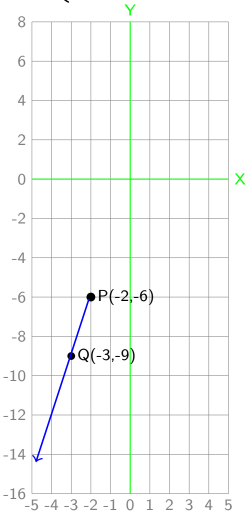

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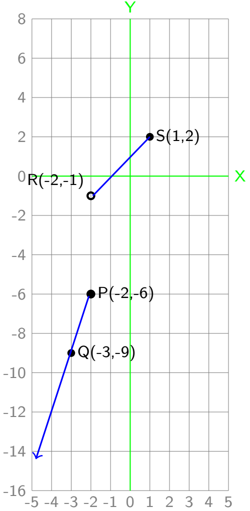

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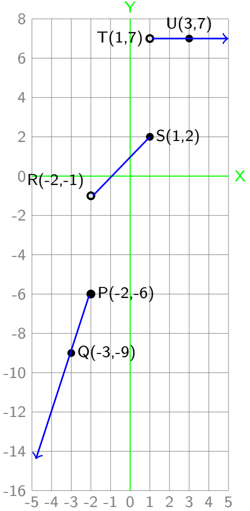

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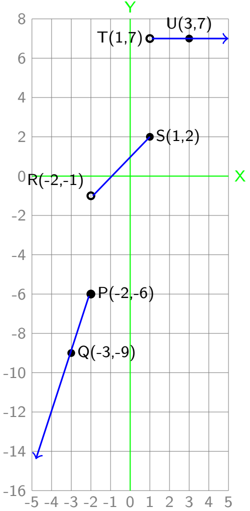

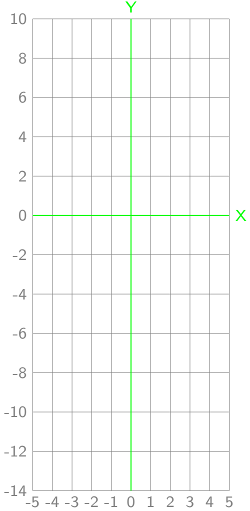

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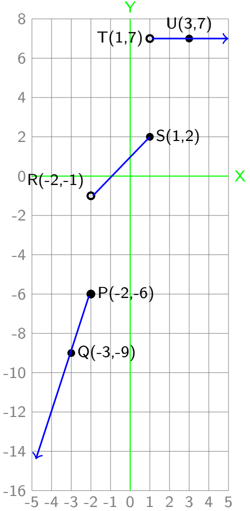

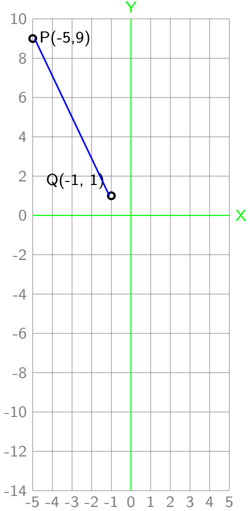

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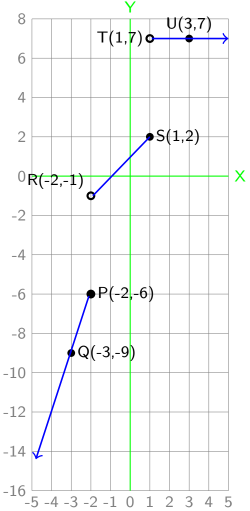

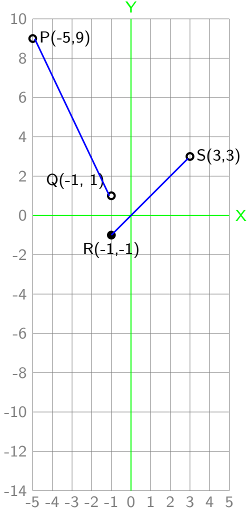

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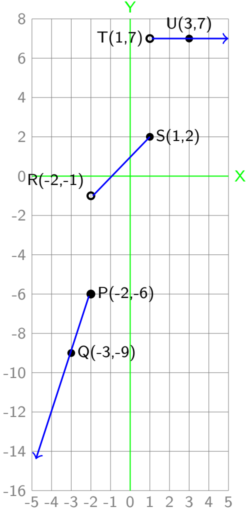

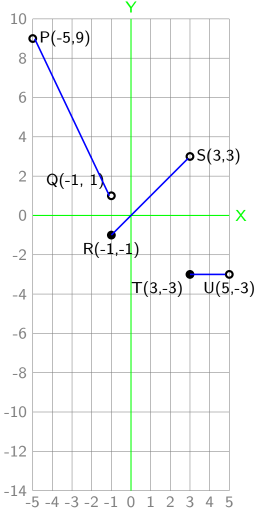

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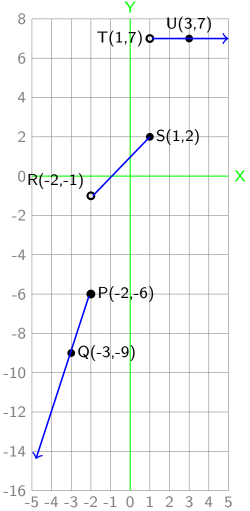

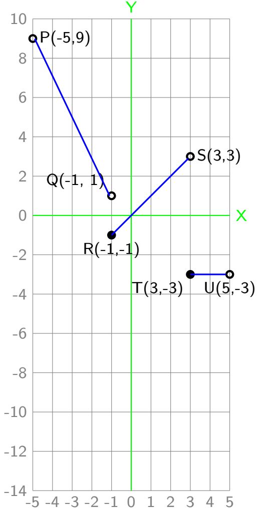

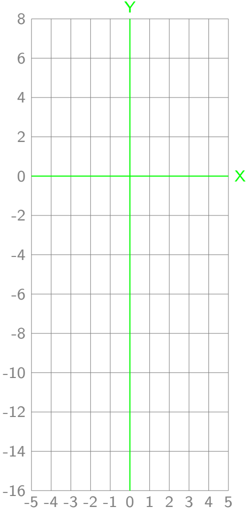

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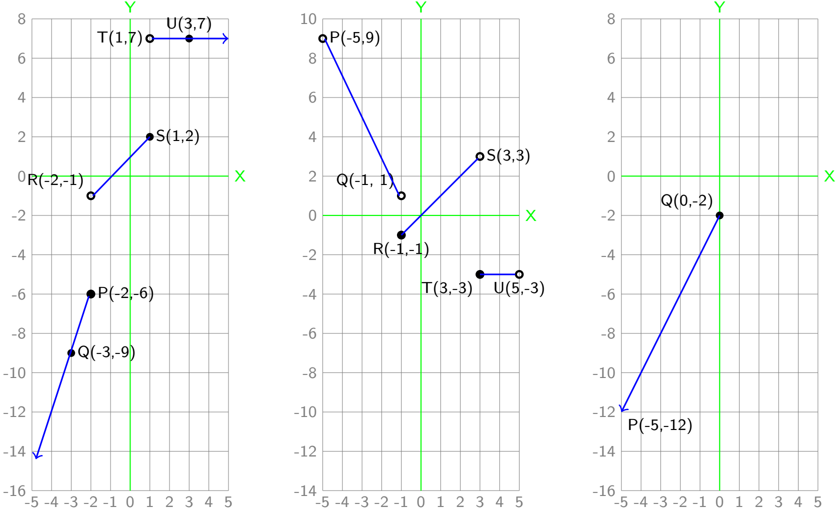

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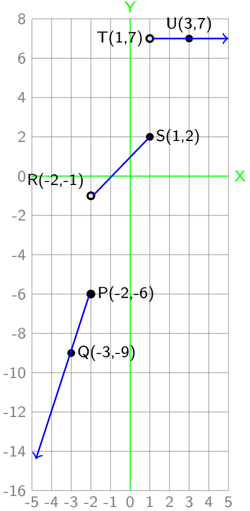

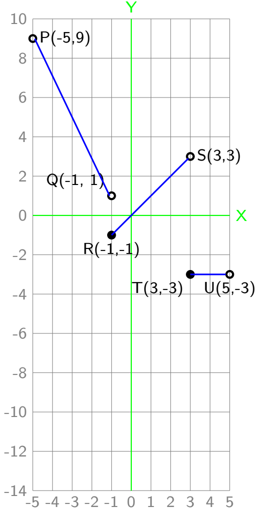

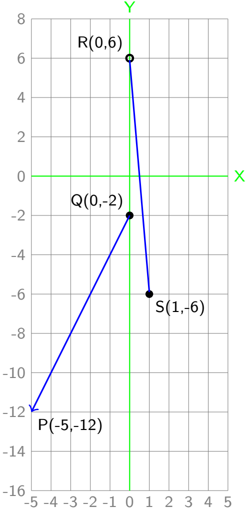

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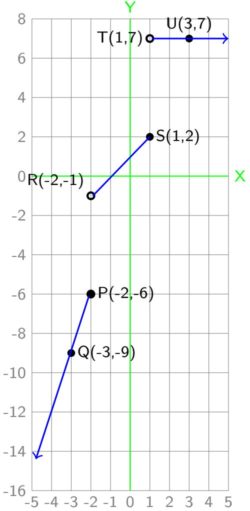

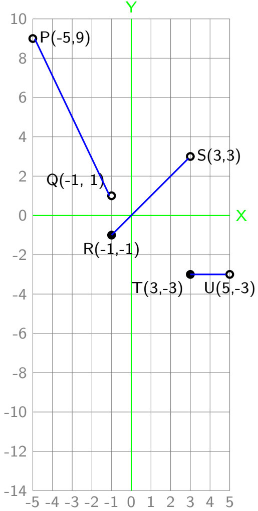

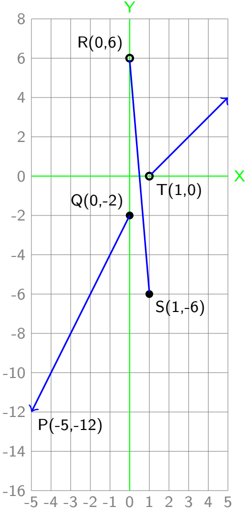

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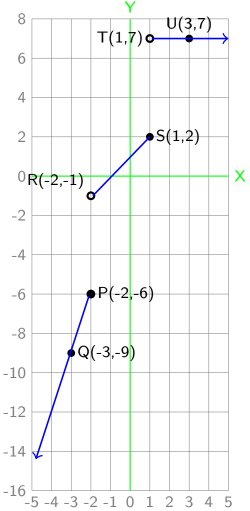

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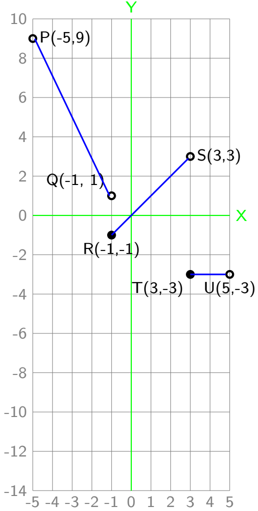

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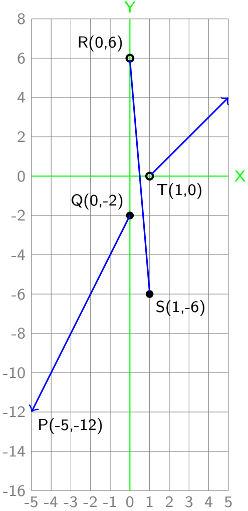

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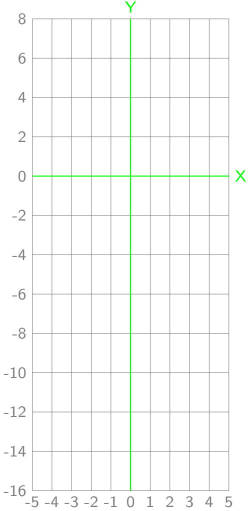

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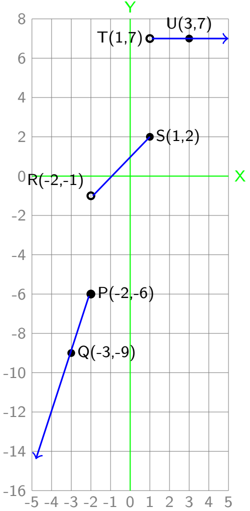

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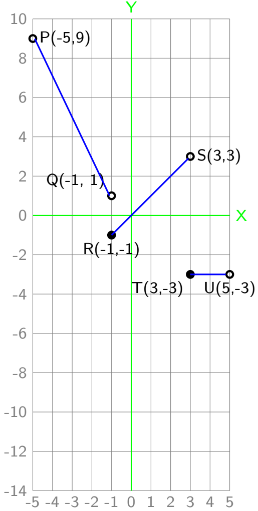

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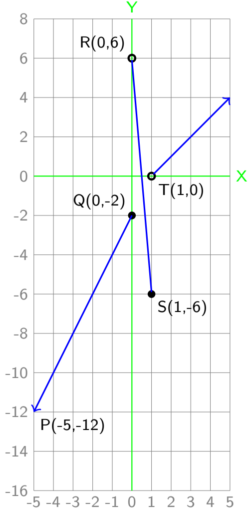

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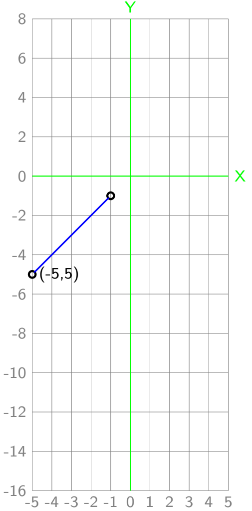

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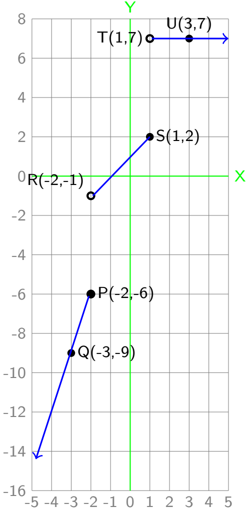

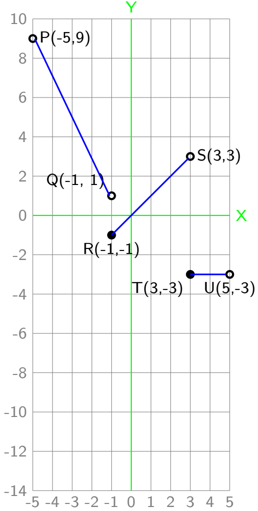

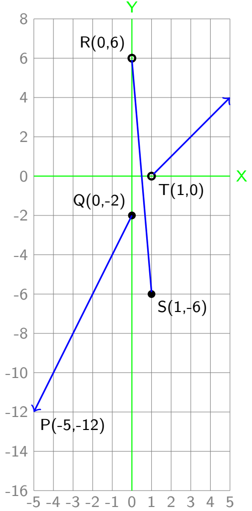

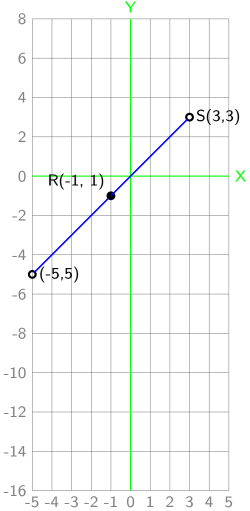

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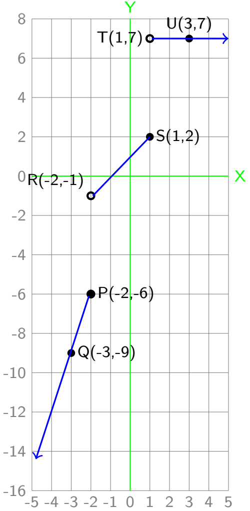

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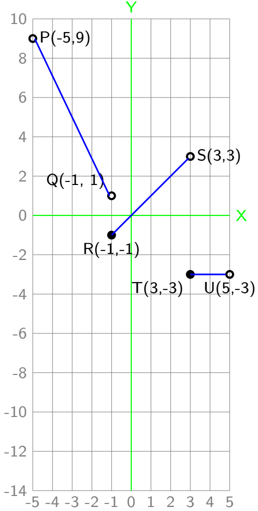

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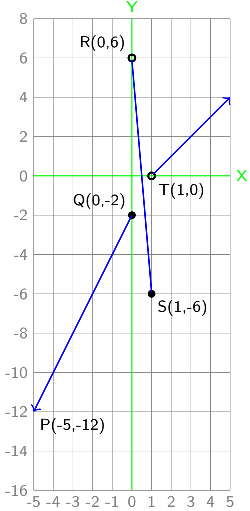

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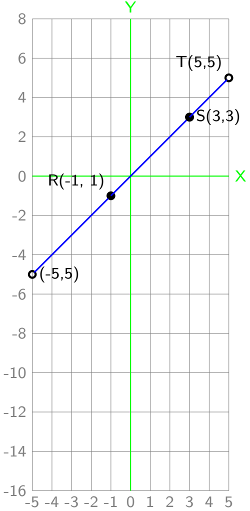
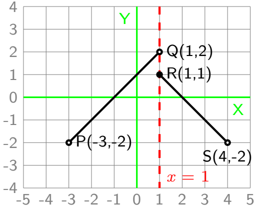


Ex. 2.2.6: Use the vertical line test to decide if the black graph below is the graph of a function.



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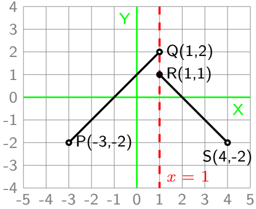
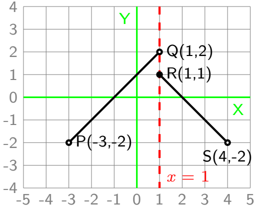
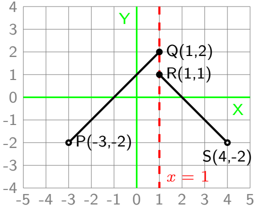


Yes. The vertical line
x = 1 meets the graph
only at point R
Any other vertical
line meets the graph
once or never.

Ex. 2.2.6: Use the vertical line test to decide if the black graph below is the graph of a function.





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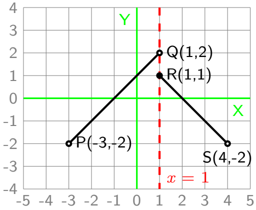

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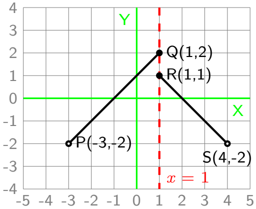
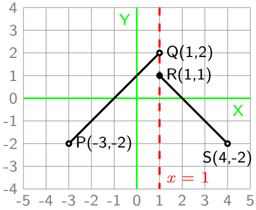
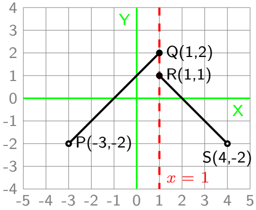
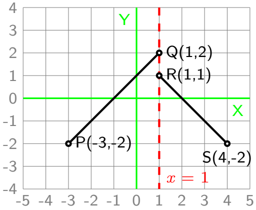


No. The vertical line
x = 1 meets the graph
at 2 points R and Q.

Ex. 2.2.6: Use the vertical line test to decide if the black graph below is the graph of a function.







Yes. The vertical line x = 1 meets the graph only at point R
Any other vertical line meets the graph once or never.

No. The vertical line x = 1 meets the graph at 2 points R and Q.

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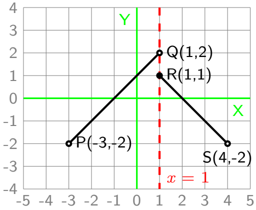

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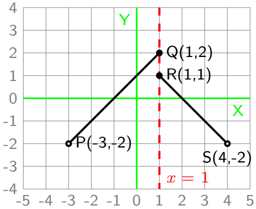

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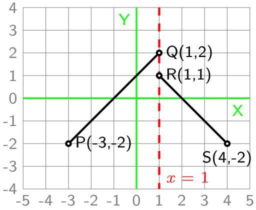
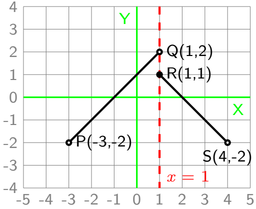


Yes. The vertical line
x = 1 meets the graph
at no points.
Any other vertical
line meets the graph
once or never.

Ex. 2.2.6: Use the vertical line test to decide if the black graph below is the graph of a function.



Yes. The vertical line
x = 1 meets the graph
only at point R
Any other vertical
line meets the graph
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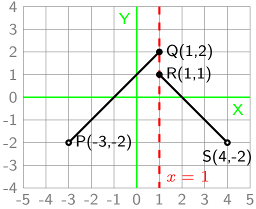

No. The vertical line
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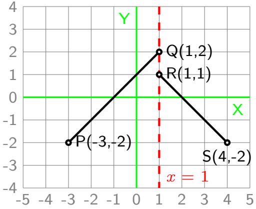
at 2 points R and Q.



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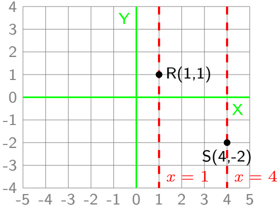

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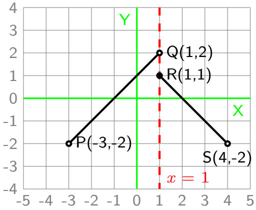

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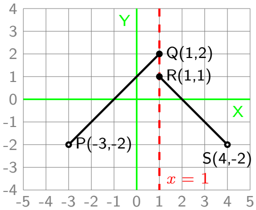

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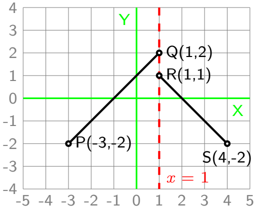

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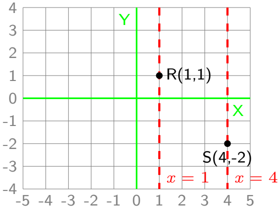


Yes. The vertical lines
$x = 1$ and $x = 4$ each meet
the graph at one point.
Any other vertical
line meets the graph
never.

Chapter 2 Section 3: Analyzing graphs of functions

- 2.3.1: Maximum and minimum points
- 2.3.2: Increasing, decreasing functions
- 2.3.3: Given $h(t)$, find t
- 2.3.4: The graph of a degree 3 polynomial
- 2.3.5: How many solutions of $h(t) = K$?
- 2.3.6: Quiz review

Section 2.3 Preview: Definitions

- Definition 2.3.1:Maximum and minimum points of a function graph
- Definition 2.3.2: Absolute or relative maximum and minimum values of a function f
- Definition 2.3.3:Increasing and decreasing functions on an interval I
- Definition 2.3.4: Rising and falling graphs on an interval I

Getting basic information from the graph of a function

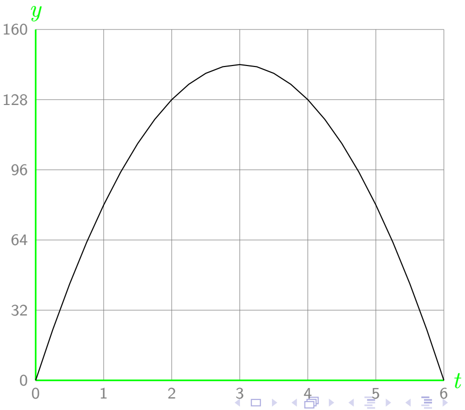
A function is a collection of number pairs (x, y), where x is an input and $y = f(x)$ is the corresponding output. Here are some models that give rise to functions.

- Manufacturing: Input x is how many tons of bricks a factory sells. Output $f(x)$ is net profit: the sale price of x bricks minus their cost.
- Saving the dolphins: Input t is the number of hours after Noon. Output $f(t)$ is the gap between the radius of a circular oil slick spreading from an oil tanker explosion and a dolphin swimming away from the explosion, as in Chapter 1.4.
- Moving particle: Input t is the number of seconds after Noon. Output $h(t)$ is the height above ground at time t of a ball thrown up from ground at Noon.

Each of these examples is modeled by a function. The information we would like to know can usually be obtained by asking related questions about the functions's graph.

- Suppose you know the output $y = h(t)$. What input or inputs t produced that output?
- What is the maximum value of $y = h(t)$? What is the minimum value?
- In what (time) interval of inputs is $y = h(t)$ increasing (getting larger) as t increases?
- In what (time) interval is $y = h(t)$ decreasing (getting smaller) as t increases?

A ball is thrown up at time $t = 0$ and released with velocity 96 feet per second. Its height above the ground (until it comes back down) is $y = h(t) = 96t - 16t^2$ feet. Below is the graph of h with domain $0 \le t \le 6$.



2.3.1 Maximum and minimum points

Maximum and minimum points of a function graph

Suppose f is a function defined on an interval I and a is a point in I. Then point $(a, f(a))$ on the graph of f is

- an absolute maximum point provided $f(a) \geq f(x)$;
- an absolute minimum point provided $f(a) \leq f(x)$;

for all x in I.

If a is NOT an endpoint of I, point $(a, f(a))$ is

- a relative maximum point provided $f(a) \geq f(x)$;
- a relative minimum point provided $f(a) \leq f(x)$;

for all x in some open interval $(a-d, a+d)$ contained in I.

Tricky point: *If a is an endpoint of I, then the graph of f does not have a relative maximum or minimum point* at $(a, f(a))$. That's because I does not contain any open interval around a.

Absolute or relative (A or R) maximum or minimum values of f

- $f(a)$ is an (A or R) maximum value of f if: $(a, f(a))$ is an (A or R) maximum point of the graph of f.
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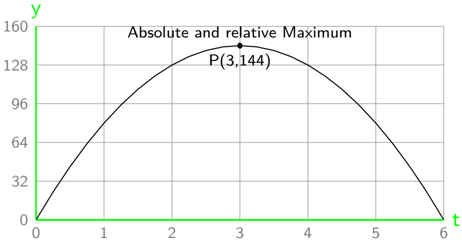

Example 1: Above is the graph of $y = f(t)$ with domain $0 \leq t \leq 6$. Find all (A or R) maximum or minimum points on the graph and all (A or R) maximum or minimum values of f.

- $(3, 144)$ is both an absolute maximum point and a relative maximum point.

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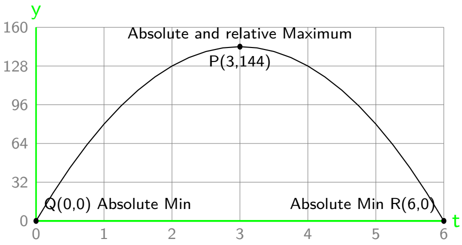

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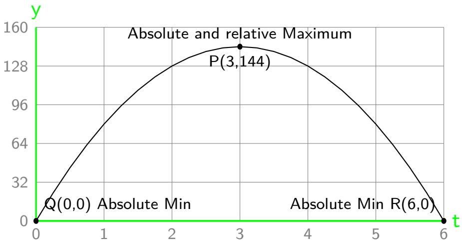

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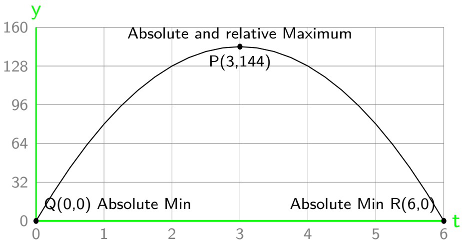

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- $f(3) = 144$ is an absolute and relative maximum value of f.
- $f(0) = f(6) = 0$, the absolute minimum value of f.

2.3.2 On what intervals is a function increasing? decreasing?

Increasing and decreasing functions on an interval I

- f is increasing on I means: $f(a) < f(b)$ for any two points a and b in I with $a < b$.
- f is decreasing on I means: $f(a) > f(b)$ for any two points a and b in I with $a < b$.

According to the above definition, a constant function $y = c$ is neither increasing nor decreasing on any interval.

Rising and falling graphs on an interval I

- The graph of f is rising on I means: the function f is increasing on I. As a point moves left to right on the graph, the point is rising.
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Example 2: On the graph at the right:

- In what interval is $h(t)$ increasing?
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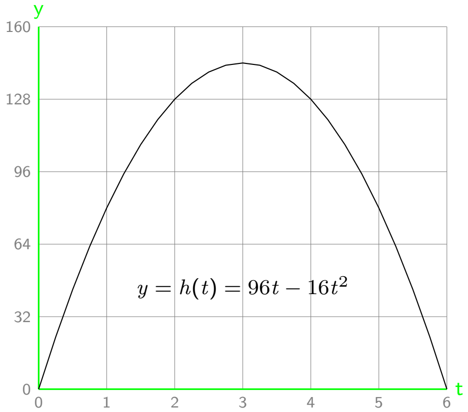

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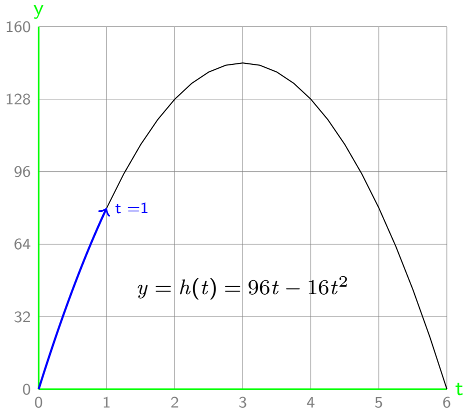

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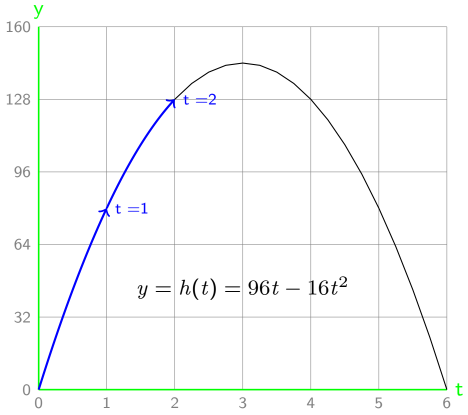

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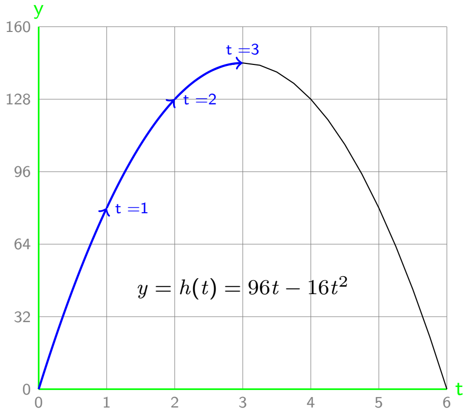

Answer:

- $h(t)$ is increasing for t in the interval $[0, 3]$.

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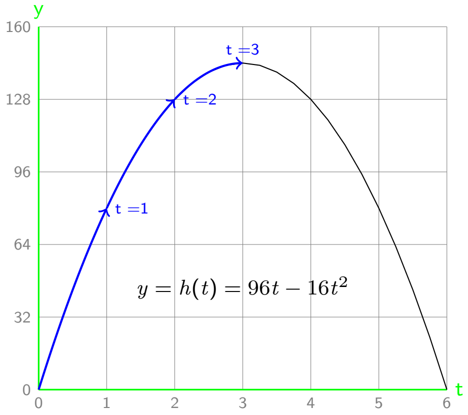

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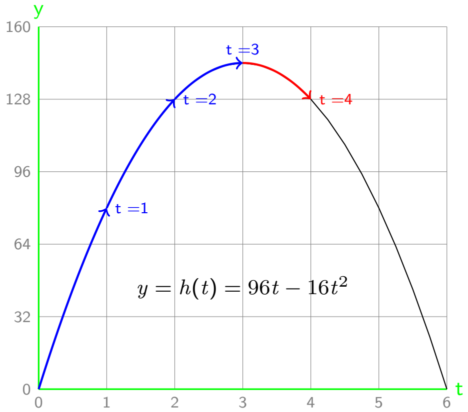

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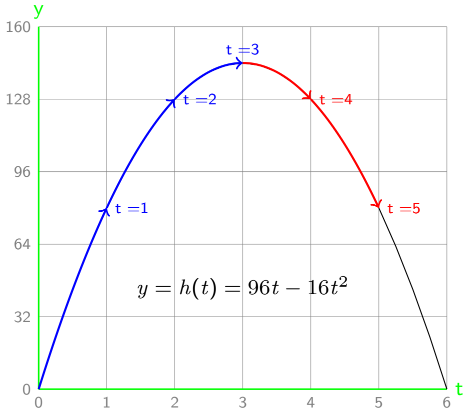

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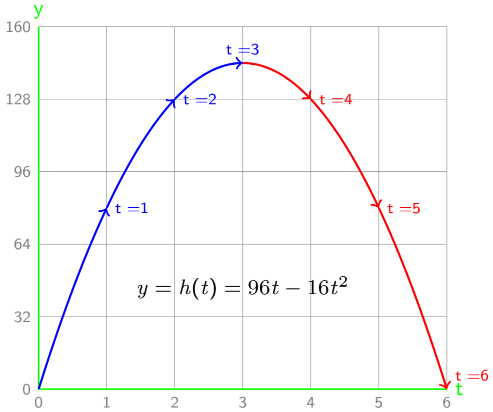


Answer:

- $h(t)$ is increasing for t in the interval $[0, 3]$.
- $h(t)$ is decreasing for t in the interval $[3, 6]$.

2.3.3 At what time(s) is a ball at a specified height?

If you want to find out how high the ball is at time t, calculate the value of $h(t)$. However, we can reverse the question as follows.

Suppose you know the output is K feet.
Find the input time or times t when $h(t) = K$.

Height is always a function of time. In this example, the reverse is false: time is not a function of height.

Indeed, if the ball gets to height K on the way up, it will be at the same height K at some time on the way down.

Example 3: Given $y = h(t) = 96t - 16t^2$, at what time t is the ball's height 112 feet?

Solution:

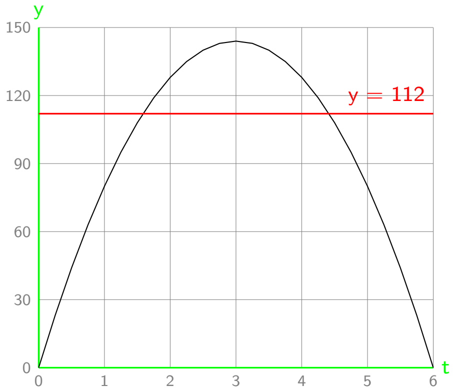
$$\begin{aligned} 96t - 16t^2 &= 112 \\ -16t^2 + 96t - 112 &= 0 \\ -16(t^2 - 6t + 7) &= 0 \\ t^2 - 6t + 7 &= 0 \end{aligned}$$

Solve by using the quadratic formula:

$$t = \frac{6 \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot 7}}{2} = \frac{6 \pm \sqrt{36-28}}{2} = \frac{6 \pm \sqrt{8}}{2} = \frac{6 \pm 2\sqrt{2}}{2} = \boxed{3 \pm \sqrt{2}}$$

Answer: The ball's height is 112 feet when $t = 3 - \sqrt{2} \approx 1.586$ seconds (on the way up) and when $t = 3 + \sqrt{2} \approx 4.414$ seconds (on the way down).

Example 4: Given $y = h(t) = 96t - 16t^2$, use the graph below to approximate the solution(s) of $h(t) = 112$.



Solution: The red line $y = 112$ hits the graph at two points P and Q, each with y-coordinate 112.

2.3.3 At what time(s) is a ball at a specified height?

If you want to find out how high the ball is at time t, calculate the value of $h(t)$. However, we can reverse the question as follows.

Suppose you know the output is K feet.
Find the input time or times t when $h(t) = K$.

Height is always a function of time. In this example, the reverse is false: time is not a function of height.

Indeed, if the ball gets to height K on the way up, it will be at the same height K at some time on the way down.

Example 3: Given $y = h(t) = 96t - 16t^2$, at what time t is the ball's height 112 feet?

Solution:

$$\begin{aligned} 96t - 16t^2 &= 112 \\ -16t^2 + 96t - 112 &= 0 \\ -16(t^2 - 6t + 7) &= 0 \\ t^2 - 6t + 7 &= 0 \end{aligned}$$

Solve by using the quadratic formula:

$$t = \frac{6 \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot 7}}{2} = \frac{6 \pm \sqrt{36-28}}{2} = \frac{6 \pm \sqrt{8}}{2} = \frac{6 \pm 2\sqrt{2}}{2} = \boxed{3 \pm \sqrt{2}}$$

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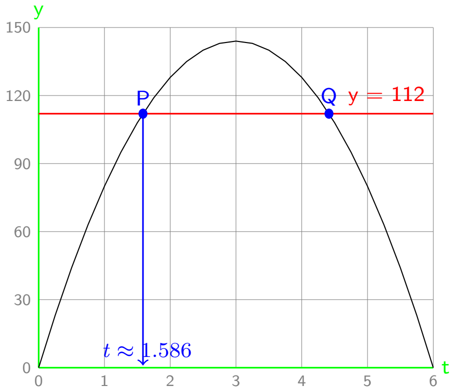

Solution: The red line $y = 112$ hits the graph at two points P and Q, each with y-coordinate 112.

To find each point's t-coordinate, draw a vertical line to the x-axis.

2.3.3 At what time(s) is a ball at a specified height?

If you want to find out how high the ball is at time t, calculate the value of $h(t)$. However, we can reverse the question as follows.

Suppose you know the output is K feet.
Find the input time or times t when $h(t) = K$.

Height is always a function of time. In this example, the reverse is false: time is not a function of height.

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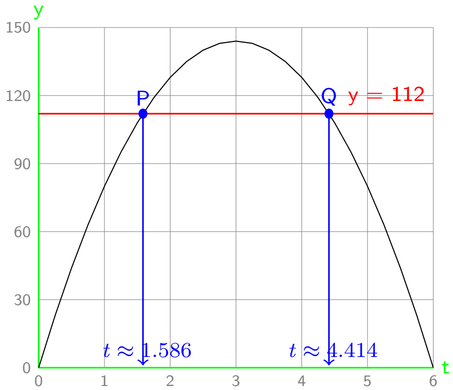


Solution: The red line $y = 112$ hits the graph at two points P and Q, each with y-coordinate 112.

To find each point's t-coordinate, draw a vertical line to the x-axis. The left vertical line's equation is $t = 1.586$ and the right vertical line's equation is $t = 4.414$ (correct to 3 decimal places) because the arrow tips are at those t-values on the t-axis.

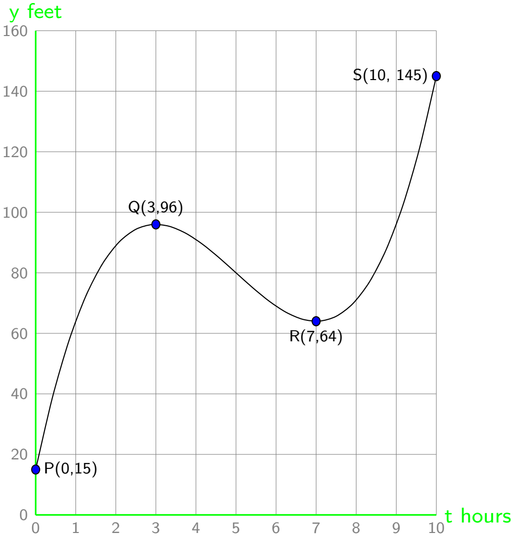
2.3.4 Analyzing the graph of a degree 3 polynomial

Example 5: At the right is the graph of function $h(t) = (t-5)^3 - 12(t-5) + 80$ feet with domain $0 \leq t \leq 10$. Use the graph to find absolute and relative minimum points of the graph of h.

Solution:
To begin, compute $h(t)$ for $t = 0, 3, 7,$ and 10 to label the special points P, Q, R and S on the graph.

Click through the following feature list:

The function $y = h(t)$



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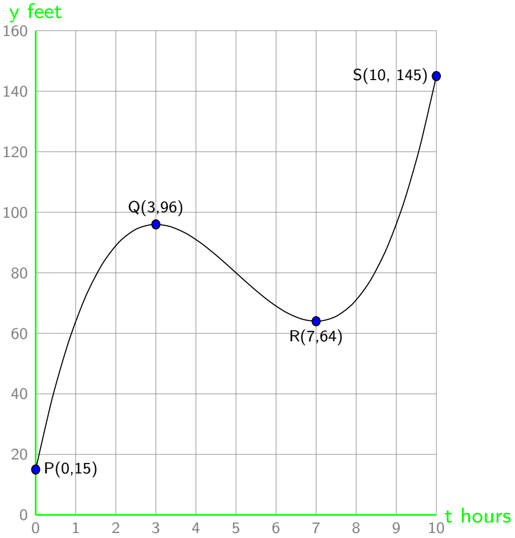

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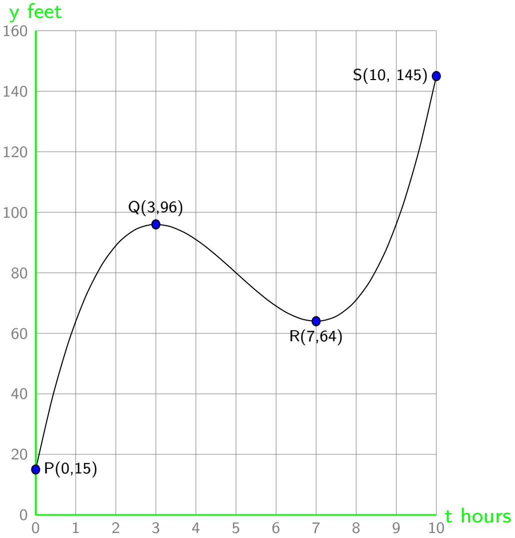
Click through the following feature list:

The function $y = h(t)$

- has domain $[0, 10]$;
- is increasing for t in $[0, 3]$ and t in $[7, 10]$;
- is decreasing for t in $[3, 7]$;



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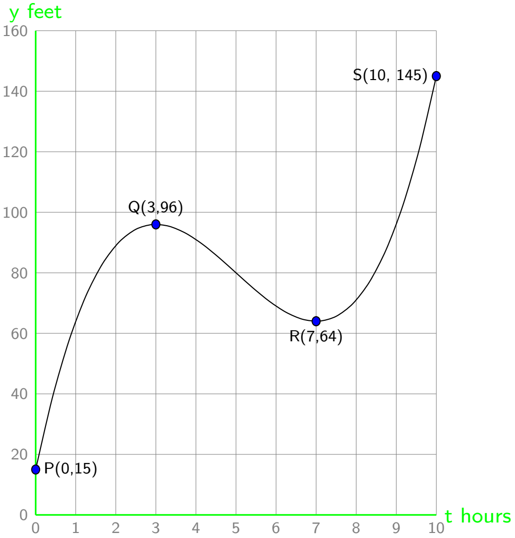

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- is decreasing for t in $[3, 7]$;
- has a relative maximum value $y = 96$ at $t = 3$;
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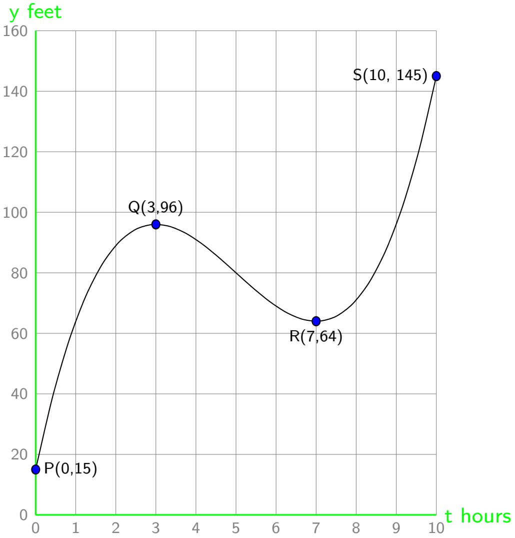

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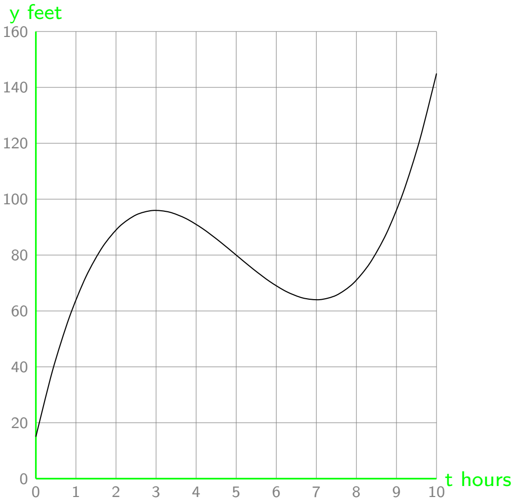
Click through the following feature list:

The function $y = h(t)$

- has domain $[0, 10]$;
- is increasing for t in $[0, 3]$ and t in $[7, 10]$;
- is decreasing for t in $[3, 7]$;
- has a relative maximum value $y = 96$ at $t = 3$;
- has a relative minimum value $y = 64$ at $t = 7$;
- has absolute minimum value 15 at $t = 0$; and
- has absolute maximum value 145 at $t = 10$.

Now rephrase the previous question as a real-life problem. A jet's height above ground at time t hours after Noon is $h(t) = (t-5)^3 - 12(t-5) + 80$ feet for times $0 \le t \le 10$. Use the graph of $y = h(t)$ at the right to describe the jet's path.

Solution: Click slowly: read and watch the graph.



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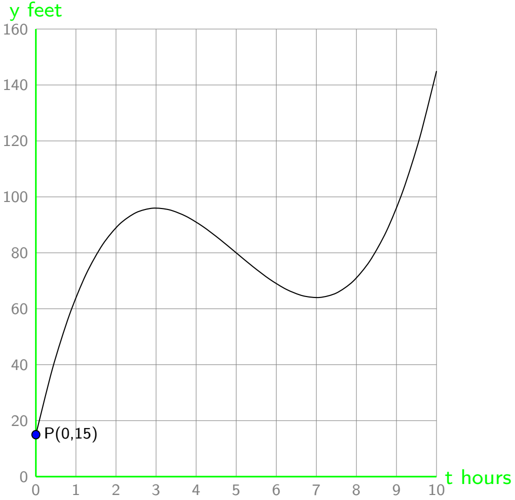

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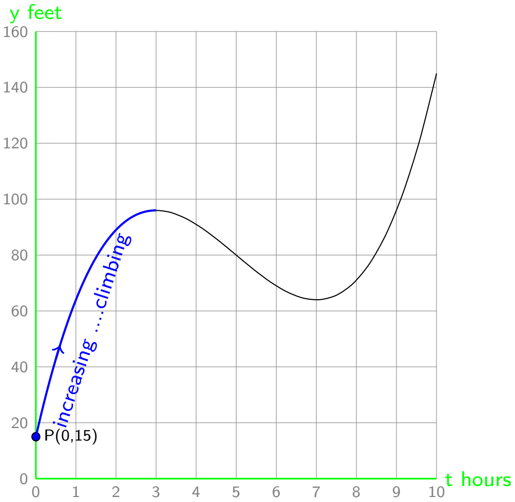

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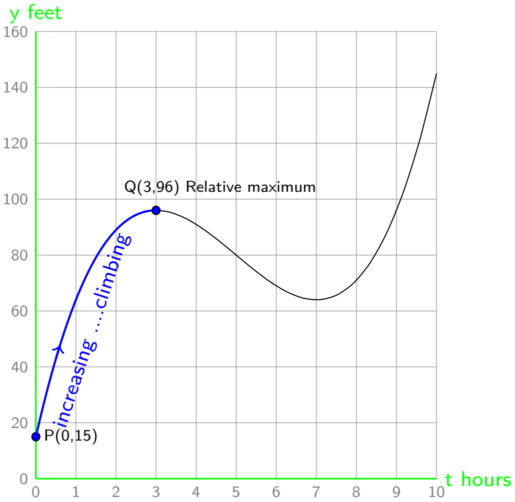
Solution: Click slowly: read and watch the graph.

- $h(t)$ is increasing for t in the interval $[0, 3]$. The jet is climbing from Noon to 3:00 P.M.
- $h(t)$ has relative maximum value $y = h(3) = 96$ at $t = 3$. $h(3) \ge h(t)$ for times t close to 3, and so $Q(3, 96)$ is a relative maximum point of the graph. The jet is at a high trajectory point at 3:00 P.M.



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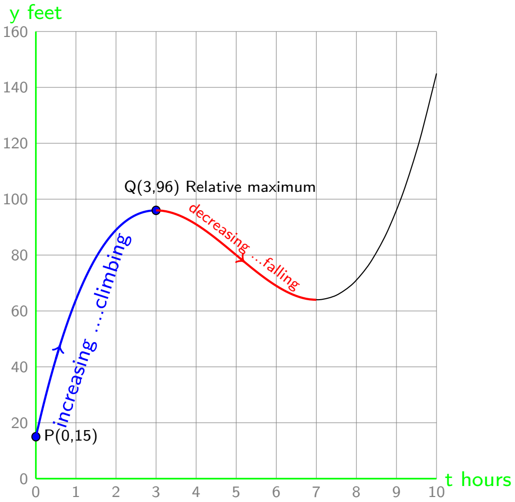
- $h(t)$ has relative maximum value $y = h(3) = 96$ at $t = 3$. $h(3) \ge h(t)$ for times t close to 3, and so $Q(3,96)$ is a relative maximum point of the graph. The jet is at a high trajectory point at 3:00 P.M.
- $h(t)$ is decreasing for t in the interval $[3,7]$. The jet is falling from 3:00 P.M. to 7:00 P.M.



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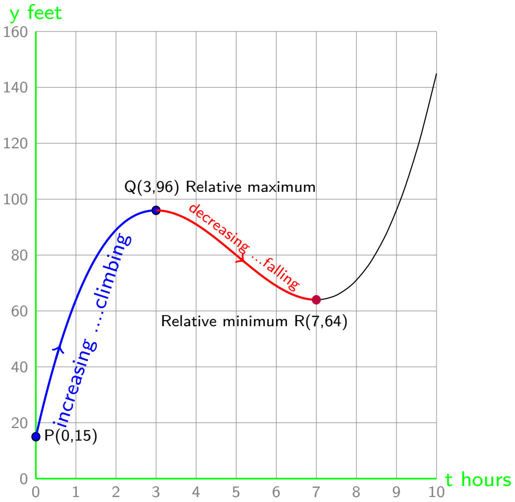
- $h(t)$ has relative maximum value $y = h(3) = 96$ at $t = 3$. $h(3) \geq h(t)$ for times t close to 3, and so $Q(3, 96)$ is a relative maximum point of the graph. The jet is at a high trajectory point at 3:00 P.M.
- $h(t)$ is decreasing for t in the interval $[3, 7]$. The jet is falling from 3:00 P.M. to 7:00 P.M.
- $h(t)$ has relative minimum value $y = h(7) = 64$ at $t = 7$. $h(7) \leq h(t)$ for times t close to 7, and so $R(7, 64)$ is a relative minimum point of the graph. The jet is at a low trajectory point at 3:00 P.M.



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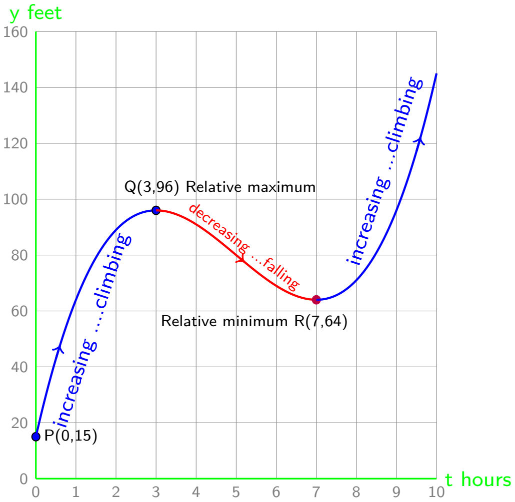
- $h(t)$ is increasing for t in the interval $[7, 10]$. the jet is climbing from 7:00 P.M. to 10:00 P.M.



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Solution: Click slowly: read and watch the graph.

- $h(t)$ is increasing for t in the interval $[0, 3]$. The jet is climbing from Noon to 3:00 P.M.
- $h(t)$ has relative maximum value $y = h(3) = 96$ at $t = 3$. $h(3) \geq h(t)$ for times t close to 3, and so $Q(3, 96)$ is a relative maximum point of the graph. The jet is at a high trajectory point at 3:00 P.M.
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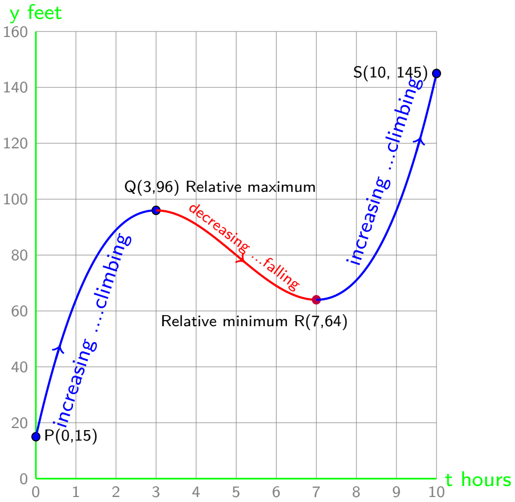
- $h(t)$ has relative minimum value $y = h(7) = 64$ at $t = 7$. $h(7) \leq h(t)$ for times t close to 7, and so $R(7, 64)$ is a relative minimum point of the graph. The jet is at a low trajectory point at 3:00 P.M.
- $h(t)$ is increasing for t in the interval $[7, 10]$. the jet is climbing from 7:00 P.M. to 10:00 P.M.
- By looking at the completed graph, we see that $h(t)$ has absolute (but not relative) minimum value 15 at endpoint $t = 0$. The jet's minimum height during this trip was 15 feet, at Noon.



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Solution: Click slowly: read and watch the graph.

- $h(t)$ is increasing for t in the interval $[0, 3]$. The jet is climbing from Noon to 3:00 P.M.
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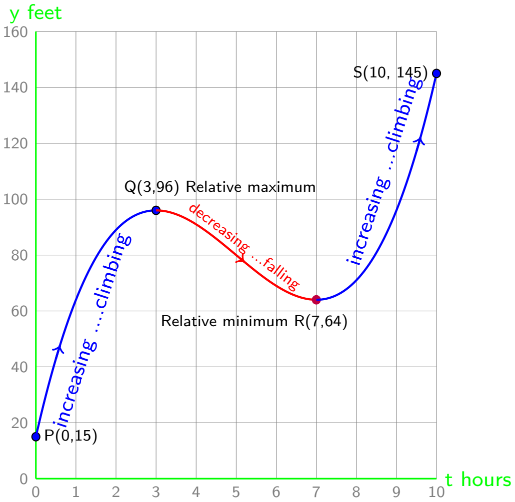
- $h(t)$ is increasing for t in the interval $[7, 10]$. the jet is climbing from 7:00 P.M. to 10:00 P.M.
- By looking at the completed graph, we see that $h(t)$ has absolute (but not relative) minimum value 15 at endpoint $t = 0$. The jet's minimum height during this trip was 15 feet, at Noon.
- $h(t)$ has absolute (but not relative) maximum value 145 at endpoint $t = 10$. The jet's maximum height during this trip was 145 feet, at 10:00 P.M.



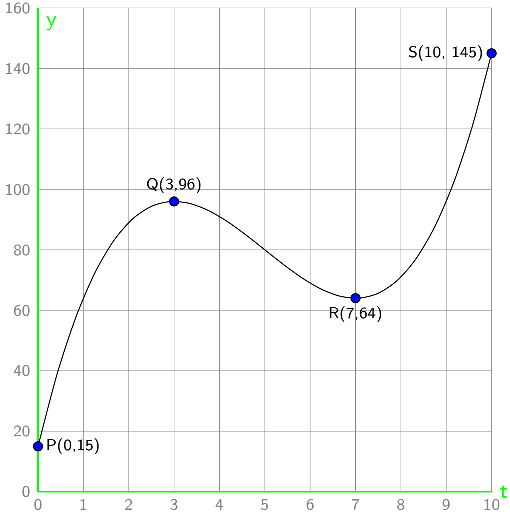
2.3.5 When is the jet at a specified height?

So far we have used time as an input and height as an output. Let's reverse the question.

Example 6: For each possible height K with $15 \leq K \leq 145$ feet,

- at *how many* times t does $h(t) = K$?;
- for each K, at what time(s) t does $h(t) = K$?

  Solution:
  The second question requires solving $(t-5)^3 - 12(t-5) + 80 = K$. That's difficult. Instead, study the graph to find *how many* solutions of $h(t) = K$ there are for different values of K.



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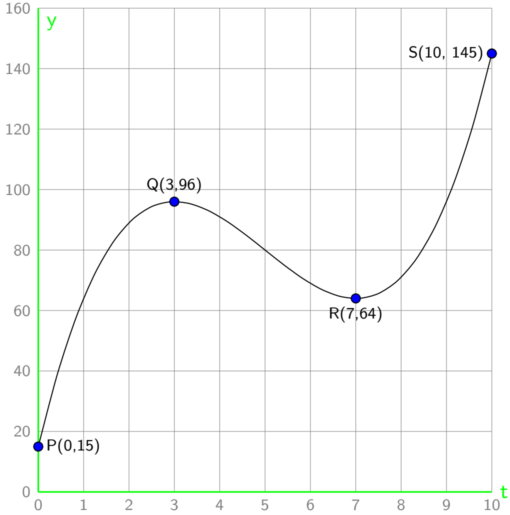
  The number of solutions of $h(t) = K$ is simply the number of times that the horizontal line $y = K$ hits the graph. Click slowly.



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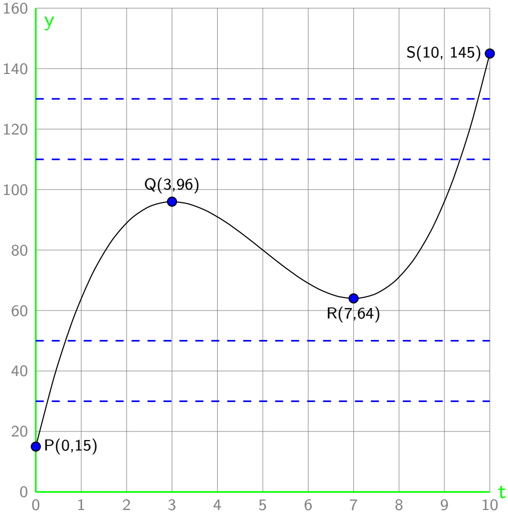
- for each height $y = K$ in $[15, 64)$ or $(96, 145]$ (sample blue lines shown are $y = 30, 50, 110, 130$) the equation $h(t) = K$ has one solution: there is only one time when the jet's height is equal to K.



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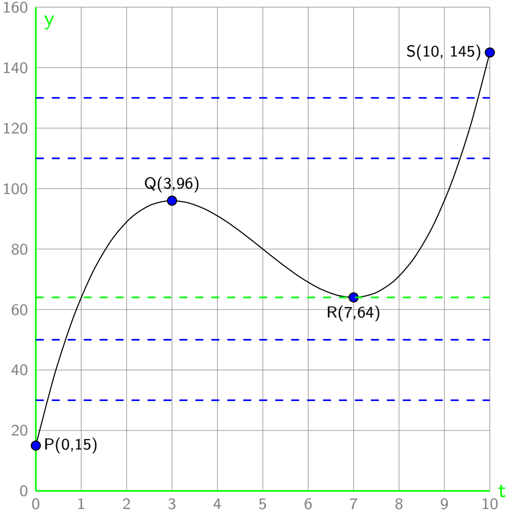
- $h(t) = 64$ (green line $y = 64$) has two solutions, $t = 1$ or 7.



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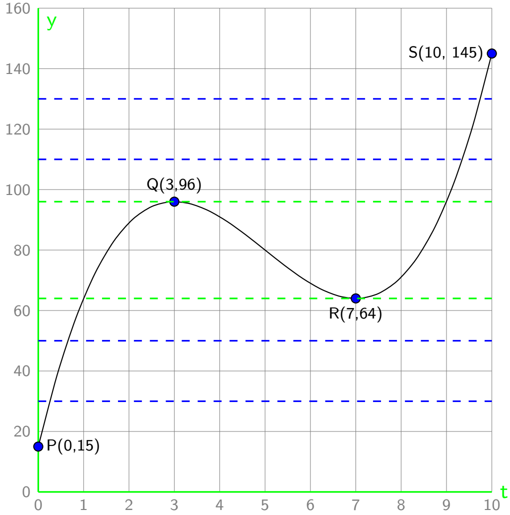
- $h(t) = 64$ (green line $y = 64$) has two solutions, $t = 1$ or 7.
- $h(t) = 96$ (green line $y = 96$) has two solutions, $t = 3$ or 9.



2.3.5 When is the jet at a specified height?

So far we have used time as an input and height as an output. Let's reverse the question.

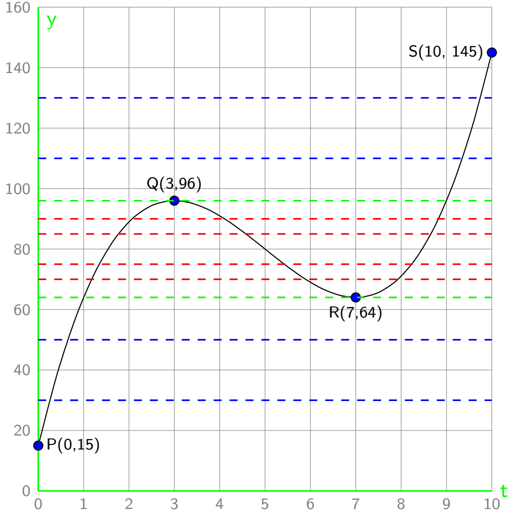
Example 6: For each possible height K with $15 \le K \le 145$ feet,

- at *how many* times t does $h(t) = K$?;
- for each K, at what time(s) t does $h(t) = K$?

  Solution:

  The second question requires solving $(t-5)^3 - 12(t-5) + 80 = K$. That's difficult. Instead, study the graph to find *how many* solutions of $h(t) = K$ there are for different values of K.

  The number of solutions of $h(t) = K$ is simply the number of times that the horizontal line $y = K$ hits the graph. Click slowly.
- for each height $y = K$ in $[15, 64)$ or $(96, 145]$ (sample blue lines shown are $y = 30, 50, 110, 130$) the equation $h(t) = K$ has one solution: there is only one time when the jet's height is equal to K.
- $h(t) = 64$ (green line $y = 64$) has two solutions, $t = 1$ or 7.
- $h(t) = 96$ (green line $y = 96$) has two solutions, $t = 3$ or 9.
- for each height $y = K$ in $(64, 96)$, sample red lines $y = 70, 75, 85, 90$, the equation $h(t) = K$ has three solutions!



2.3.6 Precalculus Section 2.3 Quiz

In Examples 1 to 4, let $h(t) = 96t - 16t^2$.

Ex. 2.3.1: Find all absolute or relative maximum or minimum points on the graph of $y = h(t)$.

Ex. 2.3.2: In what intervals is $h(t)$ increasing? In what intervals is $h(t)$ decreasing?

Ex. 2.3.3: At what time(s) t is the ball's height 112 feet?

Ex. 2.3.4: Find the approximate values of solutions of $h(t) = 112$ by using its graph to find one or more points with $y = 112$.

In Examples 5 and 6, let $h(t) = (t-5)^3 - 12(t-5) + 80$.

.

Ex. 2.3.5: A jet's height above ground at time t hours is $h(t)$ feet provided $0 \le t \le 10$. Using that function's graph, describe the path of the jet.

Ex. 2.3.6: For each height K with $h(0) \le K \le h(10)$, use the graph to determine at how many times t the height $h(t)$ is equal to K

Section 2.3 Review: Analyzing graphs

All Exercises in this section use the functions

- $f(t) = 96t - 16t^2; 0 \leq t \leq 6$
- $g(t) = 130 - 3t; t > 0$
- $h(t) = |t - 115|; 0 \leq t \leq 150$
- $s(t) = -t^2 + 3t - 2; t \geq 0$

Ex. 2.3.1: Find all absolute maximum or minimum points on the function's graph.

- f has
- g has
- h has
- s has

Section 2.3 Review: Analyzing graphs

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- $s(t) = -t^2 + 3t - 2; t \geq 0$

Ex. 2.3.1: Find all absolute maximum or minimum points on the function's graph.

- f has an absolute maximum at $(3, 144)$; an absolute minimum at $(0, 0)$ and $(6, 0)$
- g has no absolute maximum or minimum
- h has an absolute minimum at $(115, 0)$; an absolute maximum at $(0, 115)$
- s has an absolute maximum at $(\frac{3}{2}, \frac{1}{4})$

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- s has an absolute maximum at $(\frac{3}{2}, \frac{1}{4})$

Ex. 2.3.2: In what sub-intervals of the domain is the function increasing? decreasing?

- f increasing on decreasing on
- g decreasing on increasing
- h decreasing on increasing on
- s increasing on decreasing on .

Section 2.3 Review: Analyzing graphs

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Ex. 2.3.2: In what sub-intervals of the domain is the function increasing? decreasing?

- f increasing on $[0, 3]$; decreasing on $[3, 6]$;
- g decreasing on $(0, \infty)$; increasing never
- h decreasing on $[0, 115]$; increasing on $[115, 150]$;
- s increasing on $[0, \frac{3}{2}]$; decreasing on $[\frac{3}{2}, \infty)$.

Section 2.3 Review: Analyzing graphs

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Ex. 2.3.3: At what t is the function value 12?

Section 2.3 Review: Analyzing graphs

All Exercises in this section use the functions

- $f(t) = 96t - 16t^2; 0 \le t \le 6$
- $g(t) = 130 - 3t;\ t > 0$
- $h(t) = |t - 115|; 0 \le t \le 150$
- $s(t) = -t^2 + 3t - 2;\ t \ge 0$

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Ex. 2.3.3: At what t is the function value 12?

- $f(3 \pm \sqrt{2}) = 112$ • $g(6) = 112$ • $h(3) = 112$ • $s(t) = 112$ never.

Section 2.3 Review: Analyzing graphs

All Exercises in this section use the functions

- $f(t) = 96t - 16t^2; 0 \leq t \leq 6$
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Ex. 2.3.4: At what t-value(s) is the function value 0?

Section 2.3 Review: Analyzing graphs

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- $f(3 \pm \sqrt{2}) = 112$ • $g(6) = 112$ • $h(3) = 112$ • $s(t) = 112$ never.

Ex. 2.3.4: At what t-value(s) is the function value 0?

- $f(0) = f(6) = 0$ • $g(\frac{130}{3}) = 0$ • $h(115) = 0$ • $s(1) = s(2) = 0$.

Draw rough graphs of the following functions

- $f(t) = 144 - t^2$ all real t

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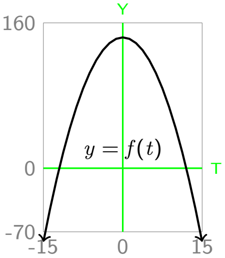

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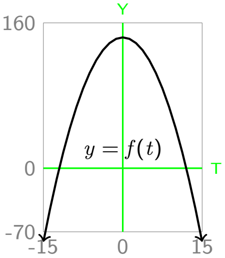

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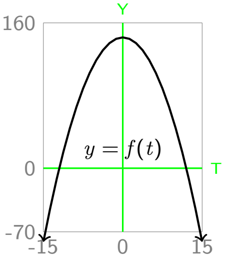

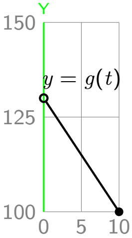

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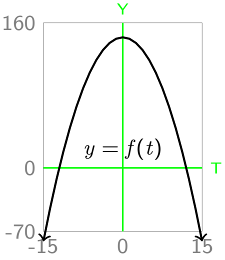

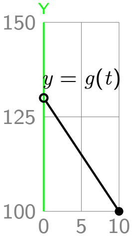

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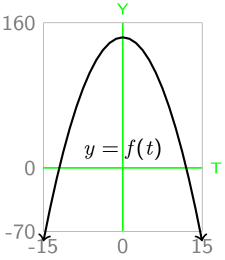

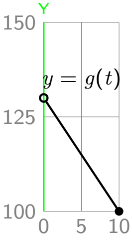

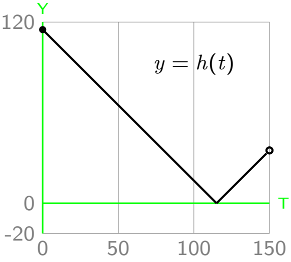

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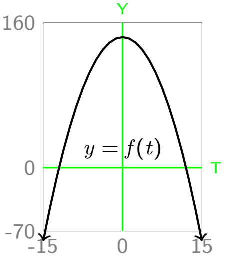

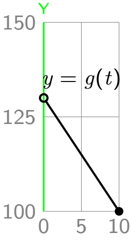

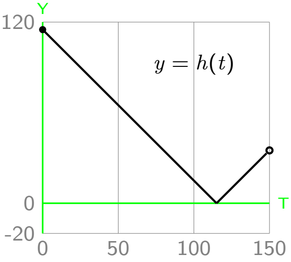

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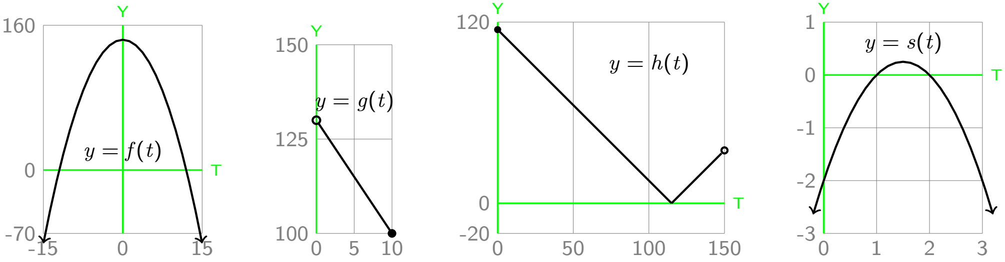

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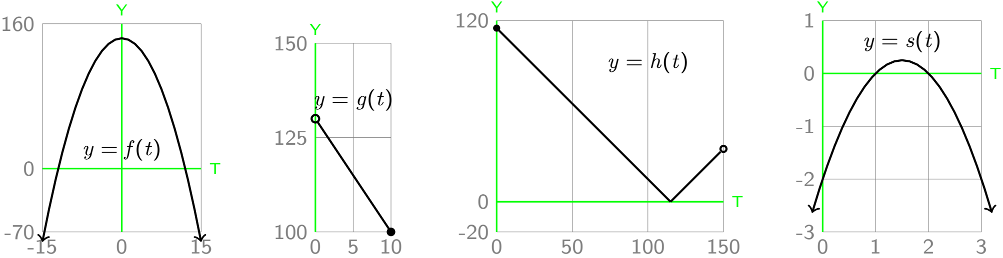

Ex. 2.3.5: Describe the path of a missile whose position is given by functions f, g, h, s. $\Rightarrow$

Draw rough graphs of the following functions

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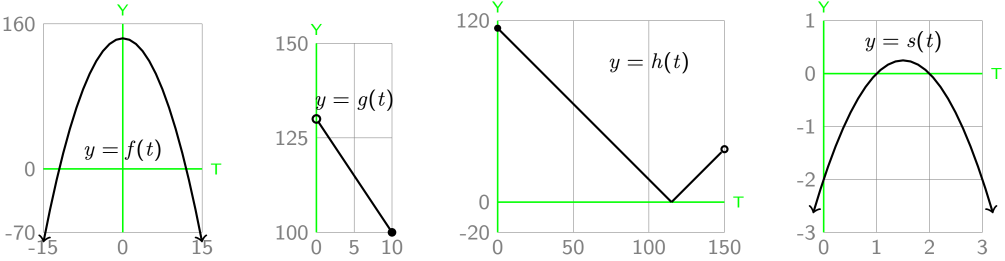

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- missile f

Draw rough graphs of the following functions

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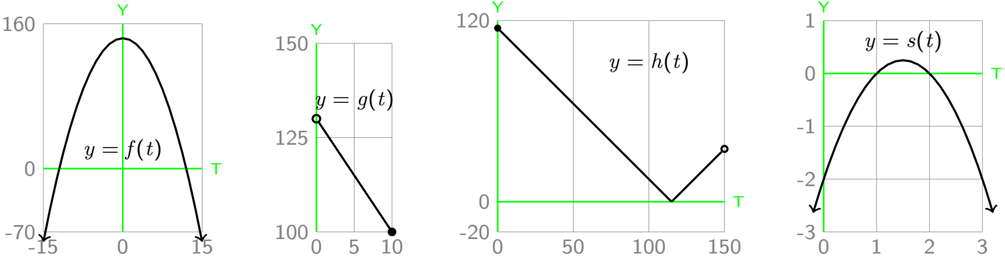

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- missile f climbing for $t \leq 0$; falling for $t \geq 0$

Draw rough graphs of the following functions

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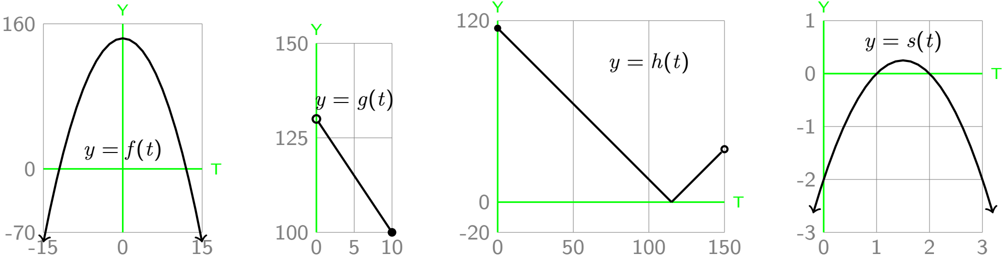

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- missile g

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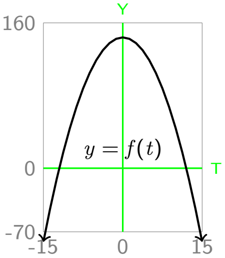

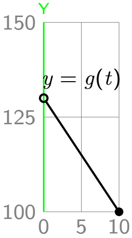

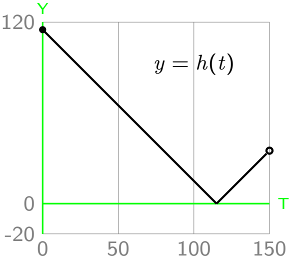

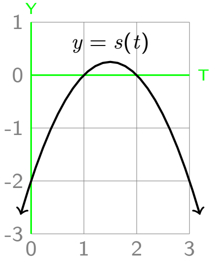

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- missile g falling for $0 < t \leq 10$

Draw rough graphs of the following functions

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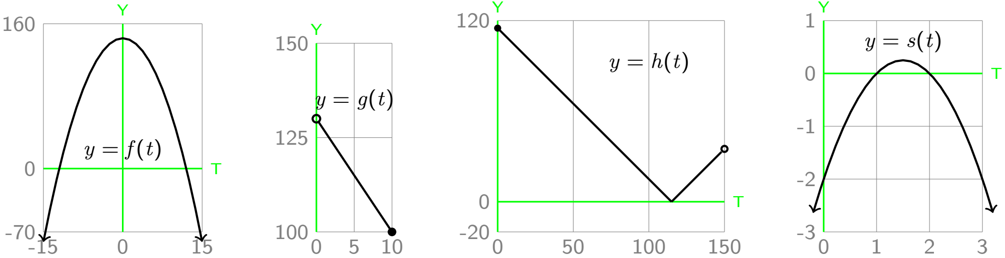

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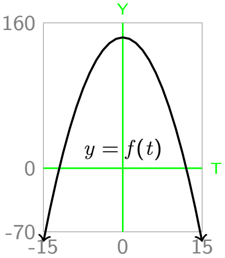

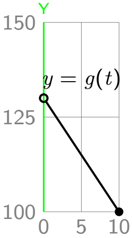

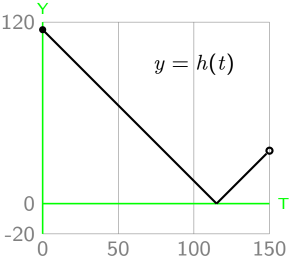

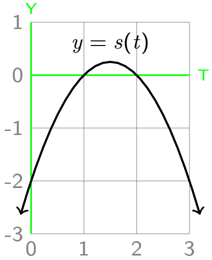

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- missile f climbing for $t \leq 0$; falling for $t \geq 0$
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- missile h falling $0 \leq t \leq 115$; rising $115 \leq t < 150$

Draw rough graphs of the following functions

- $f(t) = 144 - t^2$ all real t
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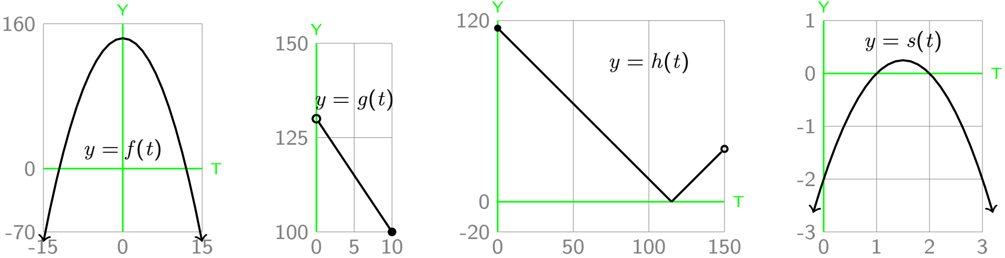

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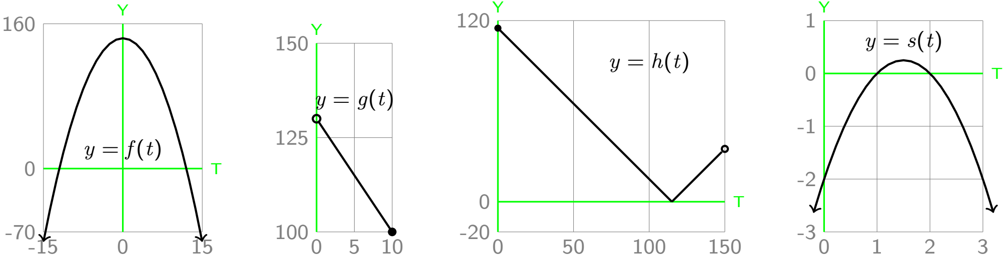
Draw rough graphs of the following functions

- $f(t) = 144 - t^2$ all real t
- $g(t) = 130 - 3t$; all $0 < t \le 10$
- $h(t) = |t - 115|; 0 \le t < 150$
- $s(t) = -t^2 + 3t - 2$; all real t.



Ex. 2.3.5: Describe the path of a missile whose position is given by functions f, g, h, s. $\Rightarrow$

- missile f climbing for $t \le 0$; falling for $t \ge 0$
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- missile h falling $0 \le t \le 115$; rising $115 \le t < 150$
- missile s climbing $t \le \frac{3}{2}$; falling $t \ge \frac{3}{2}$

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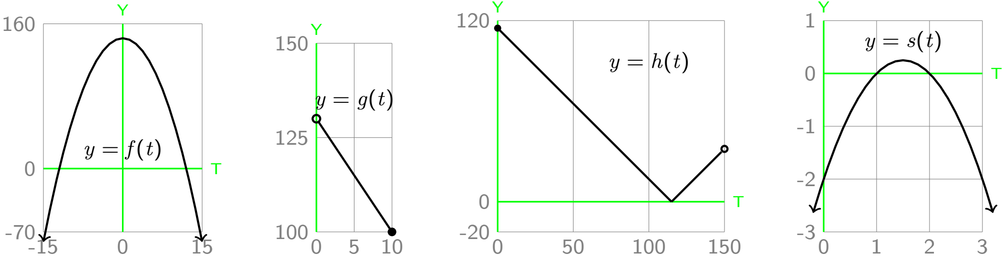

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Ex. 2.3.6: For each real number K, at how many times t does the function value equal K?

Draw rough graphs of the following functions

- $f(t) = 144 - t^2$ all real t
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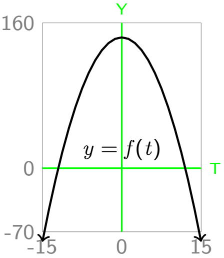

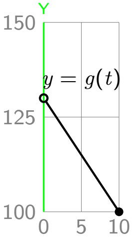

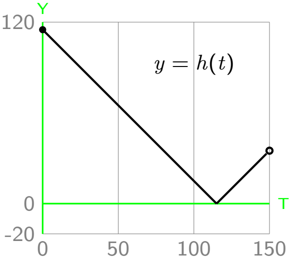

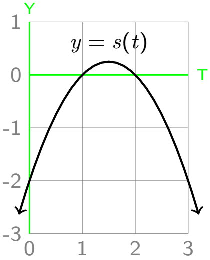

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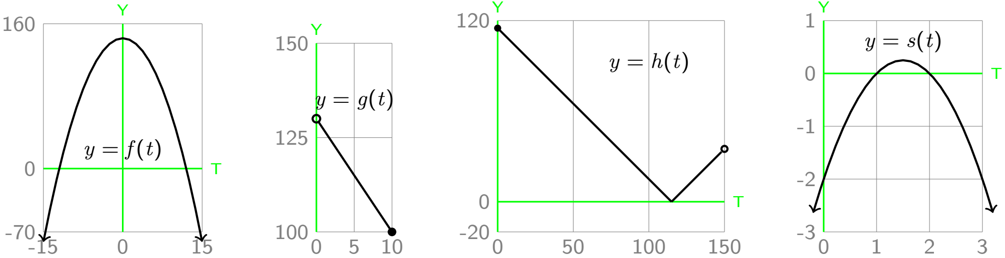

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Ex. 2.3.6: For each real number K, at how many times t does the function value equal K?

- $f(t) = K \Rightarrow$.

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- $f(t) = 144 - t^2$ all real t
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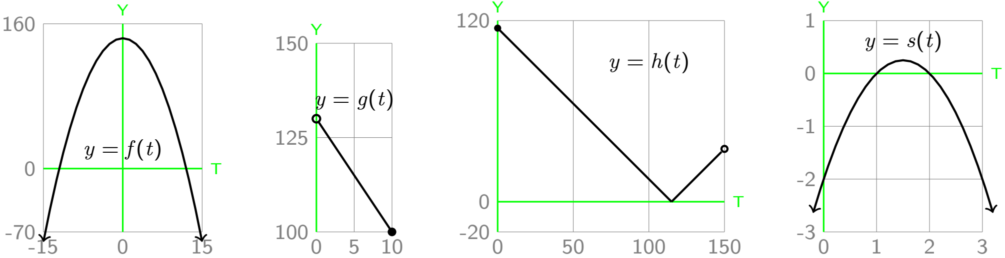

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Ex. 2.3.6: For each real number K, at how many times t does the function value equal K?

- $f(t) = K \Rightarrow$ twice if $0 < K < 144$; once if $K = 144$; never if $K > 144$.

Draw rough graphs of the following functions

- $f(t) = 144 - t^2$ all real t
- $g(t) = 130 - 3t$; all $0 < t \leq 10$
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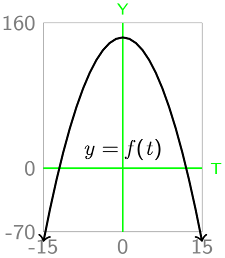

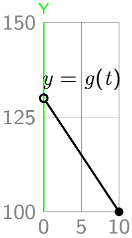

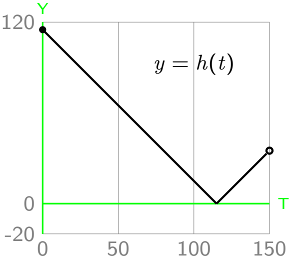

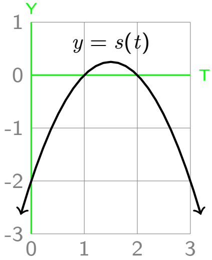

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Ex. 2.3.6: For each real number K, at how many times t does the function value equal K?

- $f(t) = K \Rightarrow$ twice if $0 < K < 144$; once if $K = 144$; never if $K > 144$.
- $g(t) = K \Rightarrow$.

Draw rough graphs of the following functions

- $f(t) = 144 - t^2$ all real t
- $h(t) = |t - 115|; 0 \le t < 150$
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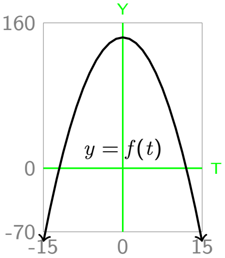

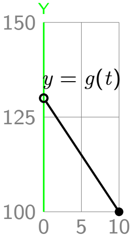

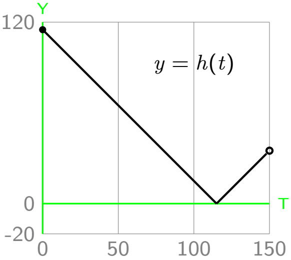

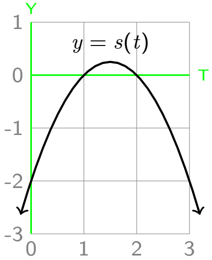

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Ex. 2.3.6: For each real number K, at how many times t does the function value equal K?

- $f(t) = K \Rightarrow$ twice if $0 < K < 144$; once if $K = 144$; never if $K > 144$.
- $g(t) = K \Rightarrow$ once if $100 \le K < 130$; never if $K \ge 130$ or $K < 100$.

Draw rough graphs of the following functions

- $f(t) = 144 - t^2$ all real t
- $h(t) = |t - 115|; 0 \leq t < 150$
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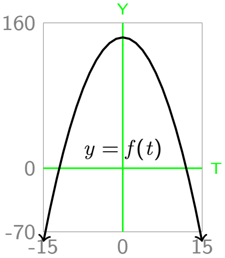

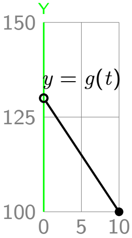

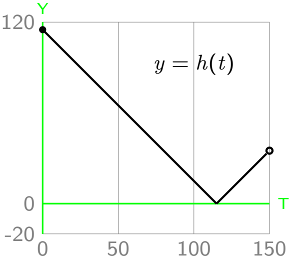

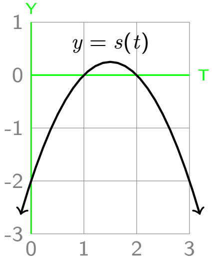

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Ex. 2.3.6: For each real number K, at how many times t does the function value equal K?

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Draw rough graphs of the following functions

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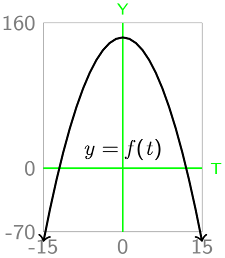

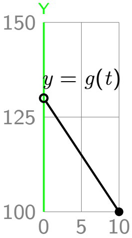

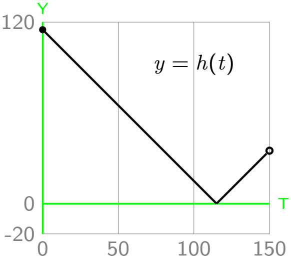

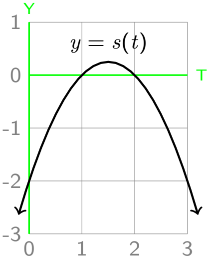

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- $g(t) = K \Rightarrow$ once if $100 \le K < 130$; never if $K \ge 130$ or $K < 100$.
- $h(t) = K \Rightarrow$ twice if $0 \le K < 35$; once if $35 \le K \le 115$; never if $K < 0$ or $K > 115$.

Draw rough graphs of the following functions

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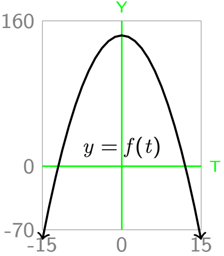

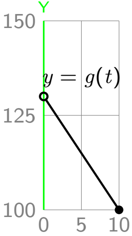

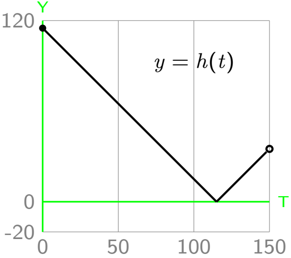

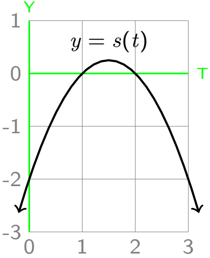

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- $h(t) = K \Rightarrow$ twice if $0 \le K < 35$; once if $35 \le K \le 115$; never if $K < 0$ or $K > 115$.
- $s(t) = K \Rightarrow$.

Draw rough graphs of the following functions

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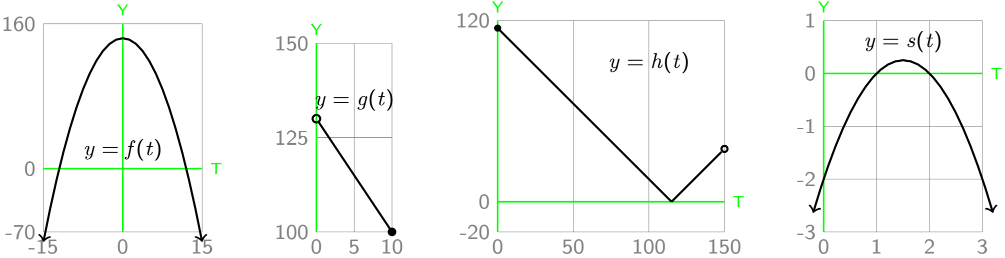

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- $h(t) = K \Rightarrow$ twice if $0 \leq K < 35$; once if $35 \leq K \leq 115$; never if $K < 0$ or $K > 115$.
- $s(t) = K \Rightarrow$ twice if $K < \frac{1}{4}$; once if $K = \frac{1}{4}$; never if $K > \frac{1}{4}$.

Chapter 2 Section 4: Quadratic functions and graphs

- 2.4.1: Graphs of quadratic functions
- 2.4.2: Graphing $y = ax^2$
- 2.4.3: Graphing $y = ax^2 + c$
- 2.4.4: Completing the square revisited
- 2.4.5: Rewriting $ax^2 + bx + c$ in standard form
- 2.4.6: Sketching the graph of $y = ax^2 + bx + c$
- 2.4.7: Transforming a quadratic function to standard form
- 2.4.8: x-intercepts of a quadratic function
- 2.4.9: Section 2.4 Review

Section 2.4 Preview: Definitions and Procedures

- Definition 2.4.1: f is a quadratic function if
- Definition 2.4.2: The vertex of the graph of $f(x) = y = a(x-h)^2 + k$
- Definition 2.4.3: Identity for completing the square

- Procedure 2.4.1: To sketch the graph of $y = ax^2$
- Procedure 2.4.2: To transform the graph of $y = x^2$ to the graph of $y = ax^2$
- Procedure 2.4.3: To transform the graph of $y = x^2$ to the graph of $y = ax^2 + c$
- Procedure 2.4.4: The standard form of $y = f(x) = ax^2 + bx + c$ is $y = f(x) = a(x-h)^2 + k$
- Procedure 2.4.5: To graph $y = f(x) = ax^2 + bx + c$; $a > 0$
- Procedure 2.4.6: To sketch the graph of $y = a(x-h)^2 + k$ by transforming the graph of $y = ax^2$
- Procedure 2.4.7: To find the intercepts of the parabola $y = ax^2 + bx + c$

2.4.1: Graphs of quadratic functions

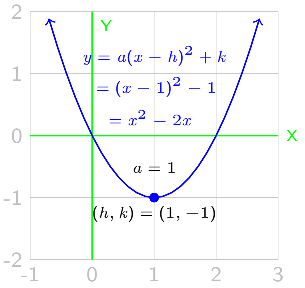
f is a quadratic function if

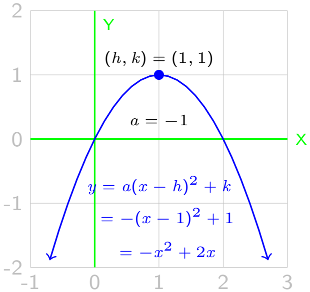
- $f(x) = ax^2 + bx + c$, where a, b, c are real numbers and $a \neq 0$.
- The graph of a quadratic function is called a *parabola*.

Every quadratic function can be rewritten in standard form as $f(x) = a(x-h)^2 + k$. Two examples are sketched at the right.

The graph of $f(x) = y = a(x-h)^2 + k$

- has *vertex* the point $(h, k) = (h, f(h))$.
- If $a > 0$ then
  - the graph $\cup$ opens upwards from the vertex;
  - the *vertex* is the absolute minimum point of the graph;
  - $f(h) = k$ is the absolute minimum value of f;
  - f is decreasing on $(-\infty, h]$ and increasing on $[h, \infty)$.
- If $a < 0$ then
  - the graph $\cap$ opens downwards from the vertex;
  - the vertex is the absolute maximum point of the graph;
  - $f(h) = k$ is the absolute maximum value of f;
  - f is increasing on $(-\infty, h]$ and decreasing on $[h, \infty)$.





2.4.2: Graphing $y = ax^2$

Sketching the graph of $y = ax^2 + bx + c$ is easy if $b = c = 0$.

To sketch the graph of $y = ax^2$

- Plot the points $(0,0), (-1, a)$ and $(1, a)$.
- Draw a smooth curve through those 3 points and extending beyond them with arrows at the ends to show behavior at infinity.

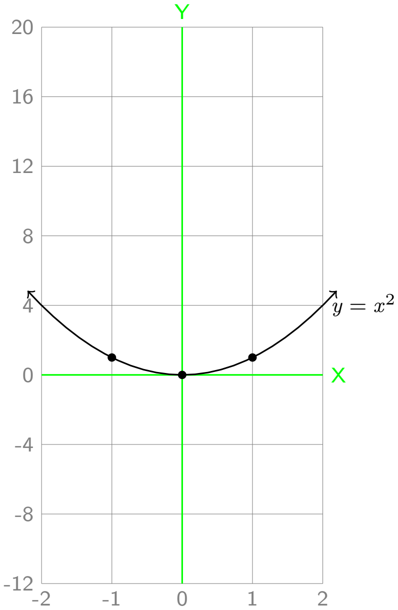
To transform the graph of $y = x^2$ to the graph of $y = ax^2$

- Stretch the graph vertically by a factor of $|a|$.
- If $a < 0$, reflect the graph across the x-axis.

Example 1: Sketch $y = ax^2$ for $a = 1, 2, 5, -\frac{1}{2}$, and $-\frac{5}{2}$.

Solution:

1. $y = x^2$ passes through $(-1, 1)$ and $(1, 1)$.



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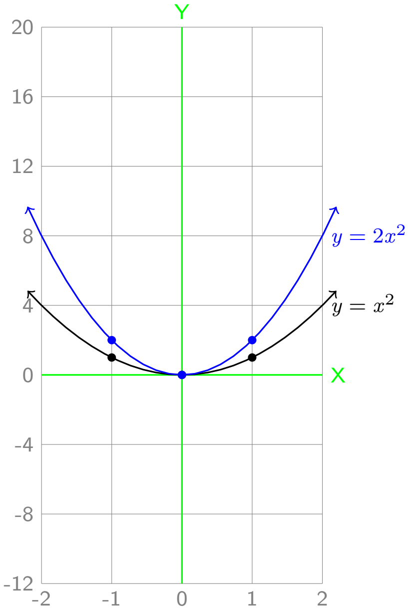


Example 1: Sketch $y = ax^2$ for $a = 1, 2, 5, -\frac{1}{2}$, and $-\frac{5}{2}$.

Solution:

1. $y = x^2$ passes through $(-1, 1)$ and $(1, 1)$.
2. $y = 2x^2$ passes through $(-1, 2)$ and $(1, 2)$.

2.4.2: Graphing $y = ax^2$

Sketching the graph of $y = ax^2 + bx + c$ is easy if $b = c = 0$.

To sketch the graph of $y = ax^2$

- Plot the points $(0, 0), (-1, a)$ and $(1, a)$.
- Draw a smooth curve through those 3 points and extending beyond them with arrows at the ends to show behavior at infinity.

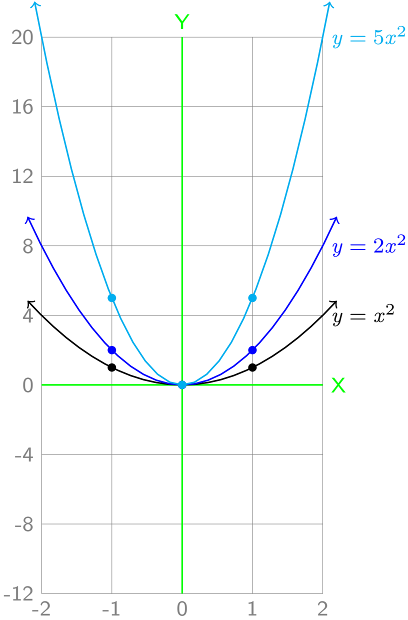
To transform the graph of $y = x^2$ to the graph of $y = ax^2$

- Stretch the graph vertically by a factor of $|a|$.
- If $a < 0$, reflect the graph across the x-axis.

Example 1: Sketch $y = ax^2$ for $a = 1, 2, 5, -\frac{1}{2}$, and $-\frac{5}{2}$.

Solution:

1. $y = x^2$ passes through $(-1, 1)$ and $(1, 1)$.
2. $y = 2x^2$ passes through $(-1, 2)$ and $(1, 2)$.
3. $y = 5x^2$ passes through $(-1, 5)$ and $(1, 5)$.



2.4.2: Graphing $y = ax^2$

Sketching the graph of $y = ax^2 + bx + c$ is easy if $b = c = 0$.

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3. $y = 5x^2$ passes through $(-1, 5)$ and $(1, 5)$.
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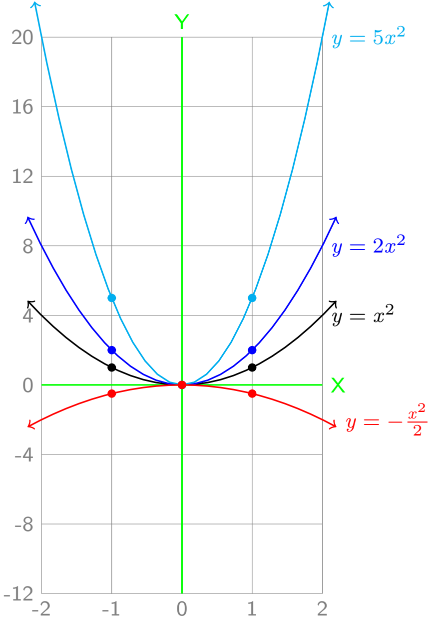

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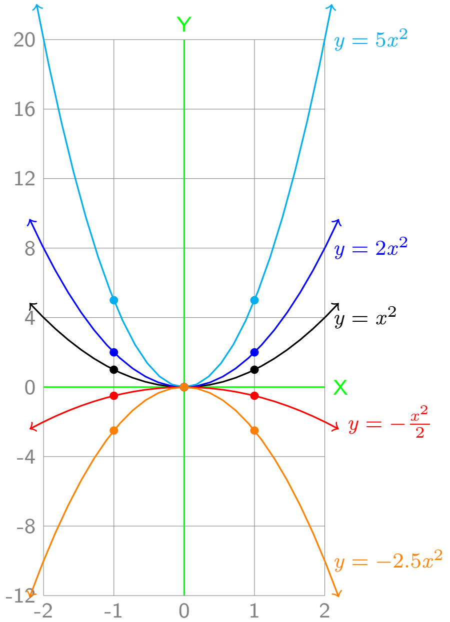

2.4.3 Graphing $y = ax^2 + c$

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- Stretch the graph vertically by a factor of $|a|$.
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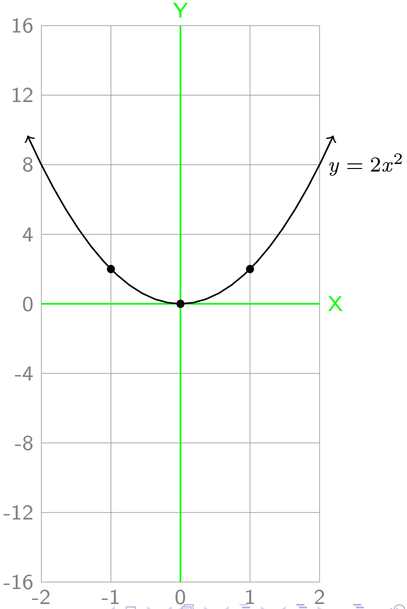
Example 2: Sketch the graph of each of the following:

• $y = 2x^2$ • $y = 2x^2 + 3$ • $y = -3x^2$ • $y = -3x^2 - 2$

1. $y = 2x^2$ has vertex $(0, 0)$ and passes through $(-1, 2)$ and $(1, 2)$.



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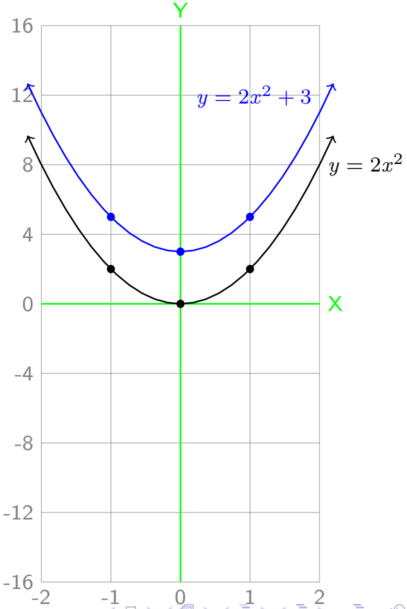

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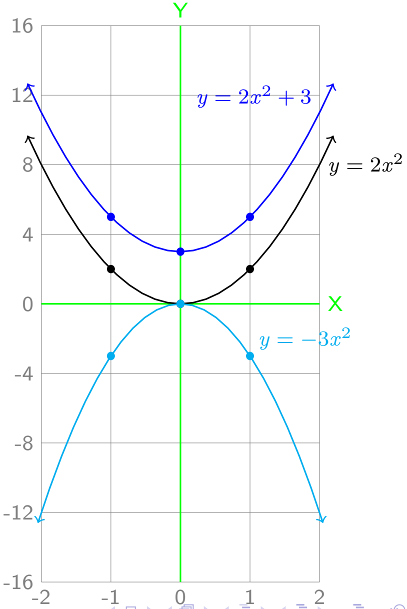
2. $y = 2x^2 + 3$ has vertex $(0, 3)$ and passes through $(-1, 5)$ and $(1, 5)$.
3. $y = -3x^2$ has vertex $(0, 0)$ and passes through $(-1, -3)$ and $(1, -3)$.



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• $y = 2x^2$ • $y = 2x^2 + 3$ • $y = -3x^2$ • $y = -3x^2 - 2$

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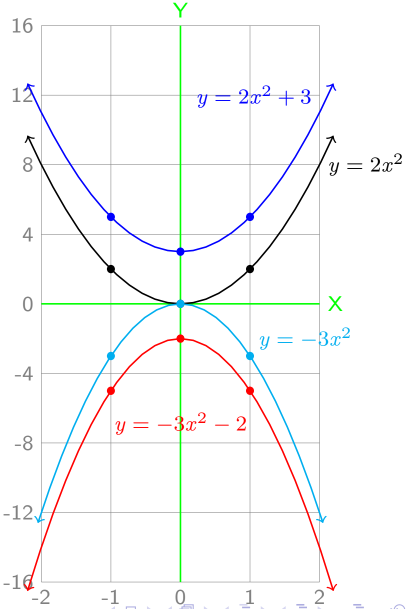
3. $y = -3x^2$ has vertex $(0, 0)$ and passes through $(-1, -3)$ and $(1, -3)$.
4. $y = -3x^2 - 2$ has vertex $(0, -2)$ and passes through $(-1, -5)$ and $(1, -5)$.



2.4.4 Completing the square revisited

In Chapter 1, we used completing the square to rewrite circle equations. In this chapter, we use it to rewrite equations of the form $y = ax^2 + bx + c$.

Identity for *completing the square*

$$x^2 + Bx = \left(x + \frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^2$$

Example 3: Use the identity to rewrite the RHS of the equation $y = x^2 + 8x + 7$.

Let $B = 8$ in	$x^2 + Bx = \left(x + \frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^2$
to obtain	$x^2 + 8x = (x + 4)^2 - 4^2$.
Therefore	$y = x^2 + 8x + 7 = (x + 4)^2 - 16 + 7$.
Answer:	$\boxed{y = (x + 4)^2 - 9}$

Example 4: Use the identity to rewrite $y = 2x^2 + 8x + 7$.
Solution: To obtain an expression of the form $x^2 + Bx$, factor out $a = 2$ from $2x^2 + 8x$ to get $y = 2(x^2 + 4x) + 7$. Then $x^2 + Bx = x^2 + 4x$ so $B = 4$.

Let $B = 4$ in	$x^2 + Bx = \left(x + \frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^2$
to obtain	$x^2 + 4x = (x + 2)^2 - 4$.
Therefore	$y = 2x^2 + 8x + 7$
becomes	$y = 2(x^2 + 4x) + 7$.
From above	$y = 2((x + 2)^2 - 4) + 7$.
Therefore	$y = 2(x + 2)^2 - 8 + 7$
Answer:	$\boxed{y = 2(x + 2)^2 - 1}$

2.4.5 Rewriting $ax^2 + bx + c$ in standard form

The following result is obtained by completing the square.

The standard form of $y = f(x) = ax^2 + bx + c$ is $y = f(x) = a(x-h)^2 + k$ where $h = -\frac{b}{2a}$ and $k = f(h)$. The graph of f is a parabola with vertex $(h, f(h))$.

- The parabola opens upward if $a > 0$, downward if $a < 0$.
- The y-intercept is $c = ah^2 + k = ah^2 + f(h)$.
- If $k = 0$ there is one x-intercept $h = -\frac{b}{2a}$.
- If $\frac{k}{a} > 0$ there are no x-intercepts.
- If $\frac{k}{a} < 0$ there are two x-intercepts $-\frac{b}{2a} \pm \sqrt{-\frac{k}{a}}$.

Example 5: Use the above result to rewrite $y = -2x^2 + 3x + 1$ in standard form.

Solution: $y = ax^2 + bx + c$ where $a = -2$; $b = 3$; $c = 1$. Then $h = -\frac{b}{2a} = \frac{3}{4}$ and $k = -2(\frac{3}{4})^2 + 3(\frac{3}{4}) + 1 = -\frac{9}{8} + \frac{9}{4} + 1 = \frac{17}{8}$.

Standard form: $y = a(x-h)^2 + k = -2(x - \frac{3}{4})^2 + \frac{17}{8}$
where $a = -2$; $h = \frac{3}{4}$, and $k = \frac{17}{8}$

Example 6: Much easier: Find the standard form for $y = -2x^2 + 3x + 1$ by using the identity $x^2 + Bx = (x + \frac{B}{2})^2 - (\frac{B}{2})^2$.

Solution: First factor out the coefficient of x^2!

$y = -2x^2 + 3x + 1$
$= -2(x^2 - \frac{3}{2}x) + 1$. Thus $x^2 + Bx = x^2 - \frac{3}{2}x$ and so $B = -\frac{3}{2}$; $\frac{B}{2} = -\frac{3}{4}$.
$= -2\left((x - \frac{3}{4})^2 - (\frac{3}{4})^2\right) + 1$
$= -2(x - \frac{3}{4})^2 + 2(\frac{9}{16}) + 1$
$= -2(x - \frac{3}{4})^2 + \frac{9}{8} + \frac{8}{8} = -2(x - \frac{3}{4})^2 + \frac{17}{8}$

$$y = -2(x-h)^2 + k = -2(x - \tfrac{3}{4})^2 + \tfrac{17}{8}$$

Examples 5,6 continued: Use the standard form of $y = -2x^2 + 3x + 1$ to describe its graph.

Since $a = -2 < 0$,
the parabola opens downward.
The vertex is $(h, k) = (\frac{3}{4}, \frac{17}{8})$.
The y-intercept (set $x = 0$) is $c = 1$.

To find x-intercepts, solve $-2x^2 + 3x + 1 = 0$:

$x = \frac{-3 \pm \sqrt{3^2 - 4(-2)(1)}}{2(-2)}$: x-intercepts are $\frac{3 \pm \sqrt{17}}{4}$

2.4.6 Graphing $y = ax^2 + bx + c$

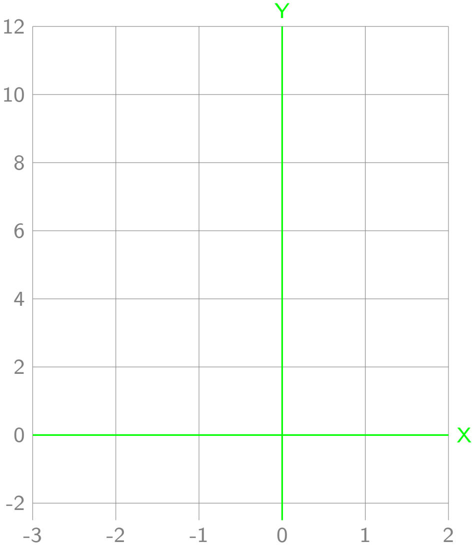
Now we graph the general quadratic function.

To graph $y = f(x) = ax^2 + bx + c$; $a > 0$

- Let $h = -\frac{b}{2a}$ and $k = f(h)$.
- $y = f(x) = a(x-h)^2 + k$ with vertex (h, k).
- $f(0) = f(2h) = ah^2 + k = c$.
- Plot y-intercept point $(0, f(0)) = (0, c)$, vertex $(h, f(h)) = (h, k)$, and point $(2h, f(2h)) = (2h, c)$.
- Draw a smooth curve through those 3 points and extending to the edges of the grid, with arrows at both ends to show behavior at infinity.

Example 7: Draw the graph of $y = f(x) = 2x^2 + 4x + 3$.

Solution: Here $a = 2$; $b = 4$; $c = 3$. Then
$y = f(x) = (x-h)^2 + k$ where
$h = -\frac{b}{2a} = -1$ and
$k = f(h) = f(-1) = 2(-1)^2 + 4(-1) + 3 = 1$.



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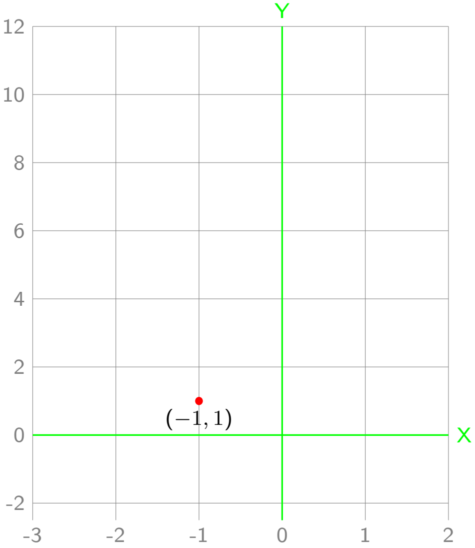
- The vertex is $(h, k) = (h, f(h)) = (-1, 1)$.



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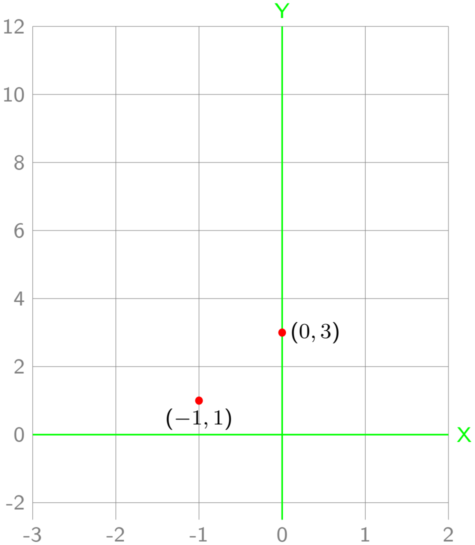
Solution: Here $a = 2$; $b = 4$; $c = 3$. Then
$y = f(x) = (x-h)^2 + k$ where
$h = -\frac{b}{2a} = -1$ and
$k = f(h) = f(-1) = 2(-1)^2 + 4(-1) + 3 = 1$.

- The vertex is $(h, k) = (h, f(h)) = (-1, 1)$.
- Plot $(0, f(0)) = (0, c) = (0, 3)$, the y-intercept point.



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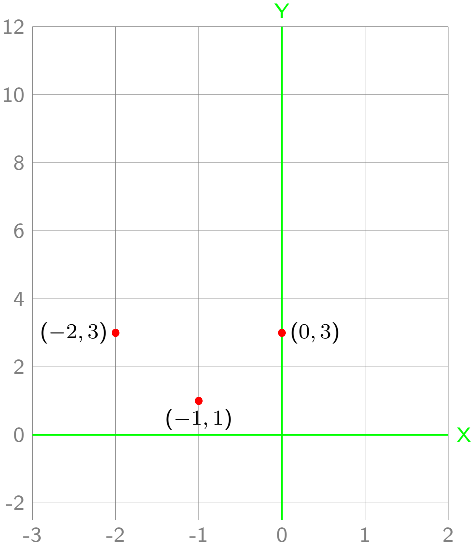
- The vertex is $(h, k) = (h, f(h)) = (-1, 1)$.
- Plot $(0, f(0)) = (0, c) = (0, 3)$, the y-intercept point.
- Plot $(2h, f(2h)) = (-2, c) = (-2, 3)$.



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- The vertex is $(h, k) = (h, f(h)) = (-1, 1)$.
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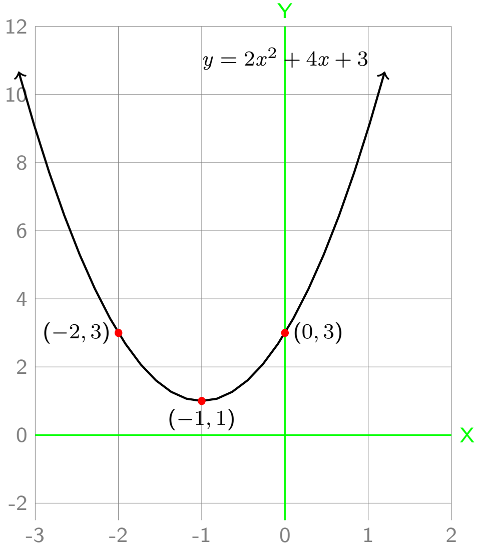
- Plot $(2h, f(2h)) = (-2, c) = (-2, 3)$.
- Draw the curve as shown.



2.4.7: Transforming a quadratic function to standard form

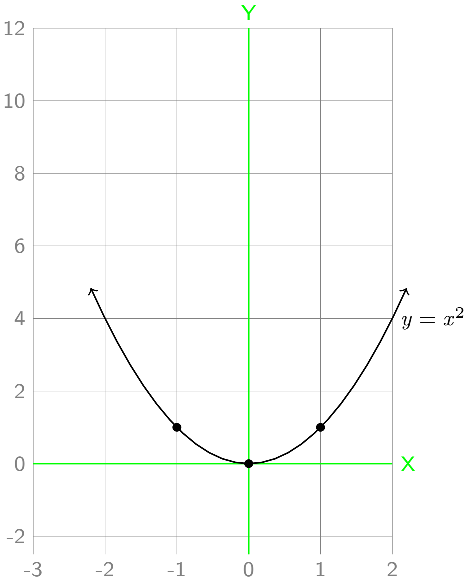
To sketch the graph of $y = a(x-h)^2 + k$ by transforming the graph of $y = ax^2$

- Draw the graph of $y = ax^2$.
- Substitute $x - h$ for x in $y = ax^2$ to get new equation $y = a(x-h)^2$. The graph shifts h units horizontally.
- Add k to the RHS of $y = a(x-h)^2$ to get new equation $y = a(x-h)^2 + k$. The graph shifts k units vertically.

Example 8: Rewrite $y = 2x^2 + 4x + 3$ in standard form and sketch its graph by transforming the graph of $y = x^2$.

Solution: Complete the square:
$2x^2 + 4x + 3 = 2(x^2 + 2x) + 3 = 2\left((x+1)^2 - 1^2\right) + 3$
$= 2(x+1)^2 + 1 : a = 2 : h = -1; k = 1.$

- Sketch $y = x^2$.



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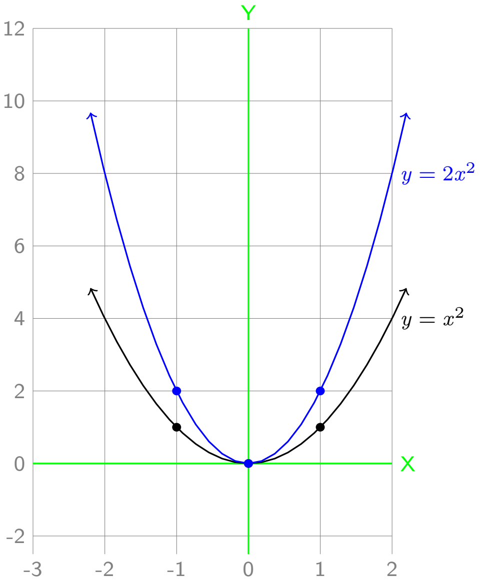
Solution: Complete the square:
$2x^2 + 4x + 3 = 2(x^2 + 2x) + 3 = 2\left((x+1)^2 - 1^2\right) + 3$
$= 2(x+1)^2 + 1 : a = 2 : h = -1; k = 1.$

- Sketch $y = x^2$.
- Stretch the graph vertically by a factor of 2 to get $y = 2x^2$ with vertex $(0,0)$ and passing through$(-1,2)$ and $(1,2)$.



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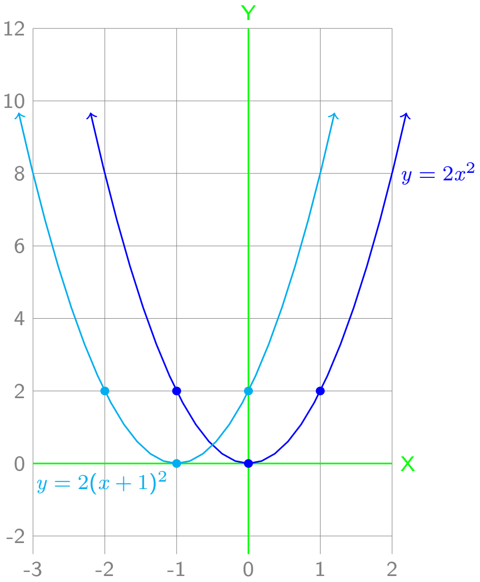
Solution: Complete the square:

$2x^2 + 4x + 3 = 2(x^2 + 2x) + 3 = 2\left((x+1)^2 - 1^2\right) + 3$

$= 2(x+1)^2 + 1 : a = 2 : h = -1; k = 1.$

- Sketch $y = x^2$.
- Stretch the graph vertically by a factor of 2 to get $y = 2x^2$ with vertex $(0,0)$ and passing through$(-1,2)$ and $(1,2)$.
- Shift the graph of $y = 2x^2$ left 1 unit to get the graph of $y = 2(x+1)^2$, with vertex $(-1,0)$



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To sketch the graph of $y = a(x-h)^2 + k$ by transforming the graph of $y = ax^2$

- Draw the graph of $y = ax^2$.
- Substitute $x - h$ for x in $y = ax^2$ to get new equation $y = a(x-h)^2$. The graph shifts h units horizontally.
- Add k to the RHS of $y = a(x-h)^2$ to get new equation $y = a(x-h)^2 + k$. The graph shifts k units vertically.

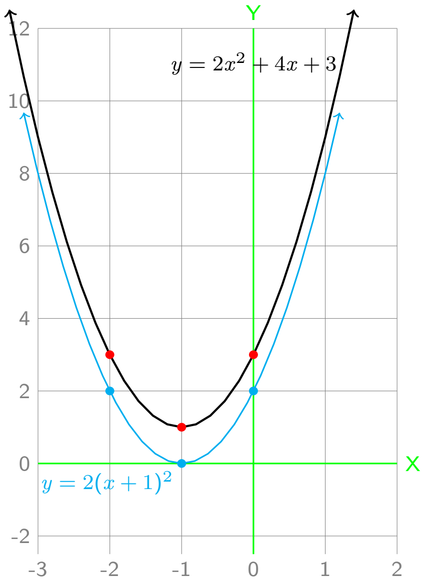
Example 8: Rewrite $y = 2x^2 + 4x + 3$ in standard form and sketch its graph by transforming the graph of $y = x^2$.

Solution: Complete the square:

$2x^2 + 4x + 3 = 2(x^2 + 2x) + 3 = 2\left((x+1)^2 - 1^2\right) + 3$

$= 2(x+1)^2 + 1 : a = 2 : h = -1; k = 1.$

- Sketch $y = x^2$.
- Stretch the graph vertically by a factor of 2 to get $y = 2x^2$ with vertex $(0,0)$ and passing through$(-1,2)$ and $(1,2)$.
- Shift the graph of $y = 2x^2$ left 1 unit to get the graph of $y = 2(x+1)^2$, with vertex $(-1,0)$
- Move the graph up 1 unit to get the graph of $y = 2(x+1)^2 + 1 = 2x^2 + 4x + 3$ with vertex $(-1,1)$.



2.4.8: Finding the x-intercepts of a quadratic function

A graph may, or may not, have x-intercepts.

To find the intercepts of the parabola $y = ax^2 + bx + c$

- Set $x = 0$ to find the y-intercept $y = c$.
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  - If $D > 0$ there are two x-intercepts $x = \frac{-b \pm \sqrt{D}}{2a}$.
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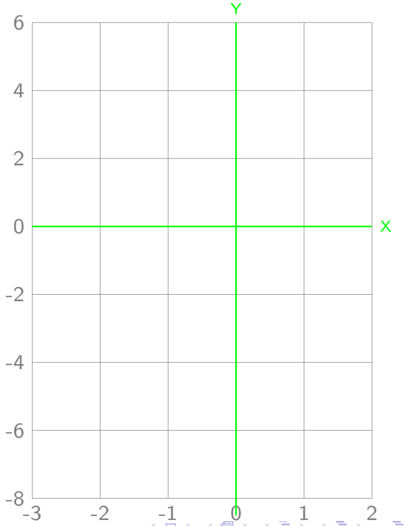
Example 9: Draw the graph of $y = 2x^2 + 4x - 3$. Color any intercept points black.

Solution: $2x^2 + 4x - 3 = 2(x^2 + 2x) - 3 = 2\left((x + 1)^2 - 1^2\right) - 3$
$= 2(x + 1)^2 - 5 : a = 2; h = -1; k = 5.$



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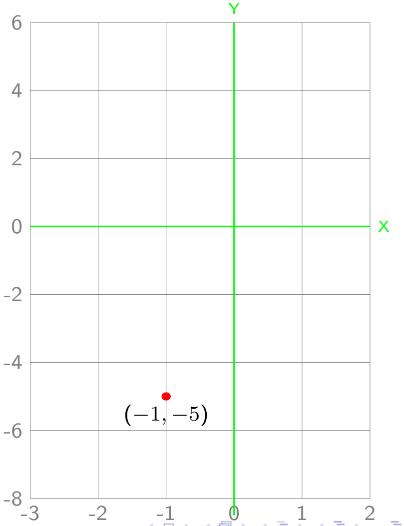
- The vertex is $(h, k) = (-1, -5)$. Plot it.



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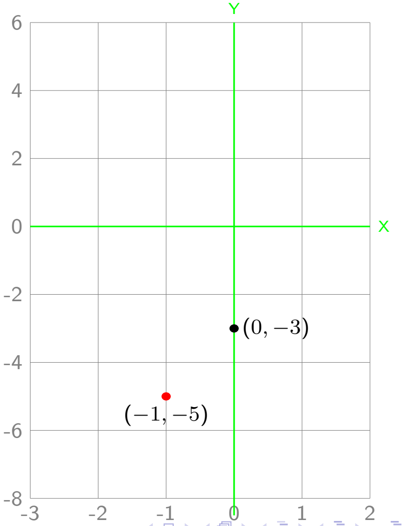

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- The vertex is $(h, k) = (-1, -5)$. Plot it.
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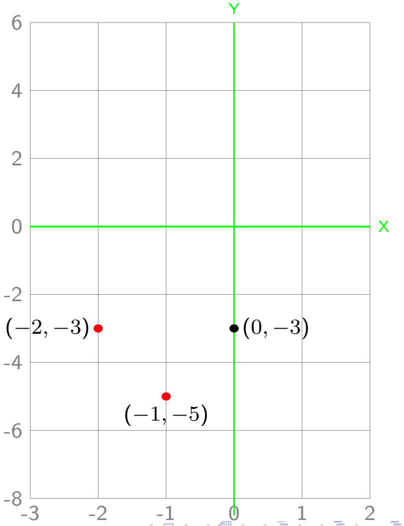
- Plot $(2h, f(2h)) = (-2, c) = (-2, -3)$.



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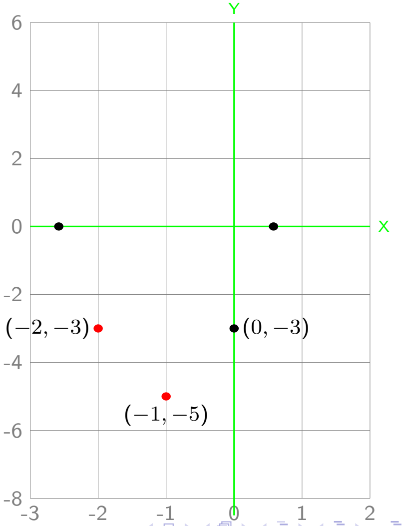
- The vertex is $(h, k) = (-1, -5)$. Plot it.
- Plot $(0, f(0)) = (0, c) = (0, -3)$, the y-intercept.
- Plot $(2h, f(2h)) = (-2, c) = (-2, -3)$.
- Since $D = b^2 - 4ac = 4^2 - 4 \cdot 2 \cdot -3 = 40$ is positive, the x-intercepts are $x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-4 \pm \sqrt{40}}{4} \approx .58$ or -2.58.



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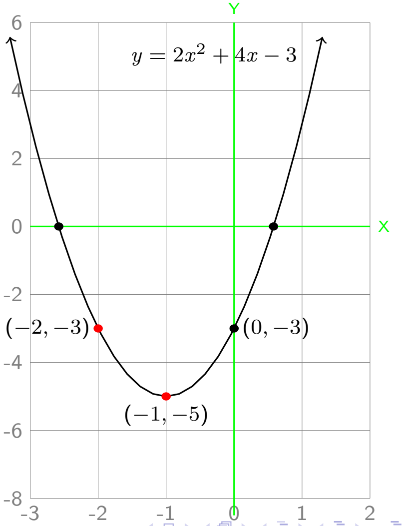
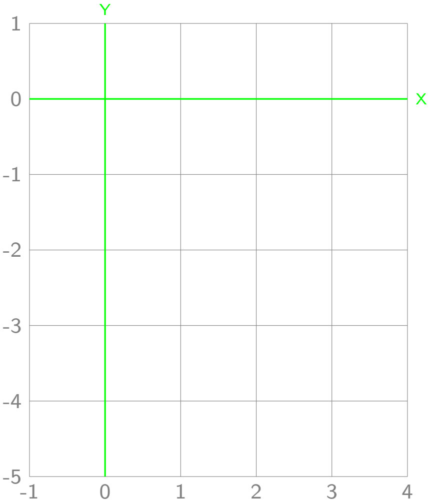


Example 10: Draw the graph of $y = -2x^2 + 6x - 4$.
Use the solution method of Example 9.

Solution: Complete the square:
$y = -2(x^2 - 3x) - 4 = -2((x - \frac{3}{2})^2 - \frac{9}{4}) - 4 =$
$-2(x - \frac{3}{2})^2 + \frac{9}{2} - 4 = -2(x - \frac{3}{2})^2 + \frac{1}{2}$.



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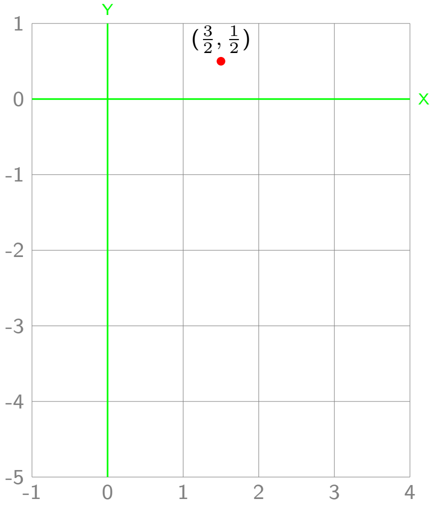
- Therefore standard form is $y = -2(x - \frac{3}{2})^2 + \frac{1}{2}$ and the vertex is $(h, k) = (\frac{3}{2}, \frac{1}{2})$. Plot it.



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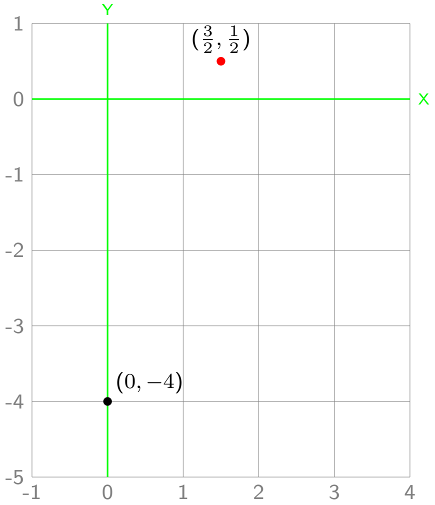

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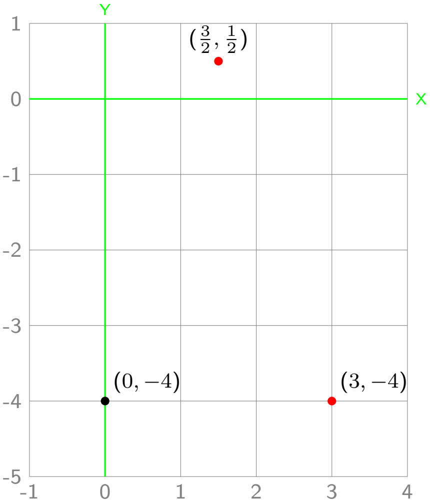

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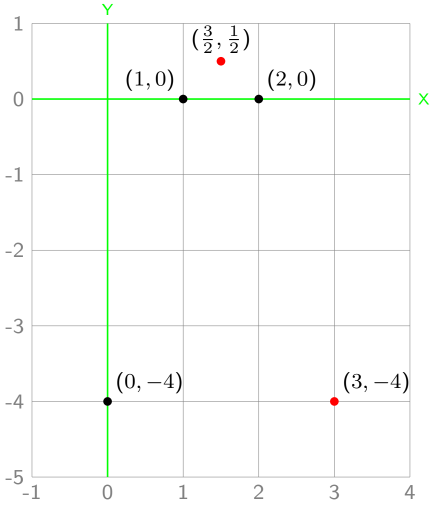

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- Therefore standard form is $y = -2(x - \frac{3}{2})^2 + \frac{1}{2}$ and the vertex is $(h, k) = (\frac{3}{2}, \frac{1}{2})$. Plot it.
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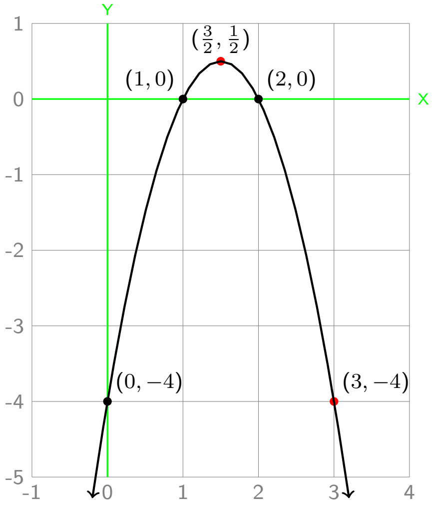
- Choose a range of y-values that includes the y-values of all these points, namely $-4, 0, 0, \frac{1}{2}$. A reasonable choice is $-5 \leq y \leq 1$.
- Join the 6 plotted points by a smooth curve, with arrows at both ends to show behavior at infinity.



$y = -2x^2 + 6x - 4$

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- Therefore standard form is $y = -2(x - \frac{3}{2})^2 + \frac{1}{2}$ and the vertex is $(h, k) = (\frac{3}{2}, \frac{1}{2})$. Plot it.
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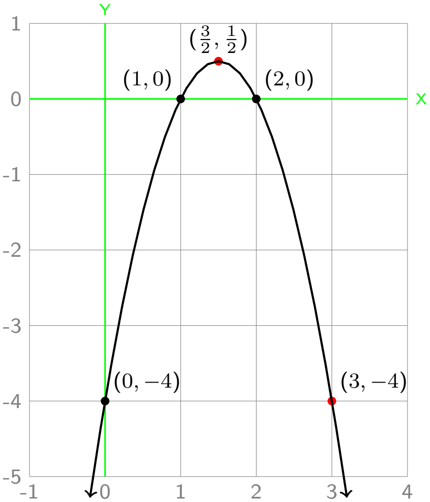
Exercise: Use the method of Example 8 to redo this problem by transforming the graph of $y = x^2$.



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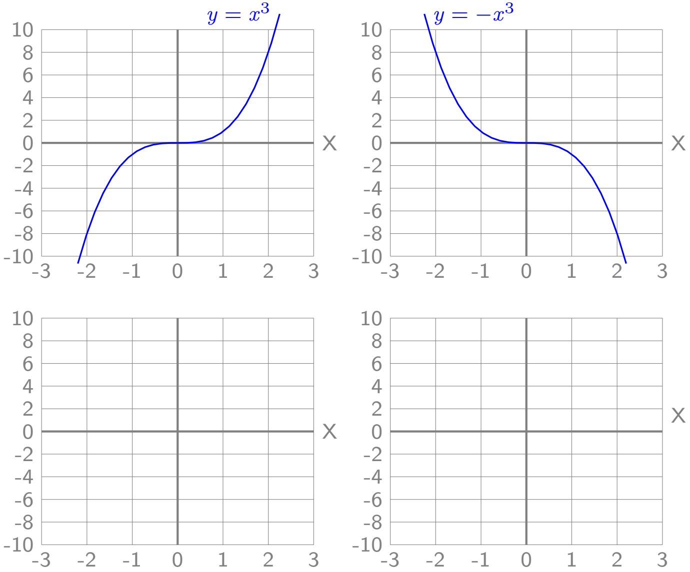
Graphs of degree 3 polynomials (preview)

We will sketch graphs of degree 3 and degree 4 polynomials in the next section.

The graph of a degree 3 polynomial

- has no absolute maximum or minimum points;
- has either no local maximum/minimum point,



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We will sketch graphs of degree 3 and degree 4 polynomials in the next section.

The graph of a degree 3 polynomial

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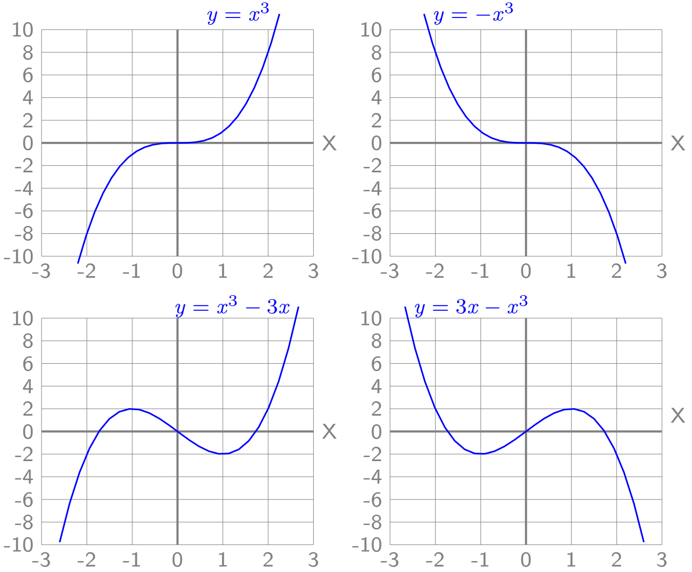
or has one local maximum point and one local minimum point.



2.4.9: Precalculus Section 2.4 Quiz

Ex. 2.4.1: Sketch $y = ax^2$ for $a = 1, 2, 5, -\frac{1}{2}$, and $-\frac{5}{2}$.

Ex. 2.4.2: Sketch the graph of each of the following:

- $y = 2x^2$
- $y = 2x^2 + 3$
- $y = -3x^2$
- $y = -3x^2 - 2$

Ex. 2.4.3: Use completing the square to rewrite the RHS of the equation $y = x^2 + 8x + 7$.

Ex. 2.4.4: Use the identity $x^2 + Bx = \left(x + \frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^2$ to rewrite $y = x^2 + 8x + 7$.

Ex. 2.4.5: Rewrite $y = f(x) = -2x^2 + 3x + 1$ in standard form and describe its graph.

Ex. 2.4.6: Rewrite $y = -2x^2 + 3x + 1$ in standard form by completing the square.

Use the identity $x^2 + Bx = (x + \frac{B}{2})^2 - (\frac{B}{2})^2$

Ex. 2.4.7: Draw the graph of $y = 2x^2 + 4x + 3$.

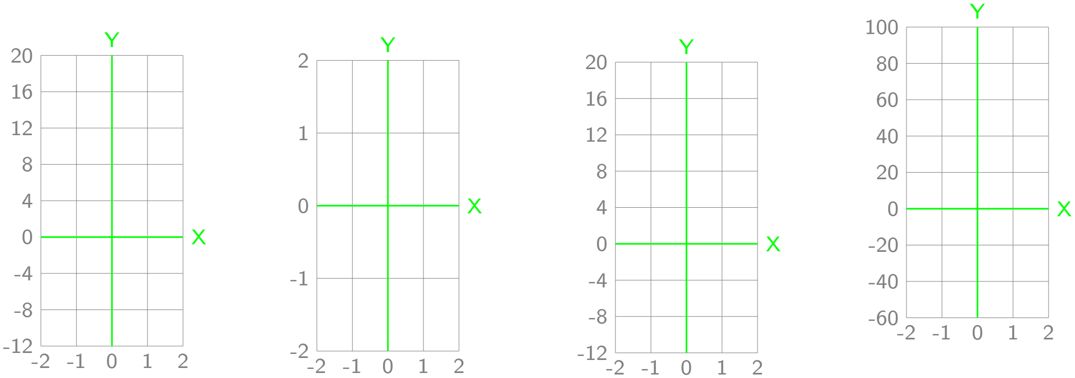
Ex. 2.4.8: Transform the equation $y = x^2$ to $y = 2x^2 + 4x + 3$ and sketch the graph at each stage.

Ex. 2.4.9: Draw the graph of $y = 2x^2 + 4x - 3$. Color intercept points black .

Ex. 2.4.10: Draw the graph of $y = -2x^2 + 6x - 4$. Color intercept points black .

Section 2.4 Review: Quadratic functions

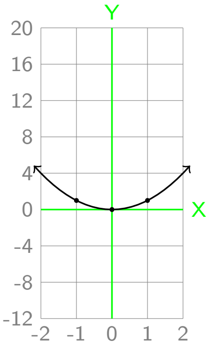
Ex. 2.4.1: Sketch the graphs of

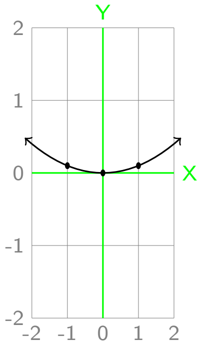


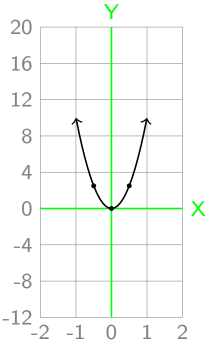
Section 2.4 Review: Quadratic functions

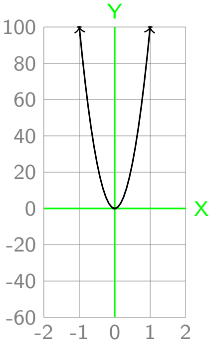
Ex. 2.4.1: Sketch the graphs of

$y = x^2$ $\quad y = \frac{x^2}{10}$ $\quad y = 10x^2$ $\quad y = 100x^2$









Section 2.4 Review: Quadratic functions

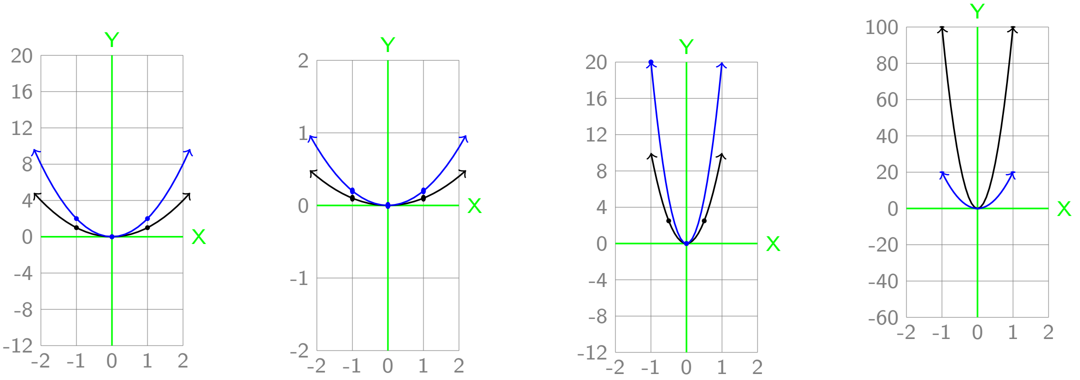
Ex. 2.4.1: Sketch the graphs of

$y = x^2$
$y = 2x^2$

$y = \frac{x^2}{10}$
$y = \frac{x^2}{5}$

$y = 10x^2$
$y = 20x^2$

$y = 100x^2$
$y = 20x^2$



Section 2.4 Review: Quadratic functions

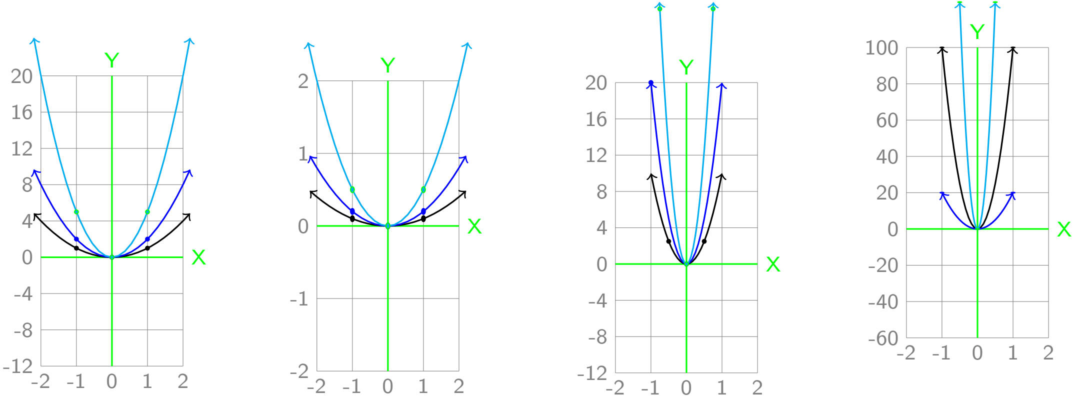
Ex. 2.4.1: Sketch the graphs of

$y = x^2$
$y = 2x^2$
$y = 5x^2$

$y = \frac{x^2}{10}$
$y = \frac{x^2}{5}$
$y = \frac{x^2}{2}$

$y = 10x^2$
$y = 20x^2$
$y = 50x^2$

$y = 100x^2$
$y = 20x^2$
$y = 500x^2$



Section 2.4 Review: Quadratic functions

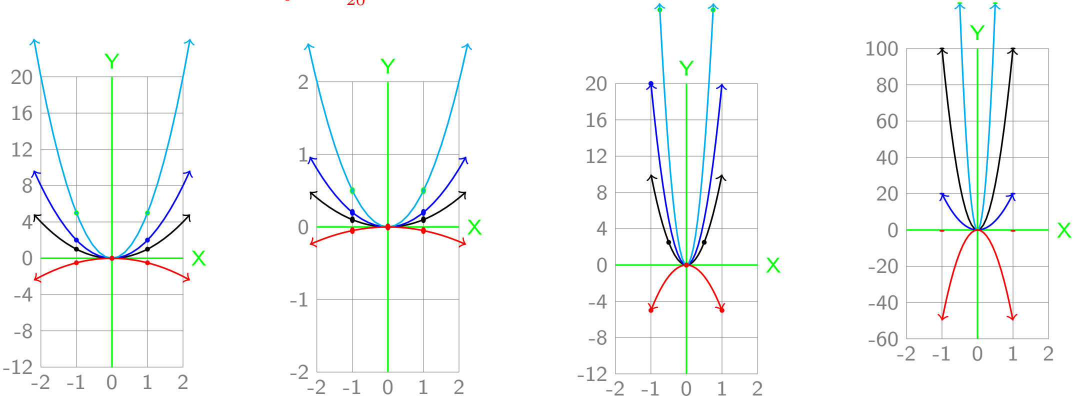
Ex. 2.4.1: Sketch the graphs of

$y = x^2$
$y = 2x^2$
$y = 5x^2$
$y = -\frac{1}{2}x^2$

$y = \frac{x^2}{10}$
$y = \frac{x^2}{5}$
$y = \frac{x^2}{2}$
$y = -\frac{x^2}{20}$

$y = 10x^2$
$y = 20x^2$
$y = 50x^2$
$y = -5x^2$

$y = 100x^2$
$y = 20x^2$
$y = 500x^2$
$y = -50x^2$



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$y = -\frac{x^2}{20}$
$y = -\frac{x^2}{4}$

$y = 10x^2$
$y = 20x^2$
$y = 50x^2$
$y = -5x^2$
$y = -25x^2$

$y = 100x^2$
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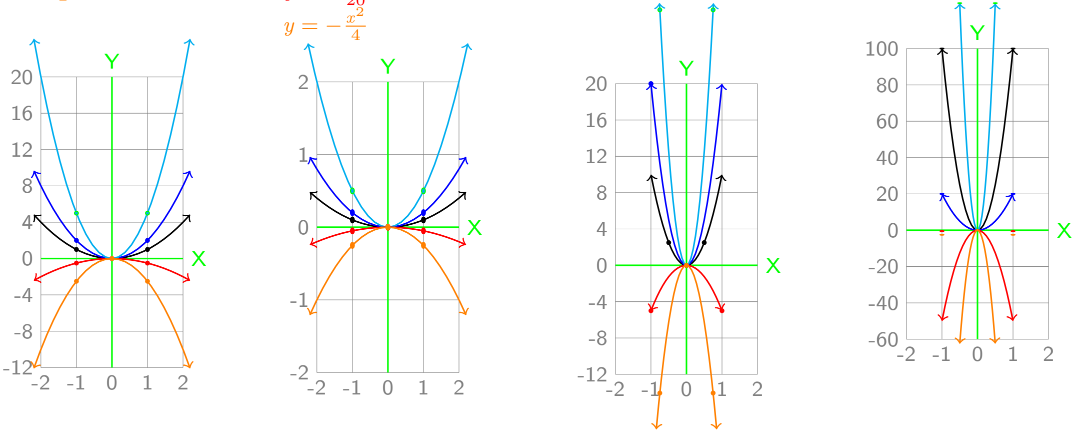
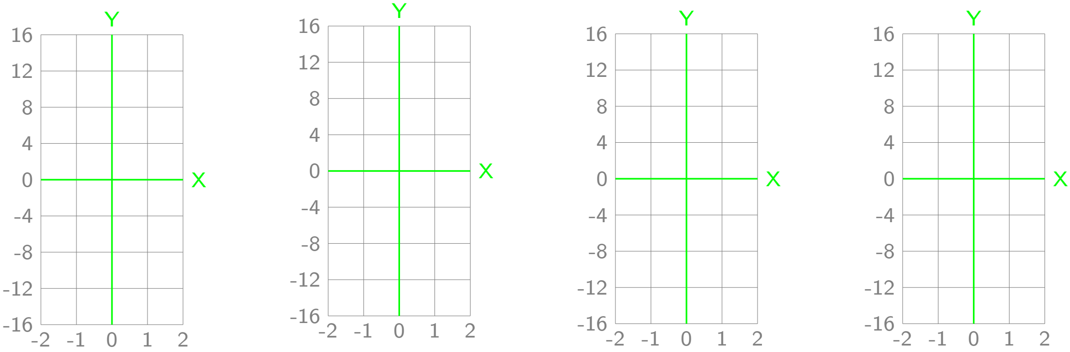
$y = 500x^2$
$y = -50x^2$
$y = -250x^2$

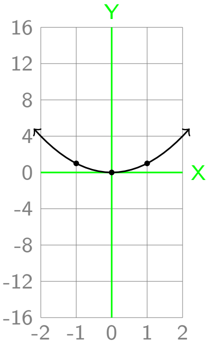


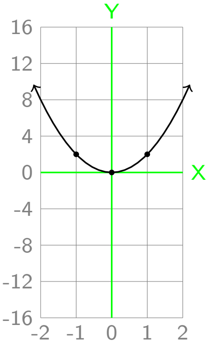
Ex. 2.4.2: Sketch the graph of each of the following. Plot points with x-coordinates $-1, 0, 1$.

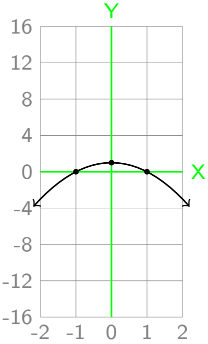


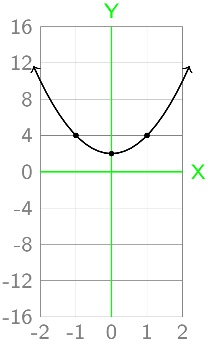
Ex. 2.4.2: Sketch the graph of each of the following. Plot points with x-coordinates $-1, 0, 1$.

- $y = x^2$
- $y = 2x^2$
- $y = 1 - x^2$
- $y = 2 + 2x^2$









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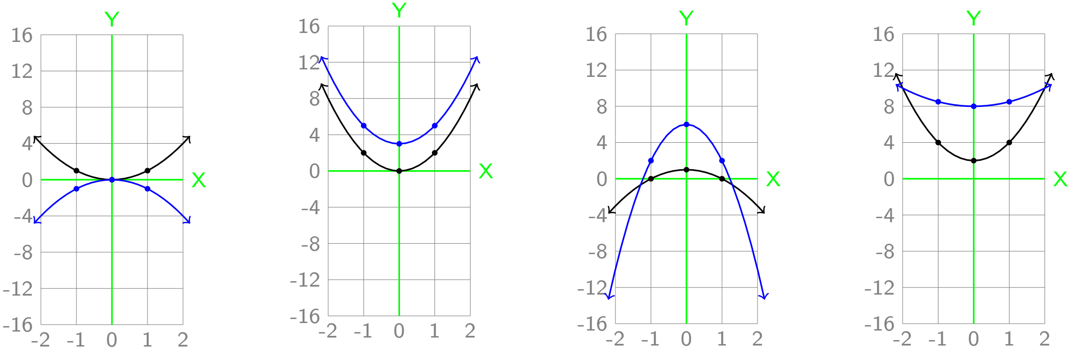
- $y = x^2$
- $y = -x^2$

- $y = 2x^2$
- $y = 2x^2 + 3$

- $y = 1 - x^2$
- $y = 6 - 4x^2$

- $y = 2 + 2x^2$
- $y = .5x^2 + 8$



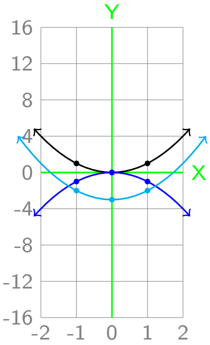
Ex. 2.4.2: Sketch the graph of each of the following. Plot points with x-coordinates $-1, 0, 1$.

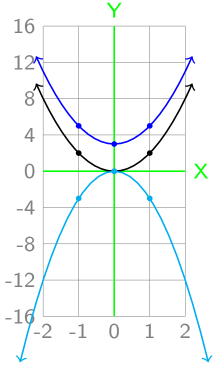
- $y = x^2$
- $y = -x^2$
- $y = x^2 - 3$

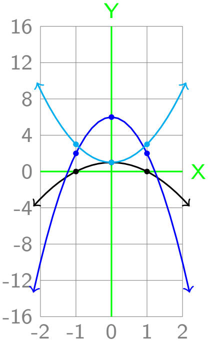
- $y = 2x^2$
- $y = 2x^2 + 3$
- $y = -3x^2$

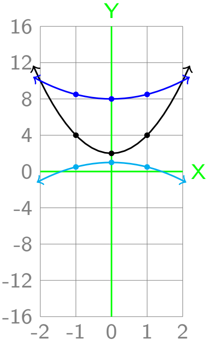
- $y = 1 - x^2$
- $y = 6 - 4x^2$
- $y = 2x^2 + 1$

- $y = 2 + 2x^2$
- $y = .5x^2 + 8$
- $y = 1 - .5x^2$









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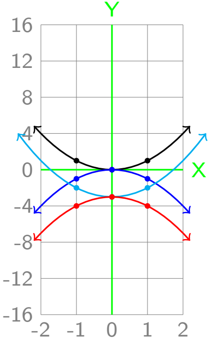
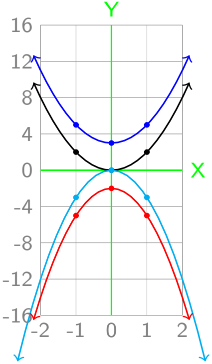
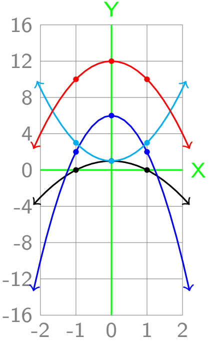


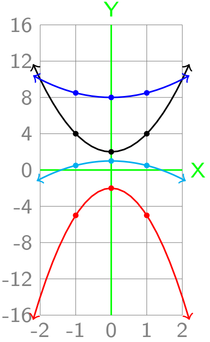
- $y = 2x^2$
- $y = 2x^2 + 3$
- $y = -3x^2$
- $y = -3x^2 - 2$



- $y = 1 - x^2$
- $y = 6 - 4x^2$
- $y = 2x^2 + 1$
- $12 - 2x^2$



- $y = 2 + 2x^2$
- $y = .5x^2 + 8$
- $y = 1 - .5x^2$
- $1 - 2x^2$



Ex. 2.4.3: Use completing the square to rewrite the RHS of the equation

- $y = x^2 + 8x + 7 \Rightarrow$
- $y = x^2 - 4x \Rightarrow$
- $y = x^2 + 9x - 3 \Rightarrow$
- $y = x^2 + 10x + 24 \Rightarrow$

Ex. 2.4.3: Use completing the square to rewrite the RHS of the equation

- $y = x^2 + 8x + 7 \Rightarrow y = (x+4)^2 - 9$
- $y = x^2 + 9x - 3 \Rightarrow y = (x + \frac{9}{2})^2 - \frac{93}{4}$
- $y = x^2 - 4x \Rightarrow y = (x-2)^2 - 4$
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Ex. 2.4.4: Redo the last Example by using the identity $x^2 + Bx = \left(x + \frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^2$ to complete the square.

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Ex. 2.4.5: Rewrite in standard form and describe the graph

- $y = f(x) = -2x^2 + 3x + 1 \Rightarrow$
- $y = f(x) = 2x^2 + 3x + 1 \Rightarrow$
- $y = f(x) = -x^2 + 3x \Rightarrow$
- $y = f(x) = x^2 + 3x \Rightarrow$

Ex. 2.4.3: Use completing the square to rewrite the RHS of the equation

- $y = x^2 + 8x + 7 \Rightarrow y = (x+4)^2 - 9$
- $y = x^2 - 4x \Rightarrow y = (x-2)^2 - 4$
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Ex. 2.4.4: Redo the last Example by using the identity $x^2 + Bx = \left(x + \frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^2$ to complete the square.

Ex. 2.4.5: Rewrite in standard form and describe the graph

- $y = f(x) = -2x^2 + 3x + 1 \Rightarrow y = -2(x - \frac{3}{4})^2 + \frac{17}{8}$ The parabola opens downward.

The vertex is $(h,k) = (\frac{3}{4}, \frac{17}{8})$. The y-intercept is $c = 1$. x-intercepts are $\frac{3 \pm \sqrt{17}}{4}$

- $y = f(x) = 2x^2 + 3x + 1 \Rightarrow y = 2(x + \frac{3}{4})^2 - \frac{1}{8}$ The parabola opens upward .

The vertex is $(h,k) = (-\frac{3}{4}, -\frac{1}{8})$. The y-intercept is $c = 1$. x-intercepts are -1 and $-\frac{1}{2}$.

- $y = f(x) = -x^2 + 3x \Rightarrow y = -(x - \frac{3}{2})^2 + \frac{9}{4}$ The parabola opens downward.

The vertex is $(h,k) = (\frac{3}{2}, \frac{9}{4})$. The y-intercept is $c = 0$. x-intercepts are 0 and 3

- $y = f(x) = x^2 + 3x \Rightarrow y = (x + \frac{3}{2})^2 - \frac{9}{4}$ The parabola opens upward.

The vertex is $(h,k) = (-\frac{3}{2}, -\frac{9}{4})$. The y-intercept is 0. x-intercepts are 0 and -3.

Ex. 2.4.3: Use completing the square to rewrite the RHS of the equation

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- $y = f(x) = -2x^2 + 3x + 1 \Rightarrow y = -2(x - \frac{3}{4})^2 + \frac{17}{8}$ The parabola opens downward.

The vertex is $(h,k) = (\frac{3}{4}, \frac{17}{8})$. The y-intercept is $c = 1$. x-intercepts are $\frac{3 \pm \sqrt{17}}{4}$

- $y = f(x) = 2x^2 + 3x + 1 \Rightarrow y = 2(x + \frac{3}{4})^2 - \frac{1}{8}$ The parabola opens upward .

The vertex is $(h,k) = (-\frac{3}{4}, -\frac{1}{8})$. The y-intercept is $c = 1$. x-intercepts are -1 and $-\frac{1}{2}$.

- $y = f(x) = -x^2 + 3x \Rightarrow y = -(x - \frac{3}{2})^2 + \frac{9}{4}$ The parabola opens downward.

The vertex is $(h,k) = (\frac{3}{2}, \frac{9}{4})$. The y-intercept is $c = 0$. x-intercepts are 0 and 3

- $y = f(x) = x^2 + 3x \Rightarrow y = (x + \frac{3}{2})^2 - \frac{9}{4}$ The parabola opens upward.

The vertex is $(h,k) = (-\frac{3}{2}, -\frac{9}{4})$. The y-intercept is 0. x-intercepts are 0 and -3.

Ex. 2.4.6: Redo the last Example by using the identity $x^2 + Bx = \left(x + \frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^2$ to complete the square.

Ex. 2.4.7: Draw the graph of each quadratic function $y = f(x) = ax^2 + bx + c$ by first rewriting it as $y = a(x-h)^2 + k$. Then plot the vertex (h, k) and points $(0, f(0))$ and $(2h, f(2h))$.

$y = 2x^2 + 4x + 3.$

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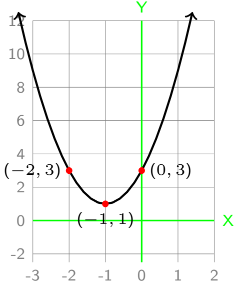

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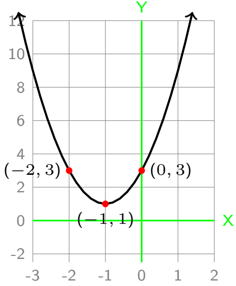

$y = -2x^2 + 3x + 1.$

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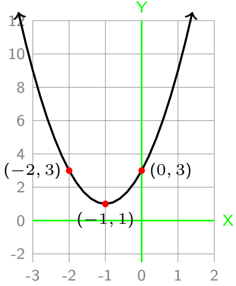

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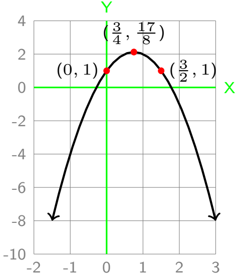
$y = -2(x - \frac{3}{4})^2 + \frac{17}{8}$



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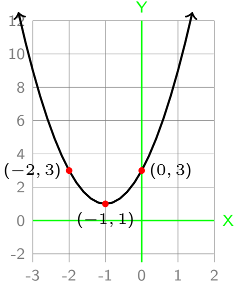

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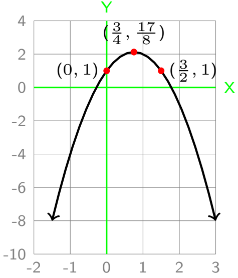

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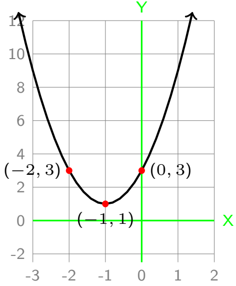

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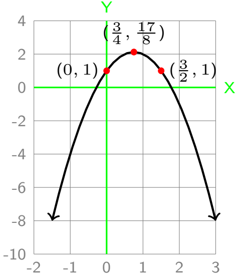
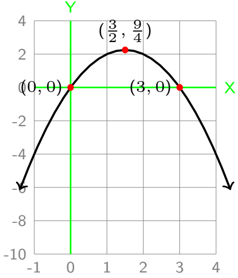


$y = -x^2 + 3x.$

$y = -(x - \frac{3}{2})^2 + \frac{9}{4}$



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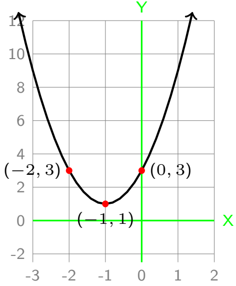

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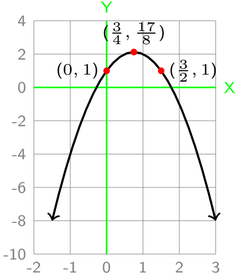

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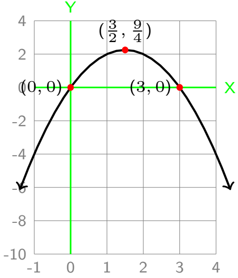

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$y = 2x^2 + 4x + 3.$

$y = 2(x+1)^2 + 1$

$y = -2x^2 + 3x + 1.$

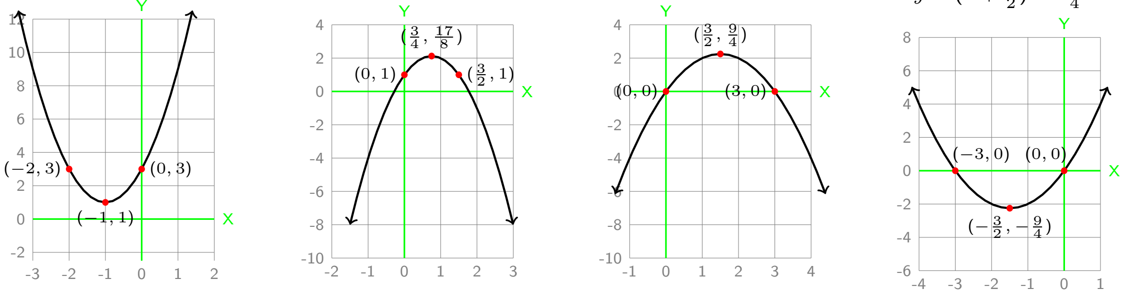
$y = -2(x - \frac{3}{4})^2 + \frac{17}{8}$

$y = -x^2 + 3x.$

$y = -(x - \frac{3}{2})^2 + \frac{9}{4}$

$y = x^2 + 3x.$

$y = (x + \frac{3}{2})^2 - \frac{9}{4}$

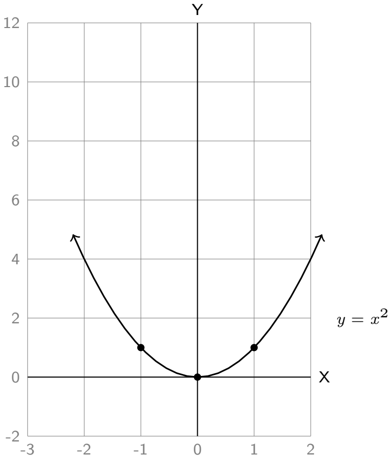


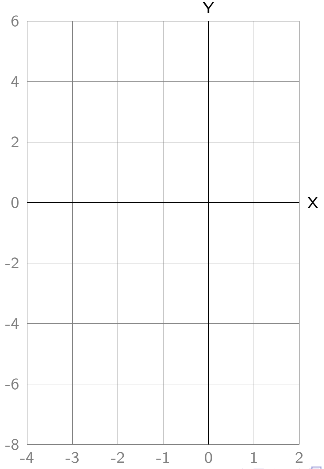
Ex. 2.4.8: Rewrite each function in standard form as $y = a(x-h)^2 + k$ and sketch its graph by transforming the graph of $y = x^2$. Plot points with x-coordinates $0, h, 2h$ if $h \neq 0$ or $(0, -1, 1)$ if $h = 0$.

- $y = 2x^2 + 4x + 3$ rewritten in standard form is:

$y = 2(x+1)^2 + 1$; $a = 2, h = -1$

- First graph $y = x^2$





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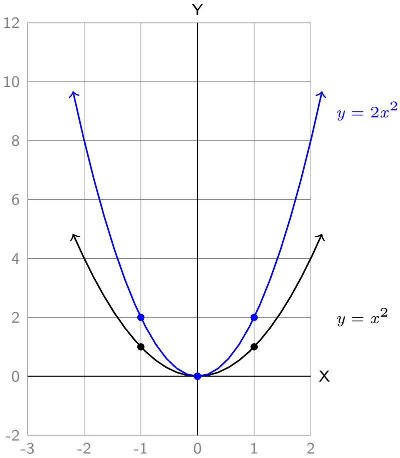
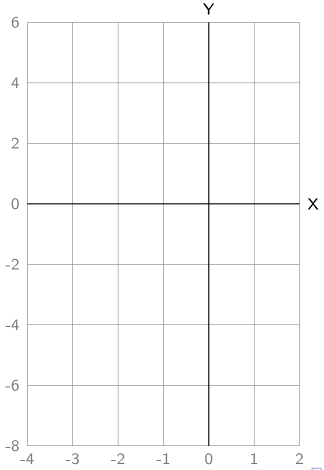
- $y = 2x^2 + 4x + 3$ rewritten in standard form is:

$y = 2(x+1)^2 + 1$; $a = 2, h = -1$

- First graph $y = x^2$
- Sketch $y = ax^2 = 2x^2$





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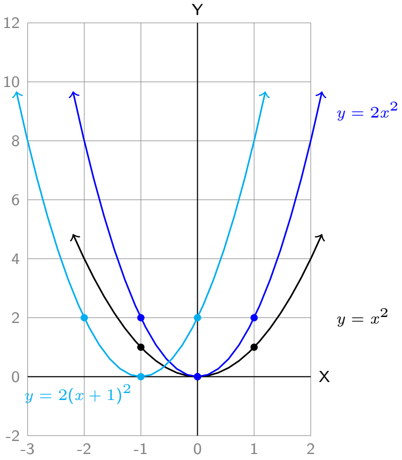

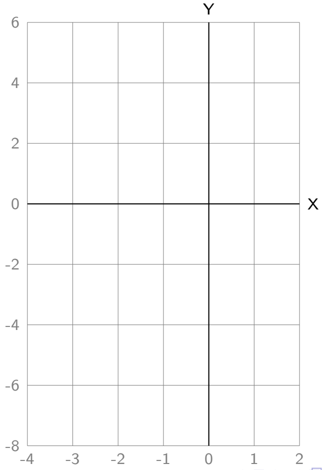

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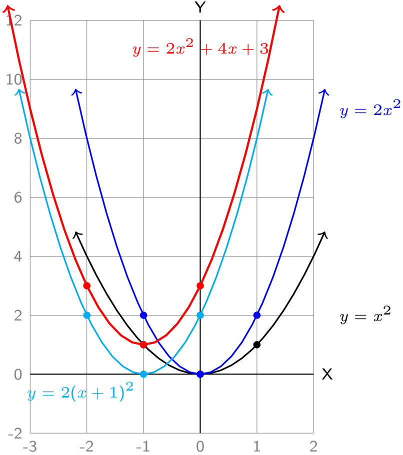

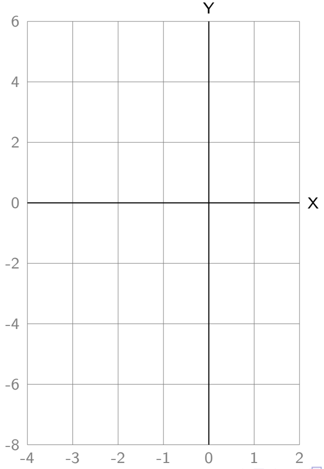

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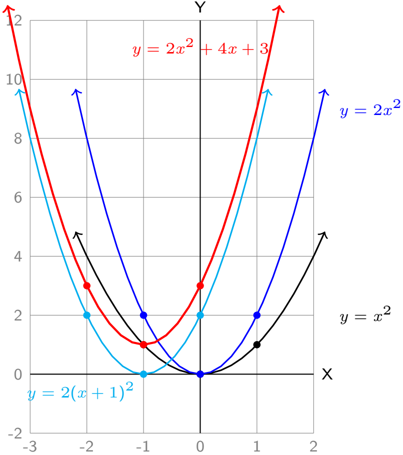

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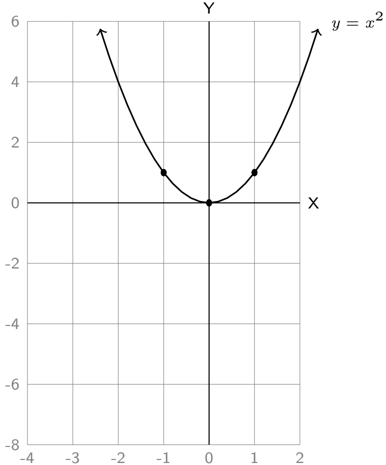

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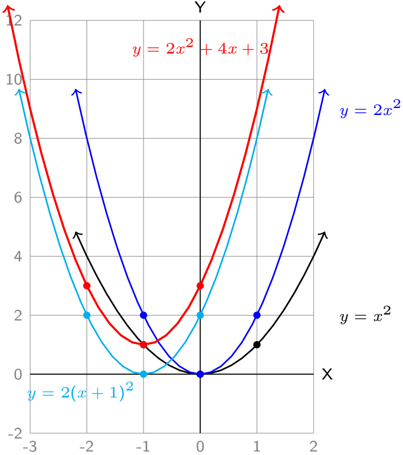

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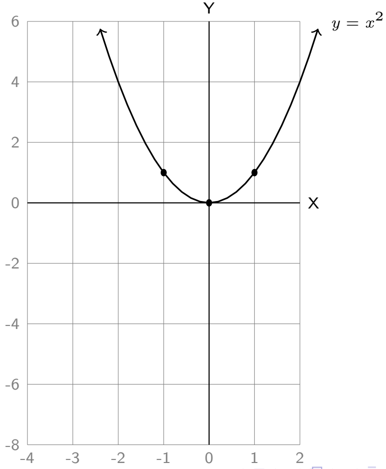

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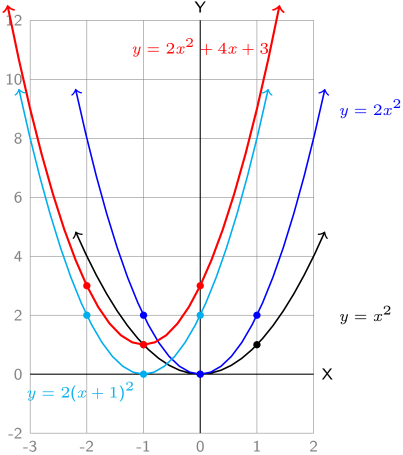

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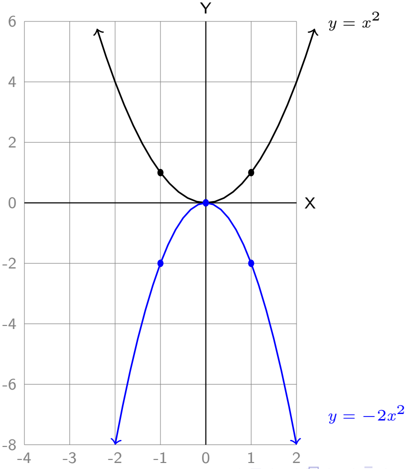

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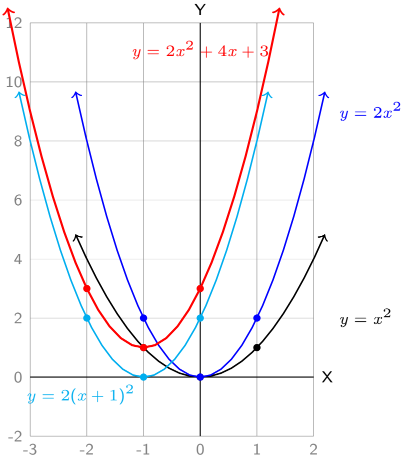

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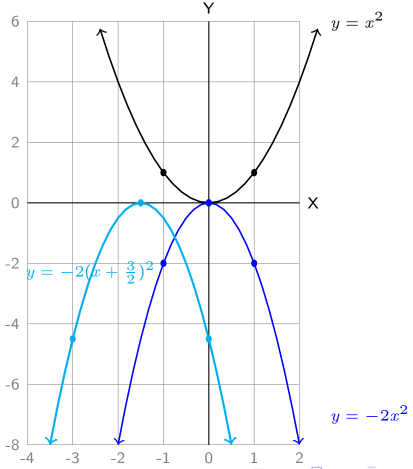

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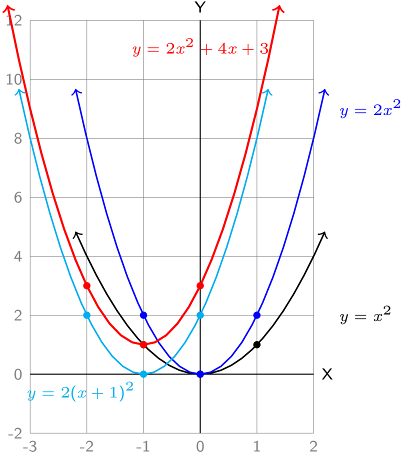

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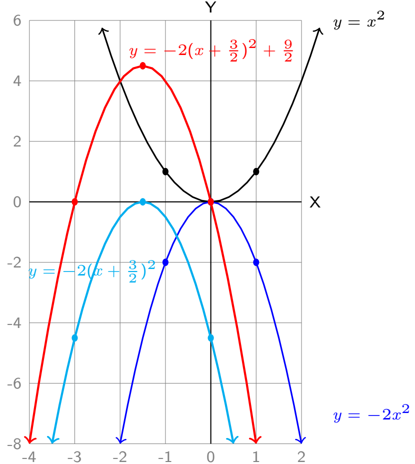

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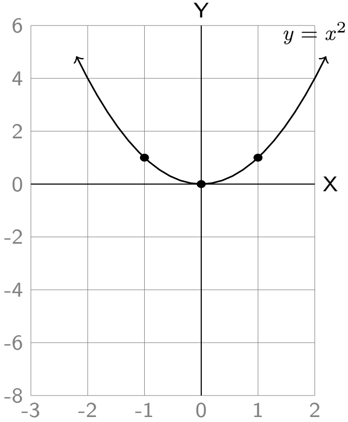

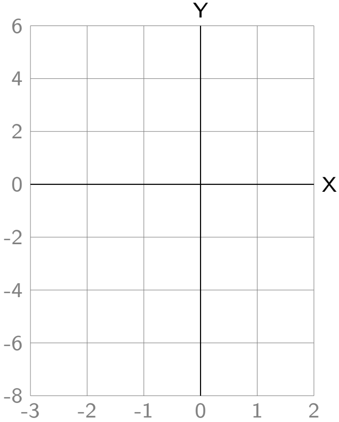

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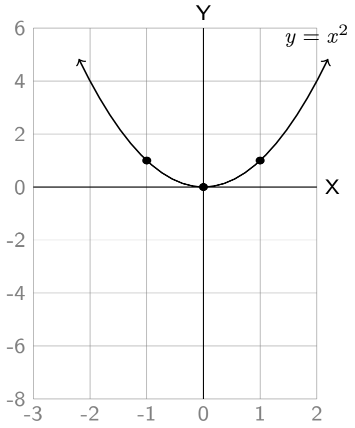

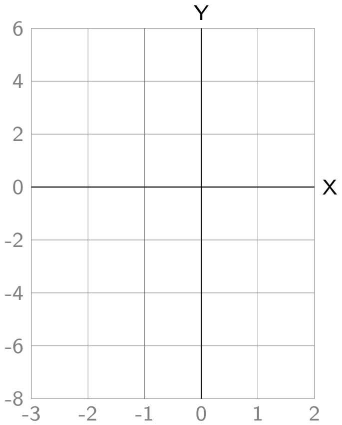

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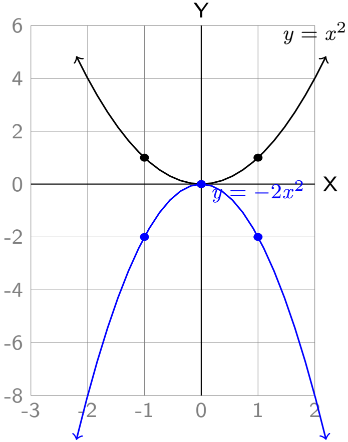

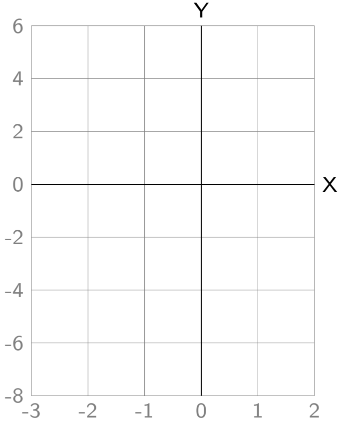

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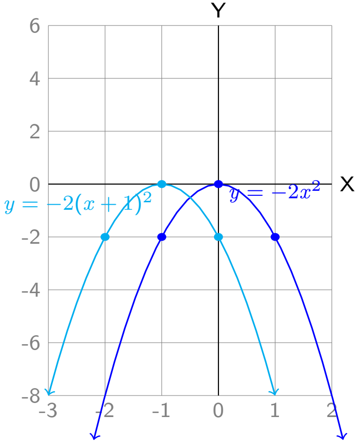

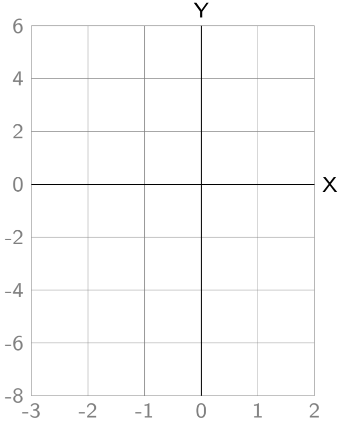

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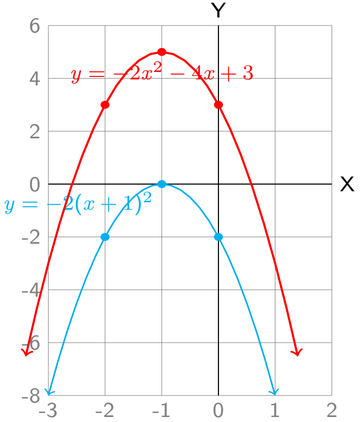

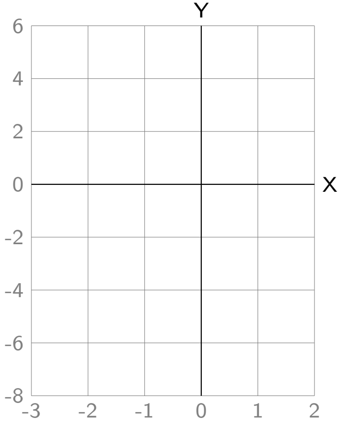

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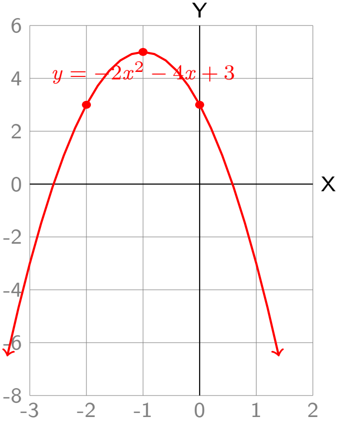

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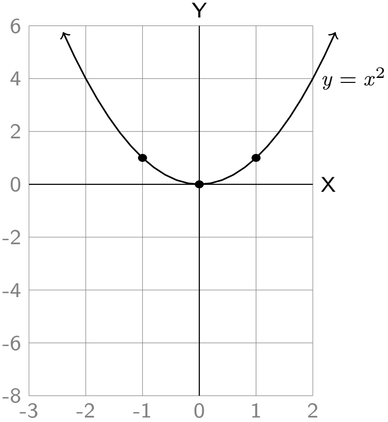

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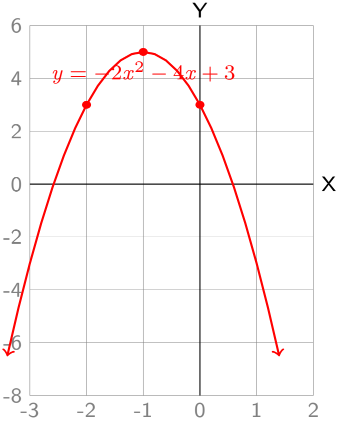

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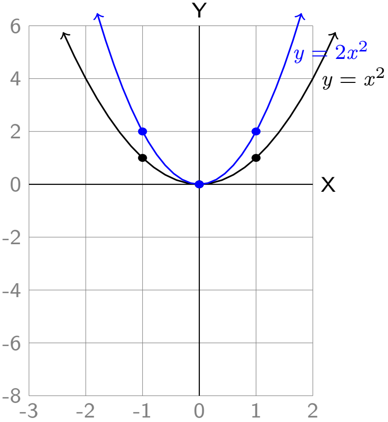

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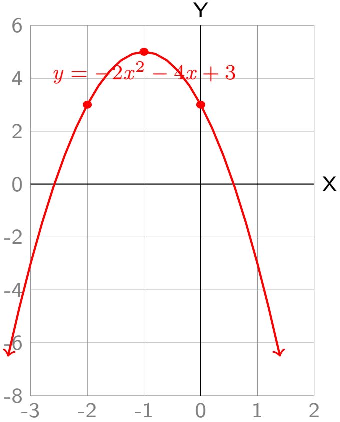

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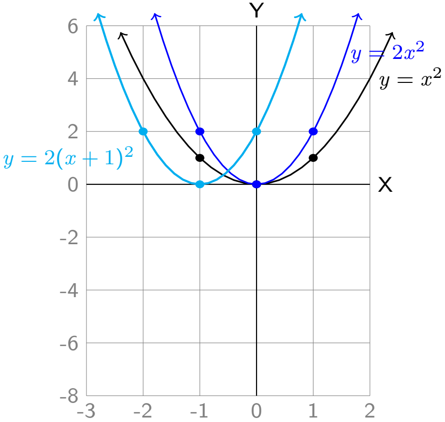

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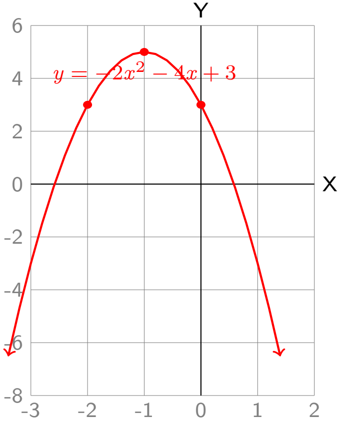

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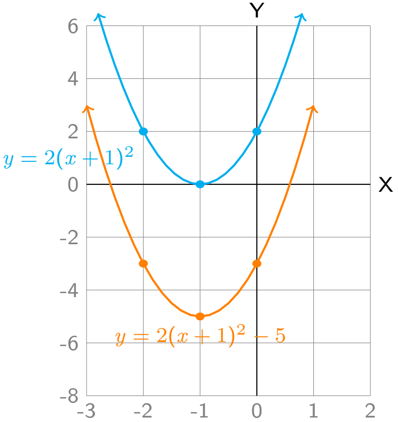

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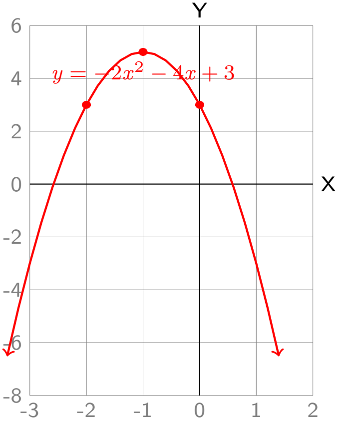

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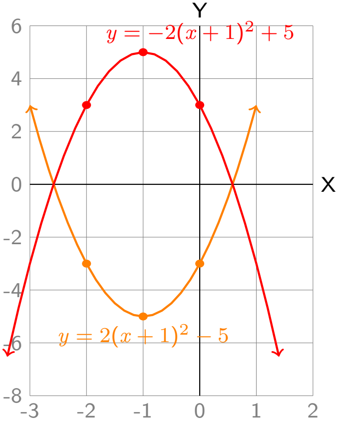
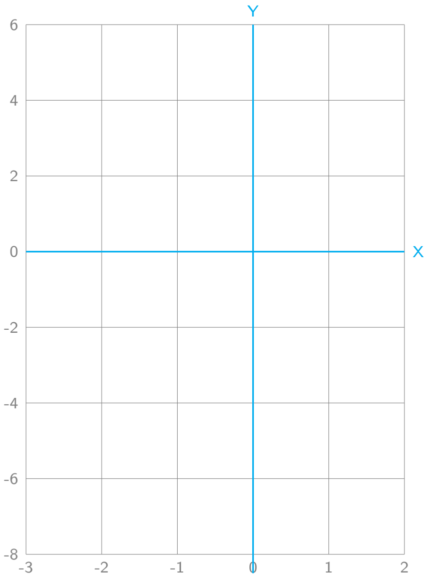
- Reflect across x- axis to $y = -2(x+1)^2 + 5$.



Ex. 2.4.9: Draw the graph of $y = 2x^2 + 4x - 3$. Plot all intercepts as black points.

Solution: $2x^2 + 4x - 3 = 2(x^2 + 2x) - 3 = 2\left((x+1)^2 - 1^2\right) - 3 = 2(x+1)^2 - 5.$



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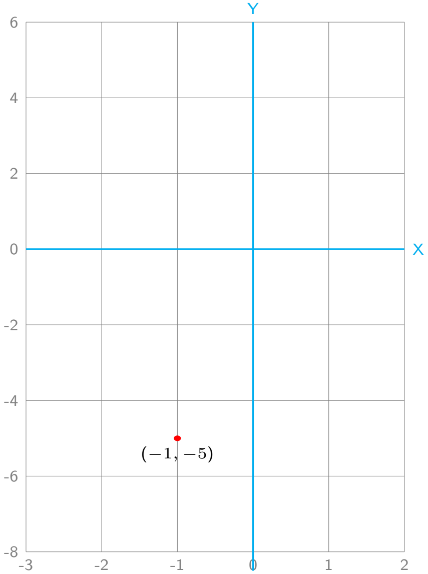
Solution: $2x^2 + 4x - 3 = 2(x^2 + 2x) - 3 =$
$2\left((x+1)^2 - 1^2\right) - 3 = 2(x+1)^2 - 5.$

- Therefore the vertex is $(h, k) = (-1, -5)$. Plot it.



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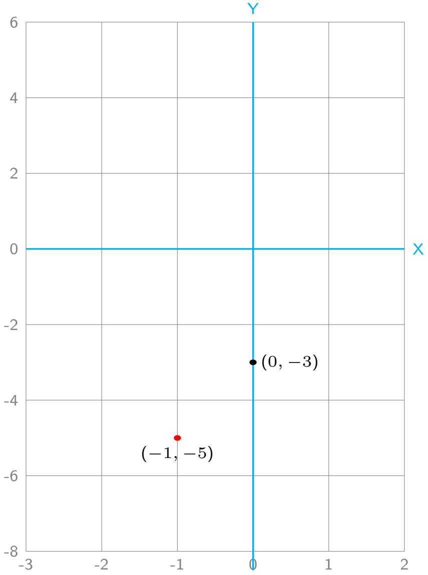

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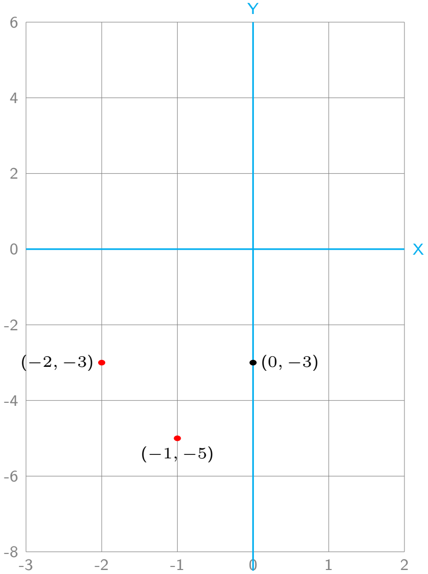

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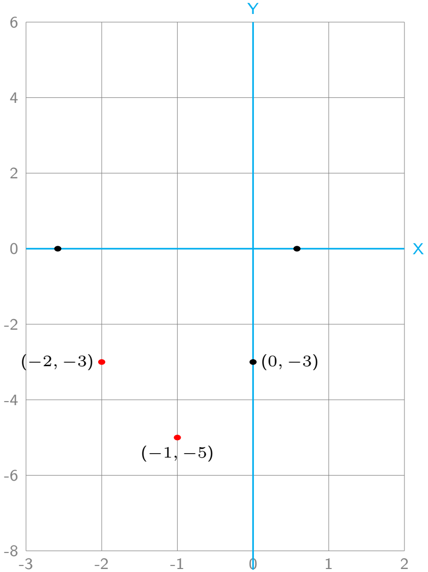
$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-4 \pm \sqrt{40}}{4} \approx .58$ or -2.58.



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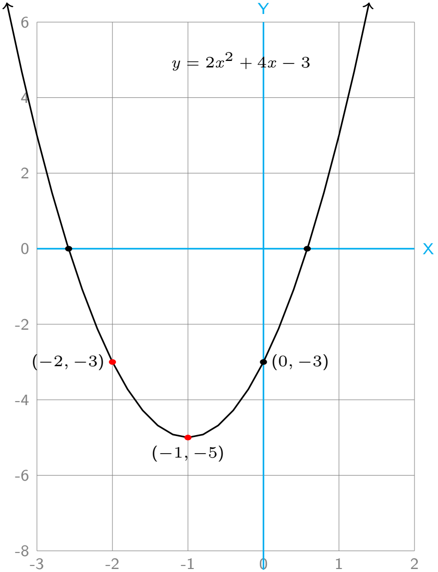
- Join the 5 plotted points by a smooth curve with arrows at both ends to show behavior at infinity.



Ex. 2.4.10: Draw the graph by completing the square.
Plot points with x-coordinate $0, h, 2h$ as red points. Plot x-intercepts as black points.

$y = -2x^2 + 6x - 4$

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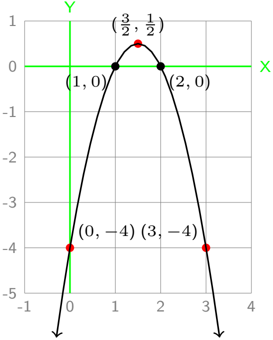

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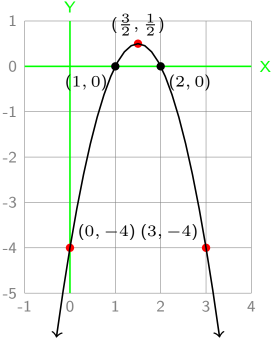

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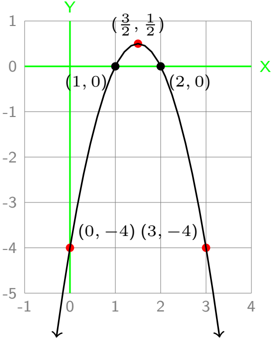

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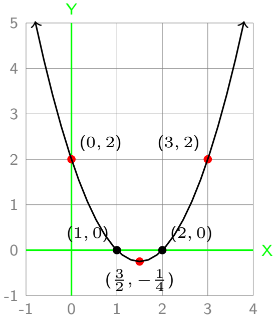

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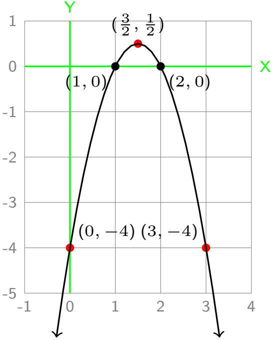

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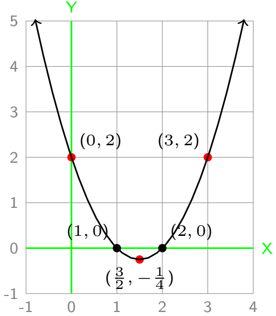

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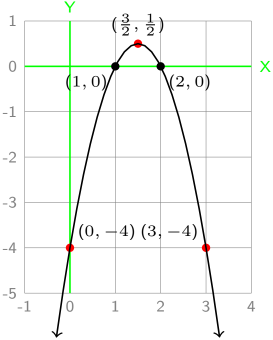

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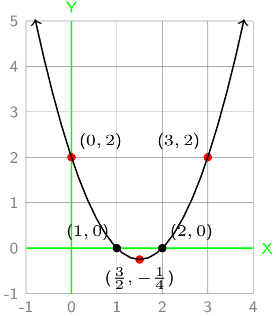

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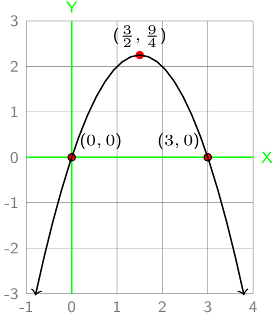

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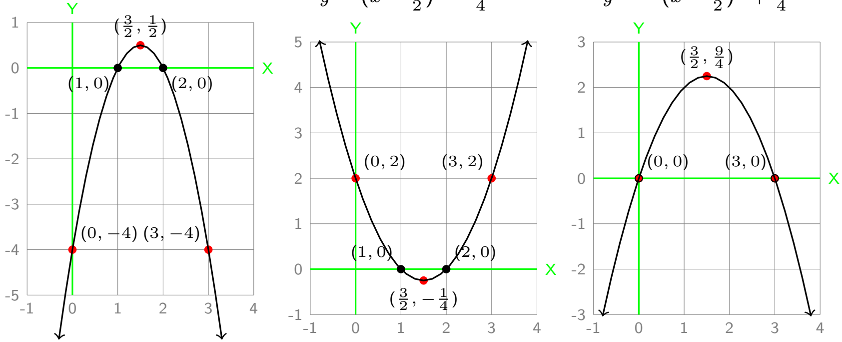
$y = x^2 - 2x - 1$



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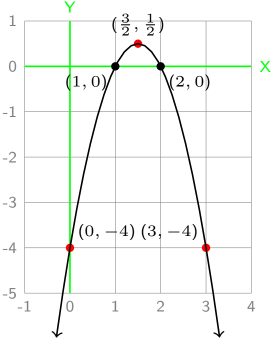

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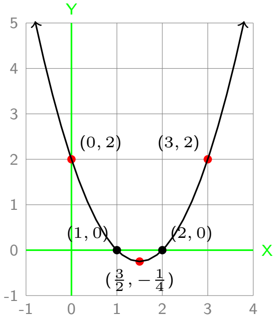

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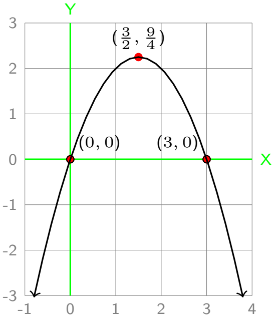

$y = x^2 - 2x - 1$

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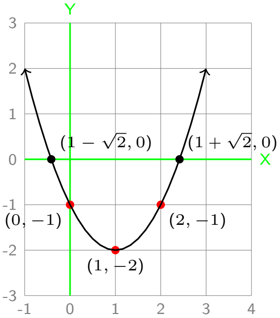

Chapter 2 Section 5: Graphing polynomials

- 2.5.1: Graphing a polynomial using sign analysis
- 2.5.2: Degree 4 polynomial sign analysis
- 2.5.3: How accurate are graphs obtained by sign analysis?
- 2.5.4: Polynomial inequalities
- 2.5.5: Rational inequalities
- 2.5.6: Section 2.5 Review and Quiz

Section 2.5 Preview: Definitions and Procedures

- Definition 2.5.1: *Critical numbers, points, intervals* of polynomial $p(x)$
- Definition 2.5.2: The test number method for sketching $y = p(x)$
- Definition 2.5.3: Sign Switch Theorem (SST) for polynomials $p(x)$
- Definition 2.5.4: The degree of the polynomial inequality $P(x) < Q(x)$ is
- Definition 2.5.5: Critical numbers and intervals of a polynomial $p(x)$
- Definition 2.5.6 :Sign Switch Theorem for linear expressions
- Definition 2.5.7: A *rational inequality* in x
- Definition 2.5.8: Critical numbers and intervals of $\frac{p(x)}{q(x)}$
- Definition 2.5.9: Sign Switch Theorem for rational polynomials $\frac{p(x)}{q(x)}$

- Procedure 2.5.1: How to solve linear inequalities
- Procedure 2.5.2:To solve a polynomial inequality, start with critical intervals
- Procedure 2.5.3: To solve a polynomial inequality: 3 steps
- Procedure 2.5.4: How to solve a rational inequality: Method 1
- Procedure 2.5.5: How to solve a rational inequality: Method 2

2.5.1 Graphing a polynomial by using sign analysis

In this chapter we draw graphs of polynomial functions that show where $p(x)$ is positive, negative, or zero. Such a sketch is of just moderate value, because it shows only the relationship of the polynomial's graph to a single line, the x-axis. To draw a reasonable accurate graph, we need to know on which x-intervals it is rising or falling.
For polynomials of degree higher than 2, this deeper analysis requires calculus.

Critical numbers, points, intervals of polynomial $p(x)$

- If $p(x) = 0$, x is a critical number of p and $(x, p(x))$ is a critical point of p. Therefore critical numbers of p are the x-intercepts of the graph of $y = p(x)$.
- Critical intervals of p are the open intervals that remain after all critical numbers of p are removed from the real number line.
- The sign of $p(x)$ is the same for all x in a critical interval: either $p(x) > 0$ for all x or $p(x) < 0$ for all x. Proving this requires calculus.
- These statements are true as well for sine, cosine, exponential, and log functions, to be studied later in this course.

The test number method for sketching $y = p(x)$

- Plot all critical points of p. These are the points where the graph of $y = p(x)$ meets the x-axis.
- In each critical interval, pick a test number x and plot test number $(x, p(x))$.
- Draw a smooth curve from left to right through all the critical and test points.
- In the left and right critical intervals, start at the graphed test point and draw an arrow extending toward the nearest corner of the grid.
- Polynomial graphs do not have sharp corners. Retouch your graph if necessary to ensure this.

Example 1: Graph $y = p(x) = x^3 - 2x^2 + x$.

Solution: Make a sign chart as follows.

- Set $p(x) = 0$ to find the critical numbers: $x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x-1)^2 = 0$, so $x = 0$ or $x = 1$.

Example 1: Graph $y = p(x) = x^3 - 2x^2 + x$.
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- Set $p(x) = 0$ to find the critical numbers: $x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x-1)^2 = 0$, so $x = 0$ or $x = 1$.
- Critical intervals are $(-\infty, 0), (0, 1), (1, \infty)$. List them on the top line of the sign analysis chart.

In critical interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$

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- Choose test numbers $x = -1$, $x = \frac{1}{2}$, $x = 2$, one in each critical interval.

In critical interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
choose test number	$x = -1$	$x = \frac{1}{2}$	$x = 2$

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- Find the signs of $p(x)$ in each critical interval.

In critical interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
choose test number	$x = -1$	$x = \frac{1}{2}$	$x = 2$
$p(x) =$	-4	$\frac{1}{8}$	2
Test point $(x, p(x))$	$(-1, -4)$	$(\frac{1}{2}, \frac{1}{8})$	$(2, 2)$
$p(x)$ is	< 0	> 0	< 0

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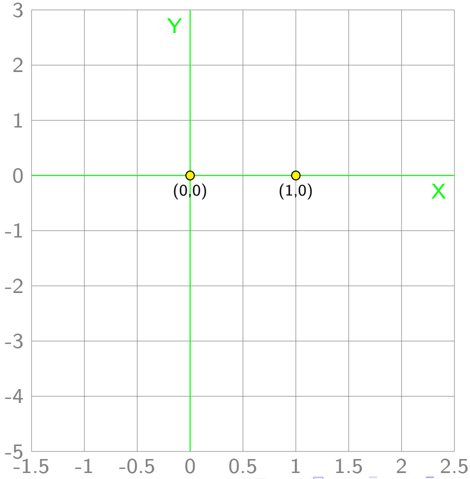
$p(x)$ is	< 0	> 0	< 0

- The yellow dots on the x-axis are the critical points of p.



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In critical interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
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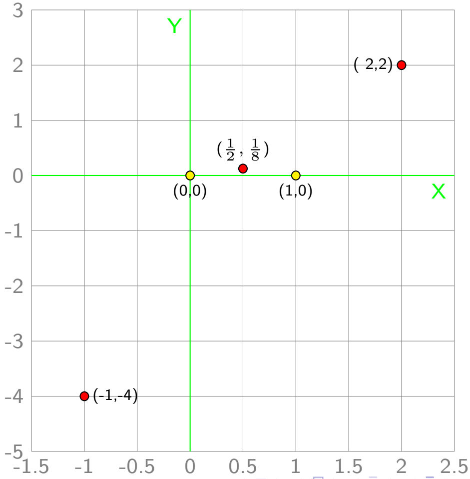
- The red dots are the test points $(x, p(x))$, calculated at test numbers $x = -1, \frac{1}{2}$, and 2.



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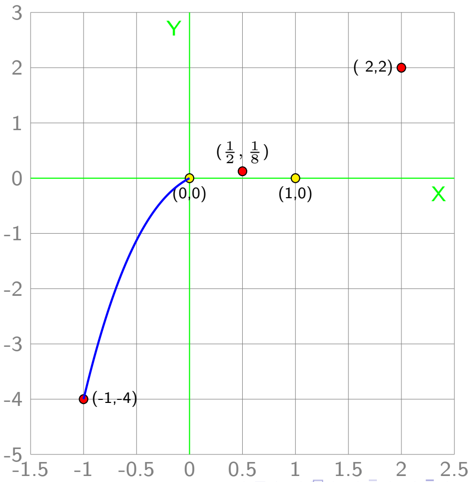
- The red dots are the test points $(x, p(x))$, calculated at test numbers $x = -1, \frac{1}{2}$, and 2.
- Now sketch the graph. In the interval $(-\infty, 0)$, $p(x) < 0$ and so the graph is below the x-axis.



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- Choose test numbers $x = -1$, $x = \frac{1}{2}$, $x = 2$, one in each critical interval.
- Find the signs of $p(x)$ in each critical interval.

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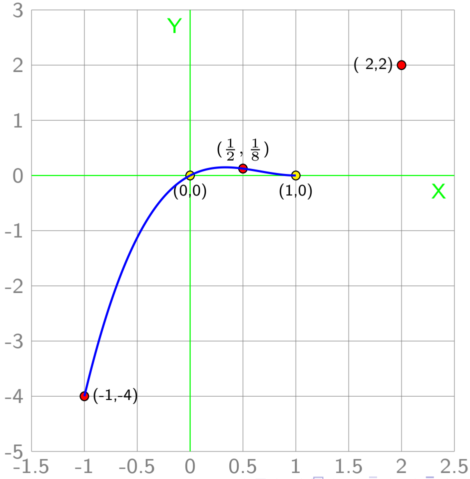
- Now sketch the graph. In the interval $(-\infty, 0)$, $p(x) < 0$ and so the graph is below the x-axis.
- The graph crosses the x-axis at $x = 0$. In interval $(0, 1)$, $p(x) > 0$, so the graph is above the x-axis.



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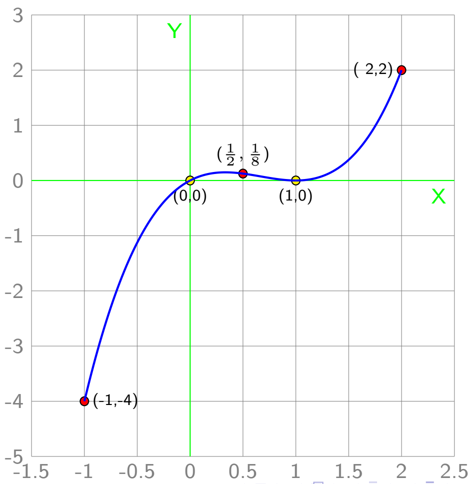
- The graph crosses the x-axis at $x = 0$. In interval $(0, 1)$, $p(x) > 0$, so the graph is above the x-axis.

- The graph bounces off the x-axis at $x = 1$. In the interval $(1, \infty)$, $p(x) > 0$ and the graph is again above the x-axis.

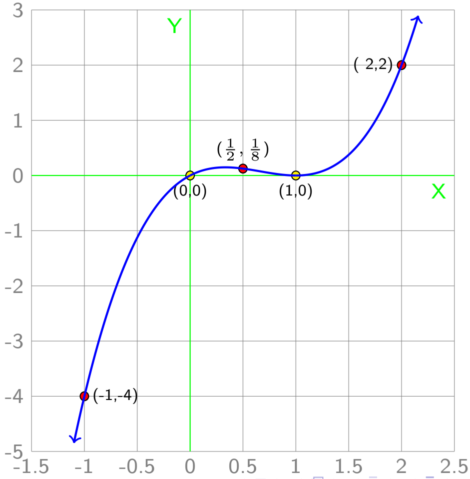


Example 1: Graph $y = p(x) = x^3 - 2x^2 + x$.
Solution: Make a sign chart as follows.

- Set $p(x) = 0$ to find the critical numbers: $x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x-1)^2 = 0$, so $x = 0$ or $x = 1$.
- Critical intervals are $(-\infty, 0), (0, 1), (1, \infty)$. List them on the top line of the sign analysis chart.
- Choose test numbers $x = -1$, $x = \frac{1}{2}$, $x = 2$, one in each critical interval.
- Find the signs of $p(x)$ in each critical interval.

In critical interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
choose test number	$x = -1$	$x = \frac{1}{2}$	$x = 2$
$p(x) =$	-4	$\frac{1}{8}$	2
Test point $(x, p(x))$	$(-1, -4)$	$(\frac{1}{2}, \frac{1}{8})$	$(2, 2)$
$p(x)$ is	< 0	> 0	< 0

- The yellow dots on the x-axis are the critical points of p.
- The red dots are the test points $(x, p(x))$, calculated at test numbers $x = -1, \frac{1}{2}$, and 2.
- Now sketch the graph. In the interval $(-\infty, 0)$, $p(x) < 0$ and so the graph is below the x-axis.
- The graph crosses the x-axis at $x = 0$. In interval $(0, 1)$, $p(x) > 0$, so the graph is above the x-axis.
- The graph bounces off the x-axis at $x = 1$. In the interval $(1, \infty)$, $p(x) > 0$ and the graph is again above the x-axis.
- The arrowheads at the ends of the graph show end behavior.
- There are no sharp corners on the graph.



2.5.2 Graphing a degree 4 polynomial requires careful sign analysis.

Example 2: Graph the polynomial $y = p(x) = x^4 - 4x^2$.

- To find the critical numbers of p, solve $p(x) = 0$ by factoring:
  $p(x) = x^4 - 4x^2 = x^2(x^2 - 4) = x^2(x-2)(x+2) = 0$ when $x = -2, 0$ or 2.

The value of $p(x)$ may, or may not, switch sign at a critical number!

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- The critical numbers split the line into critical intervals $(-\infty, -2), (-2, 0), (0, 2)$, and $(2, \infty)$.
- List the critical intervals on the top line of the sign chart below.

In interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$

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- List the critical intervals on the top line of the sign chart below.
- Pick a test number x in each interval and figure out the value of $p(x)$.

In interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
test number	-3	-1	1	3
$p(x) =$	45	-3	-3	45

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In interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
test number	-3	-1	1	3
$p(x) =$	45	-3	-3	45
test point $(x, p(x))$	$(-3, 45)$	$(-1, -3)$	$(1, -3)$	$(3, 45)$

The value of $p(x)$ may, or may not, switch sign at a critical number!

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- List the critical intervals on the top line of the sign chart below.
- Pick a test number x in each interval and figure out the value of $p(x)$.
- The sign of $p(x)$ on the interval is the same as the sign of $p(x)$ at that test number.

In interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
test number	-3	-1	1	3
$p(x) =$	45	-3	-3	45
test point $(x, p(x))$	$(-3, 45)$	$(-1, -3)$	$(1, -3)$	$(3, 45)$
$y = p(x)$ is	> 0	< 0	< 0	> 0

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test point $(x, p(x))$	$(-3, 45)$	$(-1, -3)$	$(1, -3)$	$(3, 45)$
$y = p(x)$ is	> 0	< 0	< 0	> 0

The value of $p(x)$ may, or may not, switch sign at a critical number!

In this example, $p(x)$ is

- $+$ in $(-\infty, -2)$,
- $-$ in $(-2, 0)$ and $(0, 2)$,
- $+$ in $(2, \infty)$.

Whether or not the sign switches between consecutive intervals is explained by the

Sign Switch Theorem (SST) for polynomials $p(x)$

Factor $p(x)$ completely. Suppose the highest power of factor $ax - b$ is $(ax - b)^n$. As x moves on the number line through critical number $\frac{b}{a}$

- $p(x)$ switches sign if n is odd;
- $p(x)$ keeps the same sign if n is even.

2.5.2 Graphing a degree 4 polynomial requires careful sign analysis.

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- To find the critical numbers of p, solve $p(x) = 0$ by factoring: $p(x) = x^4 - 4x^2 = x^2(x^2 - 4) = x^2(x-2)(x+2) = 0$ when $x = -2, 0$ or 2.
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In interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
test number	-3	-1	1	3
$p(x) =$	45	-3	-3	45
test point $(x, p(x))$	$(-3, 45)$	$(-1, -3)$	$(1, -3)$	$(3, 45)$
$y = p(x)$ is	> 0	< 0	< 0	> 0

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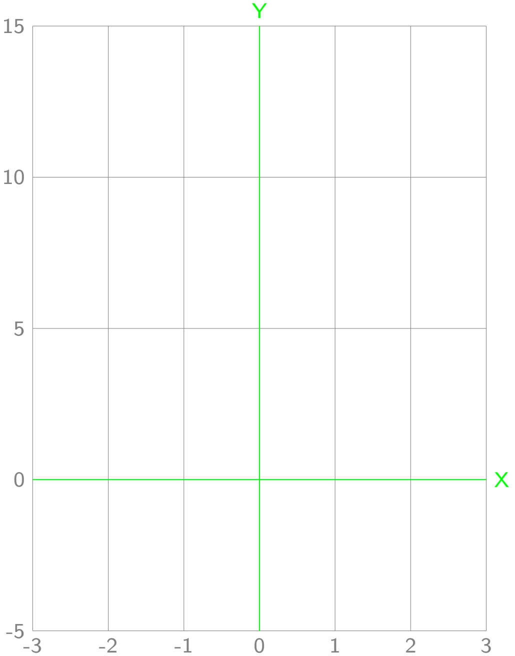
- $p(x)$ switches sign if n is odd;
- $p(x)$ keeps the same sign if n is even.

Our example: $p(x) = (x+2)^1 x^2 (x-2)^1$

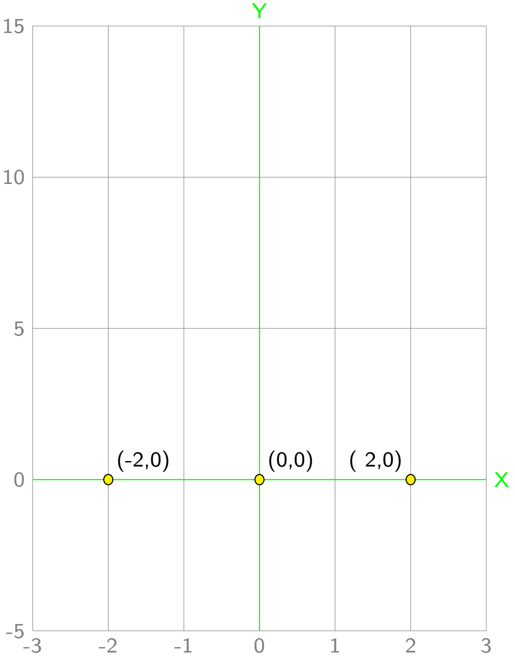
- switches sign when $x + 2 = 0$ at $x = -2$, since $(x+2)^1$ is an odd power;
- doesn't switch sign at $x = 0$, since x^2 is an even power;
- switches sign when $x - 2 = 0$ at $x = 2$, since $(x-2)^1$ is an odd power.

To graph $y = p(x) = x^4 - 4x^2 = x^2(x-2)(x+2)$, click slowly and study each step.



To graph $y = p(x) = x^4 - 4x^2 = x^2(x-2)(x+2)$, click slowly and study each step.

- Plot the x-intercepts (yellow dots) at the critical numbers $x = -2, 0, 2$.



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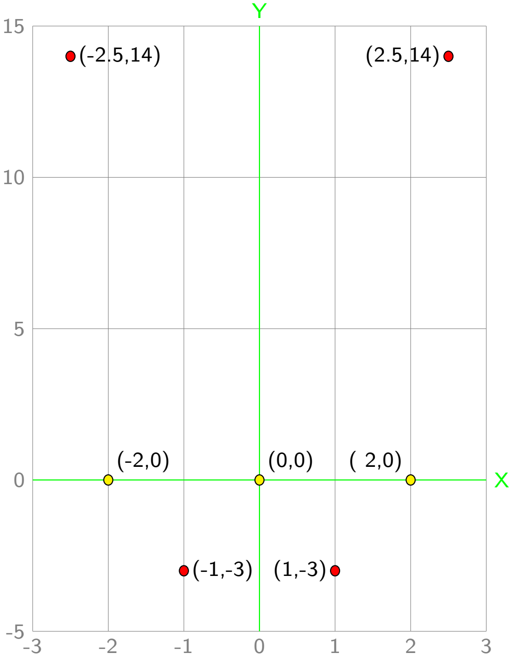
- Plot the x-intercepts (yellow dots) at the critical numbers $x = -2, 0, 2$.
- Graph test points $(x, p(x))$ (red dots) using test numbers $x = -2.5, -1, 1, 2.5$.



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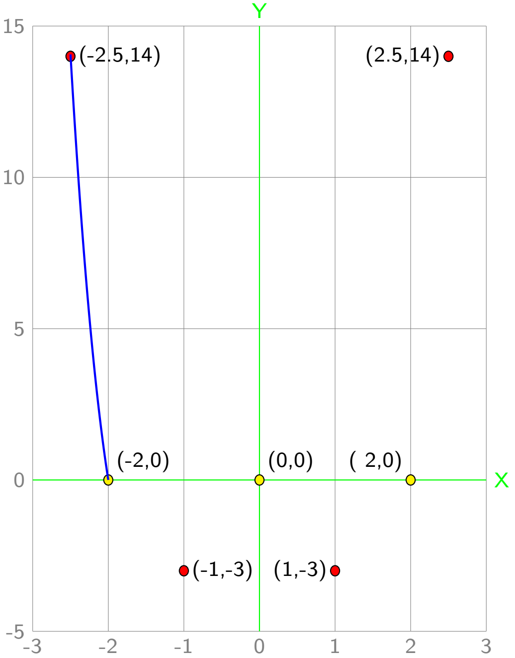
- The graph is above the x-axis in the interval $(-\infty, -2)$, where $p(x) > 0$.



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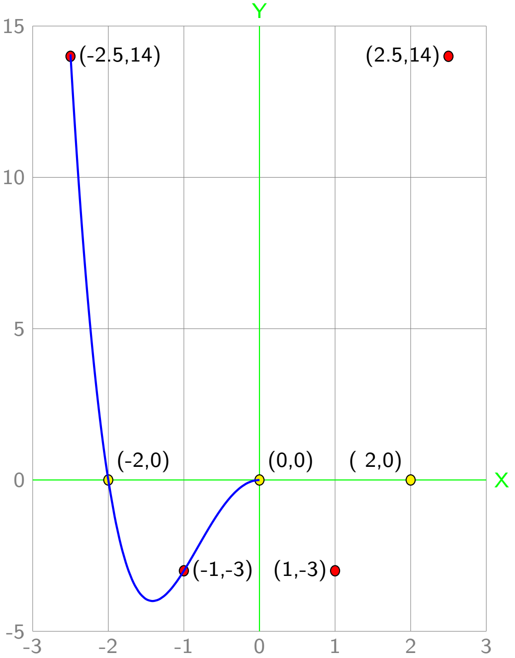

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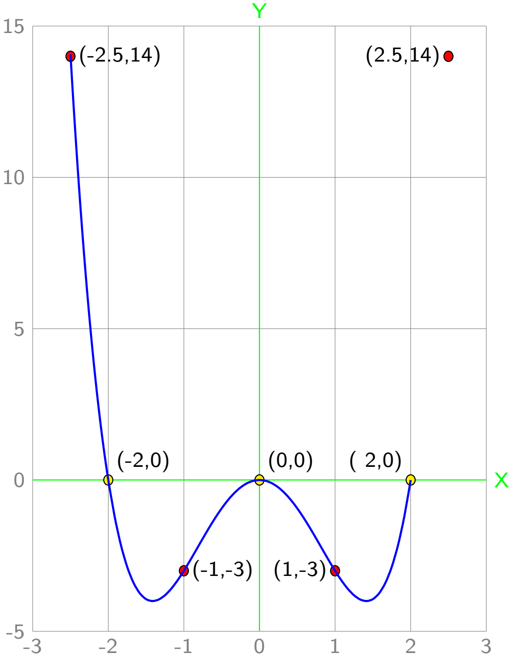
- The graph crosses the x-axis at $x = -2$ and is below it in $(-2, 0)$, where $p(x) < 0$.
- The graph bounces gently off the x-axis at $(0, 0)$ (no sharp corner, please) and stays below it in the interval $(0, 2)$, where $p(x) < 0$.



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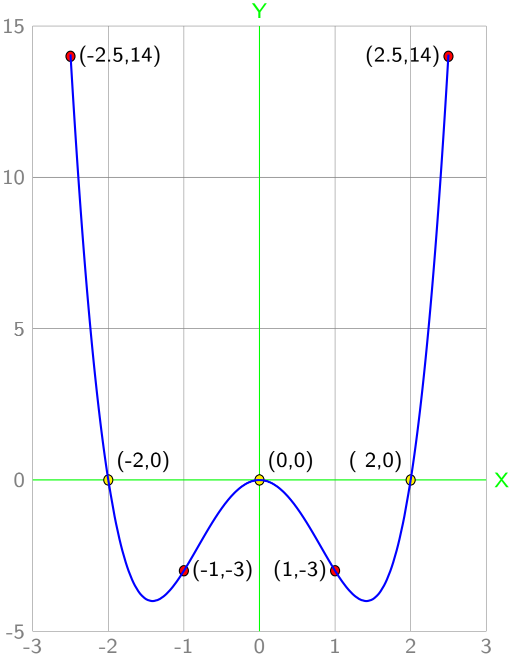

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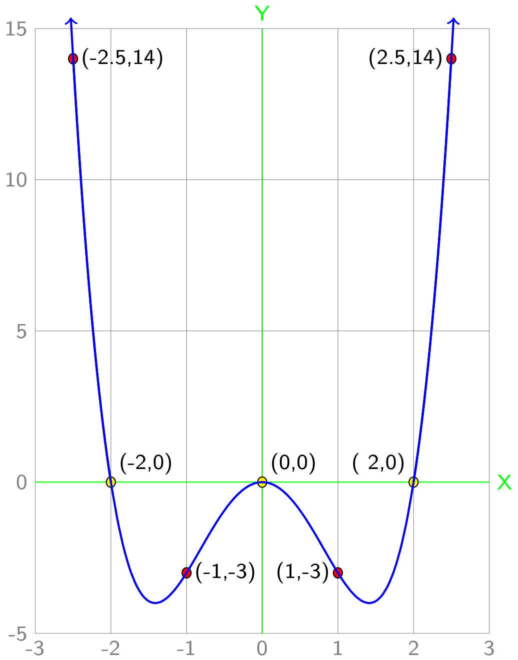
- The graph bounces gently off the x-axis at $(0, 0)$ (no sharp corner, please) and stays below it in the interval $(0, 2)$, where $p(x) < 0$.
- The graph crosses the x-axis at $x = 2$ and is above it in the interval $(2, \infty)$, where $p(x) > 0$.
- Insert arrowheads at the ends of the graph to show end behavior.



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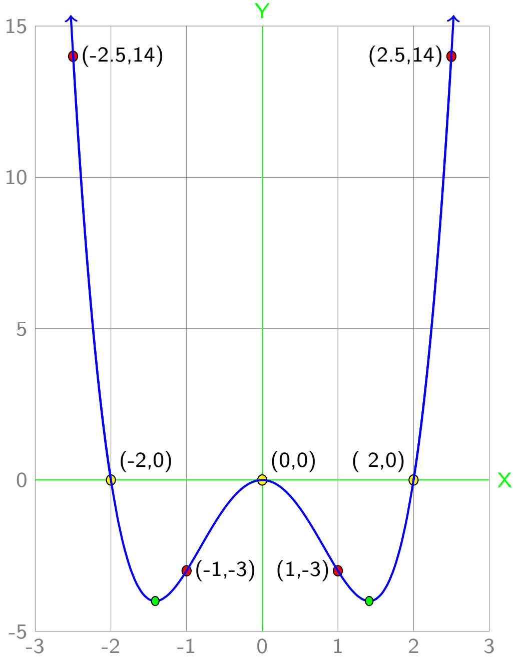
- The graph crosses the x-axis at $x = 2$ and is above it in the interval $(2, \infty)$, where $p(x) > 0$.
- Insert arrowheads at the ends of the graph to show end behavior.

If you would use only the points shown (test points and critical points), you might draw the graph with local minima (valley bottoms) at the red test points.
The blue graph, drawn with a computer plotting about 100 points, shows otherwise. The green local minimum points of $y = x^4 - 4x^2$ occur at $x = \pm\sqrt{2}$, where $y = -4$. Calculus explains why.



2.5.3 How accurate are graphs obtained by sign analysis?

The short answer is: there is no short answer. On the previous page, the graph drawn through test points is not far from the true (computer-generated) graphs. However, the error can be much more pronounced, as in the following example.

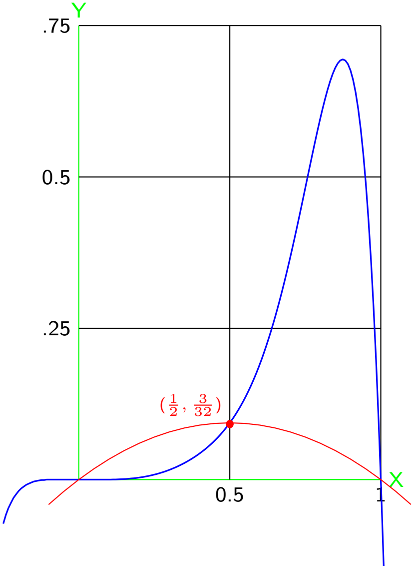
The blue curve at the right is the graph of $y = p(x) = 40x^5(x-1)(x-2)(x^2-x+.35)$ with domain $-.25 \leq x \leq 1.01$.

It's easy to solve $p(x) = 0$ to get the critical numbers $0, 1, 2$. Let's try sketching the part of the graph on the interval $(0, 1)$.

- The easiest test number is $x = \frac{1}{2} = 0.5$.
- Calculate $p(.5) = \frac{3}{32} = .09375$.
- The sign chart method we are using gives the red graph as an attempted sketch of $y = p(x)$.

This graph is far from correct. The test point method completely misses the fact that there is a steep hill between $x = 0.5$ and $x = 1$.

Sketching a reasonably accurate graph of $y = p(x)$ requires finding the coordinates of local maximum and minimum points (hilltops and valley bottoms). This is done in first semester calculus.



2.5.4 Polynomial inequalities

Section 1.1.6 showed how to solve polynomial equations $P(x) = Q(x)$. Here we solve polynomial inequalities such as $P(x) < Q(x)$.

There are four types of inequalities, one for each inequality sign:

- $P(x) > Q(x)$
- $P(x) \geq Q(x)$
- $P(x) < Q(x)$
- $P(x) \leq Q(x)$

Subtract $Q(x)$ from both sides and set $p(x) = P(x) - Q(x)$ to get four standard form polynomial inequalities:

- $p(x) > 0$;
- $p(x) \geq 0$;
- $p(x) < 0$;
- $p(x) \leq 0$.

We will often refer to just one of these types, with the understanding that the discussion applies to all of them.

Definition: The *degree of* $P(x) < Q(x)$ is

- the highest power of x in the simplified polynomial $p(x) = P(x) - Q(x)$.
- A *linear equality* is a degree one inequality.

A degree one inequality is called linear because the graph of $y = p(x)$ is a straight line.

How to solve linear inequalities

To solve inequality:	Do this to both sides. Then the answer in ↗	inequality form is:	in interval form is:
$x - 4 > 0$	add 4	$x > 4$	
	rewrite as	$4 < x$	$(4, \infty)$
$x - 4 < 0$	add 4	$x < 4$	$(-\infty, 4)$
$3x - 4 \geq 0$	add 4, divide by 3	$x \geq \frac{4}{3}$	
	rewrite as	$\frac{4}{3} \leq x$	$[\frac{4}{3}, \infty)$
$3x - 4 \leq 0$	add 4, divide by 3	$x \leq \frac{4}{3}$	$(-\infty, \frac{4}{3}]$
$4 - x > 0$	add x	$4 > x$	
	rewrite as	$x < 4$	$(-\infty, 4)$
$4 - 3x < 0$	add $3x$, divide by 3	$\frac{4}{3} < x$	$\left(\frac{4}{3}, \infty\right)$
$-5x \geq 4$	divide by -5	$x \leq \frac{4}{-5}$	$(-\infty, -\frac{4}{5}]$
$-6 < -2x$	divide by -2	$3 > x$	
	rewrite as	$x < 3$	$(-\infty, 3)$

In each of the last two examples, the inequality sign is reversed (and colored red, for emphasis) because both sides are divided by a negative number.

Higher degree polynomial inequalities

The solution of a degree one inequality is an interval of real numbers. The solutions of a polynomial inequality with degree $d > 1$ can consist of up to d intervals.

Here is a bit of review.

Critical numbers and intervals of a polynomial $p(x)$

- Critical numbers of p are the solutions of $p(x) = 0$.
- Critical intervals of p are all open intervals that remain when its critical numbers are removed from the real number line.
- The sign of $p(x)$ is the same at all x-values in a critical interval. Proving this requires calculus.

To solve a polynomial inequality:
Find the sign of $p(x)$ in each critical interval.

- Find critical numbers and intervals for $p(x)$.
- In each critical interval, choose a test number x and calculate $p(x)$.
- List critical intervals where $p(x)$ has the desired sign.
- For $p(x) \geq 0$ or $p(x) \leq 0$ include interval endpoints.
- For $p(x) > 0$ or $p(x) < 0$ omit interval endpoints.

Example: Find the sign of $p(x) = x^2 - 1$ on each of its critical intervals.
Solution: $p(x) = (x+1)(x-1) = 0$ at critical numbers -1 and 1. Choose test numbers $x = -2$ in $(-\infty, -1)$; $x = 0$ in $(-1, 1)$; and $x = 2$ in $(1, \infty)$.

- $p(x) > 0$ for x in $(-\infty, -1)$ since $p(-2) = +3$.
- $p(x) < 0$ for x in $(-1, 1)$ since $p(0) = -1$.
- $p(x) > 0$ for x in $(1, \infty)$ since $p(2) = +3$.

Here is a diagram of test points(colored blue), critical numbers (empty circles), and critical intervals.

critical number:		-1		1	
critical interval:	$(-\infty, -1)$		$(-1, 1)$		$(1, \infty)$
p(test number):	$p(-2) = 3$		$p(0) = -1$		$p(2) = 3$
$p(x)$ in interval:	> 0		< 0		> 0

Example 3: Solve $x^2 + 3 \geq 4x$.
Solution: Rewrite as $p(x) = x^2 - 4x + 3 \geq 0$.

- $p(x) = (x-3)(x-1) = 0$ at $x = 1$ and $x = 3$. These critical numbers split the number line into critical intervals $(-\infty, 1), (1, 3)$, and $(3, \infty)$.
- Choose test numbers $0, 2, 4$ (one in each interval).
- Find $p(x) = (x-3)(x-1)$ at each test number. $p(0) = 3$; $p(2) = -1$; $p(4) = 3$.

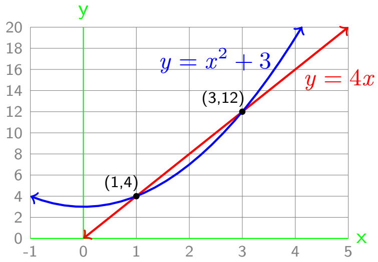
In interval	$(-\infty, 1)$	$(1, 3)$	$(3, \infty)$
Choose test number $x =$	0	2	4
$p(x) = (x-3)(x-1) =$	$+3$	-1	$+3$
In interval, $p(x)$ is	$+$	$-$	$+$

Since we are solving $p(x) \geq 0$, the answer consists of intervals on which the sign of $p(x)$ is $+$, including endpoints so that $p(x)$ can be 0.

Answer as an inequality: $x \leq 1$ or $3 \leq x$

In interval notation: x in $(-\infty, 1] \cup [3, \infty)$

The $\cup$ symbol (set theory "union") means"or."
To check the answer, graph $y = x^2 + 3$ and $y = 4x$.



Clearly $x^2 + 3 > 4x$ for $x < 1$ and for $3 < x$, while $x^2 + 3 < 4x$ for $1 < x < 3$.

The strategy in the last example was to find the value of $p(x)$ at a test number in an interval. But the only information needed is the sign of $p(x)$.

Linear Sign Switch Theorem (LSST) for linear expressions

The sign of $ax - b$

- is the same as the sign of a in critical intervals to the right of $x = \frac{b}{a}$.
- is opposite the sign of a in critical intervals to the left of $x = \frac{b}{a}$.
- switches at $x = \frac{b}{a}$.

Examples:

- $x - 3$ is positive for $x > 3$, negative for $x < 3$.
- $2x + 5$ is positive for $x > -\frac{5}{2}$; negative for $x < -\frac{5}{2}$.
- $-2x + 5$ is negative for $x > \frac{5}{2}$; positive for $x < \frac{5}{2}$.

Three steps for *solving a polynomial inequality:*
1. Set up a sign chart:

- If the coefficient of the highest power of x in polynomial $p(x)$ is negative, multiply p by -1 and reverse the inequality sign.
- Factor $p(x)$ and set each factor to 0 to find the critical numbers of p. List them in increasing numerical order at the top of the sign chart.
- Write the factors of $p(x)$ down the left side of the sign chart in the same order as their critical numbers at the top of the chart.

2. Find the sign of each factor in each critical interval. To do this, either

- Find the value of each factor at a test number OR
- Use the Linear Sign Switch Theorem.

3. Multiply signs of the factors of $p(x)$

to find the sign of $p(x)$ in each critical interval.

Example 4: Solve $x^3 > x$ by using the Linear SST.
Solution: Since $p(x) = x(x^2-1) = (x+1)x(x-1)$, rewrite as $p(x) = x^3 - x > 0$.
Set each factor to 0 to find and list the critical numbers in numerical order as $-1, 0, 1$. Then construct this chart:

Critical numbers are -1 0 1

In interval	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
Test number	$x=-2$	$x=-\frac{1}{2}$	$x=\frac{1}{2}$	$x=2$
$x+1$ is	$-$	$+$	$+$	$+$
x is	$-$	$-$	$+$	$+$
$x-1$ is	$-$	$-$	$-$	$+$
$p(x)$ is	$---=-$	$+--=+$	$+-+=-$	$+++=+$

- Notice the pattern produced by the Linear SST. Each factor switches from $-$ to $+$ at its corresponding critical number, i.e., the x-value that makes the factor zero. For example, the sign of $x+1$ in $(-\infty,-1)$ is $-$ while its sign in the three intervals to the right of $x=-1$ is $+$.
- At the bottom of each column, the sign of $p(x)$ is the product of the signs above it. For example, the bottom entry in the $(-\infty,-1)$ column is $---\ =-$ to show that the product of the 3 negative signs above it is negative.
- In this example, $p(x)$ is $+$ for x in $(-1,0)$ or $(1,\infty)$.

Answer: x in $(-1,0)\cup(1,\infty)$ | $-1 < x < 0$ or $1 < x$.

Critical interval endpoints 0 and 1 are omitted because $p(0)$ and $p(1)$ are equal to 0, not > 0 as required.

Example 5: Solve $x^3 \leq x$.

Solution: Rewrite as $p(x) = x^3 - x \leq 0$. This uses the same $p(x)$ as the last problem, but now than answer must include interval endpoints where $p(x) = 0$, since the inequality sign is $\leq$.

Answer: Intervals: x in $(-\infty, -1] \cup [0, 1]$

Inequality: $x \leq -1$ or $0 \leq x \leq 1$

Example 6: Solve $x^3 \geq 2x^2 + 3x$ by finding the value of $p(x)$ at a test number in each critical interval.

Solution: Rewrite as $p(x) = x^3 - 2x^2 - 3x \geq 0$. Then

- Factor to get $x(x^2 - 2x - 3) = x(x+1)(x-3) \geq 0$
- Solve $p(x) = x(x+1)(x-3) = 0$ to find the critical numbers $-1, 0, 3$.
- Choose a test number in each critical interval as shown below.
- List the corresponding factors $x+1, x, x-3$ down the left side of the chart.

The critical numbers are -1 0 3

In interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 3)$	$(3, \infty)$
Test number	$x = -2$	$-\frac{1}{2}$	1	4
$x+1$ is	-1	$+\frac{1}{2}$	2	$+5$
x is	-2	$-\frac{1}{2}$	1	$+4$
$x-3$ is	-5	$-\frac{7}{2}$	-2	$+1$
$p(x)$ is	$- - - = -$	$+ - - = +$	$+ + - = -$	$+ + + = +$
$p(x) \geq 0$ in		$[-1, 0]$		$[3, \infty)$

As usual, the sign of $p(x)$ is the product of the signs above it. The last line shows the intervals on which $p(x) \geq 0$, including endpoints to allow $p(x) = 0$.

Answer: x in $[-1, 0] \cup [3, \infty)$ $-1 \leq x \leq 0$ or $3 \leq x$

In the examples done so far, the signs of $p(x)$ in the bottom row of the sign chart alternated between $+$ and $-$. That is not always the case.

To find the sign of a power of a real number A

- If n is even, A^n is positive.
- If n is odd, A^n has the same sign as A.

Example 7: Solve $-x^3 + 2x^4 - x^5 < 0$.
Solution: Rearrange and multiply by -1 to get
$p(x) = x^5 - 2x^4 + x^3 > 0$ Now factor:
$p(x) = x^3(x^2 - 2x + 1) = x^3(x-1)^2 > 0$
Critical numbers: $p(x) = x^3(x-1)^2 = 0$ if
$x^3 = 0$, when $x = 0$; or if $(x-1)^2 = 0$, when $x = 1$.

The critical numbers $x = 0$ and $x = 1$ split the number line into three critical intervals.
Finding the signs of the factors x^3 and $(x-1)^2$ is easy:

In interval	$(-\infty, 0)$	$(0,1)$	$(1,\infty)$
	$x = -1$	$x = \frac{1}{2}$	$x = 2$
x^3 is	$-$	$+$	$+$
$(x-1)^2$ is	$+$	$+$	$+$
$p(x)$ is	$-+=-$	$++=+$	$++=+$
$p(x) > 0$ in		$(0,1)$	$(1,\infty)$

Answer: x in $(0,1) \cup (1,\infty)$ $0 < x < 1$ or $1 < x$

Endpoints 0 and 1 are omitted since $p(0) = p(1) = 0$.

Writing the answer as x in $(0,\infty)$ or as $0 < x$ is wrong because those answers include $x = 1$.

Now redo the last example with $\leq$ rather than $<$.

Example 8: Solve $-x^3 + 2x^4 - x^5 \leq 0$.

Solution: Rewriting as $p(x) = x^3(x-1)^2 \geq 0$ changes only the last line of the above sign chart to $p(x) \geq 0$ in $[0,1] \cup [1,\infty)$. Combining these intervals gives the single interval $[0,\infty)$. Answer: x in $[0,\infty)$ $0 \leq x$

It would be wordy, but not wrong, to write the answer as x in $[0,1] \cup [1,\infty)$ $0 \leq 1$ or $1 \leq x$

In the above two Examples, $p(x)$ switches sign at the critical number $x = 0$ but not at $x = 1$. That's because the factor x^3 is an odd power, while the factor $(x-1)^2$ is an even power. Refer to the Sign Switch Theorem for polynomials in Section 2.5.2

2.5.5 Rational inequalities

A rational inequality in x

is an inequality in x that involves a fraction.
A solution of the inequality is any real number that yields a true statement when substituted for x.

An example is $\frac{x}{x+1} + 2 > \frac{3}{x}$.
If this were an equation, you would begin by multiplying both sides by the LCD $x(x+1)$ of the fractions. Inequalities are different.

Advice for now: Don't multiply both sides of an inequality by an expression containing x.

Reason: The sign of the expression may be different for different values of x. In this example, you need to reverse the inequality sign if $x(x+1) < 0$ but leave it alone if $x(x+1) > 0$. Making this choice adds to the work.

It's safer to subtract the right side from the left side and combine fractions. The result will be one of the following:

$\frac{p(x)}{q(x)} > 0$ or $\frac{p(x)}{q(x)} < 0$ or $\frac{p(x)}{q(x)} \geq 0$ or $\frac{p(x)}{q(x)} \leq 0$
where $\frac{p(x)}{q(x)}$ is a completely reduced fraction.

Critical numbers and intervals of $\frac{p(x)}{q(x)}$

- x is a critical number of $\frac{p(x)}{q(x)}$ if $p(x) = 0$ or $q(x) = 0$.
- Critical intervals of $\frac{p(x)}{q(x)}$ are the open intervals that remain when its critical numbers are removed from the real number line.
- In each critical interval, $\frac{p(x)}{q(x)}$ has the same sign at all x.

To solve a rational inequality $\frac{p(x)}{q(x)} < 0$ (or $\leq, >, \geq 0$)

- Find critical numbers and intervals of $\frac{p(x)}{q(x)}$.
- Choose a test number in each critical interval.
- The solution consists of those critical intervals in which $\frac{p(x)}{q(x)}$ has the desired sign.
- To solve $\frac{p(x)}{q(x)} > 0$ or $\frac{p(x)}{q(x)} < 0$, omit all critical numbers from the solution.
- To solve $\frac{p(x)}{q(x)} \geq 0$ or $\frac{p(x)}{q(x)} \leq 0$, include critical number x in the solution if and only if $p(x) = 0$ *and* $q(x) \neq 0$.

Example 9, Method 1: Solve $\dfrac{x+2}{x+3} \le \dfrac{x-1}{x-2}$.

To one side: $\frac{x+2}{x+3} - \frac{x-1}{x-2} \le 0$

Subtract: $\frac{(x+2)(x-2)-(x-1)(x+3)}{(x+3)(x-2)} \le 0$

Parentheses! $\frac{x^2-4-(x^2+2x-3)}{(x+3)(x-2)} \le 0$

Distribute minus: $\frac{x^2-4-x^2-2x+3}{(x+3)(x-2)} \le 0$

Rewrite: $\frac{-2x-1}{(x+3)(x-2)} \le 0.$

Multiply by -1 $\frac{2x+1}{(x+3)(x-2)} \ge 0.$

Thus $p(x) = (2x+1)$ and $q(x) = (x+3)(x-2)$.

- Set factors $2x+1, x+3$, and $x-2$ of p and q to 0 to find the critical numbers $-3, -\frac{1}{2}$,and 2, which split the line into critical intervals $(-\infty,-3)$; $(-3,-\frac{1}{2})$; $(-\frac{1}{2},2)$; and $(2,\infty)$.
- Write these intervals at the top of the chart.
- Write factors of p and q down the left side of the chart.
- Choose a test number in each critical interval.
- Evaluate $p(x)$ at each test number.
- Find the sign of $\frac{p(x)}{q(x)} = \frac{2x+1}{(x+3)(x-2)}$ in each interval by multiplying the signs of the 3 factors.

In interval	$(-\infty,-3)$	$(-3,-\frac{1}{2})$	$(-\frac{1}{2},2)$	$(2,\infty)$
Test number	$x=-4$	$x=-1$	$x=0$	$x=3$
$x+3$ is	-1	$+2$	$+3$	$+6$
$2x+1$ is	-7	-1	$+1$	$+7$
$x-2$ is	-6	-3	-2	$+1$
$\frac{p(x)}{q(x)}$ is	$---=-$	$+--=+$	$++-=-$	$+$
$\frac{p(x)}{q(x)} > 0$ in		$(-3,-\frac{1}{2})$		$(2,\infty)$

The solution includes critical intervals $(-3,-\frac{1}{2})$ and $(2,\infty)$, in which $\frac{p(x)}{q(x)}$ is negative. Since the inequality sign is $\ge$, check endpoints x to see if $\frac{p(x)}{q(x)} = 0$. Since $p(-\frac{1}{2}) = 0$, include endpoint $-\frac{1}{2}$. However, omit endpoints 2 and -3, since $q(2) = q(-3) = 0$ and so $\frac{p(x)}{q(x)}$ is undefined at those x-values.

Answer: Intervals: x in $(-3,-\frac{1}{2}] \cup (2,\infty)$

Inequality: $-3 < x \le -\frac{1}{2}$ or $2 < x$

How to solve a rational inequality: Method 2

- Multiply both sides by the LCD of the fractions.
- Rewrite the result as a polynomial inequality $p(x) < 0$; $p(x) \le 0$; $p(x) > 0$; or $p(x) \ge 0$.
- Make a sign chart for the product LCD $\cdot p(x)$. Use as critical numbers all x-values for which LCD $\cdot p(x) = 0$.

Example 9, Method 2: Solve $\frac{x+2}{x+3} \le \frac{x-1}{x-2}$.
This method avoids adding fractions of polynomials.

The LCD is	$(x+3)(x-2)$
Inequality is	$\frac{x+2}{x+3} - \frac{x-1}{x-2} \le 0$
Multiply by its sign:	LCD even though you don't know $(x+3)(x-2)\left(\frac{x+2}{x+3} - \frac{x-1}{x-2}\right) \le 0$
Distribute	and cancel
	$(x-2)(x+2) - (x+3)(x-1) \le 0$
Use () !	$(x^2-4) - (x^2+2x-3) \le 0$
Distribute $-$:	$x^2 - 4 - x^2 - 2x + 3 \le 0$
Rewrite:	$-2x - 1 \le 0$.
Times -1	LCD $\cdot p(x) = 2x+1 \ge 0$.

Method 2 uses as critical numbers both

- The solution $x = -\frac{1}{2}$ of $p(x) = 2x+1 = 0$ and
- x-values that make the LCD $(x+3)(x-2) = 0$, namely $x = -3$ and $x = 2$.

Here we use the Sign Switch Theorem to find the signs of the factors. For example, the factor $x+3$ equals 0 when $x = -3$. Therefore $x+3 < 0$ for $x < -3$ and $x+3 > 0$ for $-3 < x$.

In interval	$(-\infty, -3)$	$(-3, -\frac{1}{2})$	$(-\frac{1}{2}, 2)$	$(2, \infty)$
$(x+3)$	$-$	$+$	$+$	$+$
$p(x) = 2x+1$	$-$	$-$	$+$	$+$
$(x-2)$	$-$	$-$	$-$	$+$
LCD $\cdot p(x)$	$---=-$	$+--=+$	$-$	$+$
LCD $\cdot p(x) > 0$ in		$(-3, -\frac{1}{2})$		$(2, \infty)$

As before, include $x = -\frac{1}{2}$ to satisfy $p(x) = 2x+1 \ge 0$.

Answer: x in $(-3, -\frac{1}{2}] \cup (2, \infty)$ | $-3 < x \le -\frac{1}{2}$ or $2 < x$

Example 10: Solve $\frac{x+2}{x+3} - \frac{x-1}{x-2} < 0$.
Solution: This is the same as the previous example, except that the inequality sign has been changed from $\le$ to $<$.

The problem becomes LCD$\cdot p(x) = 2x+1 > 0$ rather than $2x+1 \ge 0$. Now omit $x = -\frac{1}{2}$ since $p(-\frac{1}{2}) = 0$.

Answer: x in $(-3, -\frac{1}{2}) \cup (2, \infty)$ | $-3 < x < -\frac{1}{2}$ or $2 < x$

Example 11: Use Method 1 to solve $\frac{3}{x} \le 1 + \frac{2}{x^2}$

The problem is:	$\frac{3}{x} \le 1 + \frac{2}{x^2}$
Everything to one side:	$\frac{3}{x} - 1 - \frac{2}{x^2} \le 0$
Rewrite with LCD x^2:	$\frac{3x}{x^2} - \frac{x^2}{x^2} - \frac{2}{x^2} \le 0$
Combine fractions:	$\frac{3x-x^2-2}{x^2} \le 0$
Multiply by -1:	$\frac{x^2-3x+2}{x^2} \ge 0.$
Factor numerator:	$\frac{p(x)}{q(x)} = \frac{(x-1)(x-2)}{x^2} \ge 0$

- Setting $p(x) = (x-1)(x-2) = 0$ and $q(x) = x^2 = 0$ yields critical numbers 0, 1, 2.
- These split the line into critical intervals $(-\infty, 0)$; $(0, 1)$; $(1, 2)$; and $(2, \infty)$.

In interval	$(-\infty, 0)$	$(0, 1)$	$(1, 2)$	$(2, \infty)$
Choose test number $x =$	-1	$\frac{1}{2}$	$\frac{3}{2}$	3
x^2 is	$+$	$+$	$+$	$+$
$x-1$ is	-2	$-\frac{1}{2}$	$+\frac{1}{2}$	$+2$
$x-2$ is	-3	$-\frac{3}{2}$	$-\frac{1}{2}$	$+1$
$\frac{p(x)}{q(x)} = \frac{(x-1)(x-2)}{x^2}$ is	$+$	$+$	$-$	$+$
$\frac{(x-1)(x-2)}{x^2} \ge 0$ in	$(-\infty, 0)$	$(0, 1]$		$[2, \infty)$

Since the inequality sign is $\ge$, include critical numbers $x = 1$ and $x = 2$ to make the numerator of $\frac{(x-1)(x-2)}{x^2}$ equal 0. The fraction is undefined at $x = 0$, so omit it.

Answer: Inequality: $x < 0$ or $0 < x \le 1$ or $2 \le x$

Interval: $(-\infty, 0) \cup (0, 1] \cup [2, \infty)$

The Sign Switch Theorem stated earlier for polynomials $p(x)$ stays essentially the same when applied to completely reduced rational polynomials.

Sign Switch Theorem for rational polynomials $\frac{p(x)}{q(x)}$

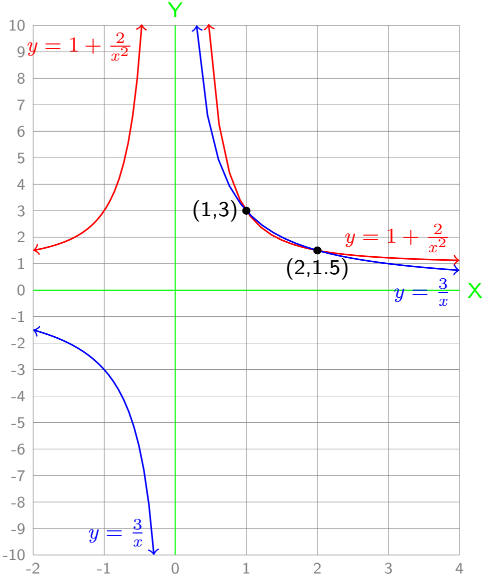
Factor $p(x)$ and $q(x)$, then completely reduce $\frac{p(x)}{q(x)}$. Suppose the highest power of factor $ax - b$ in either numerator or denominator of the reduced fraction is $(ax-b)^n$. As x moves on the number line through the critical number $\frac{b}{a}$, the value of $\frac{p(x)}{q(x)}$

- switches sign if n is odd;
- keeps the same sign if n is even.

In Example 11, the sign of $\frac{p(x)}{q(x)} = \frac{(x-1)^1(x-2)^1}{x^2}$, as x moves from left to right,

- keeps the same sign (positive)at $x = 0$.
- switches from positive to negative at $x = 1$;
- from negative to positive at $x = 2$;

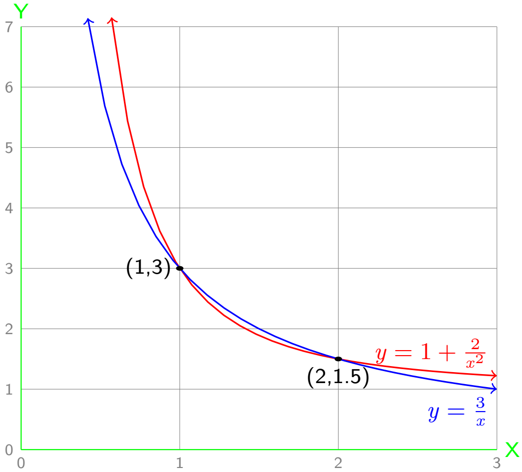
Let's check the answer to Example 11 by graphing $y = \frac{3}{x}$ and $y = 1 + \frac{2}{x^2}$ to see which value of y is larger.



The closeup below shows more detail.
The pictures confirm that $\frac{3}{x} \leq 1 + \frac{2}{x^2}$ for

$x < 0$ and $0 < x \leq 1$ and $2 \leq x$

x in $(-\infty, 0) \cup (0, 1] \cup [2, \infty)$



2.5.6 Precalculus Section 2.5 Quiz

- Ex. 2.5.1: Graph the polynomial $y = p(x) = x^3 - 2x^2 + x$.
- Ex. 2.5.2: Graph the polynomial $y = p(x) = x^4 - 4x^2$.
- Ex. 2.5.3: Solve $x^2 + 3 \geq 4x$.
- Ex. 2.5.4: Solve $x^3 > x$.
- Ex. 2.5.5: Solve $x^3 \leq x$.
- Ex. 2.5.6: Solve $x^3 \geq 2x^2 + 3x$.
- Ex. 2.5.7: Solve $-x^3 + 2x^4 - x^5 < 0$.
- Ex. 2.5.8: Solve $-x^3 + 2x^4 - x^5 \leq 0$.
- Ex. 2.5.9: Solve $\dfrac{x+2}{x+3} \leq \dfrac{x-1}{x-2}$.
- Ex. 2.5.10: Solve $\dfrac{x+2}{x+3} - \dfrac{x-1}{x-2} < 0$.
- Ex. 2.5.11: Solve $\dfrac{3}{x} \leq \dfrac{2}{x^2} - 2$.

Section 2.5 Review: Graphing polynomial functions

Ex. 2.5.1: Sketch $= x^3 - 2x^2 + x$. Color critical points (x-intercepts) yellow; test points red.

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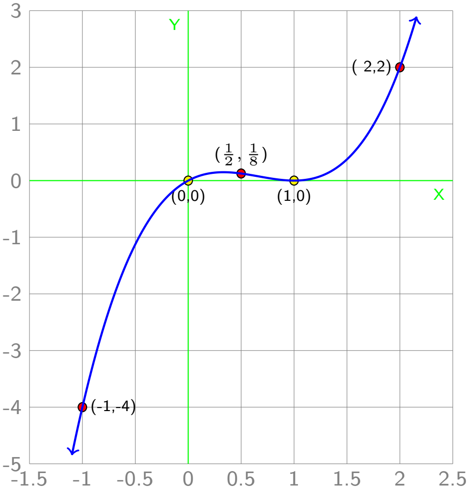

Section 2.5 Review: Graphing polynomial functions

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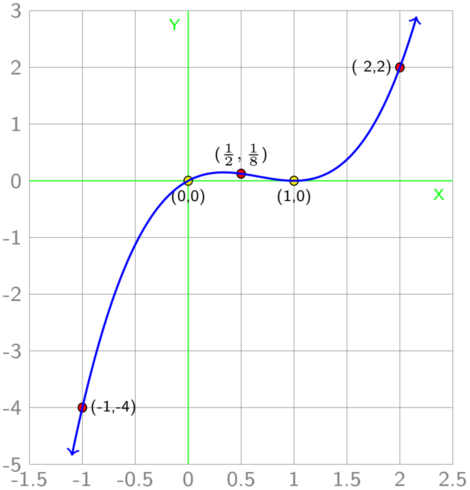
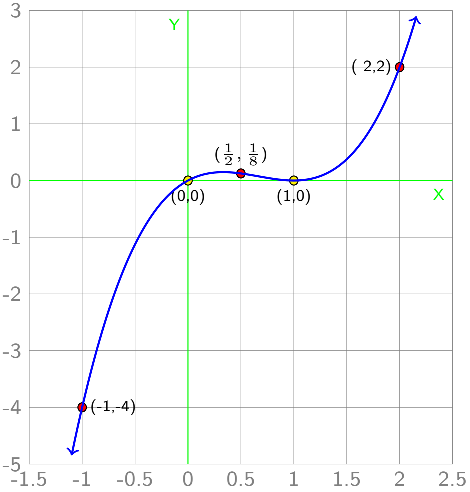


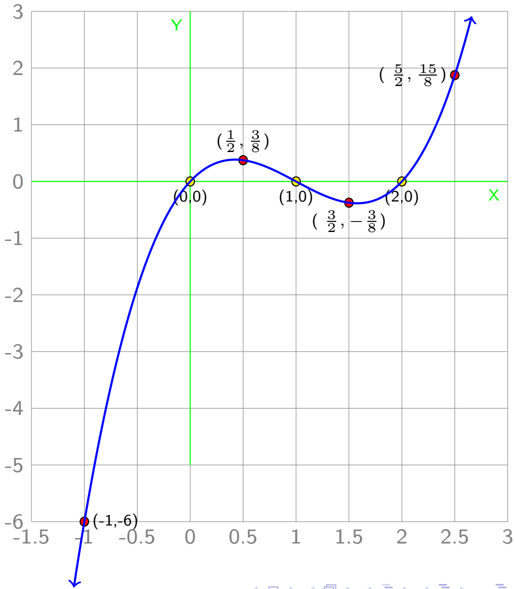
Sketch $x^3 - 3x^2 + 2x$

Section 2.5 Review: Graphing polynomial functions

Ex. 2.5.1: Sketch $= x^3 - 2x^2 + x$. Color critical points (x-intercepts) yellow; test points red.



Sketch $x^3 - 3x^2 + 2x$



Sketch $y = x^2 - x^3$

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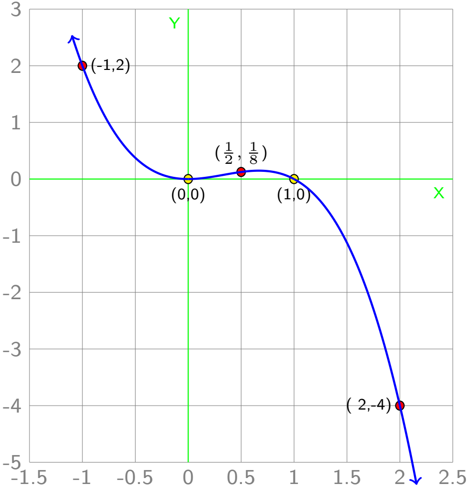
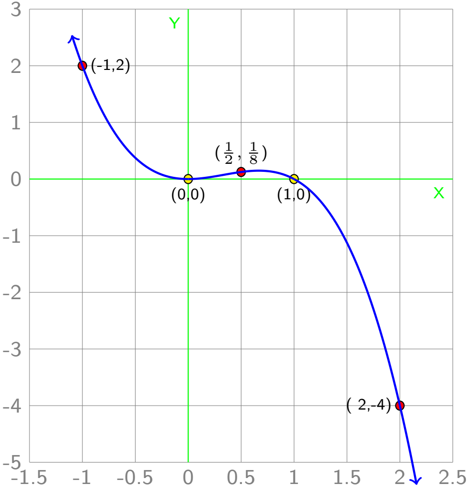
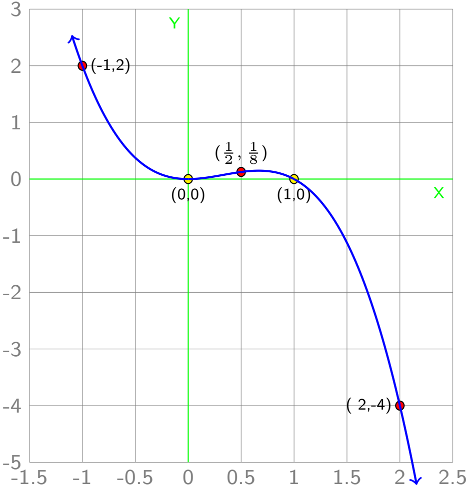


Sketch $y = x^2 - x^3$

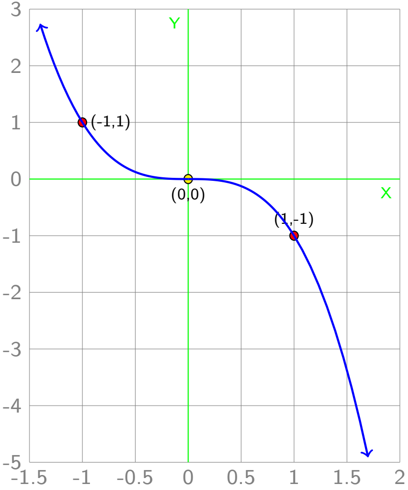


Sketch $y = -x^3$

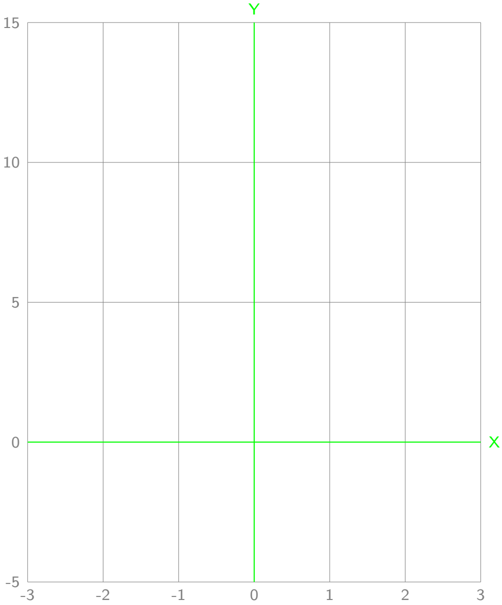
Sketch $y = x^2 - x^3$



Sketch $y = -x^3$

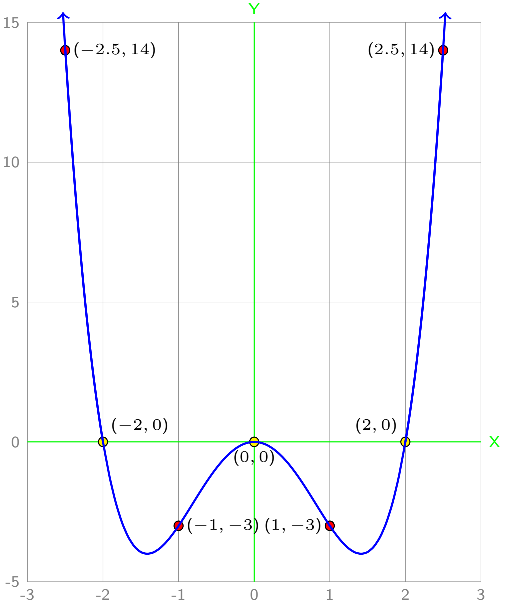


Ex. 2.5.2: Graph $y = p(x) = x^4 - 4x^2$. Color critical points (x-intercepts) yellow; test points red.

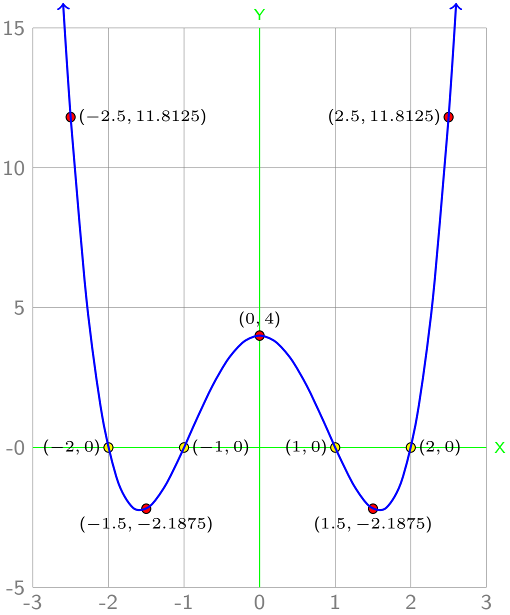


Graph $y = p(x) = (x^2 - 1)(x^2 - 4)$.

Ex. 2.5.2: Graph $y = p(x) = x^4 - 4x^2$. Color critical points (x-intercepts) yellow; test points red.



Graph $y = p(x) = (x^2 - 1)(x^2 - 4)$.



Graph $y = p(x) = x^4 - 2x^3$.

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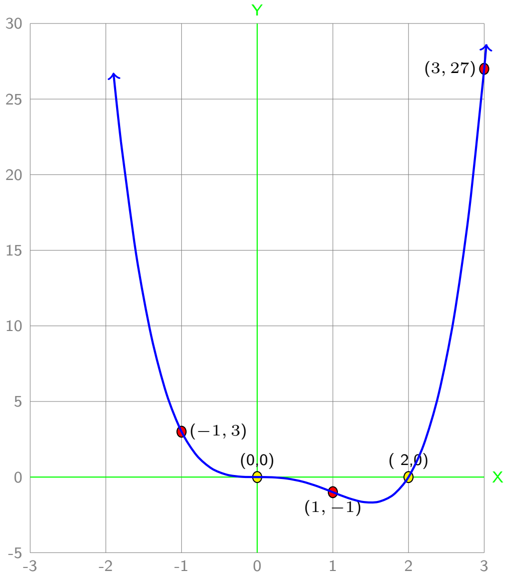

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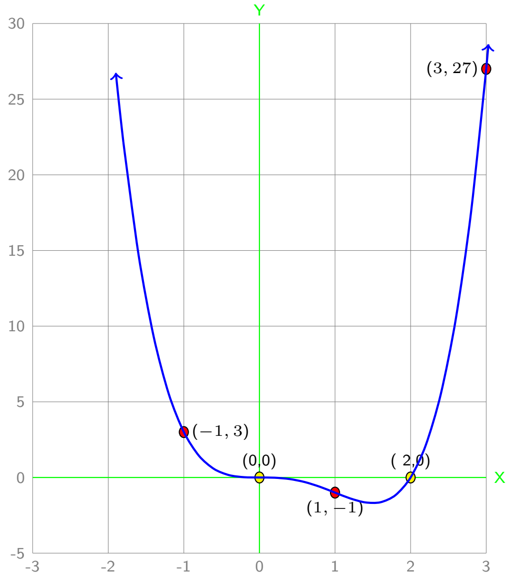

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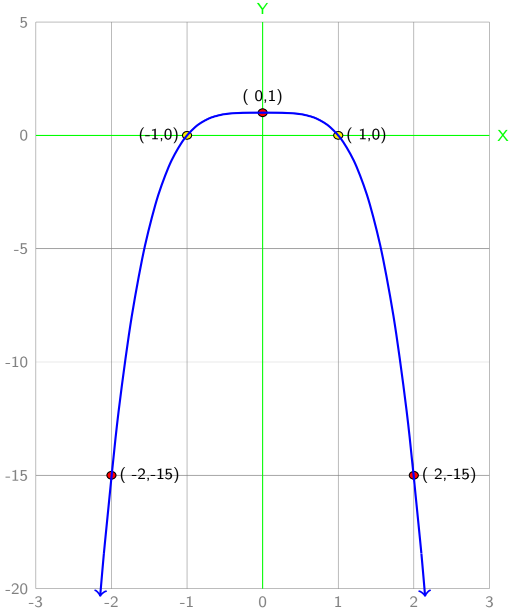

In each problem, find the critical numbers (CNs) and solve the inequality.

Ex. 2.5.3
- $x^2 + 3 \geq 4x \Rightarrow$
- $x^2 + 2x > 8 \Rightarrow$
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- $x^3 > x \Rightarrow$
- $2x^3 < x^2 + x \Rightarrow$
- $2x^3 \leq 3x^2 \Rightarrow$
- $2x^3 > 5x \Rightarrow$

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- Ex. 2.5.7 • $-x^3+2x^4-x^5<0 \Rightarrow$
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- $x^4 + 2x^3 < 3x^2 \Rightarrow$ CNs $-3, 0, 1$; x in $(-3, 0) \cup (0, 1)$; $-3 < x < 0$ or $0 < x < 1$
- $x^3 - x^4 \geq x^2 \Rightarrow$ CN 0; $x = 0$
- $x^4 + x^2 + 3 > 0 \Rightarrow$ Let $u = x^2$ No CNs; $(-\infty, \infty)$ = all real x

Ex. 2.5.8 • $-x^3 + 2x^4 - x^5 \leq 0 \Rightarrow$

- $x^3(x^2 + 5x + 4) \leq 0 \Rightarrow$
- $x^3(x^2 - 5x + 4) \leq 0 \Rightarrow$
- $x(x^2 - 1)^2 > 0 \Rightarrow$

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Ex. 2.5.9 • $\frac{x+2}{x+3} \le \frac{x-1}{x-2} \Rightarrow$

• $\frac{x+2}{x-3} \le \frac{x-5}{x-10} \Rightarrow$

• $\frac{x}{x+3} \le \frac{1}{2} \Rightarrow$

• $\frac{x^2}{x-1} > \frac{x^3}{x-2} \Rightarrow$

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Ex. 2.5.9 • $\frac{x+2}{x+3} \le \frac{x-1}{x-2} \Rightarrow$ CNs $-3,-\frac{1}{2},2$, x in $(-3,-\frac{1}{2}] \cup (2,\infty)$; $-3 < x \le -\frac{1}{2}$ or $2 < x$

• $\frac{x+2}{x-3} \le \frac{x-5}{x-10} \Rightarrow$ CNs $3,10$; x in $(-\infty,3) \cup (10,\infty)$; $x < 3$ or $10 < x$

• $\frac{x}{x+3} \le \frac{1}{2} \Rightarrow$ CNs $-3,3$; x in $(-3,3]$; $-3 < x \le 3$

• $\frac{x^2}{x-1} > \frac{x^3}{x-2} \Rightarrow$ CNs $1,2$; x in $(1,2)$; $1 < x < 2$

Ex. 2.5.8 • $-x^3+2x^4-x^5 \le 0 \Rightarrow$ CNs $0,1$; x in $[0,\infty)$; $0 \le x$

- $x^3(x^2+5x+4) \le 0 \Rightarrow$ CNs $-4,0,1$; x in $(-\infty,-4] \cup [-1,0]$; $x \le -4$ or $-1 \le x \le 0$
- $x^3(x^2-5x+4) \le 0 \Rightarrow$ CNs $0,1,4$; x in $(-\infty,0] \cup [1,4]$; $x \le 0$ or $1 \le x \le 4$
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- $\frac{x^2}{x-1} > \frac{x^3}{x-2} \Rightarrow$ CNs $1,2$; x in $(1,2)$; $1 < x < 2$

Ex. 2.5.10 • $\frac{x+2}{x+3} - \frac{x-1}{x-2} < 0 \Rightarrow$

- $\frac{x+2}{x+3} - \frac{x}{x-2} \ge 0 \Rightarrow$
- $\frac{x}{x-4} - \frac{x-6}{x-7} \le 0 \Rightarrow$
- $\frac{x}{x+10} - \frac{x-1}{x+4} < 0 \Rightarrow$

Ex. 2.5.8 • $-x^3 + 2x^4 - x^5 \le 0 \Rightarrow$ CNs $0, 1$; x in $[0, \infty)$; $0 \le x$

- $x^3(x^2 + 5x + 4) \le 0 \Rightarrow$ CNs $-4, 0, 1$; x in $(-\infty, -4] \cup [-1, 0]$; $x \le -4$ or $-1 \le x \le 0$
- $x^3(x^2 - 5x + 4) \le 0 \Rightarrow$ CNs $0, 1, 4$; x in $(-\infty, 0] \cup [1, 4]$; $x \le 0$ or $1 \le x \le 4$
- $x(x^2 - 1)^2 > 0 \Rightarrow$ CNs $-1, 0, 1$; x in $(0, 1) \cup (1, \infty)$; $0 < x < 1$ or $1 < x$

Ex. 2.5.9 • $\frac{x+2}{x+3} \le \frac{x-1}{x-2} \Rightarrow$ CNs $-3, -\frac{1}{2}, 2,$ x in $(-3, -\frac{1}{2}] \cup (2, \infty)$; $-3 < x \le -\frac{1}{2}$ or $2 < x$

- $\frac{x+2}{x-3} \le \frac{x-5}{x-10} \Rightarrow$ CNs $3, 10$; x in $(-\infty, 3) \cup (10, \infty)$; $x < 3$ or $10 < x$
- $\frac{x}{x+3} \le \frac{1}{2} \Rightarrow$ CNs $-3, 3$; x in $(-3, 3]$; $-3 < x \le 3$
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- $\frac{x+2}{x+3} - \frac{x}{x-2} \ge 0 \Rightarrow$ CNs $-3, -\frac{4}{3}, 2$; x in $(-\infty, -3) \cup [-\frac{4}{3}, 2)$; $x < -3$ or $-\frac{4}{3} \le x < 2$
- $\frac{x}{x-4} - \frac{x-6}{x-7} \le 0 \Rightarrow$ CNs $4, 7, 8$; x in $(-\infty, 4) \cup (7, 8]$; $x < 4$ or $7 < x \le 8$
- $\frac{x}{x+10} - \frac{x-1}{x+4} < 0 \Rightarrow$ CNs $-1, 0, 1$; x in $(-10, -4) \cup (2, \infty)$; $-10 < x < -4$ or $2 < x$

Ex. 2.5.8 • $-x^3 + 2x^4 - x^5 \le 0 \Rightarrow$ CNs $0, 1$; x in $[0, \infty)$; $0 \le x$
• $x^3(x^2 + 5x + 4) \le 0 \Rightarrow$ CNs $-4, 0, 1$; x in $(-\infty, -4] \cup [-1, 0]$; $x \le -4$ or $-1 \le x \le 0$
• $x^3(x^2 - 5x + 4) \le 0 \Rightarrow$ CNs $0, 1, 4$; x in $(-\infty, 0] \cup [1, 4]$; $x \le 0$ or $1 \le x \le 4$
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Ex. 2.5.9 • $\frac{x+2}{x+3} \le \frac{x-1}{x-2} \Rightarrow$ CNs $-3, -\frac{1}{2}, 2,$ x in $(-3, -\frac{1}{2}] \cup (2, \infty)$; $-3 < x \le -\frac{1}{2}$ or $2 < x$
• $\frac{x+2}{x-3} \le \frac{x-5}{x-10} \Rightarrow$ CNs $3, 10$; x in $(-\infty, 3) \cup (10, \infty)$; $x < 3$ or $10 < x$
• $\frac{x}{x+3} \le \frac{1}{2} \Rightarrow$ CNs $-3, 3$; x in $(-3, 3]$; $-3 < x \le 3$
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• $\frac{x+2}{x+3} - \frac{x}{x-2} \ge 0 \Rightarrow$ CNs $-3, -\frac{4}{3}, 2$; x in $(-\infty, -3) \cup [-\frac{4}{3}, 2)$; $x < -3$ or $-\frac{4}{3} \le x < 2$
• $\frac{x}{x-4} - \frac{x-6}{x-7} \le 0 \Rightarrow$ CNs $4, 7, 8$; x in $(-\infty, 4) \cup (7, 8]$; $x < 4$ or $7 < x \le 8$
• $\frac{x}{x+10} - \frac{x-1}{x+4} < 0 \Rightarrow$ CNs $-1, 0, 1$; x in $(-10, -4) \cup (2, \infty)$; $-10 < x < -4$ or $2 < x$

Ex. 2.5.11 • $\frac{3}{x} \le 1 + \frac{2}{x^2} \Rightarrow$
• $\frac{2}{x} + 1 < \frac{8}{x^2} \Rightarrow$
• $\frac{3}{x^2} \le \frac{1}{9} + \frac{2}{x^2} \Rightarrow$
• $\frac{3-x}{x^3-x^2} \le 0 \Rightarrow$

Ex. 2.5.8 • $-x^3+2x^4-x^5 \le 0 \Rightarrow$ CNs $0,1$; x in $[0,\infty)$; $0 \le x$
- $x^3(x^2+5x+4) \le 0 \Rightarrow$ CNs $-4,0,1$; x in $(-\infty,-4] \cup [-1,0]$; $x \le -4$ or $-1 \le x \le 0$
- $x^3(x^2-5x+4) \le 0 \Rightarrow$ CNs $0,1,4$; x in $(-\infty,0] \cup [1,4]$; $x \le 0$ or $1 \le x \le 4$
- $x(x^2-1)^2 > 0 \Rightarrow$ CNs $-1,0,1$; x in $(0,1) \cup (1,\infty)$; $0 < x < 1$ or $1 < x$

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- $\frac{x+2}{x-3} \le \frac{x-5}{x-10} \Rightarrow$ CNs $3,10$; x in $(-\infty,3) \cup (10,\infty)$; $x < 3$ or $10 < x$
- $\frac{x}{x+3} \le \frac{1}{2} \Rightarrow$ CNs $-3,3$; x in $(-3,3]$; $-3 < x \le 3$
- $\frac{x^2}{x-1} > \frac{x^3}{x-2} \Rightarrow$ CNs $1,2$; x in $(1,2)$; $1 < x < 2$

Ex. 2.5.10 • $\frac{x+2}{x+3} - \frac{x-1}{x-2} < 0 \Rightarrow$ CNs $-3,-\frac{1}{2},2,$ x in $(-3,-\frac{1}{2}) \cup (2,\infty)$; $-3 < x < -\frac{1}{2}$ or $2 < x$
- $\frac{x+2}{x+3} - \frac{x}{x-2} \ge 0 \Rightarrow$ CNs $-3,-\frac{4}{3},2$; x in $(-\infty,-3) \cup [-\frac{4}{3},2)$; $x < -3$ or $-\frac{4}{3} \le x < 2$
- $\frac{x}{x-4} - \frac{x-6}{x-7} \le 0 \Rightarrow$ CNs $4,7,8$; x in $(-\infty,4) \cup (7,8]$; $x < 4$ or $7 < x \le 8$
- $\frac{x}{x+10} - \frac{x-1}{x+4} < 0 \Rightarrow$ CNs $-1,0,1$; x in $(-10,-4) \cup (2,\infty)$; $-10 < x < -4$ or $2 < x$

Ex. 2.5.11 • $\frac{3}{x} \le 1 + \frac{2}{x^2} \Rightarrow$ CNs $0,1,2.$ $x < 0$ or $0 < x \le 1$ or $2 \le x$; $(-\infty,0) \cup (0,1] \cup [2,\infty)$
- $\frac{2}{x} + 1 < \frac{8}{x^2} \Rightarrow$ CNs $-4,0,2$; x in $(-4,0) \cup (0,2)$; $-4 < x < 0$ or $0 < x < 2$
- $\frac{3}{x^2} \le \frac{1}{9} + \frac{2}{x^2} \Rightarrow$ CNs $-3,0,3$; x in $(-\infty,-3] \cup [3,\infty)$; $x \le -3$ or $3 \le x$
- $\frac{3-x}{x^3-x^2} \le 0 \Rightarrow$ CNs $0,1,3$; x in $(-\infty,0) \cup (0,1) \cup [3,\infty)$; $x < 0$ or $0 < x < 1$ or $3 \le x$

2.5.6 Section 2.5 Review

- Ex. 2.5.1: Graph the polynomial $y = p(x) = x^3 - 2x^2 + x$.
- Ex. 2.5.2: Graph the polynomial $y = p(x) = x^4 - 4x^2$.
- Ex. 2.5.3: Solve $x^2 + 3 \geq 4x$.
- Ex. 2.5.4: Solve $x^3 > x$.
- Ex. 2.5.5: Solve $x^3 \leq x$.
- Ex. 2.5.6: Solve $x^3 \geq 2x^2 + 3x$.
- Ex. 2.5.7: Solve $-x^3 + 2x^4 - x^5 < 0$.
- Ex. 2.5.8: Solve $-x^3 + 2x^4 - x^5 \leq 0$.
- Ex. 2.5.9: Solve $\dfrac{x+2}{x+3} \leq \dfrac{x-1}{x-2}$.
- Ex. 2.5.10: Solve $\dfrac{x+2}{x+3} - \dfrac{x-1}{x-2} < 0$.
- Ex. 2.5.11: Solve $\dfrac{3}{x} \leq \dfrac{2}{x^2} - 2$.

2.5.6 Section 2.5 Quiz

Quiz 2.5.0 Solve the following inequalities

- $x - 4 > 0$
- $x - 4 < 0$
- $3x - 4 \geq 0$
- $3x - 4 \leq 0$
- $4 - x > 0$
- $4 - 3x < 0$
- $-5x \geq 4$
- $-6 < -2x$

Quiz 2.5.1: Graph the polynomial $y = p(x) = x^3 - 2x^2 + x$.

Quiz 2.5.2: Graph the polynomial $y = p(x) = x^4 - 4x^2$.

Quiz 2.5.3: Solve $x^2 + 3 \geq 4x$.

Quiz 2.5.4: Solve $x^3 > x$.

Quiz 2.5.5: Solve $x^3 \leq x$.

Quiz 2.5.6: Solve $x^3 \geq 2x^2 + 3x$.

Quiz 2.5.7: Solve $-x^3 + 2x^4 - x^5 < 0$.

Quiz 2.5.8: Solve $-x^3 + 2x^4 - x^5 \leq 0$.

Quiz 2.5.9: Solve $\dfrac{x+2}{x+3} \leq \dfrac{x-1}{x-2}$.

Quiz 2.5.10: Solve $\dfrac{x+2}{x+3} - \dfrac{x-1}{x-2} < 0$.

Quiz 2.5.11: Solve $\dfrac{3}{x} \leq \dfrac{2}{x^2} - 2$.

Chapter 2 Section 6: Rate of change of a function

- 2.6.1: Average rate of change of a function
- 2.6.2: Average rate of change as the slope of a line
- 2.6.3: A car's speedometer displays average velocity
- 2.6.4: Average velocity over a small time interval
- 2.6.5: Quiz review

Section 2.6 Preview: Definitions

- Definition 2.6.1: The average rate of change of function $f(x)$ from $x = a$ to $x = b$ is
- Definition 2.6.2: A moving object's *average velocity from time* $t = a$ *to time* $t = b$ is
- Definition 2.6.3: The *velocity* at time t of a moving object with position function $s(t)$ is
- Definition 2.6.4: *Speed* is the absolute value of velocity.
- Definition 2.6.5: A car's speedometer displays average velocity over a short time.

2.6.1 Average rate of change of a function

Assume $a \neq b$. The average rate of change of function $f(x)$ from $x = a$ to $x = b$ is

$\dfrac{f(b) - f(a)}{b - a}$ = the slope of the line in the x, y-plane through points $(a, f(a))$ and $(b, f(b))$.
That line is the secant line to the graph of $y = f(x)$ from $x = a$ to $x = b$.

The average rate of change is a pure number. However, in a physics problem, units of measurement are used and must be included in the answer.
Example 1: Let $f(x) = 96x - 16x^2$. Find the average rate of change of f:

- from $x = 1$ to $x = 3$;
- from $x = 3$ to $x = 5$;
- from $x = 1$ to $x = 5$.

Solution: Factoring makes mental calculation easier: $f(x) = 96x - 16x^2 = 16x(6 - x)$ Make a table of values:

x	0	1	2	3	4	5	6
$f(x)$	0	80	128	144	128	80	0

Answer: The average rates of change are

- from $x = 1$ to $x = 3$: $\frac{f(3)-f(1)}{3-1)} = \frac{144-80}{2} = 32$
- from $x = 3$ to $x = 5$: $\frac{f(5)-f(3)}{5-3)} = \frac{80-144}{2} = -32$
- from $x = 1$ to $x = 5$: $\frac{f(5)-f(1)}{5-1)} = \frac{80-80}{2} = 0$

Let $s(t)$ be a moving object's position at time t. The object's average velocity from time $t = a$ to time $t = b$ is

$\dfrac{s(b) - s(a)}{b - a}$ = the average rate of change of function $s(t)$ from $t = a$ to $t = b$.

If the height of a rock (in feet) at time t seconds is $s(t) = 96t - 16t^2$, the average velocity of the rock

- from time $t = 1$ to $t = 3$ is 32 feet/second.
- from time $t = 3$ to $t = 5$ is -32 feet/second
- from time $t = 1$ to $t = 5$ is 0 feet/second.

The last answer is zero for a simple reason: at times $t = 1$ and $t = 5$ the rock is at the same height. Even though it goes up, then down, its average velocity from $t = 1$ to $t = 5$ is $h(5) - h(1) = 80 - 80 = 0$.

2.6.2 Average rate of change as the slope of a line

Example 2: Find the average rate of change for the function $f(x) = 96x - 16x^2$ from $x = a = 2$ to $x = b = a + h$ for the values of h in the first column below:

h	$2+h$	$\frac{f(b)-f(a)}{b-a} = \frac{f(2+h)-f(2)}{h}$
1	3	$\frac{f(3)-f(2)}{3-2} = \frac{144-128}{1} = 16$
$.1$	2.1	$\frac{f(2.1)-f(2)}{2.1-2} = \frac{131.04-128}{.1} = 30.4$
$.01$	2.01	$\frac{f(2.01)-f(2)}{2.01-2} = \frac{128.3184-128}{.01} = 31.84$
$.001$	2.001	$\frac{f(2.001)-f(2)}{2.001-2} = \frac{128.031984-128}{.001} = 31.984$

Answer: The average rates of change for the stated elapsed times h are $16, 30.4, 31.84,$ and 31.984 .

In the corresponding motion problem, the height of a rock at time t seconds is $f(t) = 96t - 16t^2$ feet. The chart suggests that the rock's average velocity is getting very close to 32 feet/second as the elapsed time h gets smaller and smaller.

In such a situation we say that the rock's *instantaneous velocity* at time $t = 2$ is 32 feet/second.

However, the word "instantaneous" is usually omitted in this text and in most science texts.

The *velocity* at time t of a moving object with position function $s(t)$

is the number that $\dfrac{s(t+h) - s(t)}{h}$ gets close to as h becomes very close to zero.

This definition is vague, but can be made precise by the use of limits in a calculus course.

2.6.3 A car's speedometer displays average velocity

Speed is the absolute value of velocity.

A car moving 60 miles/hour (miles per hour) backward has velocity -60 miles/hour and speed 60 miles/hour.

Example 3: Convert velocity 60 miles/hour (*mph*) to
a) feet/second and to b) inches/millisecond.
Solution:
a) 1 hour = 3600 seconds; 1 mile = 5280 feet.

$$\text{Then } \frac{60 \text{ miles}}{\text{hour}} = \frac{60 \cdot 5280 \text{ feet}}{3600 \text{ seconds}} = \frac{5280 \text{ feet}}{60 \text{ seconds}} = \frac{88 \text{ feet}}{\text{second}}.$$

b) 1 foot = 12 inches; 1 second = 1000 milliseconds (ms).

$$\text{Then } \frac{88 \text{ feet}}{\text{second}} = \frac{88 \cdot 12 \text{ inches}}{1000 \text{ ms}} = \frac{1056 \text{ inches}}{1000 \text{ ms}} = 1.056 \frac{\text{inches}}{\text{ms}}.$$

A car moving 60 miles/hour moves just a bit more than one inch every thousandth of a second!

Suppose you are driving at a constant speed 50 miles/hour. You (gently) floor the accelerator to pass the car in front of you. When the speedometer registers 60 miles/hour, what does that tell you?
It is *not* predicting that the car will move

- 60 miles in the next hour, or
- 30 miles in the next 30 minutes, or even
- 1 mile in the next minute.

That's because the speedometer can't predict the future! However, *the speedometer measures your average speed from a very short time ago until now.*

For example, if the car moved 8.8 feet in the last tenth of a second, its average speed over that time interval was $\frac{8.8 \text{ ft}}{0.1 \text{ sec}} = \frac{88 \text{ ft}}{\text{sec}} \cdot \frac{1 \text{ mile}}{5280 \text{ ft}} \cdot \frac{3600 \text{ sec}}{\text{hour}} = \frac{60 \text{ miles}}{\text{hour}}$
What might your average speed be for the next tenth of a second?

Suppose the car's speed increases from 8.8 feet/second to 10 feet/second in the next 0.1 seconds. Then its acceleration (rate of change of speed) would be $\frac{1.2}{10} = 120$ feet/sec^2, subjecting the passengers to nearly four times the force of gravity. Ouch!

There is no such car. When a race car goes from 0 to 60 miles/hour in 4 seconds, its average acceleration is just 22 feet/sec^2. In real life, speed can't change very much over such a short time interval!

A car's speedometer

- displays average velocity over a short time interval that just ended.
- is a good predictor of average velocity over a small time interval that is just beginning.

Example 4: Find the average rate of change of $f(x) = x - x^2$ between $x = a - h$ and $x = a + h$. Simplify your answer.

Solution: It is safest to find the two function values $f(a+h)$ and $f(a-h)$ separately.

Put parentheses around $a + h$ and $a - h$ when you substitute for x.

$f(a+h) = (a+h) - (a+h)^2 = a + h - (a^2 + 2ah + h^2) = a + h - a^2 - 2ah - h^2$. Similarly:

$f(a-h) = (a-h) - (a-h)^2 = a - h - (a^2 - 2ah + h^2) = a - h - a^2 + 2ah - h^2$.

Subtract vertically term by term on the right: $f(a+h) - f(a-h) = h - (-h) - 2ah - 2ah = 2h - 4ah$.
The requested average rate of change is

$$\frac{f(a+h) - f(a-h)}{a + h - (a-h)} = \frac{f(a+h) - f(a-h)}{2h} = \frac{\cancel{2h}(1-2a)}{\cancel{2h}} = 1 - 2a\ .$$

The average rate of change is $1 - 2a$

You should also be able to calculate the long way, as follows:

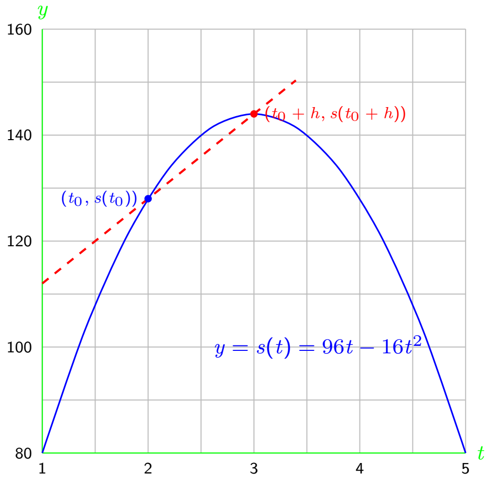
$$\frac{f(a+h) - f(a-h)}{(a+h) - (a-h)} = \frac{(a + h - a^2 - 2ah - h^2) - (a - h - a^2 + 2ah - h^2)}{2h}$$

Here, parentheses were required when substituting for function values! Now remove them

$$= \frac{a + h - a^2 - 2ah - h^2 - a + h + a^2 - 2ah + h^2}{2h}$$

$$= \frac{h - 2ah + h - 2ah}{2h} = \frac{2h - 4ah}{2h} = 1 - 2a$$ as before.

2.6.4 Instantaneous velocity is approximated by average velocity over a tiny time interval



The height at time t of a rock thrown up from the ground with velocity 96 feet/second is $s(t) = 96t - 16t^2$ feet. Let's compute the rock's average velocity between times $t = 2$ and $t = 2 + h$ as the elapsed time h gets small.

As h gets close to 0, the rock's average velocity gets close to 32 feet per second, its instantaneous velocity at time $t = 2$.

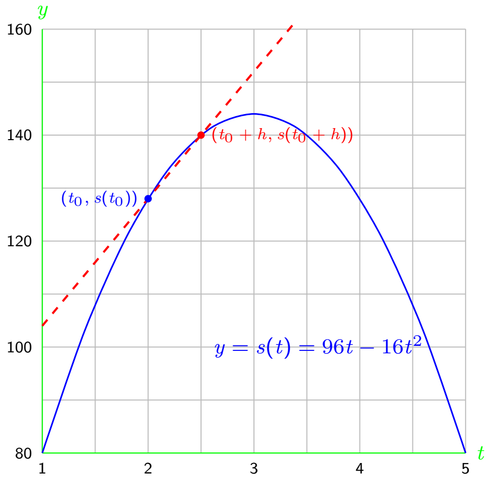
Click slowly through the following demonstration: study how the graphs at the left and the calculations below change.

When $h = 1.000\,000$, the average velocity is the slope of the red secant line, which passes through $(2, s(2)) = (2, 128.0)$ and $(2 + h, s(2 + h)) = (2 + 1.000\,000, 144.000\,000)$. That line's slope is the average rate of change of $s(t)$ between $t = 2$ and $t = 2 + h$, defined as

$$\frac{s(2+h)-s(2)}{h} = \frac{96(2+1.000\,000)-16(2+1.000\,000)^2-(96(2)-16(2)^2)}{1.000\,000}$$

$= 16.000\,000$ feet/second.

2.6.4 Instantaneous velocity is approximated by average velocity over a tiny time interval



The height at time t of a rock thrown up from the ground with velocity 96 feet/second is $s(t) = 96t - 16t^2$ feet. Let's compute the rock's average velocity between times $t = 2$ and $t = 2 + h$ as the elapsed time h gets small.

As h gets close to 0, the rock's average velocity gets close to 32 feet per second, its instantaneous velocity at time $t = 2$.

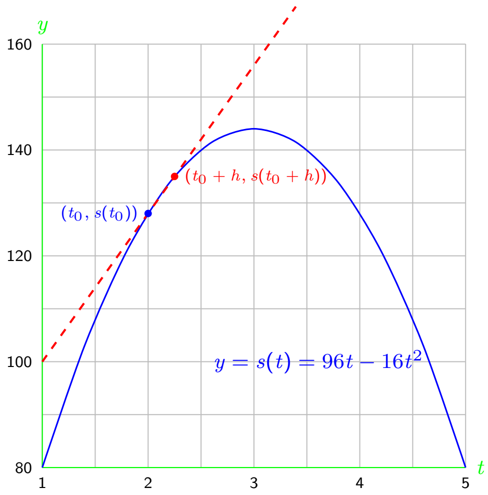
Click slowly through the following demonstration: study how the graphs at the left and the calculations below change.

When $h = 0.500\,000$, the average velocity is the slope of the red secant line, which passes through $(2, s(2)) = (2, 128.0)$ and $(2 + h, s(2 + h)) = (2 + 0.500\,000, 140.000\,000)$. That line's slope is the average rate of change of $s(t)$ between $t = 2$ and $t = 2 + h$, defined as

$$\frac{s(2+h)-s(2)}{h} = \frac{96(2+0.500\,000)-16(2+0.500\,000)^2-(96(2)-16(2)^2)}{0.500\,000}$$

$= 24.000\,000$ feet/second.

2.6.4 Instantaneous velocity is approximated by average velocity over a tiny time interval



The height at time t of a rock thrown up from the ground with velocity 96 feet/second is $s(t) = 96t - 16t^2$ feet. Let's compute the rock's average velocity between times $t = 2$ and $t = 2 + h$ as the elapsed time h gets small.

As h gets close to 0, the rock's average velocity gets close to 32 feet per second, its instantaneous velocity at time $t = 2$.

Click slowly through the following demonstration: study how the graphs at the left and the calculations below change.

When $h = 0.250\,000$, the average velocity is the slope of the red secant line, which passes through $(2, s(2)) = (2, 128.0)$ and $(2 + h, s(2 + h)) = (2 + 0.250\,000, 135.000\,000)$. That line's slope is the average rate of change of $s(t)$ between $t = 2$ and $t = 2 + h$, defined as

$$\frac{s(2+h) - s(2)}{h} = \frac{96(2+0.250\,000) - 16(2+0.250\,000)^2 - (96(2) - 16(2)^2)}{0.250\,000}$$

$= 28.000\,000$ feet/second.

Exercise: Show that $\frac{s(2+h)-s(2)}{h}$ simplifies to $32 - 16h$ and check that the above computations are correct.

2.6.5 Precalculus Section 2.6 Quiz

Ex. 2.6.1: Let $f(x) = 96x - 16x^2 = 16x(6-x)$. Find the average rate of change of f:

- from $x = 1$ to $x = 3$;
- from $x = 3$ to $x = 5$;
- from $x = 1$ to $x = 5$.

Ex. 2.6.2:Find the average rate of change for the function $f(x) = 96x - 16x^2$ from $x = a = 2$ to $x = b = a + h$ for $h = 1; h = .1; h = .01;$ and $h = .001$.

Ex. 2.6.3: Convert a velocity of 60 miles per hour to
- feet per second and to
- inches per millisecond.

Ex. 2.6.4: Find the average rate of change of $f(x) = x - x^2$ between $x = a - h$ and $x = a + h$. Simplify your answer.

Section 2.6 Review: Rate of change

Ex. 2.6.1: Find the average rate of change of :

$f(x) = 16x(6-x)$ from $x = 1$ to $x = 3 \Rightarrow$
- from $x = 3$ to $x = 5 \Rightarrow$
- from $x = 1$ to $x = 5 \Rightarrow$

$g(x) = 16x + 2345$ from $x = 1$ to $x = 3 \Rightarrow$
- from $x = 3$ to $x = 5 \Rightarrow$
- from $x = 1$ to $x = 5 \Rightarrow$

$h(x) = 2345 - 7x$ from $x = 1$ to $x = 3 \Rightarrow$
- from $x = 3$ to $x = 5 \Rightarrow$
- from $x = 1$ to $x = 5 \Rightarrow$

$s(x) = 2x^3 + 2345$ from $x = 1$ to $x = 3 \Rightarrow$
- from $x = 3$ to $x = 5 \Rightarrow$
- from $x = 1$ to $x = 5 \Rightarrow$

Section 2.6 Review: Rate of change

Ex. 2.6.1: Find the average rate of change of :

$f(x) = 16x(6-x)$ from $x = 1$ to $x = 3 \Rightarrow 64$ • from $x = 3$ to $x = 5 \Rightarrow -64$ • from $x = 1$ to $x = 5 \Rightarrow 0$

$g(x) = 16x + 2345$ from $x = 1$ to $x = 3 \Rightarrow 32$ • from $x = 3$ to $x = 5 \Rightarrow 32$ • from $x = 1$ to $x = 5 \Rightarrow 64$

$h(x) = 2345 - 7x$ from $x = 1$ to $x = 3 \Rightarrow -14$ • from $x = 3$ to $x = 5 \Rightarrow -14$ • from $x = 1$ to $x = 5 \Rightarrow -28$

$s(x) = 2x^3 + 2345$ from $x = 1$ to $x = 3 \Rightarrow 52$ • from $x = 3$ to $x = 5 \Rightarrow 196$ • from $x = 1$ to $x = 5 \Rightarrow 248$

Section 2.6 Review: Rate of change

Ex. 2.6.1: Find the average rate of change of :

$f(x) = 16x(6-x)$ from $x=1$ to $x=3 \Rightarrow 64$ • from $x=3$ to $x=5 \Rightarrow -64$ • from $x=1$ to $x=5 \Rightarrow 0$

$g(x) = 16x + 2345$ from $x=1$ to $x=3 \Rightarrow 32$ • from $x=3$ to $x=5 \Rightarrow 32$ • from $x=1$ to $x=5 \Rightarrow 64$

$h(x) = 2345 - 7x$ from $x=1$ to $x=3 \Rightarrow -14$ • from $x=3$ to $x=5 \Rightarrow -14$ • from $x=1$ to $x=5 \Rightarrow -28$

$s(x) = 2x^3 + 2345$ from $x=1$ to $x=3 \Rightarrow 52$ • from $x=3$ to $x=5 \Rightarrow 196$ • from $x=1$ to $x=5 \Rightarrow 248$

Ex. 2.6.2: Find the average rate of change for the function

$f(x) = 16x(6-x)$ from $x=2$ to $x=2+h$ for

• $h=1 \Rightarrow$ • $h=.1 \Rightarrow$ • $h=.01 \Rightarrow$ and • $h=.001 \Rightarrow$

$g(x) = 16x + 2345$ from $x=2$ to $x=2+h$ for

• $h=1 \Rightarrow$ • $h=.1 \Rightarrow$ • $h=.01 \Rightarrow$ and • $h=.001 \Rightarrow$

$h(x) = 2345 - 7x$ from $x=2$ to $x=2+h$ for

• $h=1 \Rightarrow$ • $h=.1 \Rightarrow$ • $h=.01 \Rightarrow$ and • $h=.001 \Rightarrow$

$s(x) = 2x^3$ from $x=2$ to $x=2+h$ for

• $h=1 \Rightarrow$ • $h=.1 \Rightarrow$ • $h=.01 \Rightarrow$ and • $h=.001 \Rightarrow$

Section 2.6 Review: Rate of change

Ex. 2.6.1: Find the average rate of change of :

$f(x) = 16x(6-x)$ from $x = 1$ to $x = 3 \Rightarrow 64$ • from $x = 3$ to $x = 5 \Rightarrow -64$ • from $x = 1$ to $x = 5 \Rightarrow 0$

$g(x) = 16x + 2345$ from $x = 1$ to $x = 3 \Rightarrow 32$ • from $x = 3$ to $x = 5 \Rightarrow 32$ • from $x = 1$ to $x = 5 \Rightarrow 64$

$h(x) = 2345 - 7x$ from $x = 1$ to $x = 3 \Rightarrow -14$ • from $x = 3$ to $x = 5 \Rightarrow -14$ • from $x = 1$ to $x = 5 \Rightarrow -28$

$s(x) = 2x^3 + 2345$ from $x = 1$ to $x = 3 \Rightarrow 52$ • from $x = 3$ to $x = 5 \Rightarrow 196$ • from $x = 1$ to $x = 5 \Rightarrow 248$

Ex. 2.6.2: Find the average rate of change for the function

$f(x) = 16x(6-x)$ from $x = 2$ to $x = 2 + h$ for

- $h = 1 \Rightarrow 16$ • $h = .1 \Rightarrow 30.4$ • $h = .01 \Rightarrow 31.84$ and • $h = .001 \Rightarrow 31.984$

$g(x) = 16x + 2345$ from $x = 2$ to $x = 2 + h$ for

- $h = 1 \Rightarrow 16$ • $h = .1 \Rightarrow 16$ • $h = .01 \Rightarrow 16$ and • $h = .001 \Rightarrow 16$

$h(x) = 2345 - 7x$ from $x = 2$ to $x = 2 + h$ for

- $h = 1 \Rightarrow -7$ • $h = .1 \Rightarrow -7$ • $h = .01 \Rightarrow -7$ and • $h = .001 \Rightarrow -7$

$s(x) = 2x^3$ from $x = 2$ to $x = 2 + h$ for

- $h = 1 \Rightarrow 38$ • $h = .1 \Rightarrow 25.22$ • $h = .01 \Rightarrow 24.1202$ and • $h = .001 \Rightarrow 24.012002$

Section 2.6 Review: Rate of change

Ex. 2.6.1: Find the average rate of change of :

$f(x) = 16x(6-x)$ from $x=1$ to $x=3 \Rightarrow 64$ • from $x=3$ to $x=5 \Rightarrow -64$ • from $x=1$ to $x=5 \Rightarrow 0$

$g(x) = 16x+2345$ from $x=1$ to $x=3 \Rightarrow 32$ • from $x=3$ to $x=5 \Rightarrow 32$ • from $x=1$ to $x=5 \Rightarrow 64$

$h(x) = 2345-7x$ from $x=1$ to $x=3 \Rightarrow -14$ • from $x=3$ to $x=5 \Rightarrow -14$ • from $x=1$ to $x=5 \Rightarrow -28$

$s(x) = 2x^3+2345$ from $x=1$ to $x=3 \Rightarrow 52$ • from $x=3$ to $x=5 \Rightarrow 196$ • from $x=1$ to $x=5 \Rightarrow 248$

Ex. 2.6.2: Find the average rate of change for the function

$f(x) = 16x(6-x)$ from $x=2$ to $x=2+h$ for
• $h=1 \Rightarrow 16$ • $h=.1 \Rightarrow 30.4$ • $h=.01 \Rightarrow 31.84$ and • $h=.001 \Rightarrow 31.984$

$g(x) = 16x+2345$ from $x=2$ to $x=2+h$ for
• $h=1 \Rightarrow 16$ • $h=.1 \Rightarrow 16$ • $h=.01 \Rightarrow 16$ and • $h=.001 \Rightarrow 16$

$h(x) = 2345-7x$ from $x=2$ to $x=2+h$ for
• $h=1 \Rightarrow -7$ • $h=.1 \Rightarrow -7$ • $h=.01 \Rightarrow -7$ and • $h=.001 \Rightarrow -7$

$s(x) = 2x^3$ from $x=2$ to $x=2+h$ for
• $h=1 \Rightarrow 38$ • $h=.1 \Rightarrow 25.22$ • $h=.01 \Rightarrow 24.1202$ and • $h=.001 \Rightarrow 24.012002$

Ex. 2.6.3: Convert a velocity of

60 miles per hour to • feet per second $\Rightarrow$ and to • inches per millisecond $\Rightarrow$

30 miles per hour to • miles per minute $\Rightarrow$ and to • inches per minute $\Rightarrow$

66 feet per second to • miles per hour $\Rightarrow$ and to • inches per millisecond $\Rightarrow$

630 miles per week to • feet per second $\Rightarrow$ and to • yards per second $\Rightarrow$

Section 2.6 Review: Rate of change

Ex. 2.6.1: Find the average rate of change of :

$f(x) = 16x(6-x)$ from $x = 1$ to $x = 3 \Rightarrow 64$ • from $x = 3$ to $x = 5 \Rightarrow -64$ • from $x = 1$ to $x = 5 \Rightarrow 0$

$g(x) = 16x + 2345$ from $x = 1$ to $x = 3 \Rightarrow 32$ • from $x = 3$ to $x = 5 \Rightarrow 32$ • from $x = 1$ to $x = 5 \Rightarrow 64$

$h(x) = 2345 - 7x$ from $x = 1$ to $x = 3 \Rightarrow -14$ • from $x = 3$ to $x = 5 \Rightarrow -14$ • from $x = 1$ to $x = 5 \Rightarrow -28$

$s(x) = 2x^3 + 2345$ from $x = 1$ to $x = 3 \Rightarrow 52$ • from $x = 3$ to $x = 5 \Rightarrow 196$ • from $x = 1$ to $x = 5 \Rightarrow 248$

Ex. 2.6.2: Find the average rate of change for the function

$f(x) = 16x(6-x)$ from $x = 2$ to $x = 2 + h$ for
- $h = 1 \Rightarrow 16$ • $h = .1 \Rightarrow 30.4$ • $h = .01 \Rightarrow 31.84$ and • $h = .001 \Rightarrow 31.984$

$g(x) = 16x + 2345$ from $x = 2$ to $x = 2 + h$ for
- $h = 1 \Rightarrow 16$ • $h = .1 \Rightarrow 16$ • $h = .01 \Rightarrow 16$ and • $h = .001 \Rightarrow 16$

$h(x) = 2345 - 7x$ from $x = 2$ to $x = 2 + h$ for
- $h = 1 \Rightarrow -7$ • $h = .1 \Rightarrow -7$ • $h = .01 \Rightarrow -7$ and • $h = .001 \Rightarrow -7$

$s(x) = 2x^3$ from $x = 2$ to $x = 2 + h$ for
- $h = 1 \Rightarrow 38$ • $h = .1 \Rightarrow 25.22$ • $h = .01 \Rightarrow 24.1202$ and • $h = .001 \Rightarrow 24.012002$

Ex. 2.6.3: Convert a velocity of

60 miles per hour to • feet per second $\Rightarrow 88\frac{\text{feet}}{\text{second}}$ and to • inches per millisecond $\Rightarrow 1.056\frac{\text{inches}}{\text{millisecond}}$

30 miles per hour to • miles per minute $\Rightarrow 0.5\frac{\text{miles}}{\text{minute}}$ and to • inches per minute $\Rightarrow 31680\frac{\text{inches}}{\text{minute}}$

66 feet per second to • miles per hour $\Rightarrow 45\frac{\text{miles}}{\text{hour}}$ and to • inches per millisecond $\Rightarrow 0.792\frac{\text{inches}}{\text{millisecond}}$

630 miles per week to • feet per second $\Rightarrow 5.5\frac{\text{feet}}{\text{second}}$ and to • yards per second $\Rightarrow 1\frac{5}{6}\frac{\text{yards}}{\text{second}}$

Section 2.6 Review: Rate of change

Ex. 2.6.1: Find the average rate of change of :

$f(x) = 16x(6-x)$ from $x=1$ to $x=3 \Rightarrow 64$ • from $x=3$ to $x=5 \Rightarrow -64$ • from $x=1$ to $x=5 \Rightarrow 0$

$g(x) = 16x + 2345$ from $x=1$ to $x=3 \Rightarrow 32$ • from $x=3$ to $x=5 \Rightarrow 32$ • from $x=1$ to $x=5 \Rightarrow 64$

$h(x) = 2345 - 7x$ from $x=1$ to $x=3 \Rightarrow -14$ • from $x=3$ to $x=5 \Rightarrow -14$ • from $x=1$ to $x=5 \Rightarrow -28$

$s(x) = 2x^3 + 2345$ from $x=1$ to $x=3 \Rightarrow 52$ • from $x=3$ to $x=5 \Rightarrow 196$ • from $x=1$ to $x=5 \Rightarrow 248$

Ex. 2.6.2: Find the average rate of change for the function

$f(x) = 16x(6-x)$ from $x=2$ to $x = 2+h$ for
• $h=1 \Rightarrow 16$ • $h=.1 \Rightarrow 30.4$ • $h=.01 \Rightarrow 31.84$ and • $h=.001 \Rightarrow 31.984$

$g(x) = 16x + 2345$ from $x=2$ to $x=2+h$ for
• $h=1 \Rightarrow 16$ • $h=.1 \Rightarrow 16$ • $h=.01 \Rightarrow 16$ and • $h=.001 \Rightarrow 16$

$h(x) = 2345 - 7x$ from $x=2$ to $x=2+h$ for
• $h=1 \Rightarrow -7$ • $h=.1 \Rightarrow -7$ • $h=.01 \Rightarrow -7$ and • $h=.001 \Rightarrow -7$

$s(x) = 2x^3$ from $x=2$ to $x=2+h$ for
• $h=1 \Rightarrow 38$ • $h=.1 \Rightarrow 25.22$ • $h=.01 \Rightarrow 24.1202$ and • $h=.001 \Rightarrow 24.012002$

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66 feet per second to • miles per hour $\Rightarrow 45 \frac{\text{miles}}{\text{hour}}$ and to • inches per millisecond $\Rightarrow 0.792 \frac{\text{inches}}{\text{millisecond}}$

630 miles per week to • feet per second $\Rightarrow 5.5 \frac{\text{feet}}{\text{second}}$ and to • yards per second $\Rightarrow 1\frac{5}{6} \frac{\text{yards}}{\text{second}}$

Ex. 2.6.4: Find the average rate of change of $f(x) = x^2 - x$ between

- $x = a - h$ and $x = a + h \Rightarrow$
- $x = a - h$ and $x = a \Rightarrow$
- $x = a$ and $x = a + h \Rightarrow$
- $x = a - 5h$ and $x = a + 5h \Rightarrow$

Section 2.6 Review: Rate of change

Ex. 2.6.1: Find the average rate of change of :

$f(x) = 16x(6-x)$ from $x=1$ to $x=3 \Rightarrow 64$ • from $x=3$ to $x=5 \Rightarrow -64$ • from $x=1$ to $x=5 \Rightarrow 0$

$g(x) = 16x + 2345$ from $x=1$ to $x=3 \Rightarrow 32$ • from $x=3$ to $x=5 \Rightarrow 32$ • from $x=1$ to $x=5 \Rightarrow 64$

$h(x) = 2345 - 7x$ from $x=1$ to $x=3 \Rightarrow -14$ • from $x=3$ to $x=5 \Rightarrow -14$ • from $x=1$ to $x=5 \Rightarrow -28$

$s(x) = 2x^3 + 2345$ from $x=1$ to $x=3 \Rightarrow 52$ • from $x=3$ to $x=5 \Rightarrow 196$ • from $x=1$ to $x=5 \Rightarrow 248$

Ex. 2.6.2: Find the average rate of change for the function

$f(x) = 16x(6-x)$ from $x=2$ to $x=2+h$ for
- $h=1 \Rightarrow 16$ • $h=.1 \Rightarrow 30.4$ • $h=.01 \Rightarrow 31.84$ and • $h=.001 \Rightarrow 31.984$

$g(x) = 16x + 2345$ from $x=2$ to $x=2+h$ for
- $h=1 \Rightarrow 16$ • $h=.1 \Rightarrow 16$ • $h=.01 \Rightarrow 16$ and • $h=.001 \Rightarrow 16$

$h(x) = 2345 - 7x$ from $x=2$ to $x=2+h$ for
- $h=1 \Rightarrow -7$ • $h=.1 \Rightarrow -7$ • $h=.01 \Rightarrow -7$ and • $h=.001 \Rightarrow -7$

$s(x) = 2x^3$ from $x=2$ to $x=2+h$ for
- $h=1 \Rightarrow 38$ • $h=.1 \Rightarrow 25.22$ • $h=.01 \Rightarrow 24.1202$ and • $h=.001 \Rightarrow 24.012002$

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66 feet per second to • miles per hour $\Rightarrow 45\,\frac{\text{miles}}{\text{hour}}$ and to • inches per millisecond $\Rightarrow 0.792\,\frac{\text{inches}}{\text{millisecond}}$

630 miles per week to • feet per second $\Rightarrow 5.5\,\frac{\text{feet}}{\text{second}}$ and to • yards per second $\Rightarrow 1\frac{5}{6}\,\frac{\text{yards}}{\text{second}}$

Ex. 2.6.4: Find the average rate of change of $f(x) = x^2 - x$ between

- $x = a - h$ and $x = a + h \Rightarrow 2a - 1$ • $x = a - h$ and $x = a \Rightarrow 2a - h - 1$
- $x = a$ and $x = a + h \Rightarrow 2a + h - 1$ • $x = a - 5h$ and $x = a + 5h \Rightarrow 2a - 1$

Chapter 2 Section 7: Transforming equations and graphs

- 2.7.1: Transforming equations and graphs
- 2.7.2: Modifying the equation $y = f(x)$
- 2.7.3: Modifying the argument x of $y = f(x)$
- 2.7.4: Combining transformations
- 2.7.5: Section 2.7 Review

Section 2.7 Preview: Definitions

- Definition 2.7.1: Substituting for a variable in an equation produces the opposite change to its graph.
- Definition 2.7.2: For example, substituting $3y$ for y shrinks the graph vertically by a factor of 3
- Definition 2.7.3: Changing the RHS of equation $y = f(x)$ produces the same change in its graph.

Section 2.7 Preview: Procedures

- Procedure 2.7.1: To draw the graph of $y = f(x) \pm K$ if $K > 0$
- Procedure 2.7.2: To draw the graph of $y = Kf(x)$ if $K > 0$
- Procedure 2.7.3: To draw the graph of $y = -f(x)$
- Procedure 2.7.4: To draw the graph of $y = f(x) + K$
- Procedure 2.7.5: To draw the graph of $y = f(x) - K$
- Procedure 2.7.6: To draw the graph of $y = \frac{f(x)}{K}$ if $K > 1$
- Procedure 2.7.7: To draw the graph of $y = Kf(x)$ if $K > 1$
- Procedure 2.7.8: To draw the graph of $y = -f(x)$
- Procedure 2.7.9: To draw the graph of $y = f(x - K)$
- Procedure 2.7.10: To draw the graph of $y = f(x + K)$
- Procedure 2.7.11: To draw the graph of $y = f(Kx)$ if $K > 1$
- Procedure 2.7.12: To draw the graph of $y = f(\frac{x}{K})$ if $K > 1$.
- Procedure 2.7.13: To draw the graph of $y = f(-x)$

2.7.1 Modifying an equation in two variables x,y by substituting for x or y

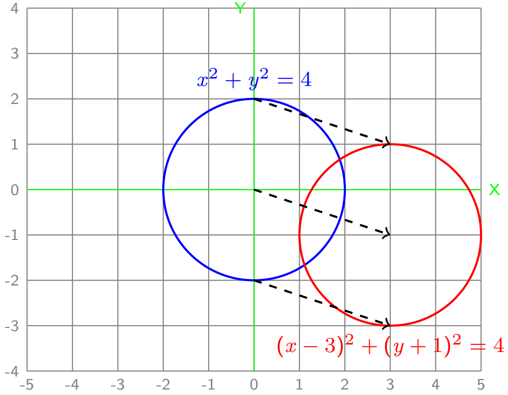
The graph of the equation

- $x^2 + y^2 = 4$ is a circle with center $(0, 0)$ and radius 2.
- $x = 4$ is a vertical line 4 units to the right of the y-axis.
- $y = x^2$ is a parabola with minimum point $(0, 0)$.

  We wish to investigate the following basic questions:
- If we change an equation, how does its graph change?
- If we want to change the graph of an equation, how should we change the equation?
  Changes in the graph might include:
- *shifting the graph* up, down, left, right;
- *reflecting* the graph across the x-axis or the y-axis;
- *squeezing* the graph toward or *stretching* it away from the x-axis or the y-axis.
- (in a later course) *rotating* the graph around a fixed point in the plane.

Example 1: Compare the graphs and equations of the circles $x^2 + y^2 = 4$ and $(x - 3)^2 + (y + 1)^2 = 4$.

Solution: Substituting $x - 3$ for x and $y + 1$ for y in equation $x^2 + y^2 = 4$ shifts that equation's graph 3 units right and 1 unit down to yield the graph of $(x - 3)^2 + (y + 1)^2 = 4$.

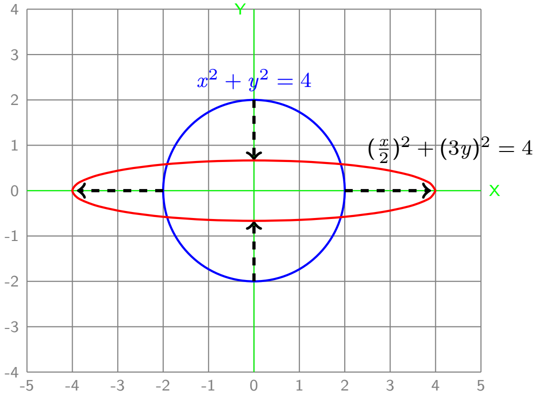


Perhaps this is the opposite of what you expect. Let's look first at the effect of replacing just one letter.

Why does substituting $x-3$ for x in equation $x^2+y^2=4$ move its graph 3 units right?

- Start with original graph $x^2+y^2=4$.
- If you are a point $P(x,y)$ on the new graph $(x-3)^2+y^2=4$, then point $Q(x-3,y)$ is 3 units to your left and is on the original graph $x^2+y^2=4$. Therefore the original graph is 3 units to the *left* of the new graph, and so
- the new graph (surprise!) is 3 units to the *right* of the original graph.
- Similarly, substituting $y+1$ for y in an equation moves its graph 1 unit down.
- Conclusion: the graph of $(x-3)^2+(y+1)^2=4$ is 3 units to the *right* of and 1 unit *down* from the graph of $x^2+y^2=4$. This makes sense: the center $(3,-1)$ of circle $(x-3)^2+(y+1)^2=4$ is 3 units to the *right* of and 1 unit *down* from the center $(0,0)$ of the circle $x^2+y^2=4$.

Example 2: Compare the graphs and equations of the circle $x^2+y^2=4$ and the ellipse $(\frac{x}{2})^2+(3y)^2=4$.
Solution:
Substituting $\frac{x}{2}$ for x and $3y$ for y in equation $x^2+y^2=4$ stretches that graph horizontally by a factor of 2 and shrinks the graph vertically by a factor of 3.



Substituting a changed variable for that variable in an equation produces the opposite change to points on the equation's graph.

- In Example 1 , substituting $y+1$ for y moves the graph down 1 unit: (x,y) moves to $(x,y-1)$.
- In Example 2 , substituting $3y$ for y shrinks the graph vertically by a factor of 3: (x,y) moves to $(x,\frac{y}{3})$.

There are five ways to substitute for x in any equation $F(x, y) = G(x, y)$.
The chart lists the effects on the graph.
H stands for Horizontally.
In each case, the point's y-coordinate stays fixed while its x-coordinate changes.

Substitute	Point (x, y) moves to	Effect on graph:
$x + K$ for x	$(x - K, y)$	H-shift left K
$x - K$ for x	$(x + K, y)$	H-shift right K
Kx for x	(Kx, y)	H-shrink by a factor of K
x/K for x	$(x/K, y)$	H-stretch by a factor of K
$-x$ for x	$(-x, y)$	Reflect across y-axis:

There are five ways to substitute for y in any equation $F(x, y) = G(x, y)$.
The chart lists the effects on the graph.
V stands for Vertically.
In each case, the point's x-coordinate stays fixed while its y-coordinate changes.

Substitute	Point (x, y) moves to	Effect on graph:
$y + K$ for y	$(x, y - K)$	V-shift up K
$y - K$ for y	$(x, y + K)$	V-shift down K
Ky for y	(x, Ky)	V-shrink by a factor of K
y/K for y	$(x, y/K)$	V-stretch by a factor of K
$-y$ for y	$(x, -y)$	Reflect across x-axis:

We will sometimes omit the words "by a factor of" and write just "V-shrink K." or "H-stretch K."
For example let $K = 4$. Notice that V-shrink 4 and V-stretch $\frac{1}{4}$ are the same operation: both move (x, y) to $(x, \frac{y}{4})$.
Similarly, H-shrink 4 and H-stretch $\frac{1}{4}$ are the same operation.

Transforming function graphs is very important. Many functions are obtained by transforming basic functions such as $y = x^2$; $y = x^3$; $y = \sqrt{x}$; $y = \sin(x)$, $y = \cos(x)$; $y = e^x$; and $y = \log(x)$.
Precalculus and calculus courses study in detail these functions and their transformations.

2.7.2: Modifying the equation $y = f(x)$.

We have seen that substituting for x produces the opposite change in the graph of $y = f(x)$. However, there is a tricky point. Substituting $y - k$ for y in equation $y = f(x)$ changes that equation to $y - k = f(x)$, that is: $y = f(x) + k$. This change adds k to the right hand side (RHS) of $y = f(x)$.

Changing the RHS of equation $y = f(x)$ produces the same change in its graph.

- Adding 1 to the RHS of $y = f(x)$ shifts its graph up 1 unit to the graph of $y = f(x) + 1$.
  (x, y) goes to $(x, y + 1)$.
- Multiplying the RHS of $y = f(x)$ by 4 V-expands its graph by a factor of 4 to the graph of $y = 4f(x)$.
  (x, y) goes to $(x, y + 1)$.

How to draw the graph of $y = f(x) \pm K$ if $K > 0$

- To draw the graph of $y = f(x) + K$, shift the graph of $y = f(x)$ vertically, up K units.
  Point $(x, f(x))$ moves to $(x, f(x) + K)$.
- To draw the graph of $y = f(x) - K$, shift the graph of $y = f(x)$ vertically, down K units.
  Point $(x, f(x))$ moves to $(x, f(x) - K)$.

For example, to obtain the graph of the parabola $y = x^2 + 2$ from the graph of $y = x^2$, shift the graph of $y = x^2$ up 2 units.
The vertex $(0, 0)$ of $y = x^2$ moves up 2 units to $(0, 2)$, which is the vertex of the shifted graph $y = x^2 + 2$.

To draw the graph of $y = Kf(x)$

- If $K > 1$, Stretch the graph of $y = f(x)$ vertically, away from the x-axis by a factor of K.
- f $K < 1$, Shrink the graph of $y = f(x)$ vertically, toward the x-axis by a factor of K.
- In all cases, point $(x, f(x))$ moves to $(x, Kf(x))$.
- Note that the graph of $y = f(x)/K$ is the graph of $Ky = f(x)$.

There is one more change to the RHS to consider.

To draw the graph of $y = -f(x)$

Reflect the graph of $y = f(x)$ across the x-axis.
Point $(x, f(x))$ moves to $(x, -f(x))$.

Modifying the Right Hand Side of equation $y = f(x)$.

Example 3: How do you obtain the graph of the parabola $y = -7x^2$ from the graph of $y = x^2$?

Solution: To stretch the graph of $y = x^2$ vertically by a factor of 7, multiply each point's y-coordinate by 7. Then reflect the graph across the x-axis.

Point $(1, 1)$ on the graph of $y = x^2$ moves to $(1, 7)$ on the graph of $y = 7x^2$, then to $(1, -7)$ on the graph of $y = -7x^2$.
The vertex $(0, 0)$ does not move.

Example 4: How do you obtain the graph of the parabola $y = \frac{x^2}{4}$ from the graph of $y = x^2$?

Solution: Rewrite $y = \frac{x^2}{4}$ as $y = \frac{1}{4}x^2$. Shrink the graph vertically by a factor of 4. Multiply every point's y-coordinate by $\frac{1}{4}$.
Point $(2, 4)$ on the graph of $y = x^2$ moves to $(2, \frac{4}{4})$ on the graph of $y = \frac{x^2}{4}$. The vertex $(0, 0)$ does not move.

Summary: there are five ways to change the output y of the equation $y = f(x)$. To do this, modify the right hand side (RHS) of the equation $y = f(x)$. The chart below lists the effects on the graph.
V stands for Vertically. In each case, the point's x-coordinate stays fixed while its y-coordinate changes.

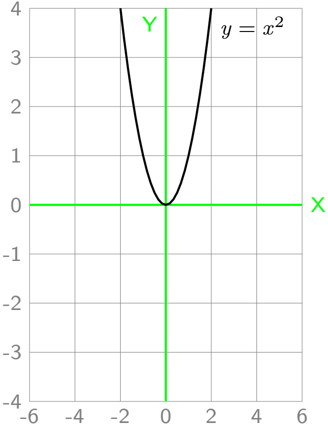
Do this to $y = f(x)$:	New equation:	Effect on graph:	Point (x, y) moves to point
Add 4 to RHS	$y = f(x) + 4$	V-shift up 4	$(x, y + 4)$
Subtract 4 from RHS	$y = f(x) - 4$	V-shift down 4	$(x, y + 4)$
Multiply RHS by 4	$y = 4f(x)$	V-stretch by a factor of 4	$(x, 4y)$
Divide RHS by 4	$y = f(x)/4$	V-shrink by a factor of 4	$(x, y/4)$
Multiply RHS by -1	$y = -f(x)$	Reflect across x-axis:	$(x, -y)$

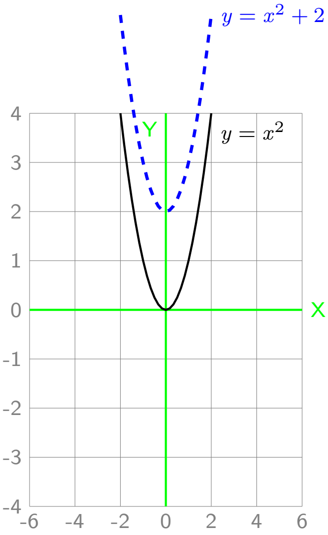
We sometimes omit the words "by a factor of" and write "V-shrink 4" or "H-stretch 3."
Note that V-shrink 4 and V-stretch $\frac{1}{4}$ are the same operation: both move (x, y) to $(x, y/4)$.

Transforming function graphs is very important. Many functions are obtained by transforming basic functions such as $y = x^2$; $y = x^3$; $y = \sqrt{x}$; $y = \sin(x)$, $y = \cos(x)$; $y = e^x$; and $y = \log(x)$.
Precalculus and calculus courses study in detail these functions and their transforms.

Transforming the equation of the parabola $y = x^2$ by modifying the right hand side: Result is another parabola.

The original graph $y = x^2$ is black. Each time you click, the transformation below will produce a blue graph at the left.

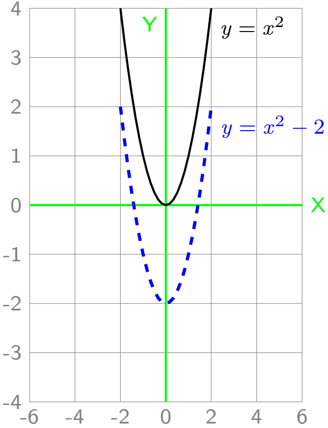




Transforming the equation of the parabola $y = x^2$ by modifying the right hand side: Result is another parabola.

The original graph $y = x^2$ is black. Each time you click, the transformation below will produce a blue graph at the left.

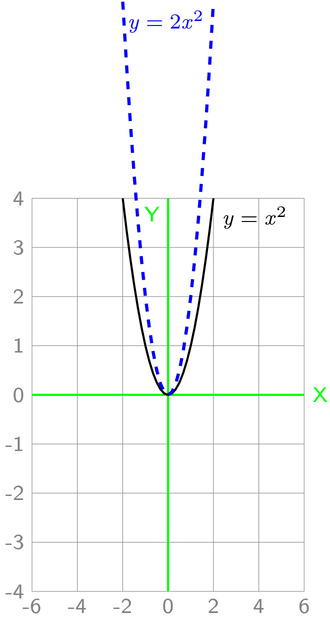
If you:	$y = x^2$ becomes	Effect on graph
• Add 2 to RHS	$y = x^2 + 2$	Shift UP 2.



Transforming the equation of the parabola $y = x^2$ by modifying the right hand side: Result is another parabola.

The original graph $y = x^2$ is black. Each time you click, the transformation below will produce a blue graph at the left.

If you:	$y = x^2$ becomes	Effect on graph
• Subtract 2 from RHS	$y = x^2 - 2$	Shift DOWN 2.



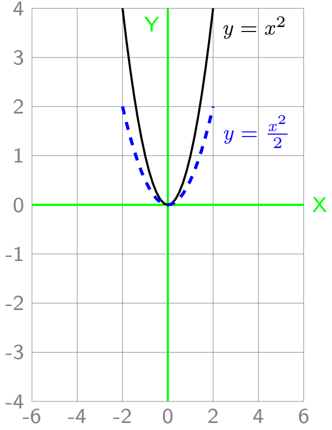
Transforming the equation of the parabola $y = x^2$ by modifying the right hand side: Result is another parabola.

The original graph $y = x^2$ is black. Each time you click, the transformation below will produce a blue graph at the left.

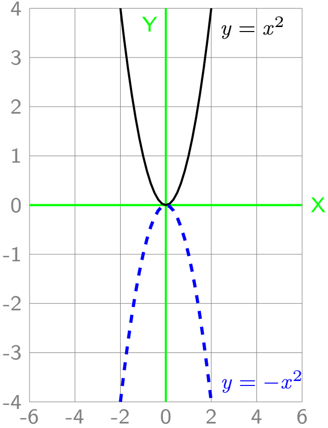
If you:	$y = x^2$ becomes	Effect on graph
• Multiply RHS by 2	$y = 2x^2$	Stretch vertically by factor of 2.

Transforming the equation of the parabola $y = x^2$ by modifying the right hand side: Result is another parabola.

The original graph $y = x^2$ is black. Each time you click, the transformation below will produce a blue graph at the left.



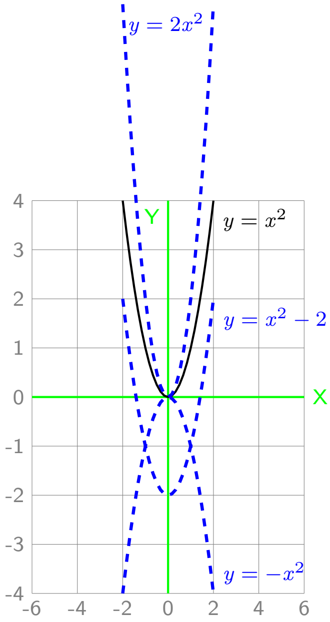
If you:	$y = x^2$ becomes	Effect on graph
• Divide RHS by 2	$y = \frac{x^2}{2}$	Shrink vertically by factor of 2.



Transforming the equation of the parabola $y = x^2$ by modifying the right hand side: Result is another parabola.

The original graph $y = x^2$ is black. Each time you click, the transformation below will produce a blue graph at the left.

If you:	$y = x^2$ becomes	Effect on graph
• Multiply RHS by -1	$y = -x^2$	Reflect across the $x-$axis.



Transforming the equation of the parabola $y = x^2$ by modifying the right hand side: Result is another parabola.

The original graph $y = x^2$ is black. Each time you click, the transformation below will produce a blue graph at the left.

If you:	$y = x^2$ becomes	Effect on graph
• Add 2 to RHS	$y = x^2 + 2$	Shift UP 2.
• Subtract 2 from RHS	$y = x^2 - 2$	Shift DOWN 2.
• Multiply RHS by 2	$y = 2x^2$	Stretch vertically by factor of 2.
• Divide RHS by 2	$y = \frac{x^2}{2}$	Shrink vertically by factor of 2.
• Multiply RHS by -1	$y = -x^2$	Reflect across the $x-$axis.

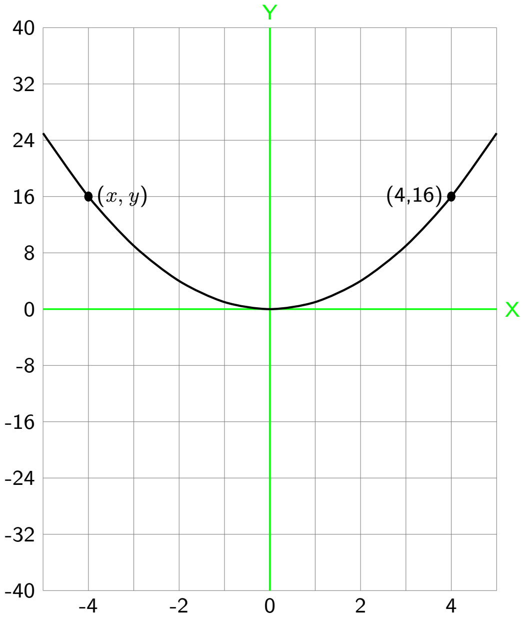
Transforming the graph of $y = f(x)$ to the graph of $y = f(x) + K$ or $y = f(x) - K$

Assume $K > 0$.

To graph $y = f(x) + K$

Shift the graph of $y = f(x)$ up K units.

Begin with the black graph of $y = f(x) = x^2$.



Transforming the graph of $y = f(x)$ to the graph of $y = f(x) + K$ or $y = f(x) - K$

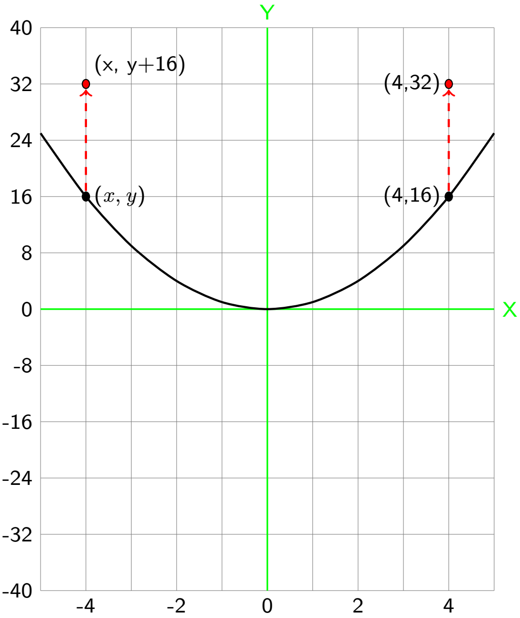
Assume $K > 0$.

To graph $y = f(x) + K$

Shift the graph of $y = f(x)$ up K units.

Begin with the black graph of $y = f(x) = x^2$.

- Shift points on that graph up 16 units by adding 16 to each point's y-coordinate .



Transforming the graph of $y = f(x)$ to the graph of $y = f(x) + K$ or $y = f(x) - K$

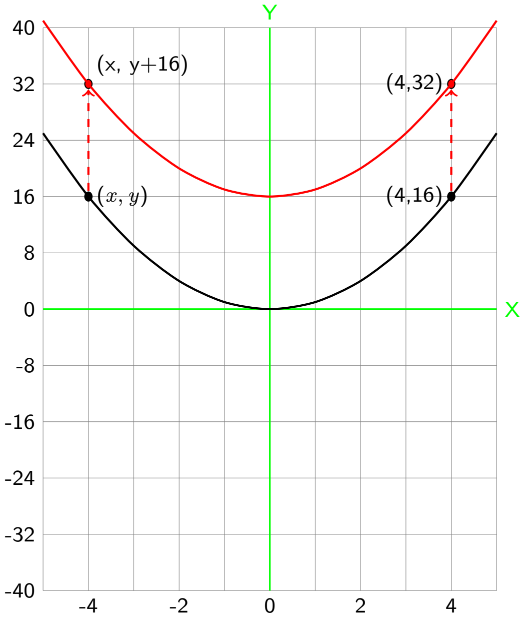
Assume $K > 0$.

To graph $y = f(x) + K$

Shift the graph of $y = f(x)$ up K units.

Begin with the black graph of $y = f(x) = x^2$.

- Shift points on that graph up 16 units by adding 16 to each point's y-coordinate .
- This gives the graph of $y = f(x) + 16 = x^2 + 16$



Transforming the graph of $y = f(x)$ to the graph of $y = f(x) + K$ or $y = f(x) - K$

Assume $K > 0$.

To graph $y = f(x) + K$

Shift the graph of $y = f(x)$ up K units.

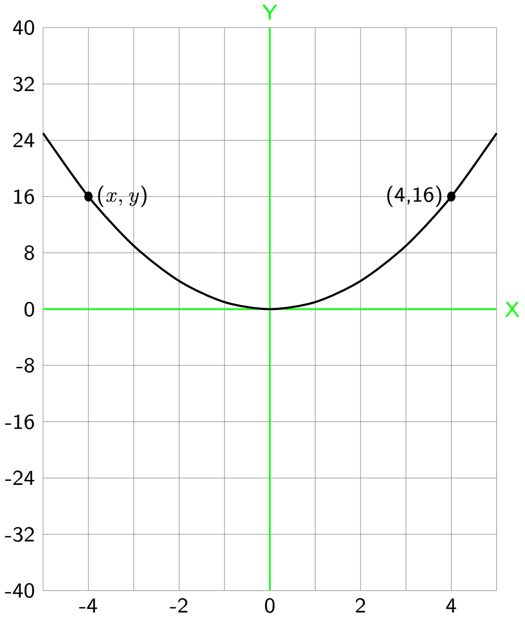
Begin with the black graph of $y = f(x) = x^2$.

- Shift points on that graph up 16 units by adding 16 to each point's y-coordinate .
- This gives the graph of $y = f(x) + 16 = x^2 + 16$

To graph $y = f(x) - K$

Shift the graph of $y = f(x)$down K units.

Start again with the black graph of $y = f(x) = x^2$.



Transforming the graph of $y = f(x)$ to the graph of $y = f(x) + K$ or $y = f(x) - K$

Assume $K > 0$.

To graph $y = f(x) + K$

Shift the graph of $y = f(x)$ up K units.

Begin with the black graph of $y = f(x) = x^2$.

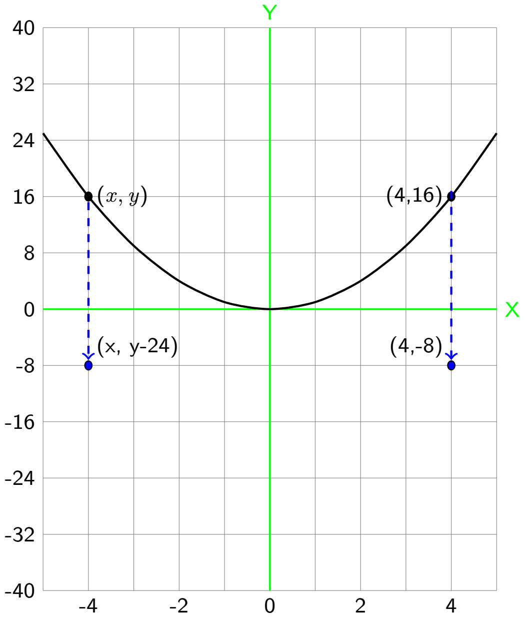
- Shift points on that graph up 16 units by adding 16 to each point's y-coordinate .
- This gives the graph of $y = f(x) + 16 = x^2 + 16$

To graph $y = f(x) - K$

Shift the graph of $y = f(x)$down K units.

Start again with the black graph of $y = f(x) = x^2$.

- Shift points on that graph down 24 units by subtracting 24 from each point's y-coordinate .



Transforming the graph of $y = f(x)$ to the graph of $y = f(x) + K$ or $y = f(x) - K$

Assume $K > 0$.

To graph $y = f(x) + K$

Shift the graph of $y = f(x)$ up K units.

Begin with the black graph of $y = f(x) = x^2$.

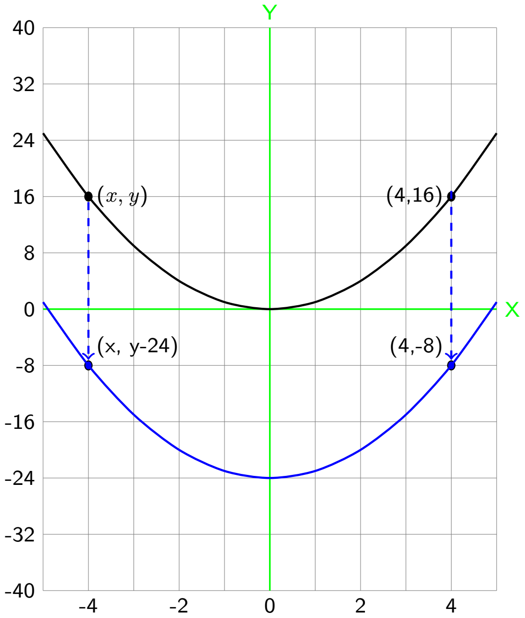
- Shift points on that graph up 16 units by adding 16 to each point's y-coordinate .
- This gives the graph of $y = f(x) + 16 = x^2 + 16$

To graph $y = f(x) - K$

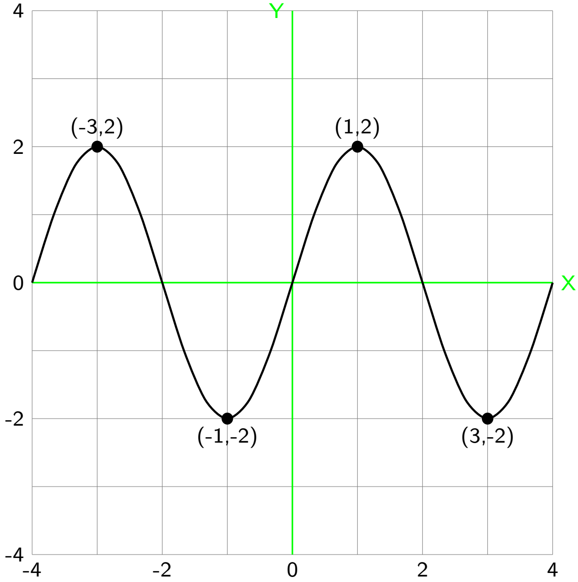
Shift the graph of $y = f(x)$down K units.

Start again with the black graph of $y = f(x) = x^2$.

- Shift points on that graph down 24 units by subtracting 24 from each point's y-coordinate .
- This gives the graph of $y = f(x) - 24 = x^2 - 24$



Transforming the graph of $y = f(x)$ to the graph of $y = Kf(x)$ or $y = \frac{f(x)}{K}$

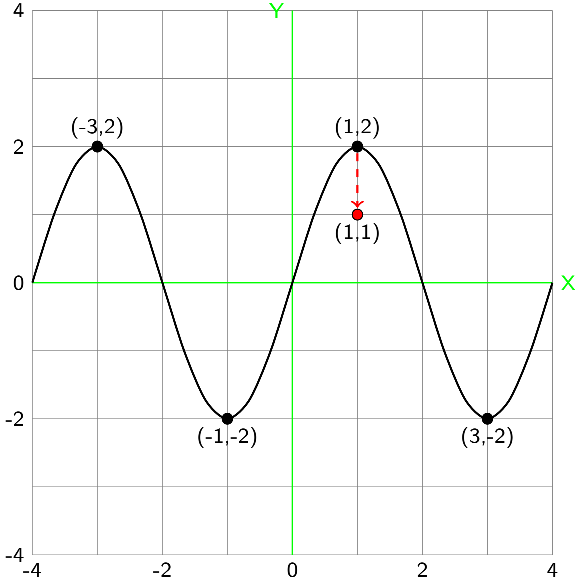


To graph $y = \frac{f(x)}{K}$ if $K > 1$

V-shrink the graph of $y = f(x)$ by a factor of K.

- Start with the black graph $y = f(x) = 2\sin\frac{\pi x}{2}$.

Transforming the graph of $y = f(x)$ to the graph of $y = Kf(x)$ or $y = \frac{f(x)}{K}$

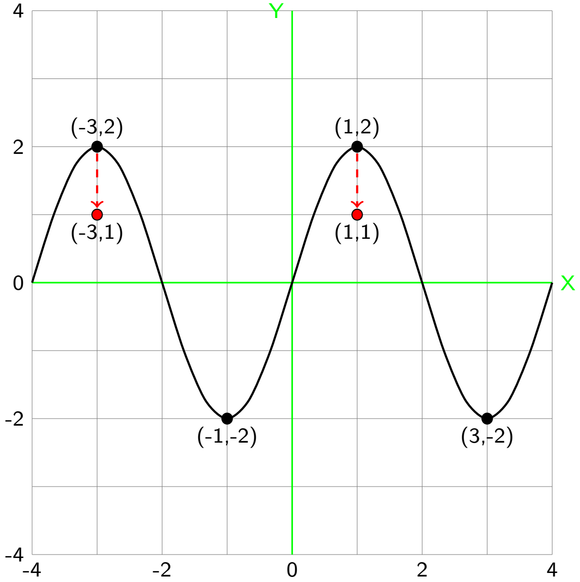


To graph $y = \frac{f(x)}{K}$ if $K > 1$

V-shrink the graph of $y = f(x)$ by a factor of K.

- Start with the black graph $y = f(x) = 2\sin\frac{\pi x}{2}$.
- Shrink the graph vertically, toward the x-axis, by a factor of 2.
- Each point's distance from the x-axis is divided by 2.
- Point $(x, f(x))$ goes to $(x, f(x)/2)$.

Transforming the graph of $y = f(x)$ to the graph of $y = Kf(x)$ or $y = \frac{f(x)}{K}$



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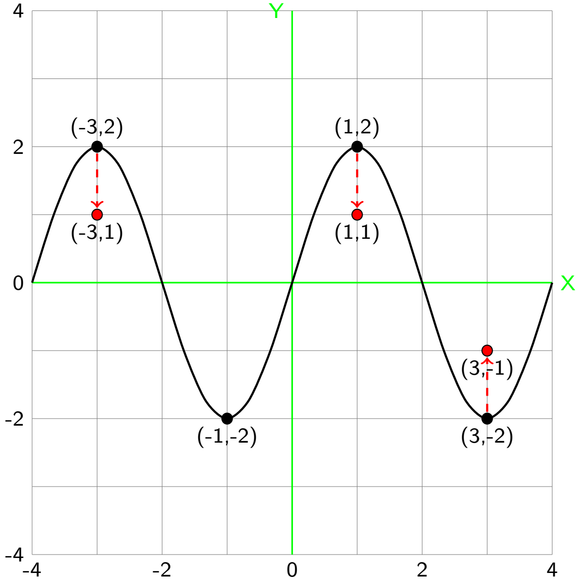
- Each point's distance from the x-axis is divided by 2.
- Point $(x, f(x))$ goes to $(x, f(x)/2)$.

Transforming the graph of $y = f(x)$ to the graph of $y = Kf(x)$ or $y = \frac{f(x)}{K}$



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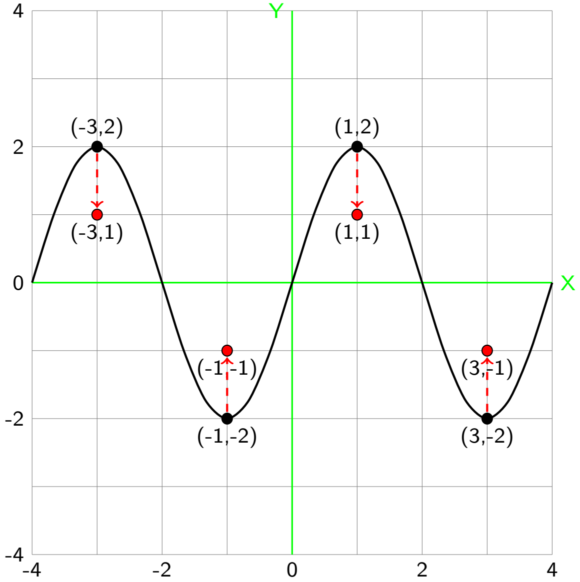
Transforming the graph of $y = f(x)$ to the graph of $y = Kf(x)$ or $y = \frac{f(x)}{K}$

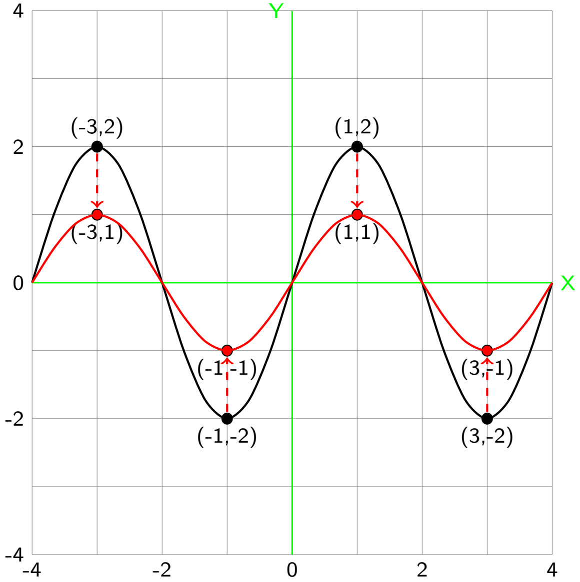


To graph $y = \frac{f(x)}{K}$ if $K > 1$

V-shrink the graph of $y = f(x)$ by a factor of K.

- Start with the black graph $y = f(x) = 2\sin\frac{\pi x}{2}$.
- Shrink the graph vertically, toward the x-axis, by a factor of 2.
- Each point's distance from the x-axis is divided by 2.
- Point $(x, f(x))$ goes to $(x, f(x)/2)$.

Transforming the graph of $y = f(x)$ to the graph of $y = Kf(x)$ or $y = \frac{f(x)}{K}$

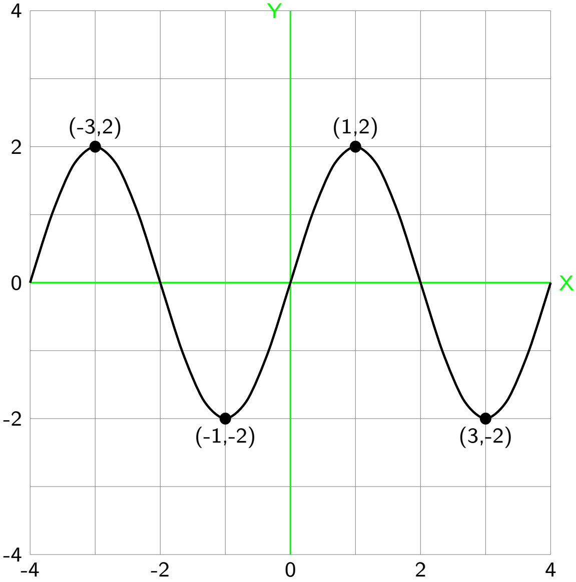


To graph $y = \frac{f(x)}{K}$ if $K > 1$

V-shrink the graph of $y = f(x)$ by a factor of K.

- Start with the black graph $y = f(x) = 2\sin\frac{\pi x}{2}$.
- Shrink the graph vertically, toward the x-axis, by a factor of 2.
- Each point's distance from the x-axis is divided by 2.
- Point $(x, f(x))$ goes to $(x, f(x)/2)$.
- This gives the red graph $y = \frac{1}{2}f(x) = \sin\frac{\pi x}{2}$.

Transforming the graph of $y = f(x)$ to the graph of $y = Kf(x)$ or $y = \frac{f(x)}{K}$



To graph $y = \frac{f(x)}{K}$ if $K > 1$

V-shrink the graph of $y = f(x)$ by a factor of K.

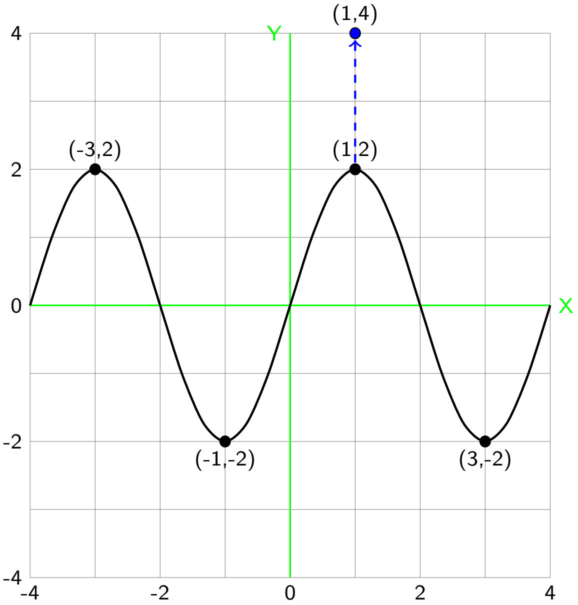
- Start with the black graph $y = f(x) = 2\sin\frac{\pi x}{2}$.
- Shrink the graph vertically, toward the x-axis, by a factor of 2.
- Each point's distance from the x-axis is divided by 2.
- Point $(x, f(x))$ goes to $(x, f(x)/2)$.
- This gives the red graph $y = \frac{1}{2}f(x) = \sin\frac{\pi x}{2}$.

To graph $y = Kf(x)$ if $K > 1$

V-stretch the graph of $y = f(x)$ by a factor of K.

Start over with the black graph $y = f(x) = 2\sin\frac{\pi x}{2}$.

Transforming the graph of $y = f(x)$ to the graph of $y = Kf(x)$ or $y = \frac{f(x)}{K}$



To graph $y = \frac{f(x)}{K}$ if $K > 1$

V-shrink the graph of $y = f(x)$ by a factor of K.

- Start with the black graph $y = f(x) = 2\,\text{sin}\,\frac{\pi x}{2}$.
- Shrink the graph vertically, toward the x-axis, by a factor of 2.
- Each point's distance from the x-axis is divided by 2.
- Point $(x, f(x))$ goes to $(x, f(x)/2)$.
- This gives the red graph $y = \frac{1}{2}f(x) = \text{sin}\,\frac{\pi x}{2}$.

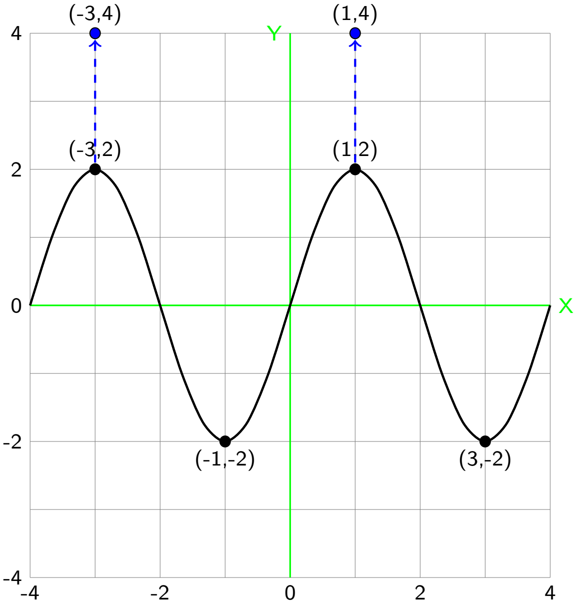
To graph $y = Kf(x)$ if $K > 1$

V-stretch the graph of $y = f(x)$ by a factor of K.

Start over with the black graph $y = f(x) = 2\,\text{sin}\,\frac{\pi x}{2}$.

- Now V-stretch the graph by a factor of 2. Each point's distance from the x-axis is doubled. Point $(x, f(x))$ goes to $(x, 2f(x))$.

Transforming the graph of $y = f(x)$ to the graph of $y = Kf(x)$ or $y = \frac{f(x)}{K}$



To graph $y = \frac{f(x)}{K}$ if $K > 1$

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- Each point's distance from the x-axis is divided by 2.
- Point $(x, f(x))$ goes to $(x, f(x)/2)$.
- This gives the red graph $y = \frac{1}{2}f(x) = \sin\frac{\pi x}{2}$.

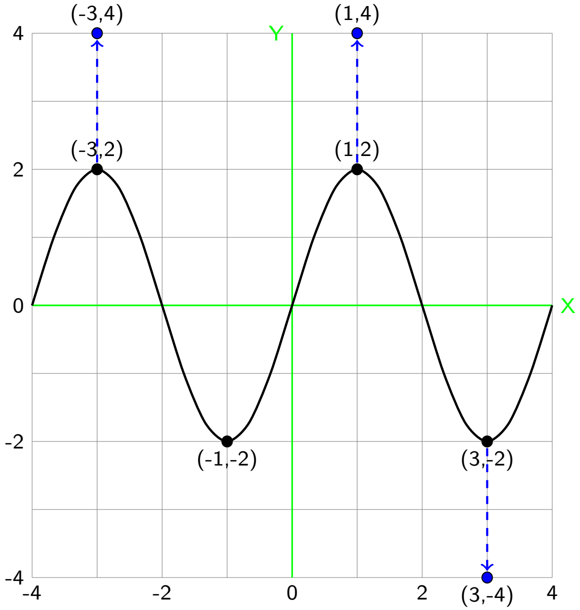
To graph $y = Kf(x)$ if $K > 1$

V-stretch the graph of $y = f(x)$ by a factor of K.

Start over with the black graph $y = f(x) = 2\sin\frac{\pi x}{2}$.

- Now V-stretch the graph by a factor of 2. Each point's distance from the x-axis is doubled. Point $(x, f(x))$ goes to $(x, 2f(x))$.

Transforming the graph of $y = f(x)$ to the graph of $y = Kf(x)$ or $y = \frac{f(x)}{K}$



To graph $y = \frac{f(x)}{K}$ if $K > 1$

V-shrink the graph of $y = f(x)$ by a factor of K.

- Start with the black graph $y = f(x) = 2\sin\frac{\pi x}{2}$.
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- This gives the red graph $y = \frac{1}{2}f(x) = \sin\frac{\pi x}{2}$.

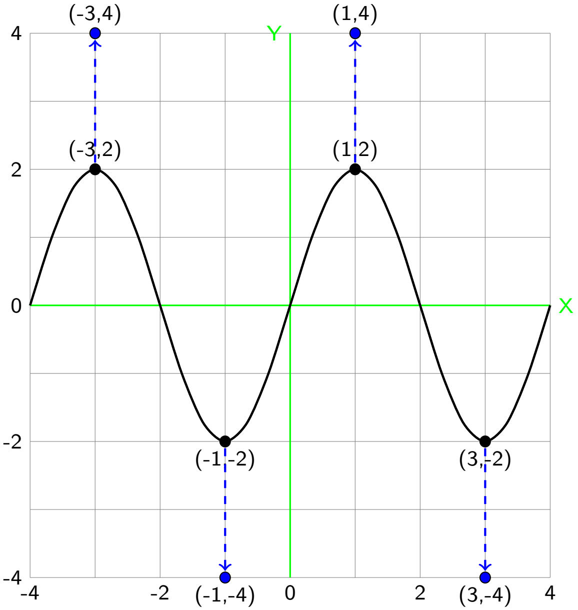
To graph $y = Kf(x)$ if $K > 1$

V-stretch the graph of $y = f(x)$ by a factor of K.

Start over with the black graph $y = f(x) = 2\sin\frac{\pi x}{2}$.

- Now V-stretch the graph by a factor of 2. Each point's distance from the x-axis is doubled. Point $(x, f(x))$ goes to $(x, 2f(x))$.

Transforming the graph of $y = f(x)$ to the graph of $y = Kf(x)$ or $y = \frac{f(x)}{K}$



To graph $y = \frac{f(x)}{K}$ if $K > 1$

V-shrink the graph of $y = f(x)$ by a factor of K.

- Start with the black graph $y = f(x) = 2\sin\frac{\pi x}{2}$.
- Shrink the graph vertically, toward the x-axis, by a factor of 2.
- Each point's distance from the x-axis is divided by 2.
- Point $(x, f(x))$ goes to $(x, f(x)/2)$.
- This gives the red graph $y = \frac{1}{2}f(x) = \sin\frac{\pi x}{2}$.

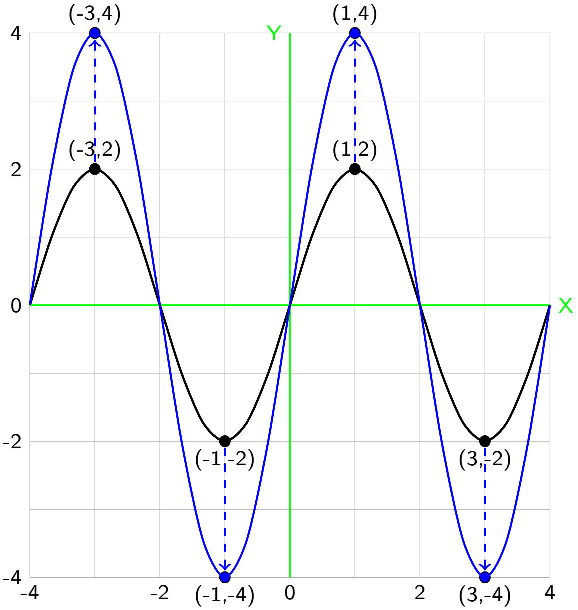
To graph $y = Kf(x)$ if $K > 1$

V-stretch the graph of $y = f(x)$ by a factor of K.

Start over with the black graph $y = f(x) = 2\sin\frac{\pi x}{2}$.

- Now V-stretch the graph by a factor of 2. Each point's distance from the x-axis is doubled. Point $(x, f(x))$ goes to $(x, 2f(x))$.

Transforming the graph of $y = f(x)$ to the graph of $y = Kf(x)$ or $y = \frac{f(x)}{K}$



To graph $y = \frac{f(x)}{K}$ if $K > 1$

V-shrink the graph of $y = f(x)$ by a factor of K.

- Start with the black graph $y = f(x) = 2\sin\frac{\pi x}{2}$.
- Shrink the graph vertically, toward the x-axis, by a factor of 2.
- Each point's distance from the x-axis is divided by 2.
- Point $(x, f(x))$ goes to $(x, f(x)/2)$.
- This gives the red graph $y = \frac{1}{2}f(x) = \sin\frac{\pi x}{2}$.

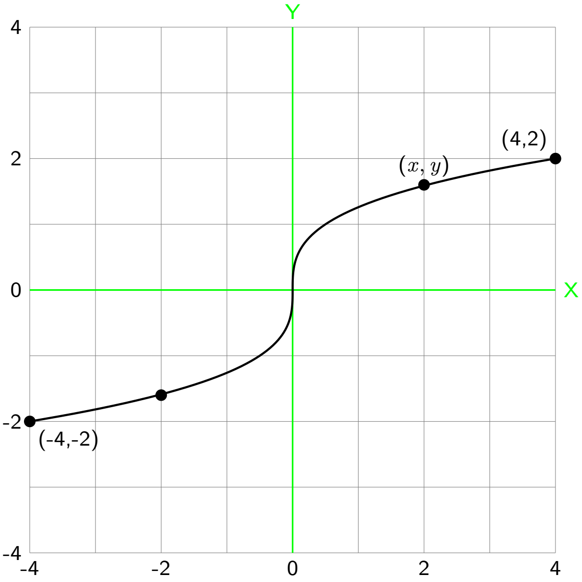
To graph $y = Kf(x)$ if $K > 1$

V-stretch the graph of $y = f(x)$ by a factor of K.

Start over with the black graph $y = f(x) = 2\sin\frac{\pi x}{2}$.

- Now V-stretch the graph by a factor of 2. Each point's distance from the x-axis is doubled. Point $(x, f(x))$ goes to $(x, 2f(x))$.
- This gives the blue graph $y = 2f(x) = 4\sin\frac{\pi x}{2}$.

Transforming the graph of $y = f(x)$ to the graph of $y = -f(x)$

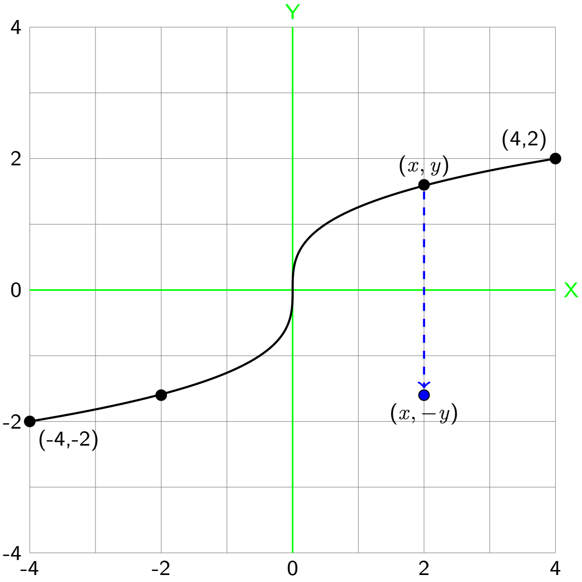


To graph $y = -f(x)$

Reflect the graph of $y = f(x)$ across the x-axis.

- Start with the graph of $y = f(x) = \sqrt[3]{2x}$.

Transforming the graph of $y = f(x)$ to the graph of $y = -f(x)$

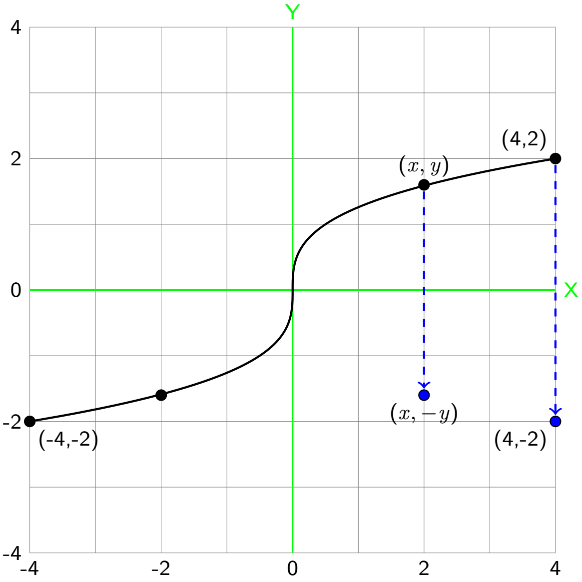


To graph $y = -f(x)$

Reflect the graph of $y = f(x)$ across the x-axis.

- Start with the graph of $y = f(x) = \sqrt[3]{2x}$.
- Reflect the graph across the x-axis.
  Reverse the sign of each point's y-coordinate.
  Point $(x, f(x))$ goes to $(x, -f(x))$.

Transforming the graph of $y = f(x)$ to the graph of $y = -f(x)$

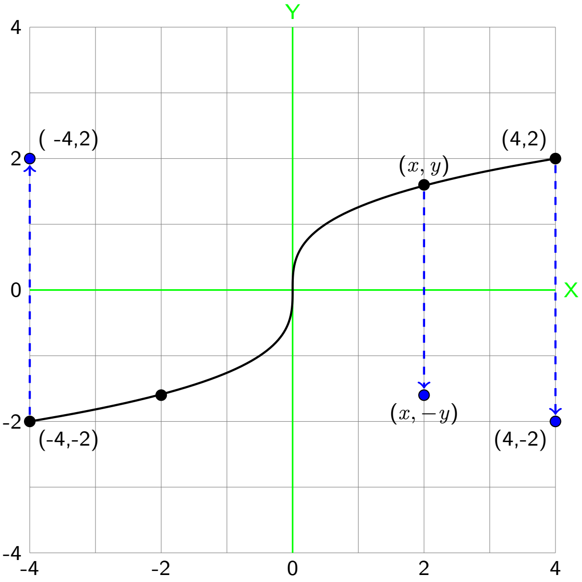


To graph $y = -f(x)$

Reflect the graph of $y = f(x)$ across the x-axis.

- Start with the graph of $y = f(x) = \sqrt[3]{2x}$.
- Reflect the graph across the x-axis.
  Reverse the sign of each point's y-coordinate.
  Point $(x, f(x))$ goes to $(x, -f(x))$.

Transforming the graph of $y = f(x)$ to the graph of $y = -f(x)$

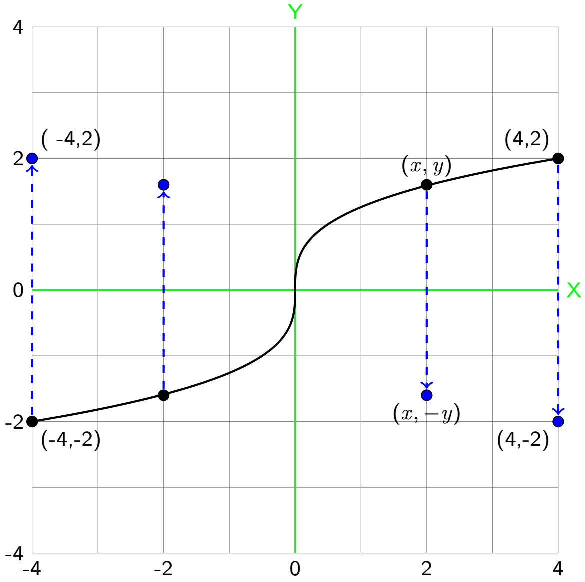


To graph $y = -f(x)$

Reflect the graph of $y = f(x)$ across the x-axis.

- Start with the graph of $y = f(x) = \sqrt[3]{2x}$.
- Reflect the graph across the x-axis.
  Reverse the sign of each point's y-coordinate.
  Point $(x, f(x))$ goes to $(x, -f(x))$.

Transforming the graph of $y = f(x)$ to the graph of $y = -f(x)$

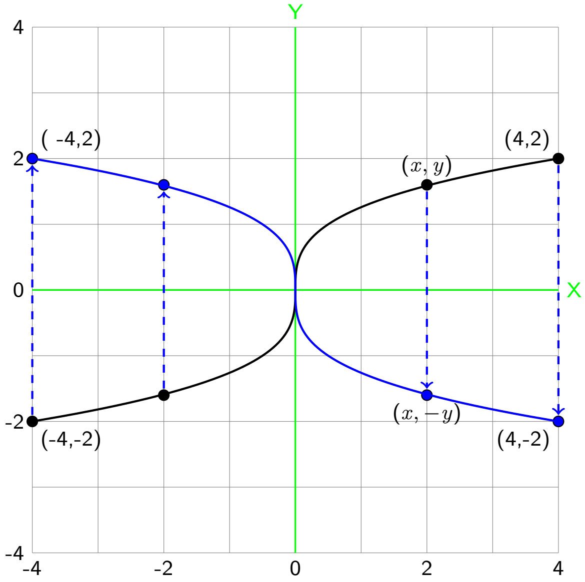


To graph $y = -f(x)$

Reflect the graph of $y = f(x)$ across the x-axis.

- Start with the graph of $y = f(x) = \sqrt[3]{2x}$.
- Reflect the graph across the x-axis.
  Reverse the sign of each point's y-coordinate.
  Point $(x, f(x))$ goes to $(x, -f(x))$.

Transforming the graph of $y = f(x)$ to the graph of $y = -f(x)$



To graph $y = -f(x)$

Reflect the graph of $y = f(x)$ across the x-axis.

- Start with the graph of $y = f(x) = \sqrt[3]{2x}$.
- Reflect the graph across the x-axis. Reverse the sign of each point's y-coordinate. Point $(x, f(x))$ goes to $(x, -f(x))$.
- This gives the graph of $y = -f(x) = -\sqrt[3]{2x}$.

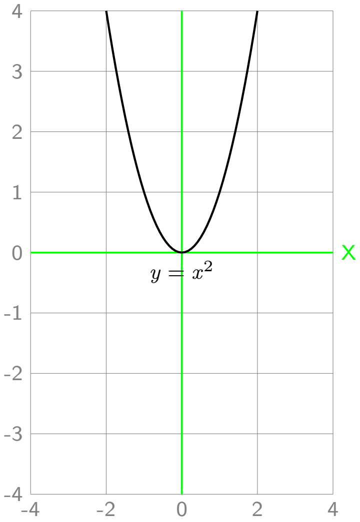
2.7.3 Modifying the graph of $y = f(x)$ by changing the argument x of f

Summary: There are five ways to change the argument x of the function $y = f(x)$.
The chart below lists the effects on the graph. H stands for Horizontally.
In each case, the point's y-coordinate stays fixed while its x-coordinate changes.

Remember that the change in the graph is the opposite of the change in the x-coordinate. For instance, replacing x by $4x$ (multiplying x by 4) *shrinks* the graph horizontally by a factor of 4.

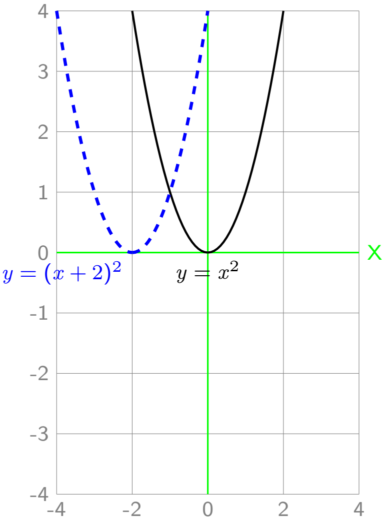
Note that H-shrink 4 and H-stretch $\frac{1}{4}$ are the same operation.

Do this to $y = f(x)$:	New equation:	Effect on graph:	Point (x, y) moves to point
Substitute $x + 4$ for x	$y = f(x + 4)$	H-shift left 4	$(x - 4, y)$
Substitute $x - 4$ for x	$y = f(x - 4)$	H-shift right 4:	$(x + 4, y)$
Substitute $4x$ for x	$y = f(4x)$	H-shrink by a factor of 4	$(x/4, y)$
Substitute $x/4$ for x	$y = f(x/4)$	H-stretch by a factor of 4	$(4x, y)$
Substitute $-x$ for x	$y = f(-x)$	Reflect across y-axis:	$(-x, y)$



Transforming the equation of the parabola $y = x^2$ by changing the input x. The result is another parabola.

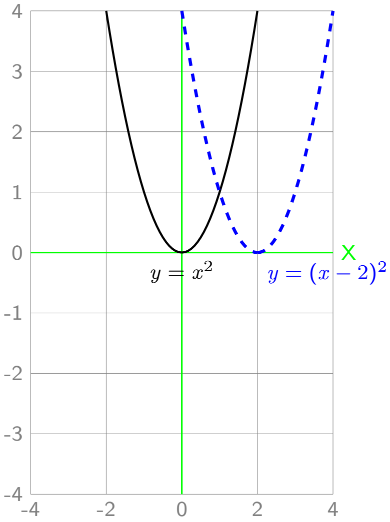
The original graph $y = x^2$ is black. Each time you click, the transformation below will produce the blue graph at the left.



Transforming the equation of the parabola $y = x^2$ by changing the input x. The result is another parabola.

The original graph $y = x^2$ is black. Each time you click, the transformation below will produce the blue graph at the left.

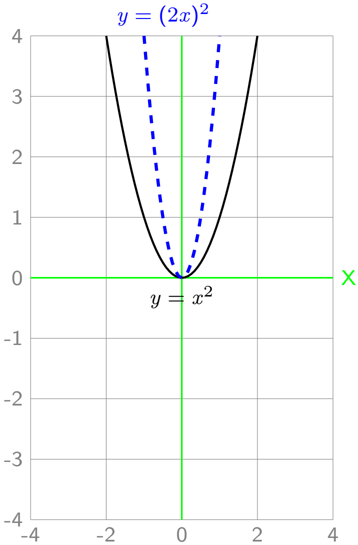
If you:	$y = x^2$ becomes	Effect on graph
• Substitute $x+2$ for x	$y = (x+2)^2$	Shift LEFT 2.



Transforming the equation of the parabola $y = x^2$ by changing the input x. The result is another parabola.

The original graph $y = x^2$ is black. Each time you click, the transformation below will produce the blue graph at the left.

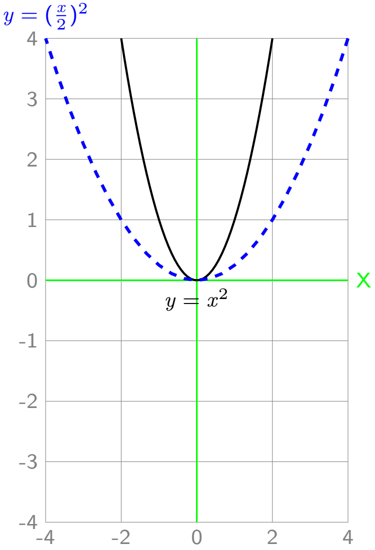
If you:	$y = x^2$ becomes	Effect on graph
• Substitute $x - 2$ for x	$y = (x-2)^2$	Shift RIGHT 2.



Transforming the equation of the parabola $y = x^2$ by changing the input x. The result is another parabola.

The original graph $y = x^2$ is black. Each time you click, the transformation below will produce the blue graph at the left.

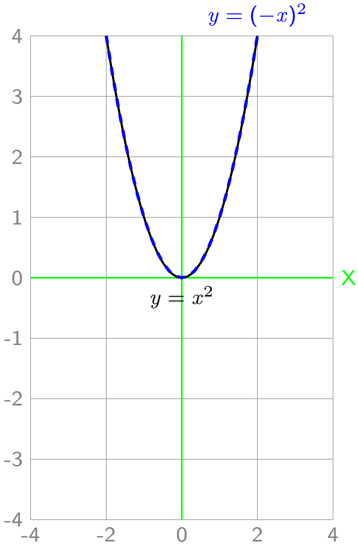
If you:	$y = x^2$ becomes	Effect on graph
• Substitute $2x$ for x	$y = (2x)^2$	Shrink horizontally by factor of 2



Transforming the equation of the parabola $y = x^2$ by changing the input x. The result is another parabola.

The original graph $y = x^2$ is black. Each time you click, the transformation below will produce the blue graph at the left.

If you:	$y = x^2$ becomes	Effect on graph
• Substitute $\frac{x}{2}$ for x	$y = (\frac{x}{2})^2$	Stretch horizontally by factor of 2.

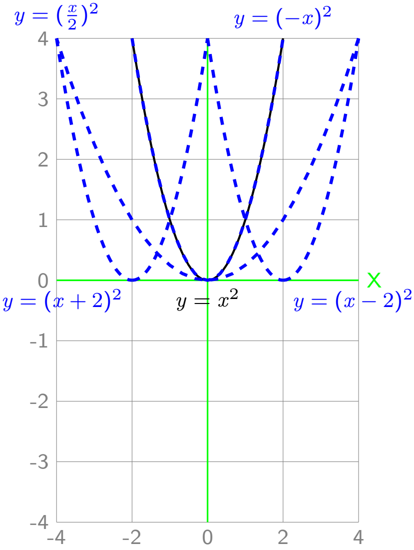


Transforming the equation of the parabola $y = x^2$ by changing the input x. The result is another parabola.

The original graph $y = x^2$ is black. Each time you click, the transformation below will produce the blue graph at the left.

If you:	$y = x^2$ becomes	Effect on graph
• Substitute $-x$ for x	$y = (-x)^2$	Reflect across y-axis.

In this example, the transformed graph and the original graph are identical. That's because the graph of $y = x^2$ is y-axis symmetric.



Transforming the equation of the parabola $y = x^2$ by changing the input x. The result is another parabola.

The original graph $y = x^2$ is black. Each time you click, the transformation below will produce the blue graph at the left.

If you:	$y = x^2$ becomes	Effect on graph
• Substitute $x+2$ for x	$y = (x+2)^2$	Shift LEFT 2.
• Substitute $x-2$ for x	$y = (x-2)^2$	Shift RIGHT 2.
• Substitute $2x$ for x	$y = (2x)^2$	Shrink horizontally by factor of 2
• Substitute $\frac{x}{2}$ for x	$y = (\frac{x}{2})^2$	Stretch horizontally by factor of 2.
• Substitute $-x$ for x	$y = (-x)^2$	Reflect across y-axis.

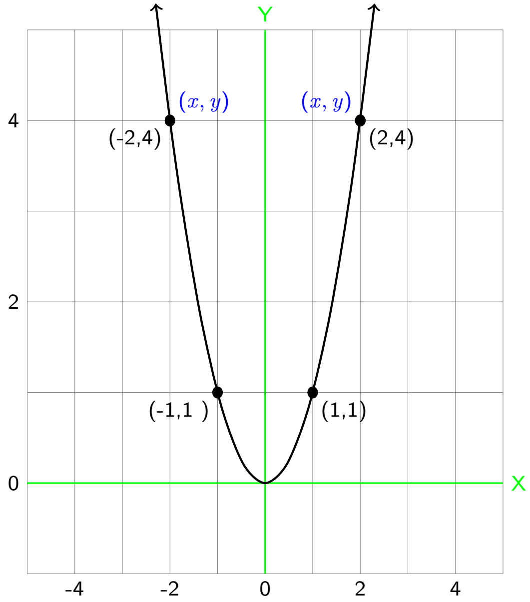
Transforming the graph of $y = f(x)$ to the graph of $y = f(x - K)$ or $y = f(x + K)$

Assume $K > 0$.

To graph $y = f(x - K)$

Shift the graph of $y = f(x)$ right K units.

Start with the graph of $y = f(x) = x^2$.



Transforming the graph of $y = f(x)$ to the graph of $y = f(x - K)$ or $y = f(x + K)$

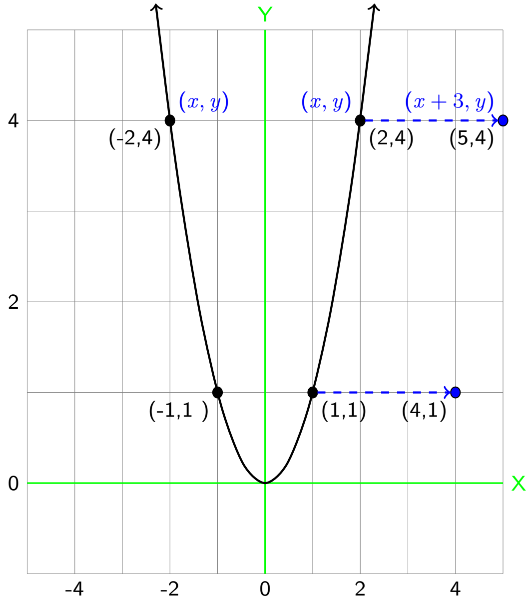
Assume $K > 0$.

To graph $y = f(x - K)$

Shift the graph of $y = f(x)$ right K units.

Start with the graph of $y = f(x) = x^2$.

- Shift points on that graph RIGHT 3 units.



Transforming the graph of $y = f(x)$ to the graph of $y = f(x - K)$ or $y = f(x + K)$

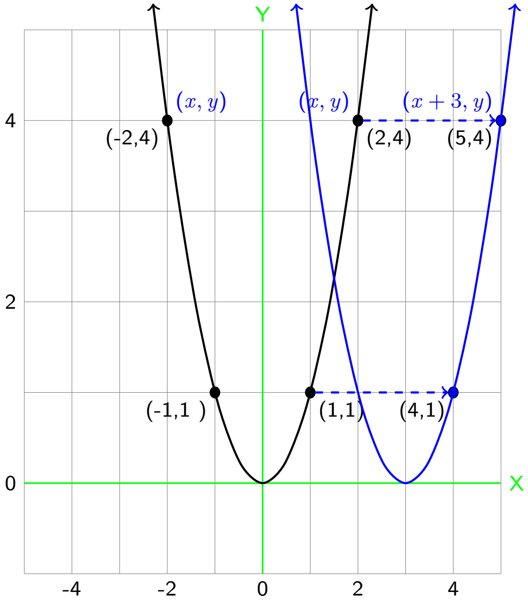
Assume $K > 0$.

To graph $y = f(x - K)$

Shift the graph of $y = f(x)$ right K units.

Start with the graph of $y = f(x) = x^2$.

- Shift points on that graph RIGHT 3 units.
- This gives the graph of $y = f(x-3) = (x-3)^2$



Transforming the graph of $y = f(x)$ to the graph of $y = f(x - K)$ or $y = f(x + K)$

Assume $K > 0$.

To graph $y = f(x - K)$

Shift the graph of $y = f(x)$ right K units.

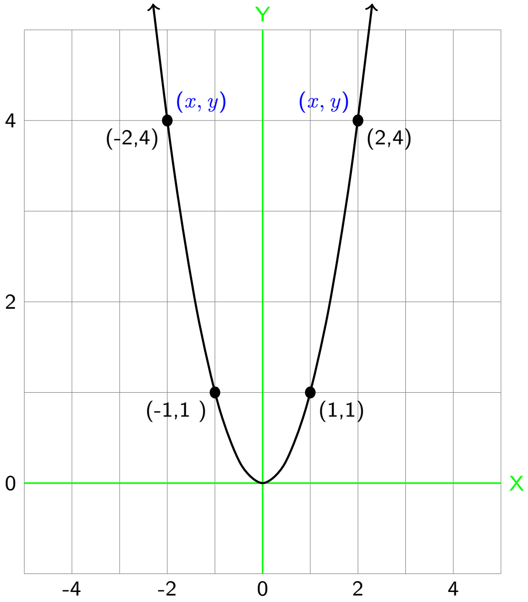
Start with the graph of $y = f(x) = x^2$.

- Shift points on that graph RIGHT 3 units.
- This gives the graph of $y = f(x - 3) = (x - 3)^2$

To graph $y = f(x + K)$

shift the graph of $y = f(x)$ left K units.

Start again with the graph $y = f(x) = x^2$.



Transforming the graph of $y = f(x)$ to the graph of $y = f(x - K)$ or $y = f(x + K)$

Assume $K > 0$.

To graph $y = f(x - K)$

Shift the graph of $y = f(x)$ right K units.

Start with the graph of $y = f(x) = x^2$.

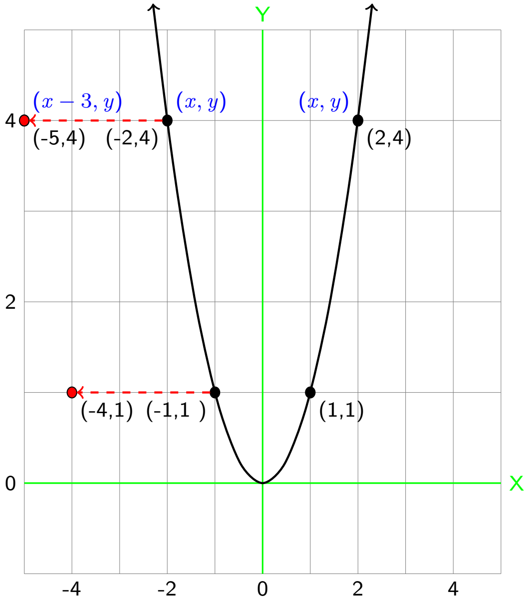
- Shift points on that graph RIGHT 3 units.
- This gives the graph of $y = f(x - 3) = (x - 3)^2$

To graph $y = f(x + K)$

shift the graph of $y = f(x)$ left K units.

Start again with the graph $y = f(x) = x^2$.

- Shift points on that graph LEFT 3 units.



Transforming the graph of $y = f(x)$ to the graph of $y = f(x - K)$ or $y = f(x + K)$

Assume $K > 0$.

To graph $y = f(x - K)$

Shift the graph of $y = f(x)$ right K units.

Start with the graph of $y = f(x) = x^2$.

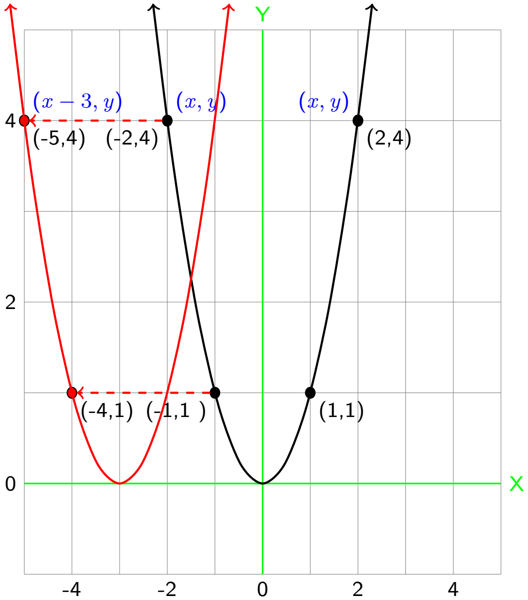
- Shift points on that graph RIGHT 3 units.
- This gives the graph of $y = f(x - 3) = (x - 3)^2$

To graph $y = f(x + K)$

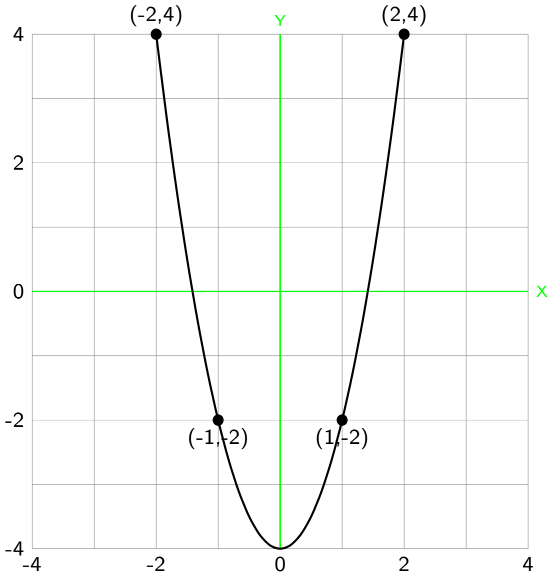
shift the graph of $y = f(x)$ left K units.

Start again with the graph $y = f(x) = x^2$.

- Shift points on that graph LEFT 3 units.
- This gives the graph of $y = f(x + 3) = (x + 3)^2$.



Transforming the graph of $y = f(x)$ to the graph of $y = f(Kx)$ or $y = f(\frac{x}{K})$

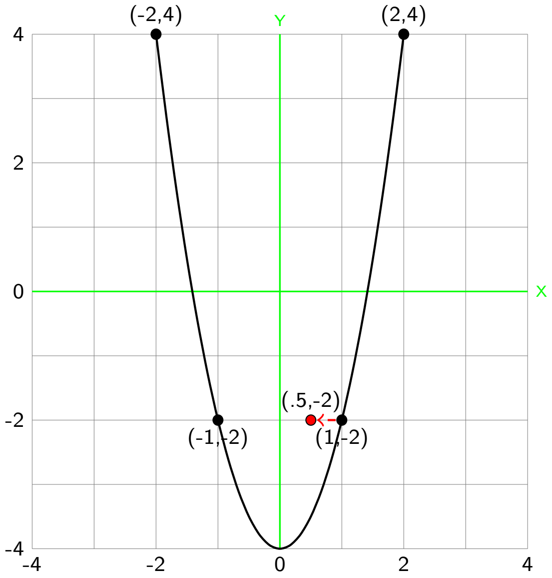


To graph $y = f(Kx)$ if $K > 1$.

H-shrink the graph of $y = f(x)$ by a factor of K.

- Start with the graph of $y = f(x) = 2x^2 - 4$.

Transforming the graph of $y = f(x)$ to the graph of $y = f(Kx)$ or $y = f(\frac{x}{K})$

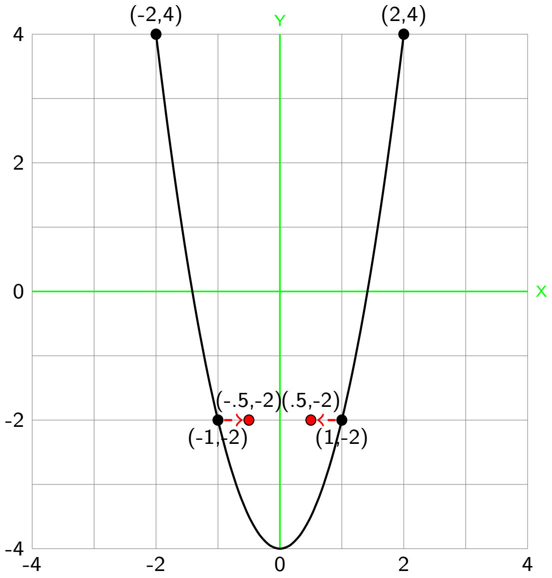


To graph $y = f(Kx)$ if $K > 1$.

H-shrink the graph of $y = f(x)$ by a factor of K.

- Start with the graph of $y = f(x) = 2x^2 - 4$.
- Shrink it horizontally by a factor of 2.
  Each point's distance from the y-axis is divided by 2. Point $(x, f(x))$ goes to $(x/2, f(x))$.

Transforming the graph of $y = f(x)$ to the graph of $y = f(Kx)$ or $y = f(\frac{x}{K})$



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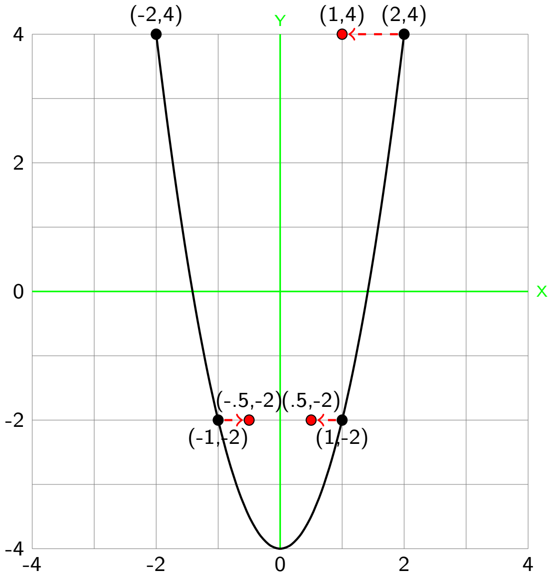
H-shrink the graph of $y = f(x)$ by a factor of K.

- Start with the graph of $y = f(x) = 2x^2 - 4$.
- Shrink it horizontally by a factor of 2.
  Each point's distance from the y-axis is divided by 2. Point $(x, f(x))$ goes to $(x/2, f(x))$.

Transforming the graph of $y = f(x)$ to the graph of $y = f(Kx)$ or $y = f(\frac{x}{K})$



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H-shrink the graph of $y = f(x)$ by a factor of K.

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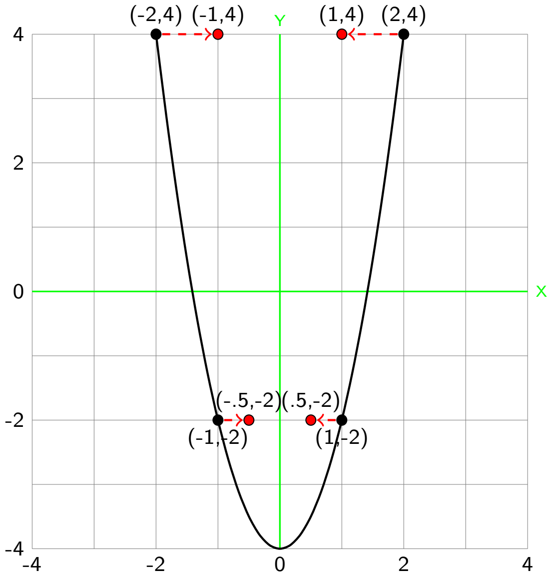
Transforming the graph of $y = f(x)$ to the graph of $y = f(Kx)$ or $y = f(\frac{x}{K})$



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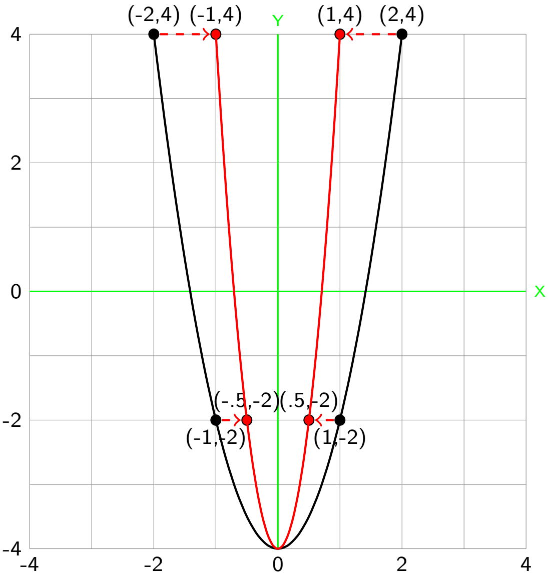
- Shrink it horizontally by a factor of 2.
  Each point's distance from the y-axis is divided by 2. Point $(x, f(x))$ goes to $(x/2, f(x))$.

Transforming the graph of $y = f(x)$ to the graph of $y = f(Kx)$ or $y = f(\frac{x}{K})$

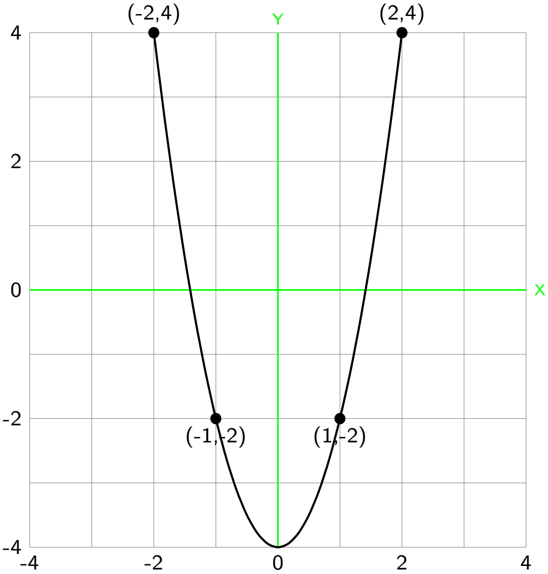


To graph $y = f(Kx)$ if $K > 1$.

H-shrink the graph of $y = f(x)$ by a factor of K.

- Start with the graph of $y = f(x) = 2x^2 - 4$.
- Shrink it horizontally by a factor of 2.
  Each point's distance from the y-axis is divided by 2. Point $(x, f(x))$ goes to $(x/2, f(x))$.
- This gives the graph
  $y = f(2x) = 2(2x)^2 - 4 = 8x^2 - 4.$

Transforming the graph of $y = f(x)$ to the graph of $y = f(Kx)$ or $y = f(\frac{x}{K})$



To graph $y = f(Kx)$ if $K > 1$.

H-shrink the graph of $y = f(x)$ by a factor of K.

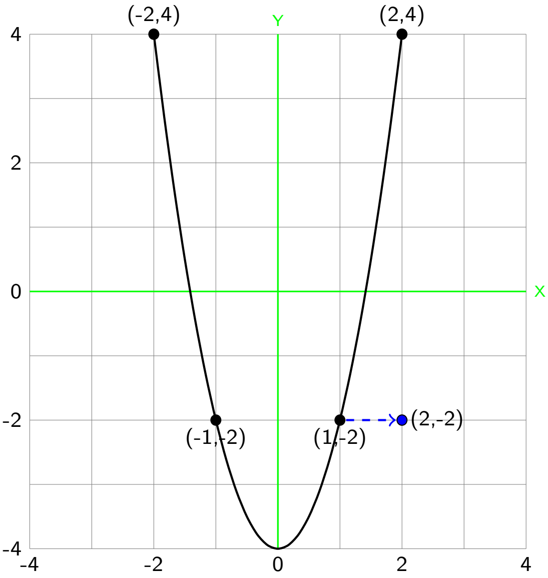
- Start with the graph of $y = f(x) = 2x^2 - 4$.
- Shrink it horizontally by a factor of 2.
  Each point's distance from the y-axis is divided by 2. Point $(x, f(x))$ goes to $(x/2, f(x))$.
- This gives the graph
  $y = f(2x) = 2(2x)^2 - 4 = 8x^2 - 4.$

To graph $y = f(\frac{x}{K})$ if $K > 1$.

Stretch the graph of $y = f(x)$ horizontally, toward the y-axis, by a factor of K.

Start again with the graph of $y = f(x) = 2x^2 - 4$.

Transforming the graph of $y = f(x)$ to the graph of $y = f(Kx)$ or $y = f(\frac{x}{K})$



To graph $y = f(Kx)$ if $K > 1$.

H-shrink the graph of $y = f(x)$ by a factor of K.

- Start with the graph of $y = f(x) = 2x^2 - 4$.
- Shrink it horizontally by a factor of 2.
  Each point's distance from the y-axis is divided by 2. Point $(x, f(x))$ goes to $(x/2, f(x))$.
- This gives the graph
  $y = f(2x) = 2(2x)^2 - 4 = 8x^2 - 4.$

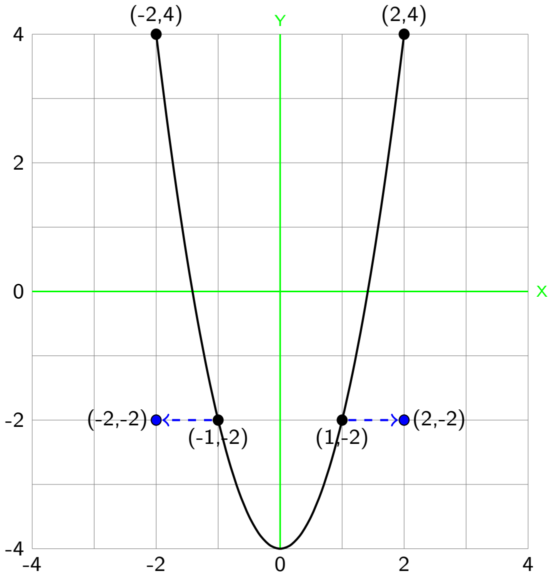
To graph $y = f(\frac{x}{K})$ if $K > 1$.

Stretch the graph of $y = f(x)$ horizontally, toward the y-axis, by a factor of K.

Start again with the graph of $y = f(x) = 2x^2 - 4$.

- Stretch it horizontally by a factor of 2.
  Each point's distance from the y-axis is multiplied by 2. Point $(x, f(x))$ goes to $(2x, f(x))$.

Transforming the graph of $y = f(x)$ to the graph of $y = f(Kx)$ or $y = f(\frac{x}{K})$



To graph $y = f(Kx)$ if $K > 1$.

H-shrink the graph of $y = f(x)$ by a factor of K.

- Start with the graph of $y = f(x) = 2x^2 - 4$.
- Shrink it horizontally by a factor of 2.
  Each point's distance from the y-axis is divided by 2. Point $(x, f(x))$ goes to $(x/2, f(x))$.
- This gives the graph
  $y = f(2x) = 2(2x)^2 - 4 = 8x^2 - 4.$

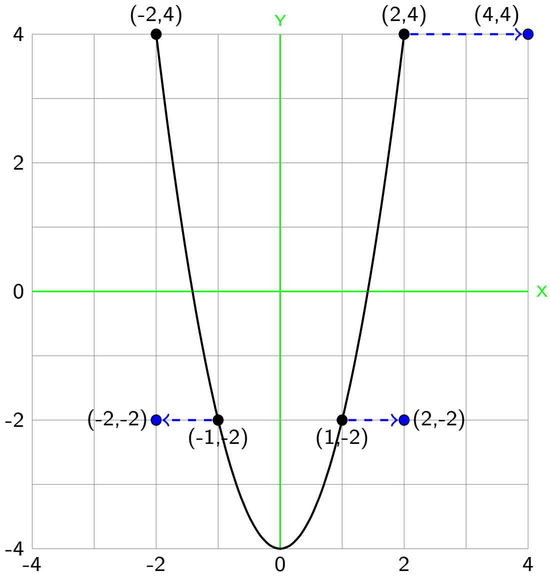
To graph $y = f(\frac{x}{K})$ if $K > 1$.

Stretch the graph of $y = f(x)$ horizontally, toward the y-axis, by a factor of K.

Start again with the graph of $y = f(x) = 2x^2 - 4$.

- Stretch it horizontally by a factor of 2.
  Each point's distance from the y-axis is multiplied by 2. Point $(x, f(x))$ goes to $(2x, f(x))$.

Transforming the graph of $y = f(x)$ to the graph of $y = f(Kx)$ or $y = f(\frac{x}{K})$



To graph $y = f(Kx)$ if $K > 1$.

H-shrink the graph of $y = f(x)$ by a factor of K.

- Start with the graph of $y = f(x) = 2x^2 - 4$.
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  Each point's distance from the y-axis is divided by 2. Point $(x, f(x))$ goes to $(x/2, f(x))$.
- This gives the graph
  $y = f(2x) = 2(2x)^2 - 4 = 8x^2 - 4.$

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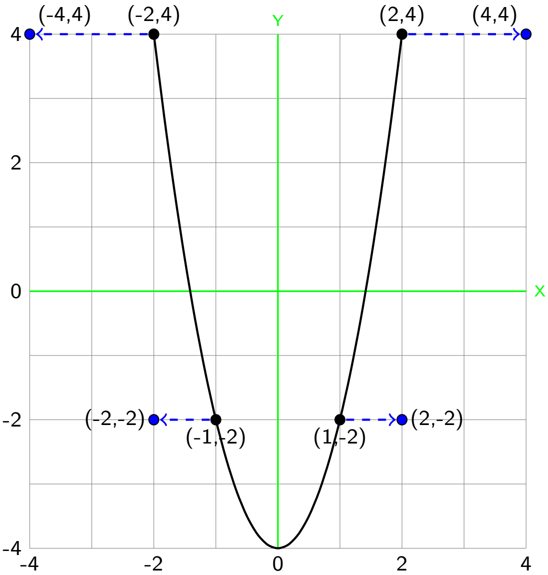

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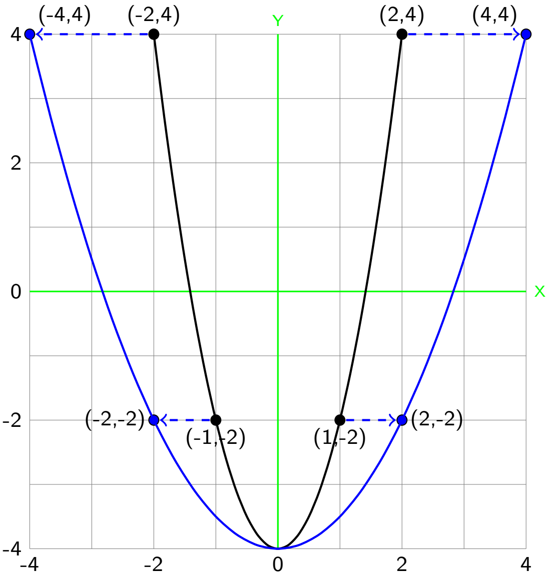

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- Stretch it horizontally by a factor of 2.
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- This gives the graph
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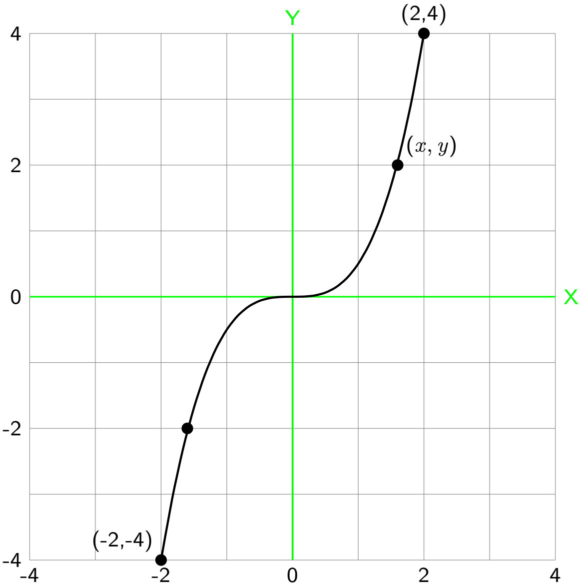
Transforming the graph of $y = f(x)$ to the graph of $y = f(-x)$

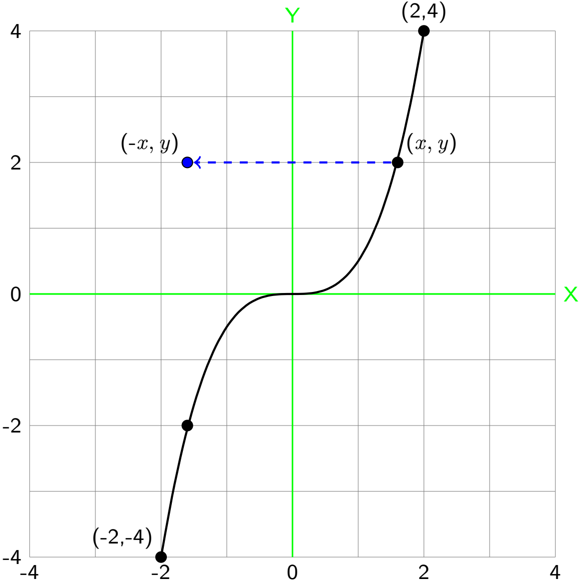


To graph $y = f(-x)$

Reflect the graph of $y = f(x)$ across the y-axis.

- Start with the graph of $y = f(x) = \frac{x^3}{2}$.

Transforming the graph of $y = f(x)$ to the graph of $y = f(-x)$



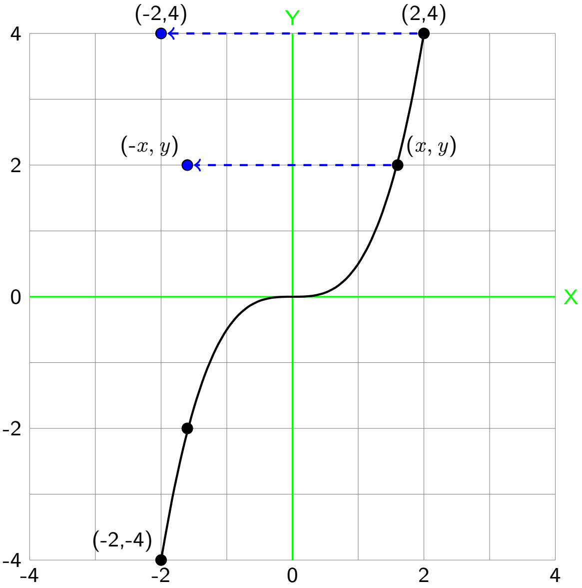
To graph $y = f(-x)$

Reflect the graph of $y = f(x)$ across the y-axis.

- Start with the graph of $y = f(x) = \frac{x^3}{2}$.
- Reflect the graph across the y-axis:

  Reverse the sign of each point's x-coordinate . Point $(x, f(x))$ goes to $(-x, f(x))$.

Transforming the graph of $y = f(x)$ to the graph of $y = f(-x)$



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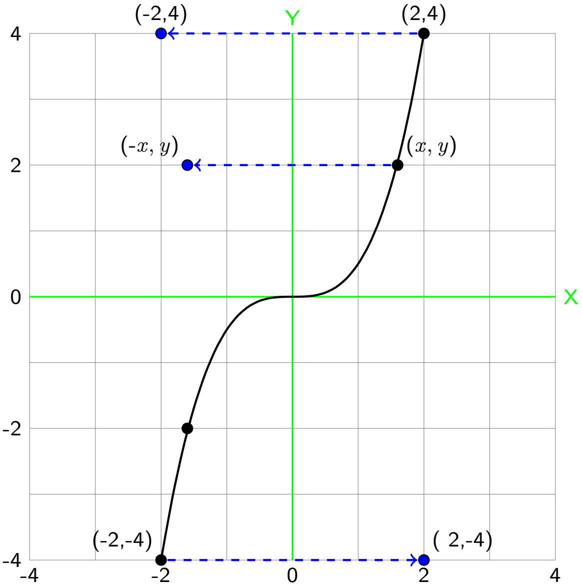

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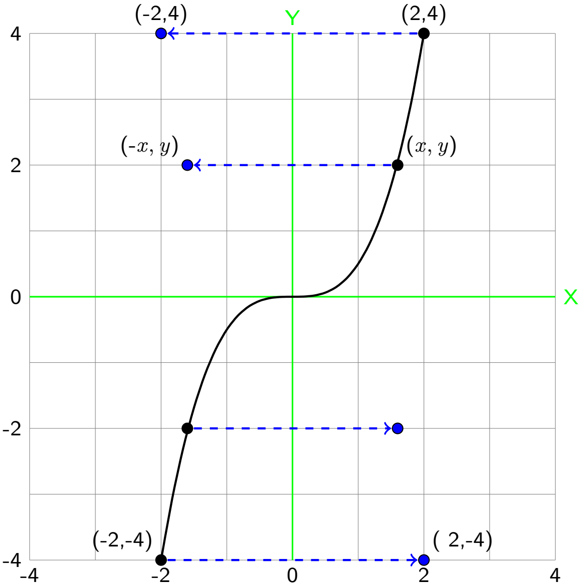

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Reflect the graph of $y = f(x)$ across the y-axis.

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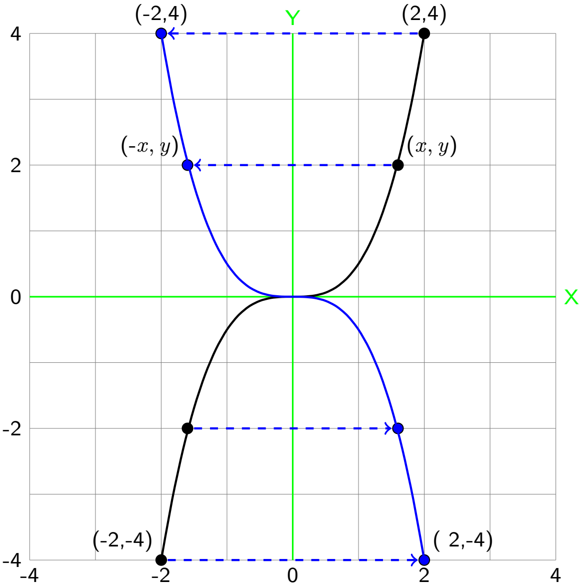
Transforming the graph of $y = f(x)$ to the graph of $y = f(-x)$



To graph $y = f(-x)$

Reflect the graph of $y = f(x)$ across the y-axis.

- Start with the graph of $y = f(x) = \frac{x^3}{2}$.
- Reflect the graph across the y-axis:

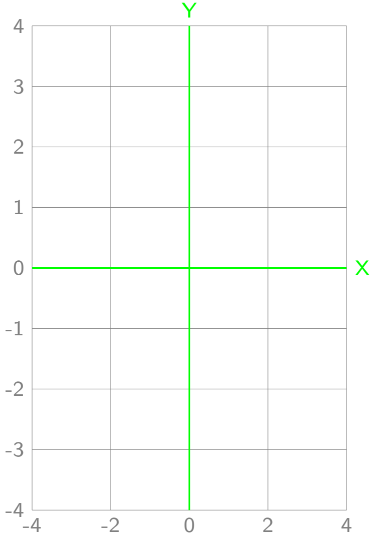
  Reverse the sign of each point's x-coordinate . Point $(x, f(x))$ goes to $(-x, f(x))$.
- This gives the graph of

  $y = f(-x) = \frac{(-x)^3}{2} = -\frac{x^3}{2}$.

2.7.4 Combining transformations

Example 5. What sequence of equation transformations changes $y = x^2$ to $y = 4 - x^2$? What is the effect on the graph at each stage?

Solution:



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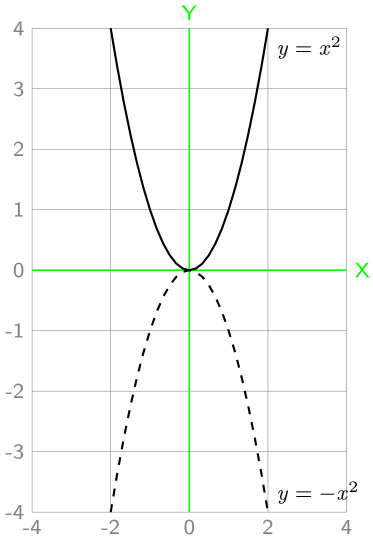
Example 5. What sequence of equation transformations changes $y = x^2$ to $y = 4 - x^2$? What is the effect on the graph at each stage?

Solution:

- Start with equation $y = x^2$. Multiply the RHS by -1 to get equation $y = -x^2$, whose graph is obtained by reflecting the graph of $y = x^2$ across the $x-$axis.



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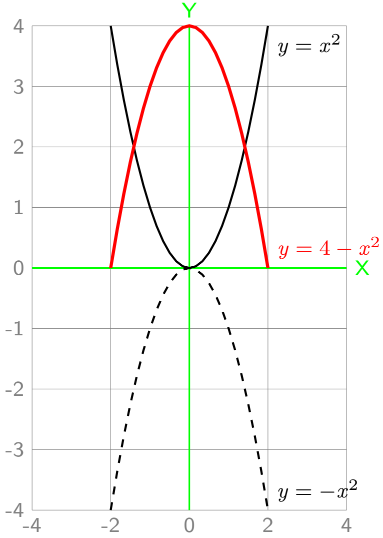
- Add 4 to the RHS of $y = -x^2$ to get equation $y = -x^2 + 4 = 4 - x^2$, whose graph is obtained by shifting the graph of $y = -x^2$ up 4 .



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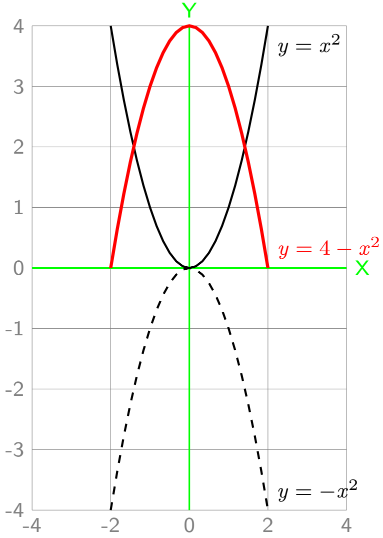
- Add 4 to the RHS of $y = -x^2$ to get equation $y = -x^2 + 4 = 4 - x^2$, whose graph is obtained by shifting the graph of $y = -x^2$ up 4 .

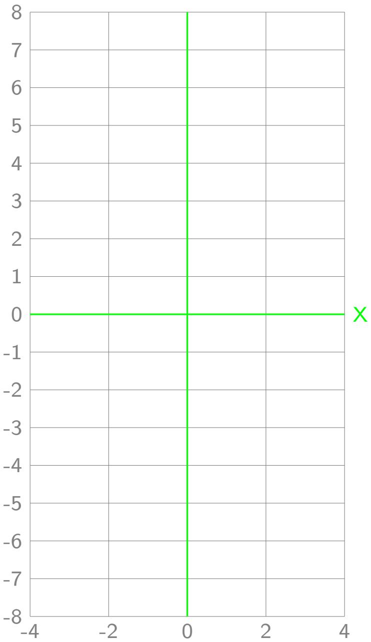
Answer: The graph of $y = 4 - x^2$ is obtained by reflecting the graph of $y = x^2$ across the x-axis, then shifting the result up 4 units. The resulting graph is a parabola with maximum point at $(0, 4)$



Let's reverse the order of the graph transformations in Example 5.

Example 6. What is the equation of the graph obtained by shifting the graph of $y = x^2$ up 4 units, and then reflecting the resulting graph across the x-axis?

- Start with equation $y = x^2$.

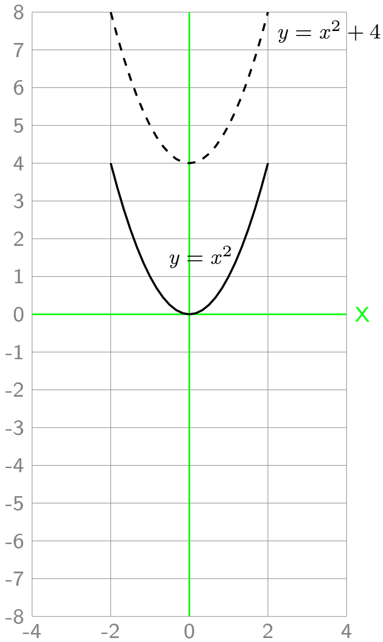


Let's reverse the order of the graph transformations in Example 5.

Example 6. What is the equation of the graph obtained by shifting the graph of $y = x^2$ up 4 units, and then reflecting the resulting graph across the x-axis?

Solution:

- Start with equation $y = x^2$.
- To shift the graph up 4 units, add 4 to the RHS of $y = x^2$ to get $y = x^2 + 4$.



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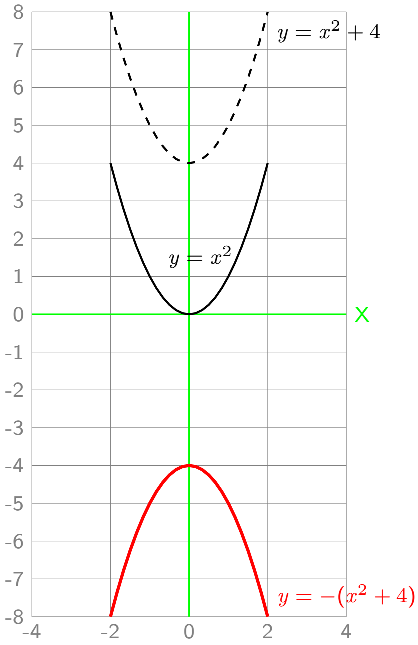
Example 6. What is the equation of the graph obtained by shifting the graph of $y = x^2$ up 4 units, and then reflecting the resulting graph across the x-axis?

Solution:

- Start with equation $y = x^2$.
- To shift the graph up 4 units, add 4 to the RHS of $y = x^2$ to get $y = x^2 + 4$.
- To reflect the resulting graph across the x-axis, multiply the RHS of $y = x^2 + 4$ by -1 to get equation $y = -x^2 - 4$.



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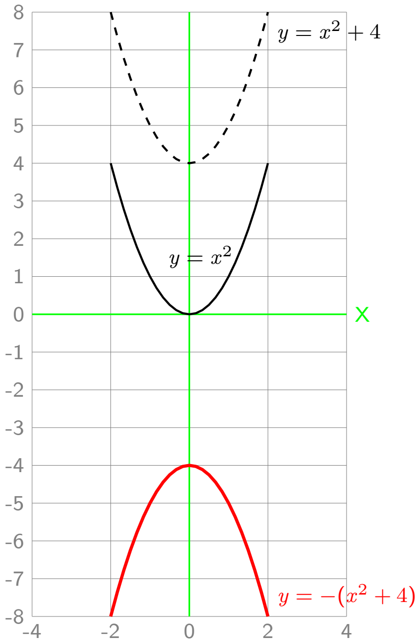
- Start with equation $y = x^2$.
- To shift the graph up 4 units, add 4 to the RHS of $y = x^2$ to get $y = x^2 + 4$.
- To reflect the resulting graph across the x-axis, multiply the RHS of $y = x^2 + 4$ by -1 to get equation $y = -x^2 - 4$.

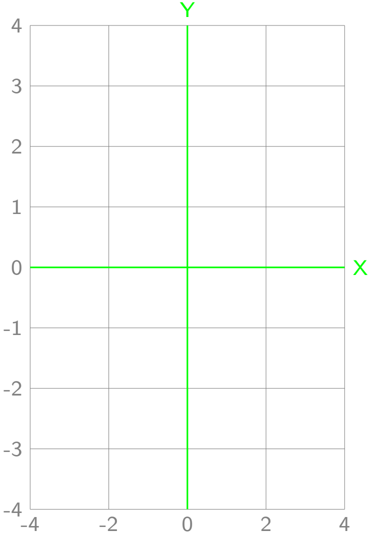
Answer: The graph of $y = -x^2 - 4$ is a parabola with maximum point $(0, -4)$, and is certainly not the same as the graph of $y = 4 - x^2$ obtained in Example 6.

Shifting, then reflecting gives a different result than reflecting, then shifting!



Example 7. What sequence of equation transformations changes $y = x^2$ to $y = (x + 4)^2 - 5$? What is the effect on the graph at each stage?

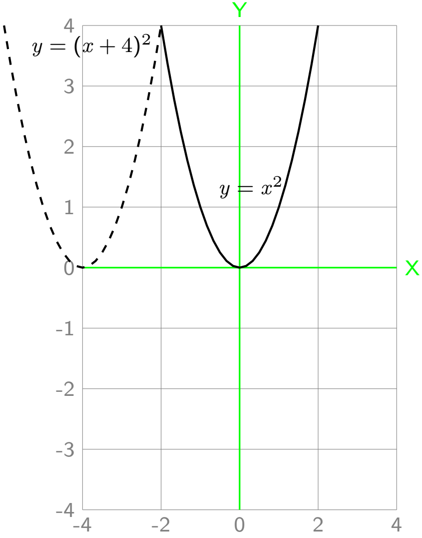
Solution:



Example 7. What sequence of equation transformations changes $y = x^2$ to $y = (x+4)^2 - 5$? What is the effect on the graph at each stage?

Solution:

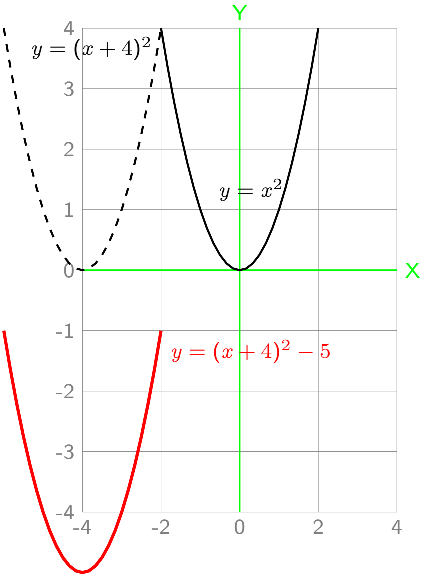
- Start with equation $y = x^2$. Substitute $x + 4$ for x to get $y = (x+4)^2$, whose graph is obtained by shifting the graph of $y = x^2$ left 4 units.



Example 7. What sequence of equation transformations changes $y = x^2$ to $y = (x+4)^2 - 5$? What is the effect on the graph at each stage?

Solution:

- Start with equation $y = x^2$. Substitute $x + 4$ for x to get $y = (x+4)^2$, whose graph is obtained by shifting the graph of $y = x^2$ left 4 units.
- Subtract 5 from the RHS of $y = (x+4)^2$ to get equation $y = (x+4)^2 - 5$, whose graph is obtained by shifting the graph of $y = (x+4)^2$ down 5 units.



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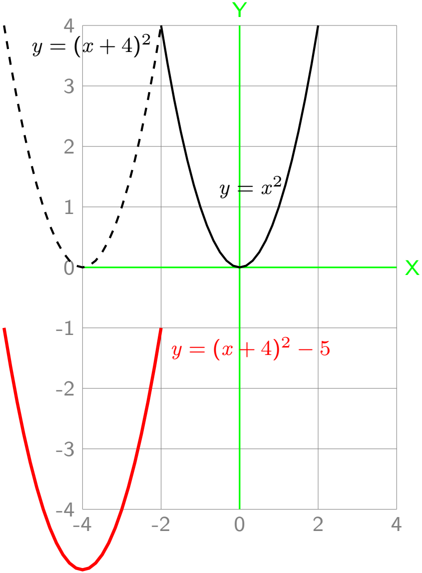
Solution:

- Start with equation $y = x^2$. Substitute $x + 4$ for x to get $y = (x+4)^2$, whose graph is obtained by shifting the graph of $y = x^2$ left 4 units.
- Subtract 5 from the RHS of $y = (x+4)^2$ to get equation $y = (x+4)^2 - 5$, whose graph is obtained by shifting the graph of $y = (x+4)^2$ down 5 units.

Answer: To get the graph of $y = (x+4)^2 - 5$, shift the graph of $y = x^2$ left 4 units, then down 5 units.

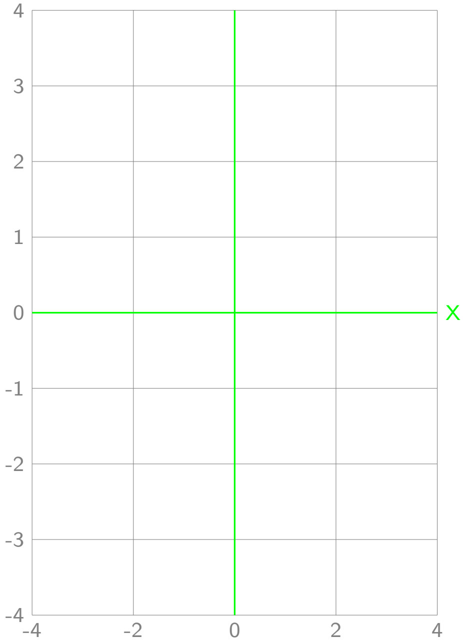
Reversing the order of shifts, by shifting the graph of $y = x^2$ down 5 units, then left 4 units, will produce the same result. This is not the case with shifting and reflecting, as shown in the last two Examples.



Example 8: What sequence of equation transformations changes $y = x^2$ to $y = 4 - (x+2)^2$? What is the effect on the graph at each stage?

Solution:

- Start with the equation $y = f(x) = x^2$.



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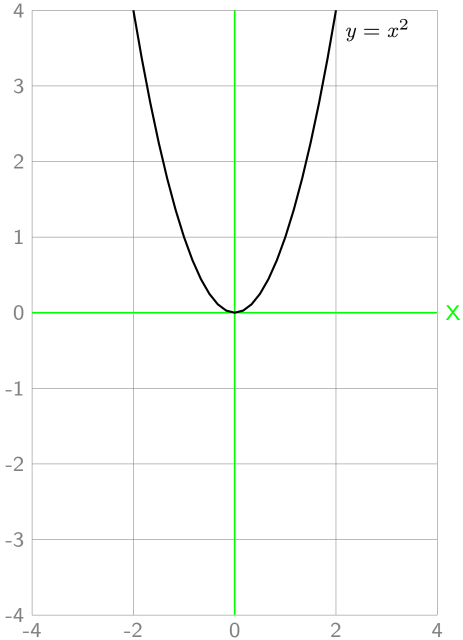

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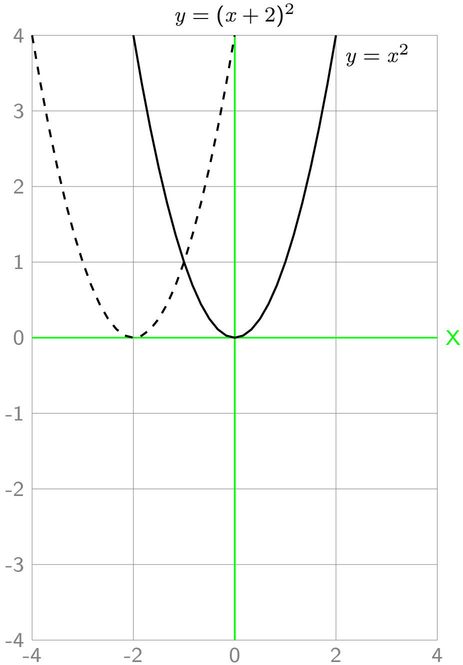
Solution:

- Start with the equation $y = f(x) = x^2$.
- Substitute $x + 2$ for x to get the new equation $y = (x+2)^2$. To graph this equation, shift the graph of $y = x^2$ left 2 units.



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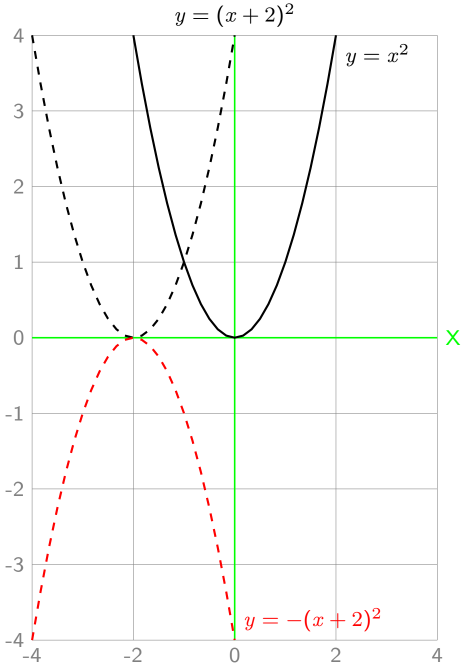
Solution:

- Start with the equation $y = f(x) = x^2$.
- Substitute $x + 2$ for x to get the new equation $y = (x+2)^2$. To graph this equation, shift the graph of $y = x^2$ left 2 units.
- To graph $y = -(x+2)^2$, reflect the graph of $y = (x+2)^2$ across the x-axis.



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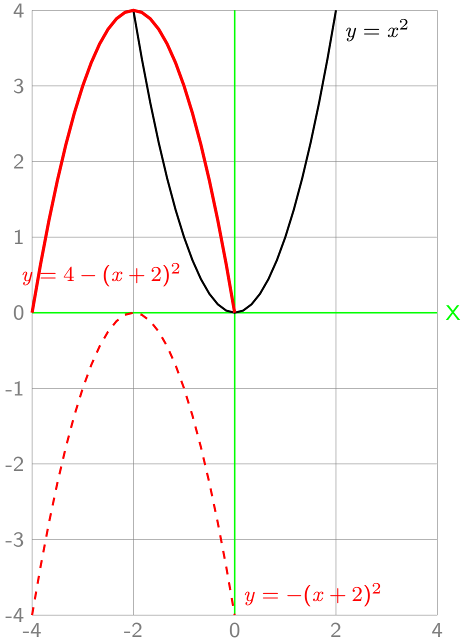
- Substitute $x + 2$ for x to get the new equation $y = (x+2)^2$. To graph this equation, shift the graph of $y = x^2$ left 2 units.
- To graph $y = -(x+2)^2$, reflect the graph of $$y = (x+2)^2$$ across the x-axis.
- To graph $y = -(x+2)^2 + 4 = 4 - (x+2)^2$, shift the graph of $y = -(x+2)^2$ up 4 units.



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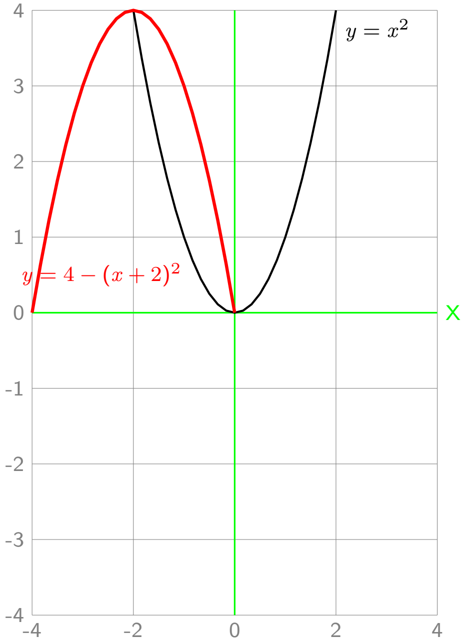
- To graph $y = -(x+2)^2$, reflect the graph of

  $y = (x+2)^2$ across the x-axis.
- To graph $y = -(x+2)^2 + 4 = 4 - (x+2)^2$, shift the graph of $y = -(x+2)^2$ up 4 units.

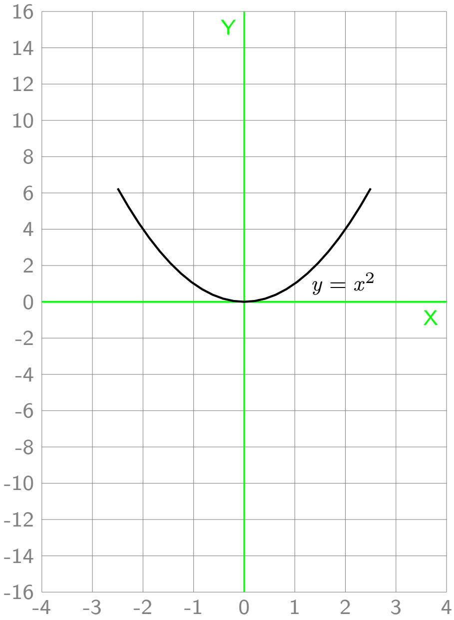
Answer: Start with the graph of $y = x^2$. Shift it left 2 units, then reflect across the x-axis, then shift up 4 units to obtain the graph of $y = 4 - (x+2)^2$.



Example 9: What sequence of equation transformations changes $y = x^2$ to $y = -2(2x + 1)^2$? What is the effect on the graph at each stage?

Solution:

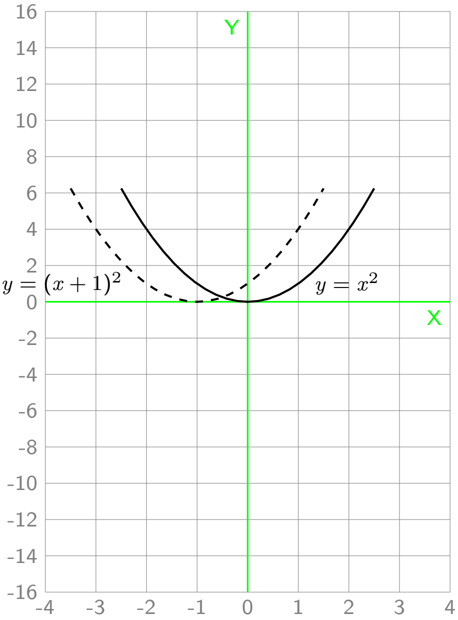
- Start with the equation $y = f(x) = x^2$.



Example 9: What sequence of equation transformations changes $y = x^2$ to $y = -2(2x+1)^2$? What is the effect on the graph at each stage?

Solution:

- Start with the equation $y = f(x) = x^2$.
- Substitute $x + 1$ for x to get the new equation $y = (x+1)^2$. To graph this equation, shift the graph of $y = x^2$ left 1 .



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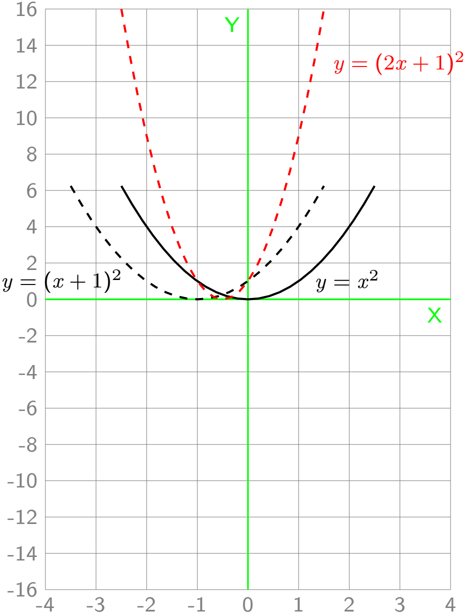
- Start with the equation $y = f(x) = x^2$.
- Substitute $x + 1$ for x to get the new equation $y = (x+1)^2$. To graph this equation, shift the graph of $y = x^2$ left 1 .
- To graph $y = (2x+1)^2$, substitute $2x$ for x in $y = (x+1)^2$, whose graph H-shrinks by 2.



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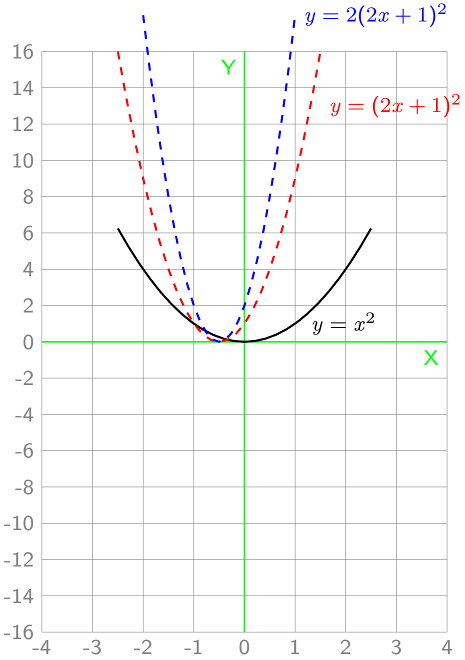
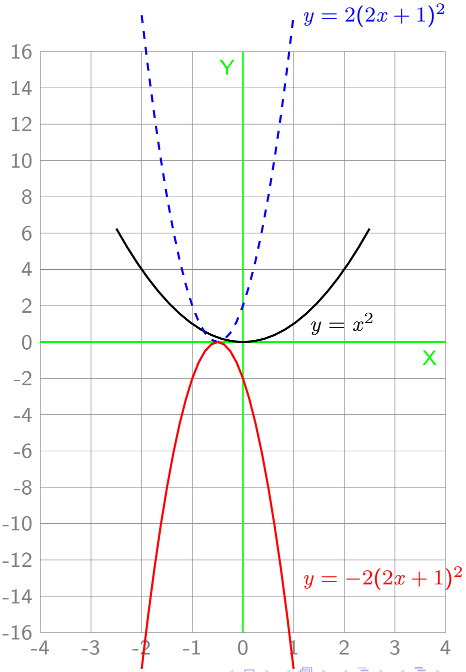
- To graph $y = (2x+1)^2$, substitute $2x$ for x in $y = (x+1)^2$, whose graph H-shrinks by 2.
- To graph $y = 2(2x+1)^2$, V-stretch the graph of $y = (2x+1)^2$ by 2.



Example 9: What sequence of equation transformations changes $y = x^2$ to $y = -2(2x+1)^2$? What is the effect on the graph at each stage?

Solution:

- Start with the equation $y = f(x) = x^2$.
- Substitute $x + 1$ for x to get the new equation $y = (x+1)^2$. To graph this equation, shift the graph of $y = x^2$ left 1 .
- To graph $y = (2x+1)^2$, substitute $2x$ for x in $y = (x+1)^2$, whose graph H-shrinks by 2.
- To graph $y = 2(2x+1)^2$, V-stretch the graph of $y = (2x+1)^2$ by 2.
- To graph $y = -2(2x+1)^2$, reflect the graph of $y = 2(2x+1)^2$ through the x-axis.

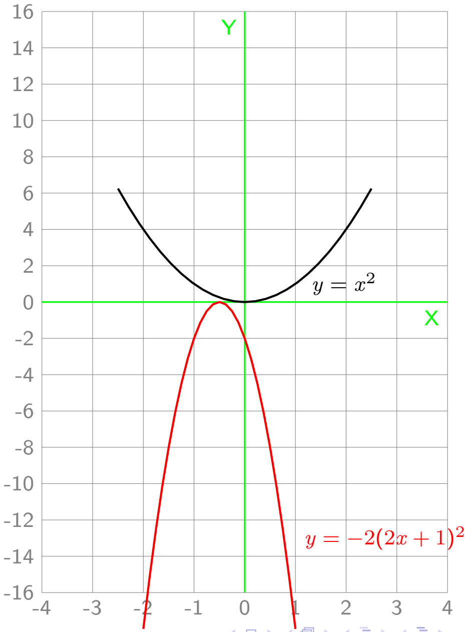


Example 9: What sequence of equation transformations changes $y = x^2$ to $y = -2(2x+1)^2$? What is the effect on the graph at each stage?

Solution:

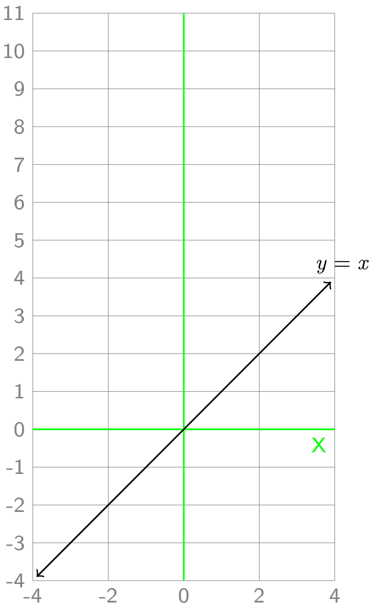
- Start with the equation $y = f(x) = x^2$.
- Substitute $x + 1$ for x to get the new equation $y = (x+1)^2$. To graph this equation, shift the graph of $y = x^2$ left 1 .
- To graph $y = (2x+1)^2$, substitute $2x$ for x in $y = (x+1)^2$, whose graph H-shrinks by 2.
- To graph $y = 2(2x+1)^2$, V-stretch the graph of $y = (2x+1)^2$ by 2.
- To graph $y = -2(2x+1)^2$, reflect the graph of $y = 2(2x+1)^2$ through the x-axis.

Answer: Start with the graph of $y = x^2$. Shift it left 1 unit, shrink the result horizontally toward the y-axis by a factor of 2, stretch the result vertically from the x-axis by a factor of 2, then reflect the result through the x-axis.



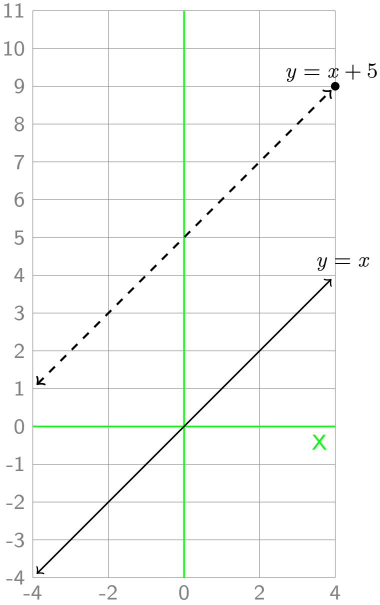
Example 10: What sequence of equation transformations changes $y = x$ to $y = 3x + 5$? What is the effect on the graph at each stage?

- Start by sketching the graph of $y = x$.



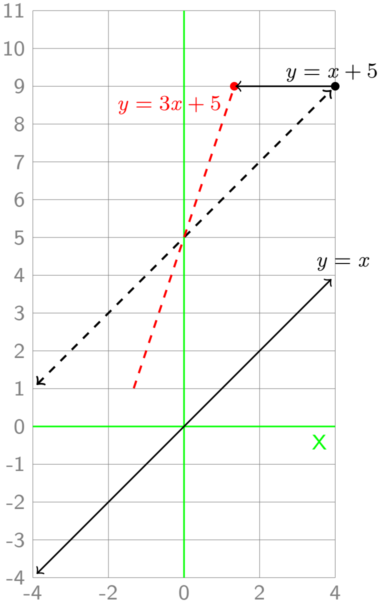
Example 10: What sequence of equation transformations changes $y = x$ to $y = 3x + 5$? What is the effect on the graph at each stage?

- Start by sketching the graph of $y = x$.
- Substitute $x + 5$ for x to get $y = x + 5$. (This could also be described as adding 5 to the right side of $y = f(x) = x$.) The graph shifts up 5.



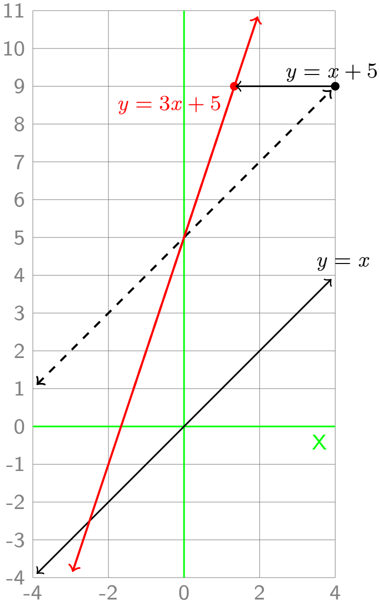
Example 10: What sequence of equation transformations changes $y = x$ to $y = 3x + 5$? What is the effect on the graph at each stage?

- Start by sketching the graph of $y = x$.
- Substitute $x + 5$ for x to get $y = x + 5$. (This could also be described as adding 5 to the right side of $y = f(x) = x$.) The graph shifts up 5.
- Substitute $3x$ for x to get $y = 3x + 5$. The graph shrinks toward the y-axis by a factor of 3. To check this, note that the point $(2, 7)$ moves left to point $(\frac{2}{3}, 7)$ and $(4, 9)$ moves to $(\frac{4}{3}, 9)$.



Example 10: What sequence of equation transformations changes $y = x$ to $y = 3x + 5$? What is the effect on the graph at each stage?

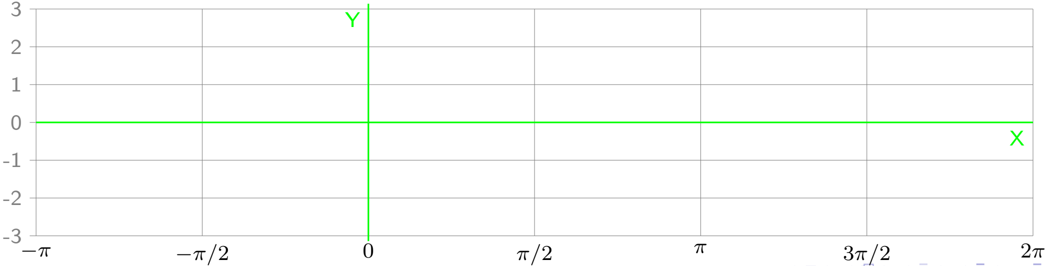
- Start by sketching the graph of $y = x$.
- Substitute $x + 5$ for x to get $y = x + 5$. (This could also be described as adding 5 to the right side of $y = f(x) = x$.) The graph shifts up 5.
- Substitute $3x$ for x to get $y = 3x + 5$. The graph shrinks toward the y-axis by a factor of 3. To check this, note that the point $(2, 7)$ moves left to point $(\frac{2}{3}, 7)$ and $(4, 9)$ moves to $(\frac{4}{3}, 9)$.
- Extend the straight line $y = 3x + 5$ to the largest possible domain that fits in the grid.



Example 11: What sequence of equation transformations changes $y = \sin(x)$ to $y = 3\sin(2x + \pi)$. What is the effect on the graph at each stage?

Graphs are shown below, but solving the problem doesn't require them.

Solution:

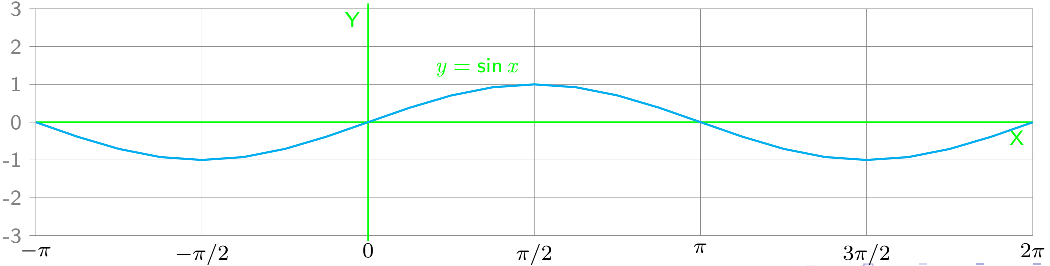


Example 11: What sequence of equation transformations changes $y = \sin(x)$ to $y = 3\sin(2x + \pi)$. What is the effect on the graph at each stage?

Graphs are shown below, but solving the problem doesn't require them.

Solution:

- Start with the equation $y = \sin(x)$.

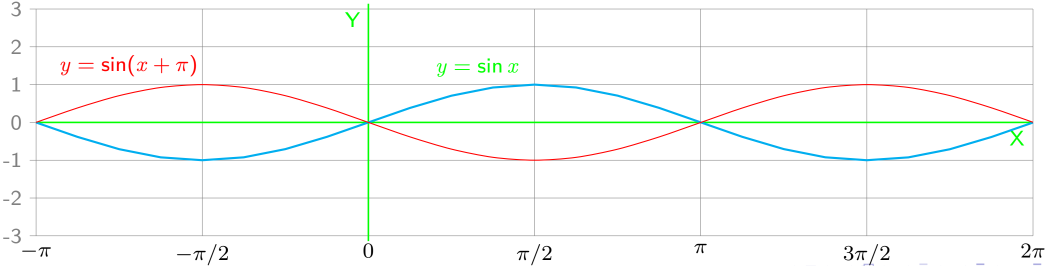


Example 11: What sequence of equation transformations changes $y = \sin(x)$ to $y = 3\sin(2x + \pi)$. What is the effect on the graph at each stage?

Graphs are shown below, but solving the problem doesn't require them.

Solution:

- Start with the equation $y = \sin(x)$.
- Substitute $x + \pi$ for x to get the new equation $y = \sin(x + \pi)$.
  To graph this equation, shift the graph of $y = \sin(x)$ left π units.

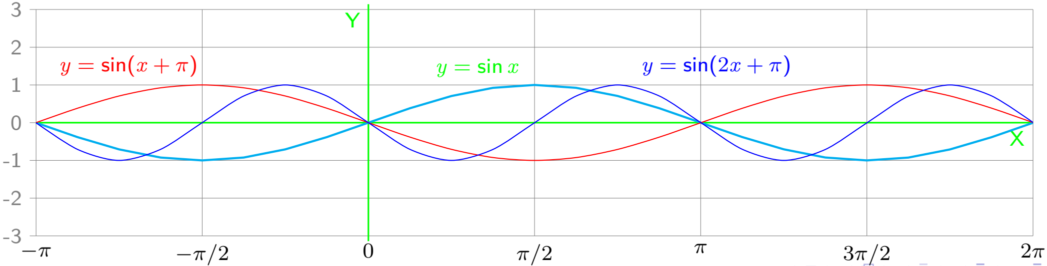


Example 11: What sequence of equation transformations changes $y = \sin(x)$ to $y = 3\sin(2x + \pi)$. What is the effect on the graph at each stage?

Graphs are shown below, but solving the problem doesn't require them.

Solution:

- Start with the equation $y = \sin(x)$.
- Substitute $x + \pi$ for x to get the new equation $y = \sin(x + \pi)$.
  To graph this equation, shift the graph of $y = \sin(x)$ left π units.
- Substitute $2x$ for x in the equation $y = \sin(x + \pi)$ to get $y = \sin(2x + \pi)$.
  To graph this equation, shrink the graph of $y = \sin(x + \pi)$ horizontally toward the y-axis by a factor of 2.

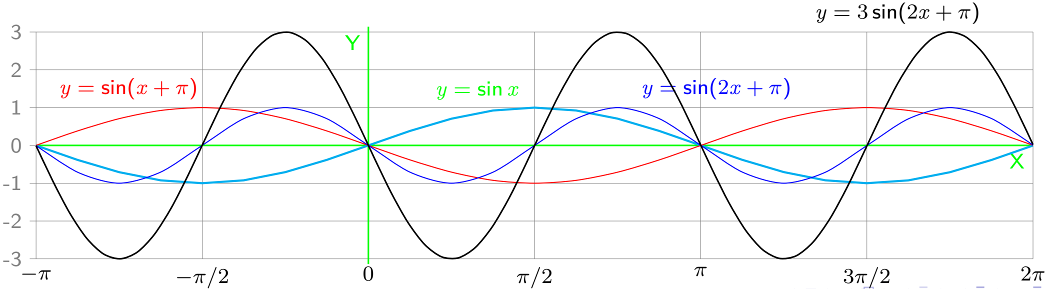


Example 11: What sequence of equation transformations changes $y = \sin(x)$ to $y = 3\sin(2x + \pi)$. What is the effect on the graph at each stage?

Graphs are shown below, but solving the problem doesn't require them.

Solution:

- Start with the equation $y = \sin(x)$.
- Substitute $x + \pi$ for x to get the new equation $y = \sin(x + \pi)$.
  To graph this equation, shift the graph of $y = \sin(x)$ left π units.
- Substitute $2x$ for x in the equation $y = \sin(x + \pi)$ to get $y = \sin(2x + \pi)$.
  To graph this equation, shrink the graph of $y = \sin(x + \pi)$ horizontally toward the y-axis by a factor of 2.
- To graph $y = 3\sin(2x + \pi)$, stretch the graph of $y = \sin(2x + \pi)$ vertically away from the x-axis by a factor of 3.

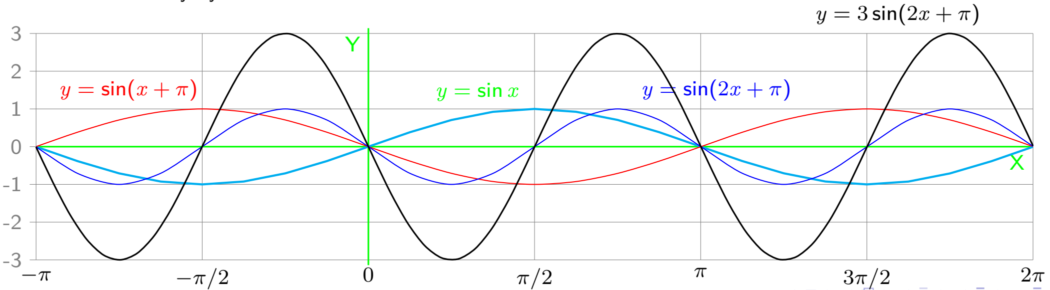


Example 11: What sequence of equation transformations changes $y = \sin(x)$ to $y = 3\sin(2x + \pi)$. What is the effect on the graph at each stage?

Graphs are shown below, but solving the problem doesn't require them.

Solution:

- Start with the equation $y = \sin(x)$.
- Substitute $x + \pi$ for x to get the new equation $y = \sin(x + \pi)$.
  To graph this equation, shift the graph of $y = \sin(x)$ left π units.
- Substitute $2x$ for x in the equation $y = \sin(x + \pi)$ to get $y = \sin(2x + \pi)$.
  To graph this equation, shrink the graph of $y = \sin(x + \pi)$ horizontally toward the y-axis by a factor of 2.
- To graph $y = 3\sin(2x + \pi)$, stretch the graph of $y = \sin(2x + \pi)$ vertically away from the x-axis by a factor of 3.
- Answer: Shift the graph of $y = \sin(x)$ left π units, shrink horizontally toward the y-axis by a factor of 2, then stretch vertically by a factor of 3.



Section 2.7 Quiz

Ex. 2.7.1: Compare the graphs and equations of the circles $x^2 + y^2 = 4$ and $(x-3)^2 + (y+1)^2 = 4$.

Ex. 2.7.2: Compare the graphs and equations of the circles $x^2 + y^2 = 4$ and $(\frac{x}{2})^2 + (3y)^2 = 4$.

Ex. 2.7.3: How do you obtain the graph of $y = -7x^2$ from the graph of $y = x^2$?

Ex. 2.7.4: How do you obtain the graph of $y = \frac{x^2}{4}$ from the graph of $y = x^2$?

In the following problems, sketch the graph that is produced after each transformation.

Ex. 2.7.5: What sequence of equation transformations changes $y = x^2$ to $y = 4 - x^2$?

Ex. 2.7.6: What is the equation of the graph obtained by shifting the graph of $y = x^2$ up 4 units, and then reflecting the resulting graph across the y-axis?

Ex. 2.7.7:What sequence of equation transformations changes $y = x^2$ to $y = (x+4)^2 - 5$?

Ex. 2.7.8: What sequence of equation transformations changes the graph of $y = x^2$ to the graph of $y = 4 - (x+2)^2$?

Ex. 2.7.9: What sequence of equation transformations changes the graph of $y = x^2$ to the graph of graph of $y = -2(2x+1)^2$?

Ex. 2.7.10: What sequence of equation transformations changes $y = x$ to $y = 3x + 5$? What is the effect on the graph at each stage?

Ex. 2.7.11: What sequence of equation transformations changes $y = \sin(x)$ to the graph of $y = 3\sin(2x + \pi)$.
What is the effect on the graph at each stage?

Section 2.7 Review: Transformations

Ex. 2.7.1: How do you obtain the graph of the circle

- $(x-3)^2+(y+1)^2=4$ from the graph of $x^2+y^2=4$?
  $\Rightarrow$
- $x^2+y^2=4$ from the graph of $(x-3)^2+(y+1)^2=4$?
  $\Rightarrow$
- $(x+7)^2+(y+6)^2=4$ from the graph of $x^2+y^2=4$?
  $\Rightarrow$
- $(x+7)^2+(y+6)^2=4$ from the graph of $(x-3)^2+(y+1)^2=4$?
  $\Rightarrow$

Section 2.7 Review: Transformations

Ex. 2.7.1: How do you obtain the graph of the circle

- $(x-3)^2+(y+1)^2=4$ from the graph of $x^2+y^2=4$?
  $\Rightarrow$ Shift the graph of $x^2+y^2=4$ 3 units right and 1 unit down.
- $x^2+y^2=4$ from the graph of $(x-3)^2+(y+1)^2=4$?
  $\Rightarrow$ Shift the graph of $(x-3)^2+(y+1)^2=4$ 3 units left and 1 unit up.
- $(x+7)^2+(y+6)^2=4$ from the graph of $x^2+y^2=4$?
  $\Rightarrow$ Shift the graph of $x^2+y^2=4$ 7 units left and 6 units down.
- $(x+7)^2+(y+6)^2=4$ from the graph of $(x-3)^2+(y+1)^2=4$?
  $\Rightarrow$ Shift the graph of $(x-3)^2+(y+1)^2=4$ 10 units left and 5 units down.

Ex. 2.7.1: How do you obtain the graph of the circle

- $(x-3)^2+(y+1)^2=4$ from the graph of $x^2+y^2=4$?
  $\Rightarrow$ Shift the graph of $x^2+y^2=4$ 3 units right and 1 unit down.
- $x^2+y^2=4$ from the graph of $(x-3)^2+(y+1)^2=4$?
  $\Rightarrow$ Shift the graph of $(x-3)^2+(y+1)^2=4$ 3 units left and 1 unit up.
- $(x+7)^2+(y+6)^2=4$ from the graph of $x^2+y^2=4$?
  $\Rightarrow$ Shift the graph of $x^2+y^2=4$ 7 units left and 6 units down.
- $(x+7)^2+(y+6)^2=4$ from the graph of $(x-3)^2+(y+1)^2=4$?
  $\Rightarrow$ Shift the graph of $(x-3)^2+(y+1)^2=4$ 10 units left and 5 units down.

Ex. 2.7.2: How do you obtain the graph of the parabola

- $(\frac{x}{2})^2+(3y)^2=4$ from the graph of $y=x^2$?
  $\Rightarrow$
- $y=(2x)^4$ from the graph of $y=x^4$?
  $\Rightarrow$
- $y=(x+7)^2+6$ from the graph of $y=x^2$?
  $\Rightarrow$
- the graph of $y=(2x+7)^2$ from the graph of $y=x^2$?
  $\Rightarrow$

Section 2.7 Review: Transformations

Ex. 2.7.1: How do you obtain the graph of the circle

- $(x-3)^2+(y+1)^2=4$ from the graph of $x^2+y^2=4$?
  $\Rightarrow$ Shift the graph of $x^2+y^2=4$ 3 units right and 1 unit down.
- $x^2+y^2=4$ from the graph of $(x-3)^2+(y+1)^2=4$?
  $\Rightarrow$ Shift the graph of $(x-3)^2+(y+1)^2=4$ 3 units left and 1 unit up.
- $(x+7)^2+(y+6)^2=4$ from the graph of $x^2+y^2=4$?
  $\Rightarrow$ Shift the graph of $x^2+y^2=4$ 7 units left and 6 units down.
- $(x+7)^2+(y+6)^2=4$ from the graph of $(x-3)^2+(y+1)^2=4$?
  $\Rightarrow$ Shift the graph of $(x-3)^2+(y+1)^2=4$ 10 units left and 5 units down.

Ex. 2.7.2: How do you obtain the graph of the parabola

- $(\frac{x}{2})^2+(3y)^2=4$ from the graph of $y=x^2$?
  $\Rightarrow$ Stretch the graph by a factor of 2 horizontally and shrink it vertically by a factor of 3.
- $y=(2x)^4$ from the graph of $y=x^4$?
  $\Rightarrow$ Shrink the graph horiontally by a factor of 2.
- $y=(x+7)^2+6$ from the graph of $y=x^2$?
  $\Rightarrow$ Shift the graph 7 units left and 6 units up.
- the graph of $y=(2x+7)^2$ from the graph of $y=x^2$?
  $\Rightarrow$ Shift the graph 7 units left and then shrink horizontally by a factor of 2.

Ex. 2.7.3: How do you obtain the graph of

- $y = -7x^2$ from the graph of $y = x^2$?
  $\Rightarrow$
- $y = -\frac{x^2}{3}$ from the graph of $y = x^2$?
  $\Rightarrow$
- $y = 1 + 3x^2$ from the graph of $y = x^2$?
  $\Rightarrow$
- $y = 1 - 3x^2$ from the graph of $y = 3x^2$?
  $\Rightarrow$

Ex. 2.7.3: How do you obtain the graph of

- $y = -7x^2$ from the graph of $y = x^2$?
  $\Rightarrow$ Stretch the graph vertically by a factor of 7, then reflect the graph across the x-axis.
- $y = -\frac{x^2}{3}$ from the graph of $y = x^2$?
  $\Rightarrow$ Shrink the graph vertically by a factor of 3 and reflect through the $x-$axis .
- $y = 1 + 3x^2$ from the graph of $y = x^2$?
  $\Rightarrow$ Stretch the graph vertically: multiply every point's y-coordinate by 3. Then shift the resulting graph up 1.
- $y = 1 - 3x^2$ from the graph of $y = 3x^2$?
  $\Rightarrow$ Reflect the graph through the $x-$axis: multiply every point's y-coordinate by -1.Then shift the graph up 1.

Ex. 2.7.3: How do you obtain the graph of

- $y = -7x^2$ from the graph of $y = x^2$?
  $\Rightarrow$ Stretch the graph vertically by a factor of 7, then reflect the graph across the x-axis.
- $y = -\frac{x^2}{3}$ from the graph of $y = x^2$?
  $\Rightarrow$ Shrink the graph vertically by a factor of 3 and reflect through the $x-$axis .
- $y = 1 + 3x^2$ from the graph of $y = x^2$?
  $\Rightarrow$ Stretch the graph vertically: multiply every point's y-coordinate by 3. Then shift the resulting graph up 1.
- $y = 1 - 3x^2$ from the graph of $y = 3x^2$?
  $\Rightarrow$ Reflect the graph through the $x-$axis: multiply every point's y-coordinate by -1.Then shift the graph up 1.

Ex. 2.7.4: How do you obtain the graph of

- $y = \frac{x^2}{4}$ from the graph of $y = x$?
  $\Rightarrow$
- $y = 7x + 14$ from the graph of $y = x$?
  $\Rightarrow$ Shift the graph up 14.
- $y = 7(x + 2)^4$ from the graph of $y = x^4$?
  $\Rightarrow$
- $y = -2x$ from the graph of $y = \frac{x}{5}$?
  $\Rightarrow$ Then shift the graph up 7.

Ex. 2.7.3: How do you obtain the graph of

- $y = -7x^2$ from the graph of $y = x^2$?
  $\Rightarrow$ Stretch the graph vertically by a factor of 7, then reflect the graph across the x-axis.
- $y = -\frac{x^2}{3}$ from the graph of $y = x^2$?
  $\Rightarrow$ Shrink the graph vertically by a factor of 3 and reflect through the $x-$axis .
- $y = 1 + 3x^2$ from the graph of $y = x^2$?
  $\Rightarrow$ Stretch the graph vertically: multiply every point's y-coordinate by 3. Then shift the resulting graph up 1.
- $y = 1 - 3x^2$ from the graph of $y = 3x^2$?
  $\Rightarrow$ Reflect the graph through the $x-$axis: multiply every point's y-coordinate by -1.Then shift the graph up 1.

Ex. 2.7.4: How do you obtain the graph of

- $y = \frac{x^2}{4}$ from the graph of $y = x$?
  $\Rightarrow$ Shrink the graph vertically by a factor of 4.
- $y = 7x + 14$ from the graph of $y = x$?
  $\Rightarrow$ Shift the graph up 14. Stretch the graph vertically by a factor of 7: multiply each point's y-coordinate by 7.
  $\Rightarrow$
- $y = 7(x + 2)^4$ from the graph of $y = x^4$?
  $\Rightarrow$ Shift the graph left 2 $\Rightarrow$ Then stretch the graph vertically by a factor of 7.
- $y = -2x$ from the graph of $y = \frac{x}{5}$?
  $\Rightarrow$ Then shift the graph up 7. Substitute $10x$ for x: stretch the graph vertically by a factor of 10 (or shrink the graph horizontally by a factor of 10) to get the graph of $y = 2x$.
  $\Rightarrow$ Then reflect the graph across the x-axis

Ex. 2.7.5: What sequence of equation transformations changes $y = x^2$ to $y = 4 - x^2$? What is the effect on the graph at each stage?

Solution:

Ex. 2.7.5: What sequence of equation transformations changes $y = x^2$ to $y = 4 - x^2$? What is the effect on the graph at each stage?

Solution:

- Start with equation $y = x^2$. Multiply the RHS by -1 to get equation $y = -x^2$, whose graph is obtained by reflecting the graph of $y = x^2$ across the $x-$axis.

Ex. 2.7.5: What sequence of equation transformations changes $y = x^2$ to $y = 4 - x^2$? What is the effect on the graph at each stage?

Solution:

- Start with equation $y = x^2$. Multiply the RHS by -1 to get equation $y = -x^2$, whose graph is obtained by reflecting the graph of $y = x^2$ across the $x-$axis.
- Add 4 to the RHS of $y = -x^2$ to get equation $y = -x^2 + 4 = 4 - x^2$, whose graph is obtained by shifting the graph of $y = -x^2$ up 4 units.

Ex. 2.7.5: What sequence of equation transformations changes $y = x^2$ to $y = 4 - x^2$? What is the effect on the graph at each stage?

Solution:

- Start with equation $y = x^2$. Multiply the RHS by -1 to get equation $y = -x^2$, whose graph is obtained by reflecting the graph of $y = x^2$ across the $x-$axis.
- Add 4 to the RHS of $y = -x^2$ to get equation $y = -x^2 + 4 = 4 - x^2$, whose graph is obtained by shifting the graph of $y = -x^2$ up 4 units.

Answer: The graph of $y = 4 - x^2$ is obtained by reflecting the graph of $y = x^2$ across the x-axis, then shifting the result up 4 units. The resulting graph is a parabola with maximum point at $(0, 4)$

Ex. 2.7.6: What is the equation of the graph obtained if you

- shift the graph of $y = x^2$ up 4 units, then reflect the resulting graph across the x-axis?
  $\Rightarrow$

Ex. 2.7.6: What is the equation of the graph obtained if you

- shift the graph of $y = x^2$ up 4 units, then reflect the resulting graph across the x-axis?
  $\Rightarrow$ Add 4 to the RHS of $y = x^2$ to get $y = x^2 + 4$.
  $\Rightarrow$ Then multiply the RHS of $y = x^2 + 4$ by -1 to get equation $y = -x^2 - 4$.

Ex. 2.7.6: What is the equation of the graph obtained if you

- shift the graph of $y = x^2$ up 4 units, then reflect the resulting graph across the x-axis?
  $\Rightarrow$ Add 4 to the RHS of $y = x^2$ to get $y = x^2 + 4$.
  $\Rightarrow$ Then multiply the RHS of $y = x^2 + 4$ by -1 to get equation $y = -x^2 - 4$.
- shift the graph of $y = x^2$ up 4 units, then shift the result left 3 units?
  $\Rightarrow$

Ex. 2.7.6: What is the equation of the graph obtained if you

- shift the graph of $y = x^2$ up 4 units, then reflect the resulting graph across the x-axis?
  $\Rightarrow$ Add 4 to the RHS of $y = x^2$ to get $y = x^2 + 4$.
  $\Rightarrow$ Then multiply the RHS of $y = x^2 + 4$ by -1 to get equation $y = -x^2 - 4$.
- shift the graph of $y = x^2$ up 4 units, then shift the result left 3 units?
  $\Rightarrow$ Add 4 to the RHS of $y = x^2$ to get $y = x^2 + 4$.
  $\Rightarrow$ Then substitute $x + 3$ for x to get $y = (x + 3)^2 + 4$.

Ex. 2.7.6: What is the equation of the graph obtained if you

- shift the graph of $y = x^2$ up 4 units, then reflect the resulting graph across the x-axis?
  $\Rightarrow$ Add 4 to the RHS of $y = x^2$ to get $y = x^2 + 4$.
  $\Rightarrow$ Then multiply the RHS of $y = x^2 + 4$ by -1 to get equation $y = -x^2 - 4$.
- shift the graph of $y = x^2$ up 4 units, then shift the result left 3 units?
  $\Rightarrow$ Add 4 to the RHS of $y = x^2$ to get $y = x^2 + 4$.
  $\Rightarrow$ Then substitute $x + 3$ for x to get $y = (x + 3)^2 + 4$.
- shift the graph of $y = x^2$ right 4 units, then shrink the result towards the x-axis by a factor of 7?
  $\Rightarrow$

Ex. 2.7.6: What is the equation of the graph obtained if you

- shift the graph of $y = x^2$ up 4 units, then reflect the resulting graph across the x-axis?
  $\Rightarrow$ Add 4 to the RHS of $y = x^2$ to get $y = x^2 + 4$.
  $\Rightarrow$ Then multiply the RHS of $y = x^2 + 4$ by -1 to get equation $y = -x^2 - 4$.
- shift the graph of $y = x^2$ up 4 units, then shift the result left 3 units?
  $\Rightarrow$ Add 4 to the RHS of $y = x^2$ to get $y = x^2 + 4$.
  $\Rightarrow$ Then substitute $x + 3$ for x to get $y = (x + 3)^2 + 4$.
- shift the graph of $y = x^2$ right 4 units, then shrink the result towards the x-axis by a factor of 7?
  $\Rightarrow$ Add 4 Substitute $x - 4$ for x to get $y = (x - 4)^2$.
  $\Rightarrow$ Then divide the RHS by 7 to get $y = \frac{1}{7}(x - 4)^2$.

Ex. 2.7.6: What is the equation of the graph obtained if you

- shift the graph of $y = x^2$ up 4 units, then reflect the resulting graph across the x-axis?
  $\Rightarrow$ Add 4 to the RHS of $y = x^2$ to get $y = x^2 + 4$.
  $\Rightarrow$ Then multiply the RHS of $y = x^2 + 4$ by -1 to get equation $y = -x^2 - 4$.
- shift the graph of $y = x^2$ up 4 units, then shift the result left 3 units?
  $\Rightarrow$ Add 4 to the RHS of $y = x^2$ to get $y = x^2 + 4$.
  $\Rightarrow$ Then substitute $x + 3$ for x to get $y = (x + 3)^2 + 4$.
- shift the graph of $y = x^2$ right 4 units, then shrink the result towards the x-axis by a factor of 7?
  $\Rightarrow$ Add 4 Substitute $x - 4$ for x to get $y = (x - 4)^2$.
  $\Rightarrow$ Then divide the RHS by 7 to get $y = \frac{1}{7}(x - 4)^2$.
- stretch the graph of $y = x^2$ away from the $x-$axis by a factor of 7, then shrink the result toward the y-axis by a factor of 3?
  $\Rightarrow$

Ex. 2.7.6: What is the equation of the graph obtained if you

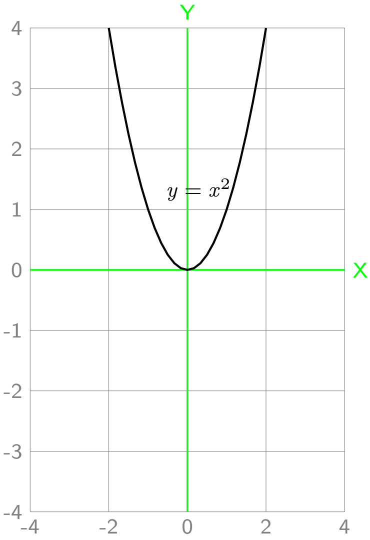
- shift the graph of $y = x^2$ up 4 units, then reflect the resulting graph across the x-axis?
  $\Rightarrow$ Add 4 to the RHS of $y = x^2$ to get $y = x^2 + 4$.
  $\Rightarrow$ Then multiply the RHS of $y = x^2 + 4$ by -1 to get equation $y = -x^2 - 4$.
- shift the graph of $y = x^2$ up 4 units, then shift the result left 3 units?
  $\Rightarrow$ Add 4 to the RHS of $y = x^2$ to get $y = x^2 + 4$.
  $\Rightarrow$ Then substitute $x + 3$ for x to get $y = (x + 3)^2 + 4$.
- shift the graph of $y = x^2$ right 4 units, then shrink the result towards the x-axis by a factor of 7?
  $\Rightarrow$ Add 4 Substitute $x - 4$ for x to get $y = (x - 4)^2$.
  $\Rightarrow$ Then divide the RHS by 7 to get $y = \frac{1}{7}(x - 4)^2$.
- stretch the graph of $y = x^2$ away from the $x-$axis by a factor of 7, then shrink the result toward the y-axis by a factor of 3?
  $\Rightarrow$ Multiply the RHS of $y = x^2$ by 7 to get $y = 7x^2$.
  $\Rightarrow$ Then substitute $3x$ for x to get $y = 7(3x)^2 = 63x^3$.

The answers to the following problems use abbreviations:

- "V-shrink by 2" means "shrink vertically, toward the x-axis, by a factor of 2."
  Point (x, y) goes to $(x, \frac{y}{2})$.
- "V-stretch by 2" means "stretch vertically, away from the x-axis by a factor of 2."
  Point (x, y) goes to $(x, 2y)$.
- "H-shrink by 2" means "shrink horizontally, toward the y-axis, by a factor of 2."
  Point (x, y) goes to $(\frac{x}{2}, y)$.
- "H-stretch by 2" means "stretch horizontally, away from the y-axis by a factor of 2."
  Point (x, y) goes to $(2x, y)$.
- The word " unit"is omitted from a shift operation: "shift left 4" means "shift left 4 units." The word "unit" is used in deference to science problems, where the unit of measurement must be stated. In math problems, the only unit is the number 1, and so it need not be mentioned explicitly.

Ex. 2.7.7: Example 7a. What sequence of equation transformations changes $y = x^2$ to $y = (x + 4)^2 - 5$? What is the effect on the graph at each stage?

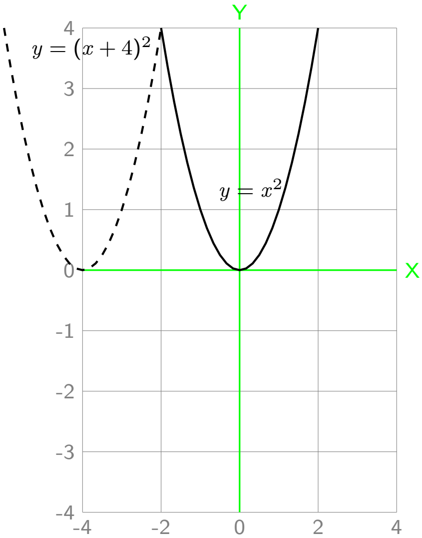
Solution:



Ex. 2.7.7: Example 7a. What sequence of equation transformations changes $y = x^2$ to $y = (x+4)^2 - 5$? What is the effect on the graph at each stage?

Solution:

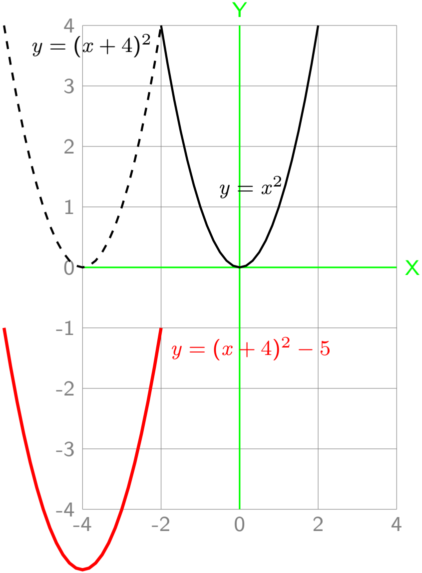
- Substitute $x + 4$ for x in $y = x^2$, whose graph shifts left 4 to the graph of $y = (x+4)^2$.



Ex. 2.7.7: Example 7a. What sequence of equation transformations changes $y = x^2$ to $y = (x+4)^2 - 5$? What is the effect on the graph at each stage?

Solution:

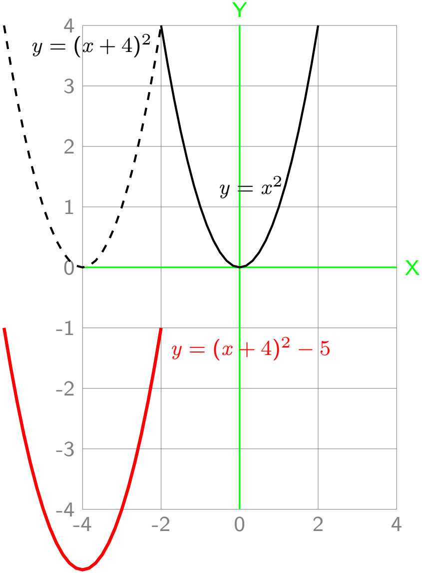
- Substitute $x + 4$ for x in $y = x^2$, whose graph shifts left 4 to the graph of $y = (x+4)^2$.
- Subtract 5 from the RHS of $y = (x+4)^2$, whose graph shifts down 5 to the graph of $y = (x+4)^2 - 5$.



Ex. 2.7.7: Example 7a. What sequence of equation transformations changes $y = x^2$ to $y = (x + 4)^2 - 5$? What is the effect on the graph at each stage?

Solution:

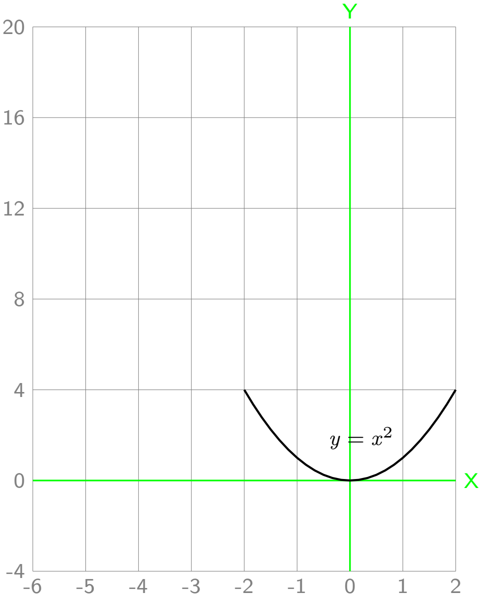
- Substitute $x + 4$ for x in $y = x^2$, whose graph shifts left 4 to the graph of $y = (x + 4)^2$.
- Subtract 5 from the RHS of $y = (x + 4)^2$, whose graph shifts down 5 to the graph of $y = (x + 4)^2 - 5$.
- Answer: To get the graph of $y = (x + 4)^2 - 5$,
  - shift the graph of $y = x^2$ left 4;
  - shift the result down 5.



Ex. 2.7.7: Example7b. What sequence of equation transformations changes $y = x^2$ to $y = 5(3x+4)^2$? What is the effect on the graph at each stage?

Solution:

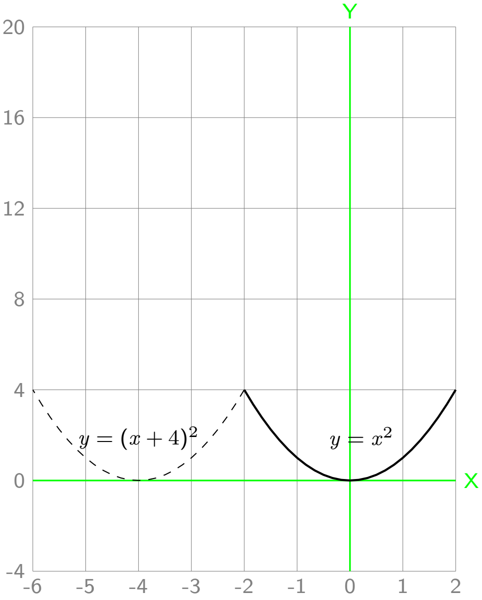
- Substitute $x + 4$ for x in $y = x^2$, whose graph shifts left 4 units to the graph of $y = (x+4)^2$.



Ex. 2.7.7: Example7b. What sequence of equation transformations changes $y = x^2$ to $y = 5(3x + 4)^2$? What is the effect on the graph at each stage?

Solution:

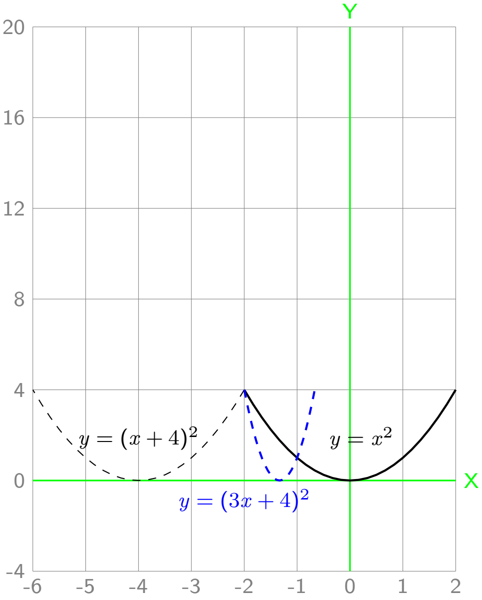
- Substitute $x + 4$ for x in $y = x^2$, whose graph shifts left 4 units to the graph of $y = (x + 4)^2$.
- Substitute $3x$ for x in the RHS of $y = (x + 4)^2$ to H-shrink its graph by 3 to the graph of $y = (3x + 4)^2$.



Ex. 2.7.7: Example7b. What sequence of equation transformations changes $y = x^2$ to $y = 5(3x + 4)^2$? What is the effect on the graph at each stage?

Solution:

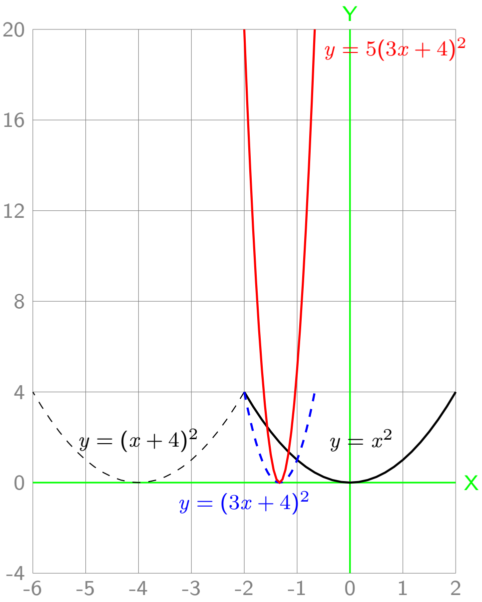
- Substitute $x + 4$ for x in $y = x^2$, whose graph shifts left 4 units to the graph of $y = (x + 4)^2$.
- Substitute $3x$ for x in the RHS of $y = (x + 4)^2$ to H-shrink its graph by 3 to the graph of $y = (3x + 4)^2$.
- Multiply the RHS of $y = (3x + 4)^2$ by 5 to V-stretch its graph by 5 to the graph of $y = 5(3x + 4)^2$.



Ex. 2.7.7: Example7b. What sequence of equation transformations changes $y = x^2$ to $y = 5(3x+4)^2$? What is the effect on the graph at each stage?

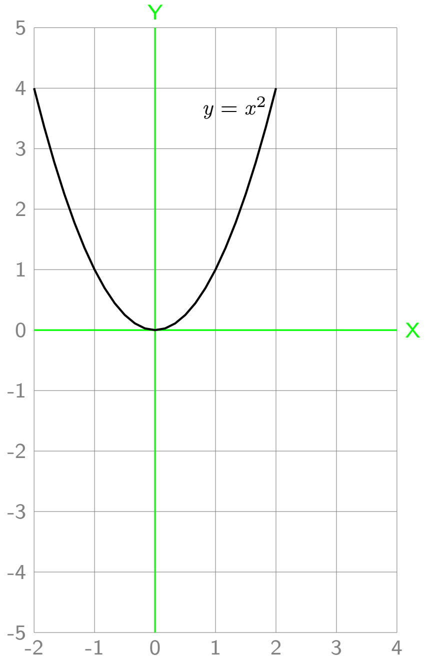
Solution:

- Substitute $x + 4$ for x in $y = x^2$, whose graph shifts left 4 units to the graph of $y = (x+4)^2$.
- Substitute $3x$ for x in the RHS of $y = (x+4)^2$ to H-shrink its graph by 3 to the graph of $y = (3x+4)^2$.
- Multiply the RHS of $y = (3x+4)^2$ by 5 to V-stretch its graph by 5 to the graph of $y = 5(3x+4)^2$.
- Answer: to get the graph of $y = 5(3x+4)^2$,
  - shift the graph of $y = x^2$ left 4 units;
  - H-shrink the result by 3;
  - V-stretch the result by 5.



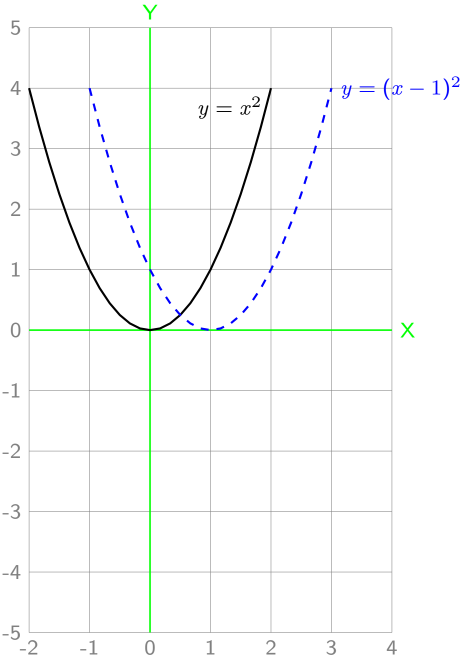
Ex. 2.7.7: Example 7c. What sequence of equation transformations changes $y = x^2$ to $y = 5 - (x - 1)^2$? What is the effect on the graph at each stage?

- Start with the graph of $y = x^2$



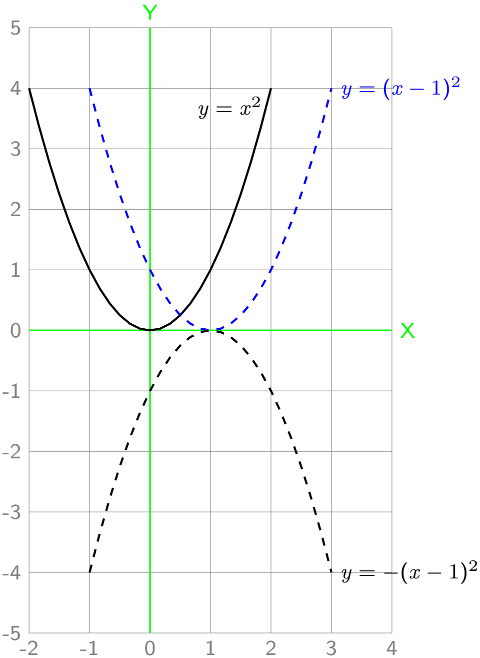
Ex. 2.7.7: Example 7c. What sequence of equation transformations changes $y = x^2$ to $y = 5 - (x-1)^2$? What is the effect on the graph at each stage?

- Start with the graph of $y = x^2$
- Substitute $x - 1$ for x in $y = x^2$, whose graph shifts right 1 to the graph of $y = (x-1)^2$.



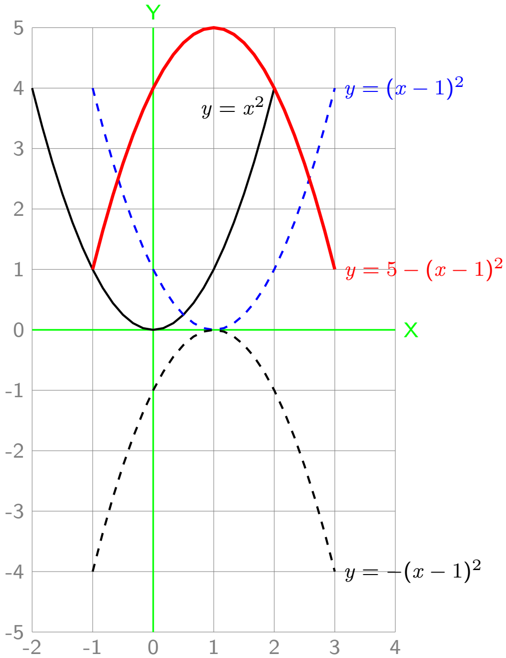
Ex. 2.7.7: Example 7c. What sequence of equation transformations changes $y = x^2$ to $y = 5 - (x-1)^2$? What is the effect on the graph at each stage?

- Start with the graph of $y = x^2$
- Substitute $x - 1$ for x in $y = x^2$, whose graph shifts right 1 to the graph of $y = (x-1)^2$.
- Multiply RHS of $y = (x-1)^2$ by -1 to reflect its graph across the x-axis to the graph of $y = -(x-1)^2$.



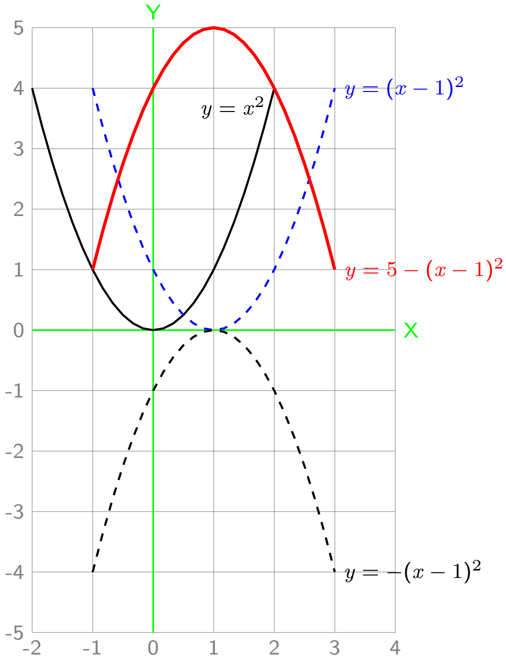
Ex. 2.7.7: Example 7c. What sequence of equation transformations changes $y = x^2$ to $y = 5 - (x-1)^2$? What is the effect on the graph at each stage?

- Start with the graph of $y = x^2$
- Substitute $x - 1$ for x in $y = x^2$, whose graph shifts right 1 to the graph of $y = (x-1)^2$.
- Multiply RHS of $y = (x-1)^2$ by -1 to reflect its graph across the x-axis to the graph of $y = -(x-1)^2$.
- Add 5 to the RHS of $y = -(x-1)^2$. to shift its graph up 5 to the graph of $y = 5 - (x-1)^2$.



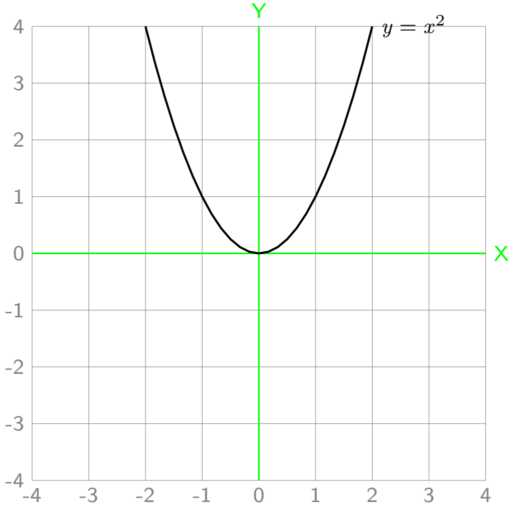
Ex. 2.7.7: Example 7c. What sequence of equation transformations changes $y = x^2$ to $y = 5 - (x-1)^2$? What is the effect on the graph at each stage?

- Start with the graph of $y = x^2$
- Substitute $x - 1$ for x in $y = x^2$, whose graph shifts right 1 to the graph of $y = (x-1)^2$.
- Multiply RHS of $y = (x-1)^2$ by -1 to reflect its graph across the x-axis to the graph of $y = -(x-1)^2$.
- Add 5 to the RHS of $y = -(x-1)^2$. to shift its graph up 5 to the graph of $y = 5 - (x-1)^2$.
- Summary: To obtain the graph of $y = 5 - (x-1)^2$,
  - shift the graph of $y = x^2$ right 1;
  - reflect the result across the x-axis;
  - shift the result up 5.



Ex. 2.7.7: Example 7d. What sequence of equation transformations changes $y = x^2$ to $y = -\frac{(2x+1)^2}{2}$? What is the effect on the graph at each stage?

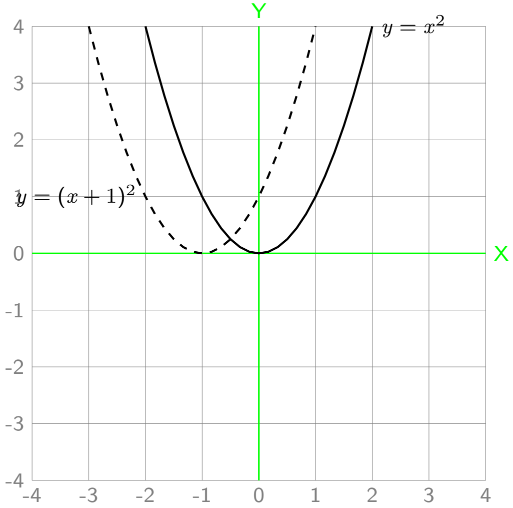
Solution:



Ex. 2.7.7: Example 7d. What sequence of equation transformations changes $y = x^2$ to $y = -\frac{(2x+1)^2}{2}$? What is the effect on the graph at each stage?

Solution:

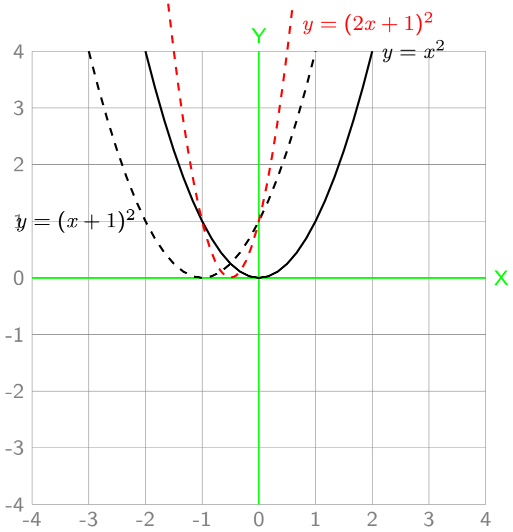
- Start with equation $y = x^2$. Substitute $x + 1$ for x in RHS of $y = x^2$ by 2 to shift its graph left 1 to the graph of $y = (x + 1)^2$.



Ex. 2.7.7: Example 7d. What sequence of equation transformations changes $y = x^2$ to $y = -\frac{(2x+1)^2}{2}$? What is the effect on the graph at each stage?

Solution:

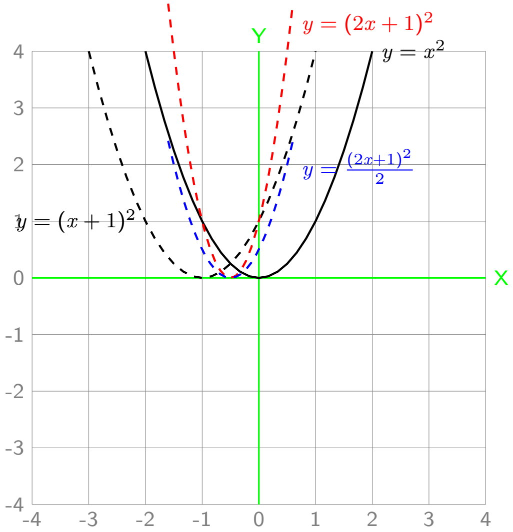
- Start with equation $y = x^2$. Substitute $x + 1$ for x in RHS of $y = x^2$ by 2 to shift its graph left 1 to the graph of $y = (x + 1)^2$.
- Substitute $2x$ for x in RHS of$y = (x + 1)^2$ to H-shrink its graph by 2 to the graph of $y = (2x + 1)^2$.



Ex. 2.7.7: Example 7d. What sequence of equation transformations changes $y = x^2$ to $y = -\frac{(2x+1)^2}{2}$? What is the effect on the graph at each stage?

Solution:

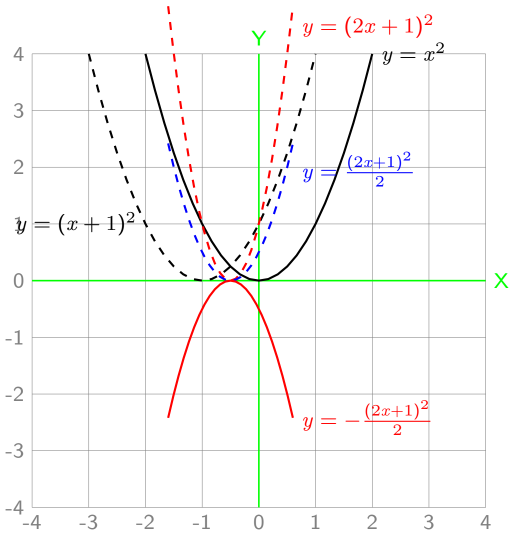
- Start with equation $y = x^2$. Substitute $x + 1$ for x in RHS of $y = x^2$ by 2 to shift its graph left 1 to the graph of $y = (x + 1)^2$.
- Substitute $2x$ for x in RHS of$y = (x + 1)^2$ to H-shrink its graph by 2 to the graph of $y = (2x + 1)^2$.
- Divide RHS of $y = (2x + 1)^2$ by 2 to V-shrink its graph by 2 to the graph of $y = \frac{(2x+1)^2}{2}$.



Ex. 2.7.7: Example 7d. What sequence of equation transformations changes $y = x^2$ to $y = -\frac{(2x+1)^2}{2}$? What is the effect on the graph at each stage?

Solution:

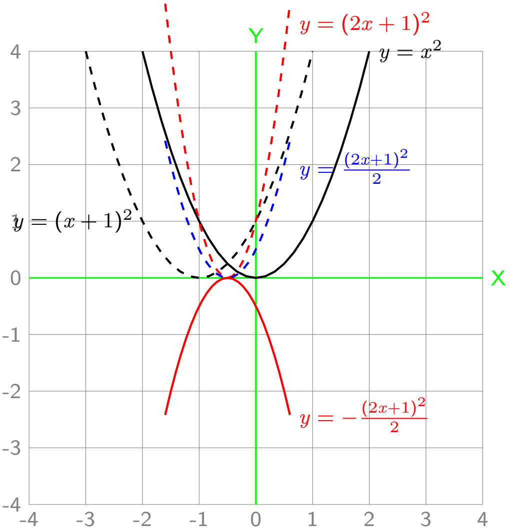
- Start with equation $y = x^2$. Substitute $x + 1$ for x in RHS of $y = x^2$ by 2 to shift its graph left 1 to the graph of $y = (x + 1)^2$.
- Substitute $2x$ for x in RHS of$y = (x + 1)^2$ to H-shrink its graph by 2 to the graph of $y = (2x + 1)^2$.
- Divide RHS of $y = (2x + 1)^2$ by 2 to V-shrink its graph by 2 to the graph of $y = \frac{(2x+1)^2}{2}$.
- Multiply RHS of $y = \frac{(2x+1)^2}{2}$ by -1 to reflect its graph across the x-axis to the graph of $y = -\frac{(2x+1)^2}{2}$.
- Answer: To get the graph of $-\frac{(2x+1)^2}{2}$,
  - shift the graph of $y = x^2$ right 1;
  - H-shrink the result by 2;
  - V-shrink the result by 2;
  - reflect the result across the x-axis.



Ex. 2.7.7: Example 7d. What sequence of equation transformations changes $y = x^2$ to $y = -\frac{(2x+1)^2}{2}$? What is the effect on the graph at each stage?

Solution:

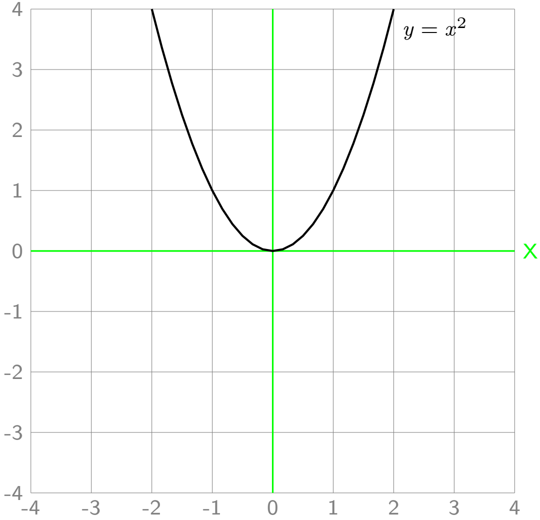
- Start with equation $y = x^2$. Substitute $x + 1$ for x in RHS of $y = x^2$ by 2 to shift its graph left 1 to the graph of $y = (x + 1)^2$.
- Substitute $2x$ for x in RHS of$y = (x + 1)^2$ to H-shrink its graph by 2 to the graph of $y = (2x + 1)^2$.
- Divide RHS of $y = (2x + 1)^2$ by 2 to V-shrink its graph by 2 to the graph of $y = \frac{(2x+1)^2}{2}$.
- Multiply RHS of $y = \frac{(2x+1)^2}{2}$ by -1 to reflect its graph across the x-axis to the graph of $y = -\frac{(2x+1)^2}{2}$.
- Answer: To get the graph of $-\frac{(2x+1)^2}{2}$,
    - shift the graph of $y = x^2$ right 1;
    - H-shrink the result by 2;
    - V-shrink the result by 2;
    - reflect the result across the x-axis.



Ex. 2.7.8a: What sequence of transformations applied to the equation $y = x^2$ yields the graph of $y = 4 - (x + 2)^2$? What is the effect on the graph at each stage?

Solution:

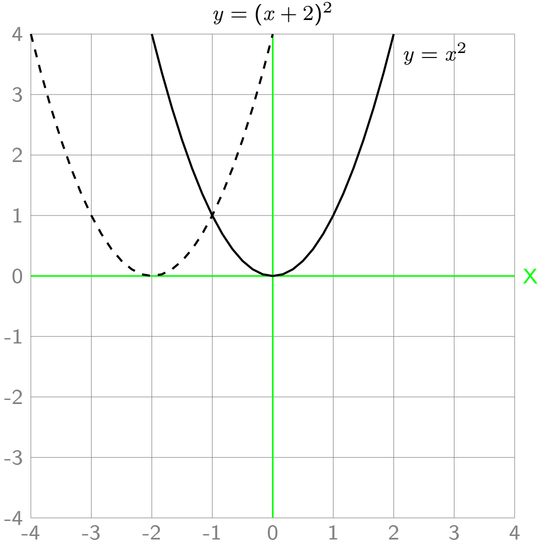
- Start with the equation $y = x^2$.



Ex. 2.7.8a: What sequence of transformations applied to the equation $y = x^2$ yields the graph of $y = 4 - (x+2)^2$? What is the effect on the graph at each stage?

Solution:

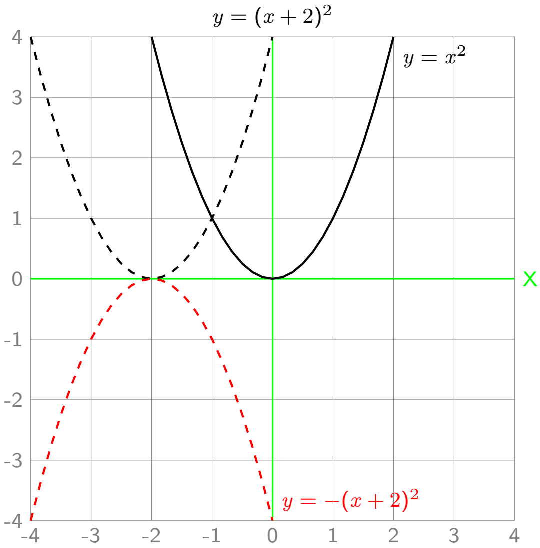
- Start with the equation $y = x^2$.
- Substitute $x + 2$ for x to shift its graph left 2 to the graph of $y = (x+2)^2$.



Ex. 2.7.8a: What sequence of transformations applied to the equation $y = x^2$ yields the graph of $y = 4 - (x+2)^2$? What is the effect on the graph at each stage?

Solution:

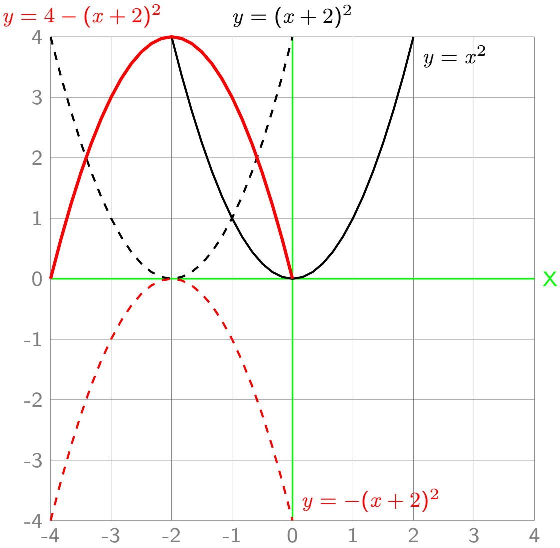
- Start with the equation $y = x^2$.
- Substitute $x + 2$ for x to shift its graph left 2 to the graph of $y = (x+2)^2$.
- To graph $y = -(x+2)^2$, reflect the graph of $y = (x+2)^2$ across the x-axis.



Ex. 2.7.8a: What sequence of transformations applied to the equation $y = x^2$ yields the graph of $y = 4 - (x+2)^2$? What is the effect on the graph at each stage?

Solution:

- Start with the equation $y = x^2$.
- Substitute $x + 2$ for x to shift its graph left 2 to the graph of $y = (x+2)^2$.
- To graph $y = -(x+2)^2$, reflect the graph of $y = (x+2)^2$ across the x-axis.
- To graph $y = -(x+2)^2 + 4 = 4 - (x+2)^2$, shift the graph of $y = -(x+2)^2$ up 4 .

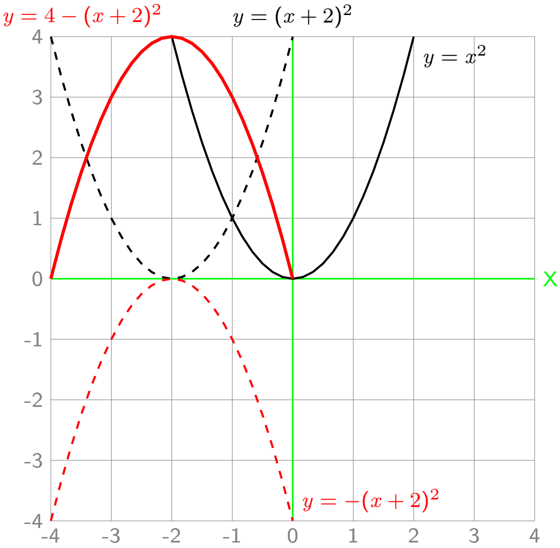


Ex. 2.7.8a: What sequence of transformations applied to the equation $y = x^2$ yields the graph of $y = 4 - (x+2)^2$? What is the effect on the graph at each stage?

Solution:

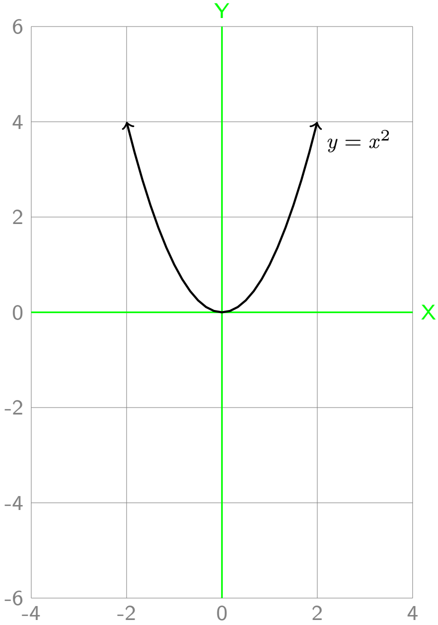
- Start with the equation $y = x^2$.
- Substitute $x + 2$ for x to shift its graph left 2 to the graph of $y = (x+2)^2$.
- To graph $y = -(x+2)^2$, reflect the graph of $y = (x+2)^2$ across the x-axis.
- To graph $y = -(x+2)^2 + 4 = 4 - (x+2)^2$, shift the graph of $y = -(x+2)^2$ up 4 .

  Answer:
  - Start with the graph of $y = x^2$. Shift it left 2 units;
  - reflect across the x-axis;
  - shift up 4.



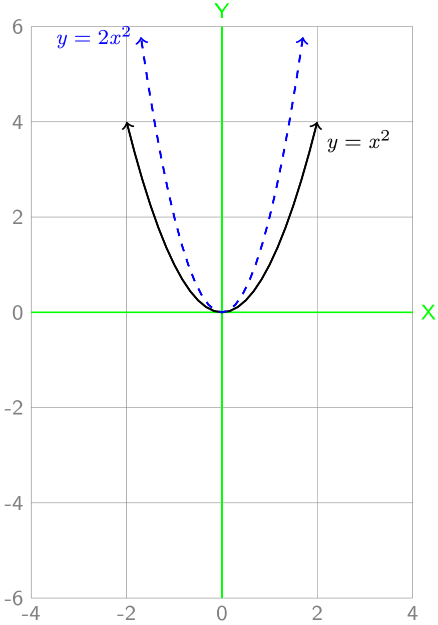
Ex. 2.7.8b:

- Describe the graph at each stage as you transform equation $y = x^2$ to $y = 5 - 2x^2 \Rightarrow$



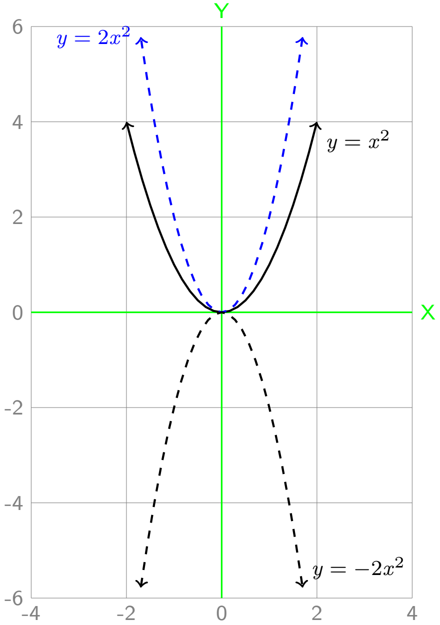
Ex. 2.7.8b:

- Describe the graph at each stage as you transform equation $y = x^2$ to $y = 5 - 2x^2 \Rightarrow$
  - Multiply RHS of $y = x^2$ by 2 to V-stretch its graph by 2 to the graph of $y = 2x^2$.



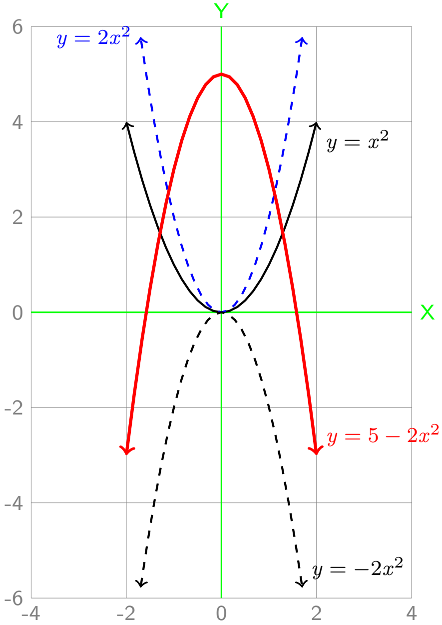
Ex. 2.7.8b:

- Describe the graph at each stage as you transform equation $y = x^2$ to $y = 5 - 2x^2 \Rightarrow$
  - Multiply RHS of $y = x^2$ by 2 to V-stretch its graph by 2 to the graph of $y = 2x^2$.
  - Multiply the RHS of $y = 2x^2$ by -1 to reflect its graph across the x-axis to the graph of $y = -2x^2$.



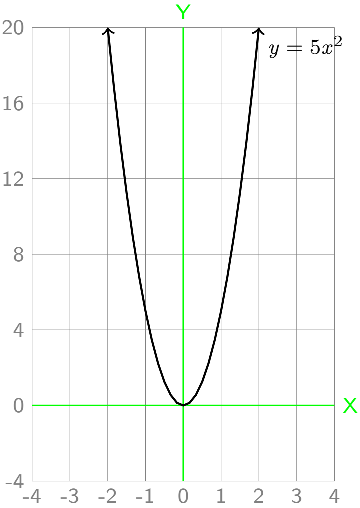
Ex. 2.7.8b:

- Describe the graph at each stage as you transform equation $y = x^2$ to $y = 5 - 2x^2 \Rightarrow$
  - Multiply RHS of $y = x^2$ by 2 to V-stretch its graph by 2 to the graph of $y = 2x^2$.
  - Multiply the RHS of $y = 2x^2$ by -1 to reflect its graph across the x-axis to the graph of $y = -2x^2$.
  - Add 5 to the RHS of $y = -2x^2$ to shifts its graph up 2 to the graph of $y = 5 - 2x^2$.
  - The resulting graph is a parabola with vertex with maximum point at vertex $(0, 5)$.



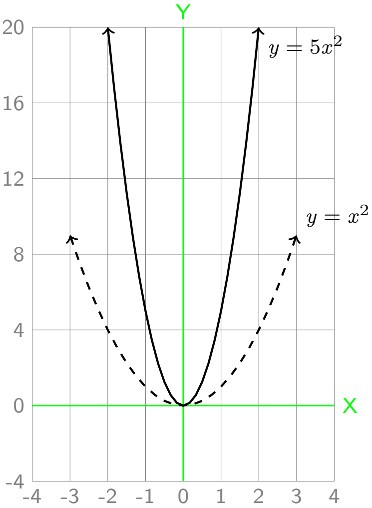
Ex. 2.7.8c:

- Describe the graph at each stage as you transform equation $y = 5x^2$ to $y = 4 - x^2 \Rightarrow$



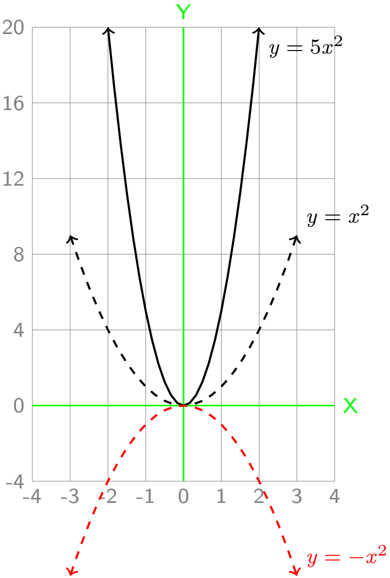
Ex. 2.7.8c:

- Describe the graph at each stage as you transform equation $y = 5x^2$ to $y = 4 - x^2 \Rightarrow$
  - Divide the RHS by 5 to get $y = x^2$. The graph shrinks toward the x-axis by a factor of 5.



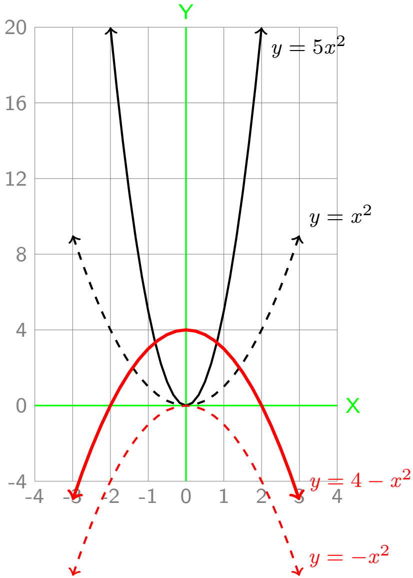
Ex. 2.7.8c:

- Describe the graph at each stage as you transform equation $y = 5x^2$ to $y = 4 - x^2$ $\Rightarrow$
  - Divide the RHS by 5 to get $y = x^2$. The graph shrinks toward the x-axis by a factor of 5.
  - Multiply the RHS of $y = x^2$ by -1 to get $y = -x^2$. The graph reflects across the x-axis.

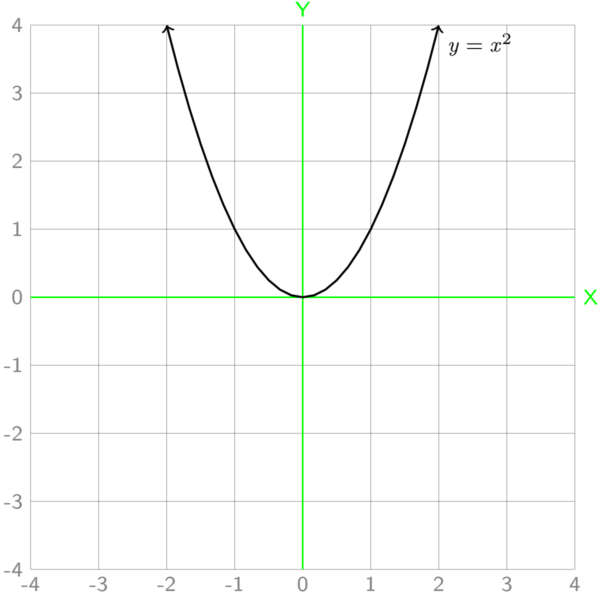


Ex. 2.7.8c:

- Describe the graph at each stage as you transform equation $y = 5x^2$ to $y = 4 - x^2 \Rightarrow$
  - Divide the RHS by 5 to get $y = x^2$. The graph shrinks toward the x-axis by a factor of 5.
  - Multiply the RHS of $y = x^2$ by -1 to get $y = -x^2$. The graph reflects across the x-axis.
  - Add 4 to the RHS to get $y = -x^2 + 4 = 4 - x^2$. The graph shifts up 4 units.
  - The resulting graph is a parabola with maximum point at vertex $(0, 4)$.

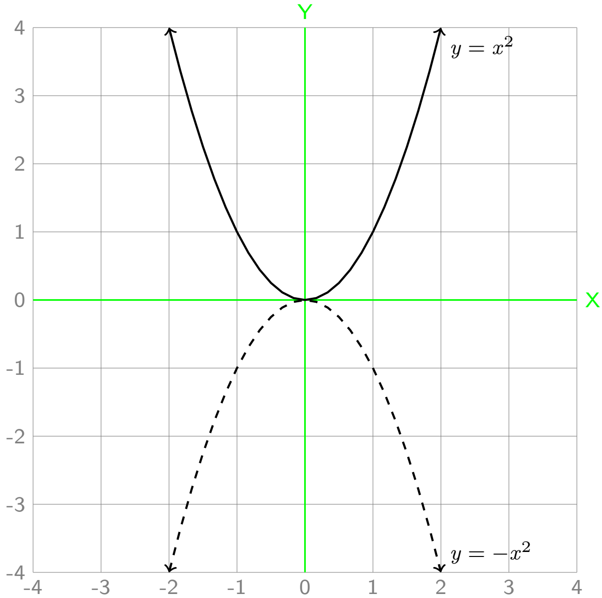


Ex. 2.7.8d: Describe the graph at each stage as you apply transformations to the graph of $y = x^2$ to yield the graph of $y = 4 - x^2 \Rightarrow$



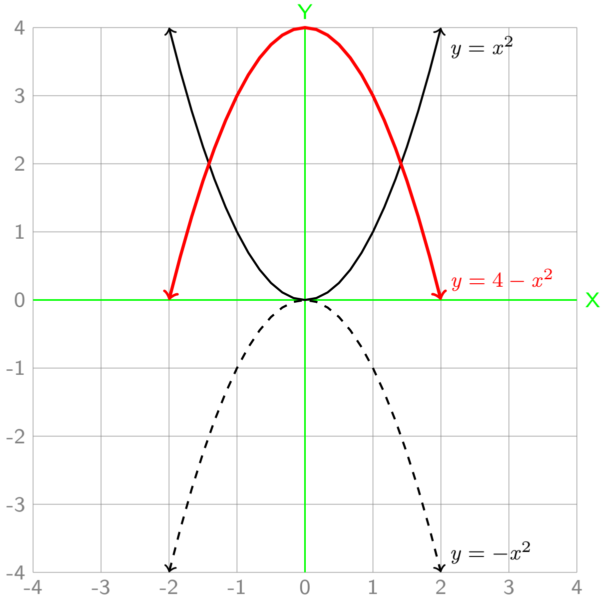
Ex. 2.7.8d: Describe the graph at each stage as you apply transformations to the graph of $y = x^2$ to yield the graph of $y = 4 - x^2 \Rightarrow$

- Multiply RHS of $y = x^2$ by -1 to get $y = -x^2$.The graph reflects across the x-axis.



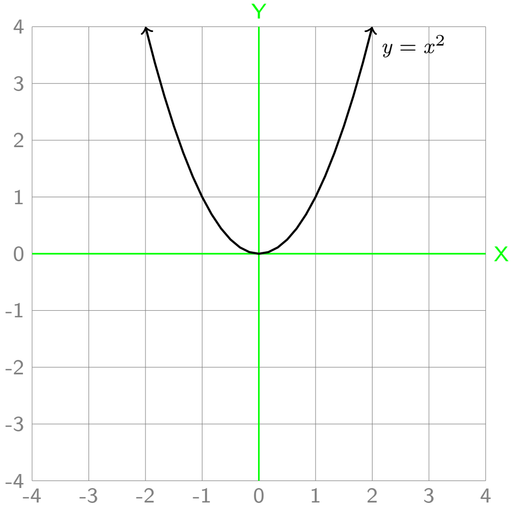
Ex. 2.7.8d: Describe the graph at each stage as you apply transformations to the graph of $y = x^2$ to yield the graph of $y = 4 - x^2 \Rightarrow$

- Multiply RHS of $y = x^2$ by -1 to get $y = -x^2$.The graph reflects across the x-axis.
- Add 4 to the RHS to get $y = -x^2 + 4 = 4 - x^2$. The graph shifts up 4 units.
- The resulting graph is a parabola with maximum point at vertex $(0, 4)$.



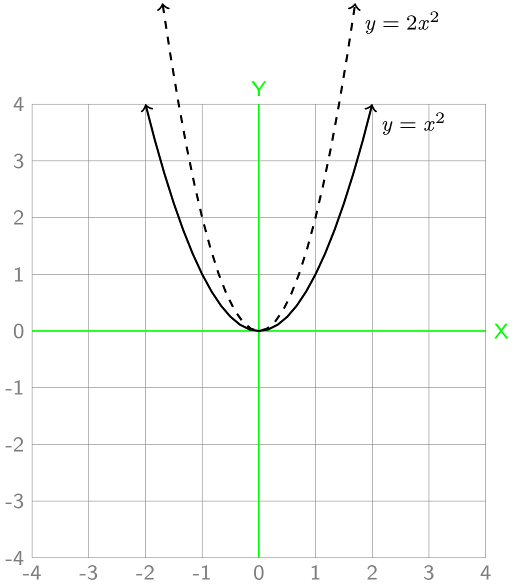
Ex. 2.7.8e: Describe the graph at each stage as you apply transformations to the graph of $y = x^2$ to yield the graph of $y = 2x^2 - 4 \Rightarrow$

- Multiply RHS of $y = x^2$ by 2 to get $y = 2x^2$. The graph stretches away from the x-axis by a factor of 2.



Ex. 2.7.8e: Describe the graph at each stage as you apply transformations to the graph of $y = x^2$ to yield the graph of $y = 2x^2 - 4 \Rightarrow$

- Multiply RHS of $y = x^2$ by 2 to get $y = 2x^2$. The graph stretches away from the x-axis by a factor of 2.
- Subtract 4 from the RHS to get $y = 2x^2 - 4$. The graph shifts down 4 units.
- The resulting graph is a parabola with minimum point at vertex $(0, -4)$.



Ex. 2.7.8e: Describe the graph at each stage as you apply transformations to the graph of $y = x^2$ to yield the graph of $y = 2x^2 - 4 \Rightarrow$

- Multiply RHS of $y = x^2$ by 2 to get $y = 2x^2$. The graph stretches away from the x-axis by a factor of 2.
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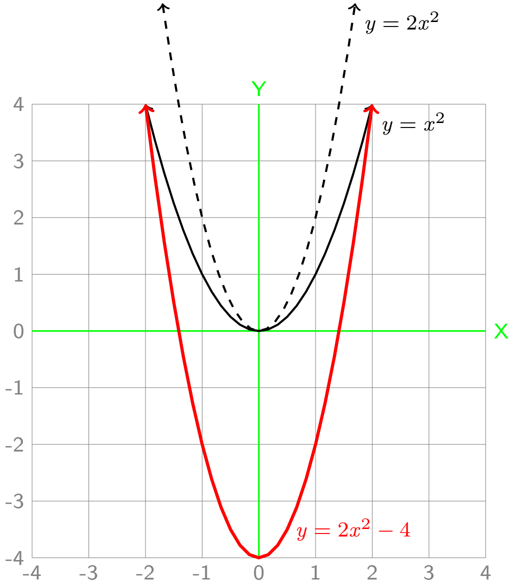
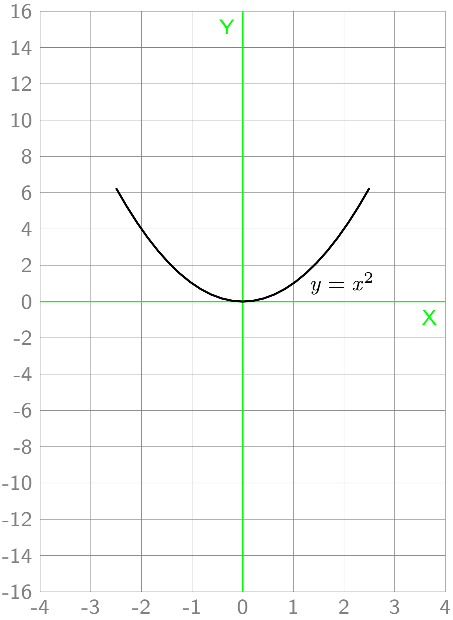


Ex. 2.7.9: What sequence of equation transformations will produce the graph of $y = -2(2x+1)^2$ from the graph of $y = x^2$? What is the effect on the graph at each stage?

Solution:

- Start with the equation $y = f(x) = x^2$.



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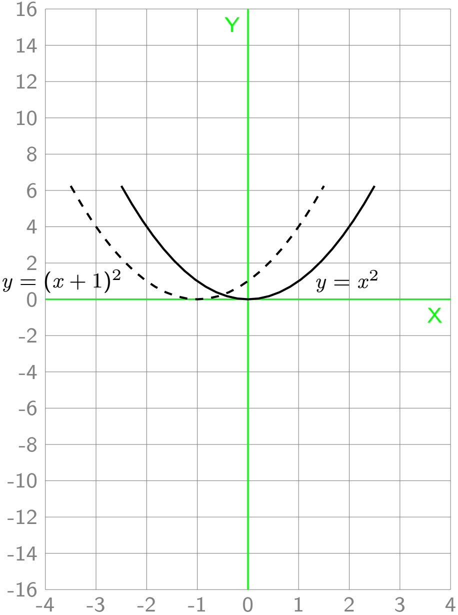
Solution:

- Start with the equation $y = f(x) = x^2$.
- Substitute $x + 1$ for x to get the new equation $y = (x+1)^2$. To graph this equation, shift the graph of $y = x^2$ left 1 unit.



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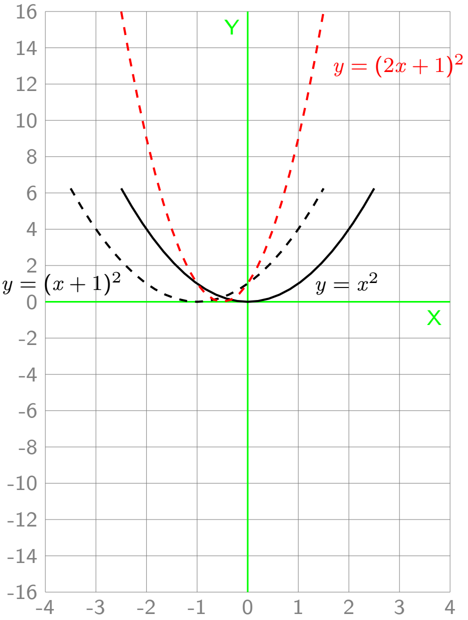
- Substitute $x + 1$ for x to get the new equation $y = (x+1)^2$. To graph this equation, shift the graph of $y = x^2$ left 1 unit.
- To graph $y = (2x+1)^2$, substitute $2x$ for x in $y = (x+1)^2$, whose graph H-shrinks by 2.



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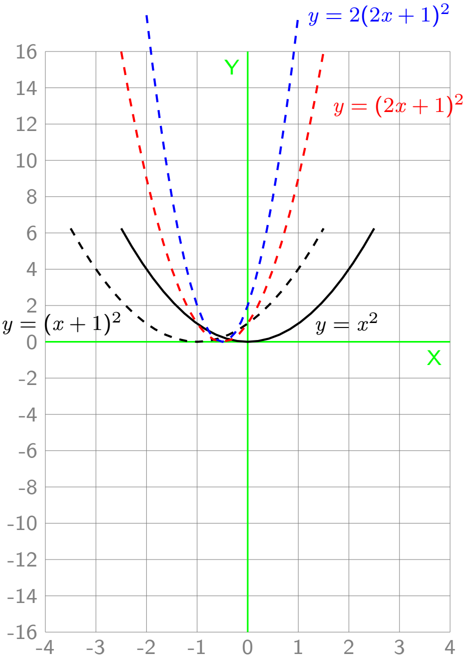
Solution:

- Start with the equation $y = f(x) = x^2$.
- Substitute $x + 1$ for x to get the new equation $y = (x+1)^2$. To graph this equation, shift the graph of $y = x^2$ left 1 unit.
- To graph $y = (2x+1)^2$, substitute $2x$ for x in $y = (x+1)^2$, whose graph H-shrinks by 2.
- To graph $y = 2(2x+1)^2$, V-stretch the graph of $y = (2x+1)^2$ by 2.



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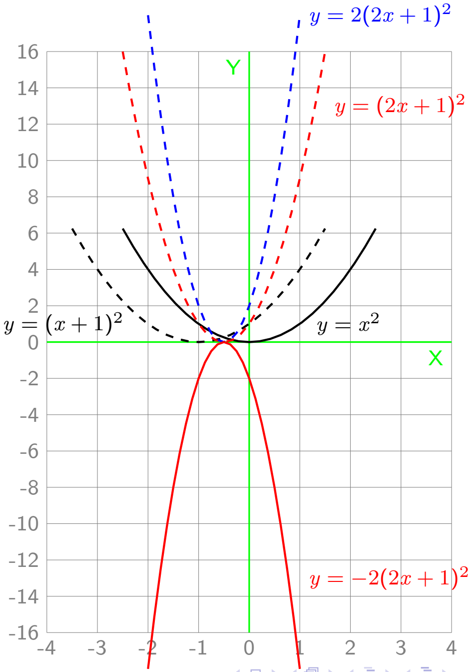
- To graph $y = (2x+1)^2$, substitute $2x$ for x in $y = (x+1)^2$, whose graph H-shrinks by 2.
- To graph $y = 2(2x+1)^2$, V-stretch the graph of $y = (2x+1)^2$ by 2.
- To graph $y = -2(2x+1)^2$, reflect the graph of $y = 2(2x+1)^2$ through the x-axis.



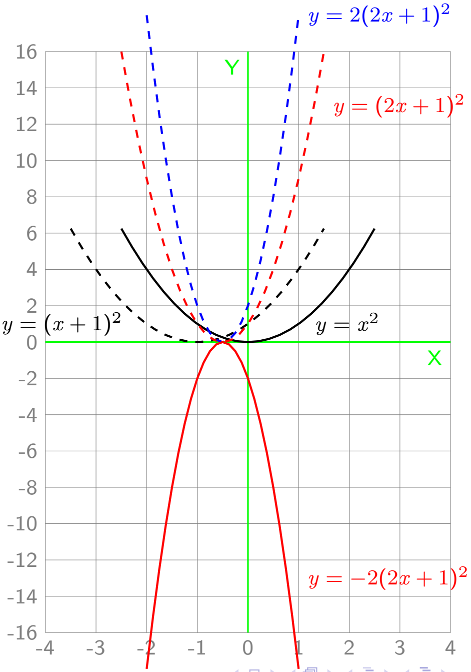
Ex. 2.7.9: What sequence of equation transformations will produce the graph of $y = -2(2x+1)^2$ from the graph of $y = x^2$? What is the effect on the graph at each stage?

Solution:

- Start with the equation $y = f(x) = x^2$.
- Substitute $x+1$ for x to get the new equation $y = (x+1)^2$. To graph this equation, shift the graph of $y = x^2$ left 1 unit.
- To graph $y = (2x+1)^2$, substitute $2x$ for x in $y = (x+1)^2$, whose graph H-shrinks by 2.
- To graph $y = 2(2x+1)^2$, V-stretch the graph of $y = (2x+1)^2$ by 2.
- To graph $y = -2(2x+1)^2$, reflect the graph of $y = 2(2x+1)^2$ through the x-axis.

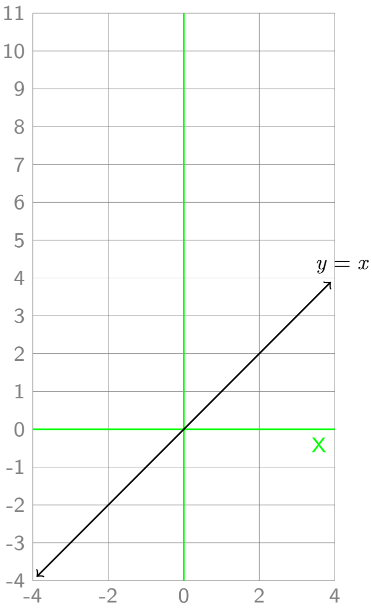
  Answer:

  - Shift the graph of $y = x^2$ left 1;
  - H-shrink the result by 2;
  - V-stretch the result by 2;
  - reflect the result through the x-axis.



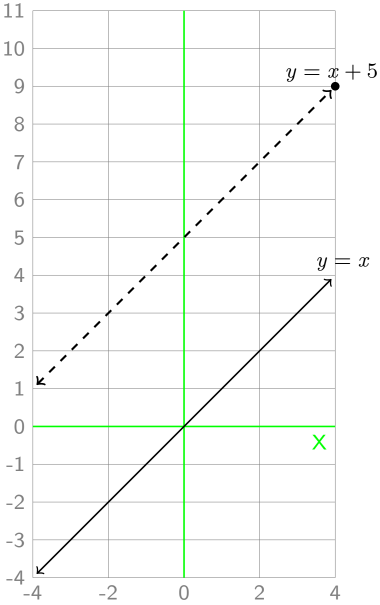
Ex. 2.7.10: What sequence of equation transformations changes $y = x$ to $y = 3x + 5$? What is the effect on the graph at each stage?

- Start by sketching the graph of $y = x$.



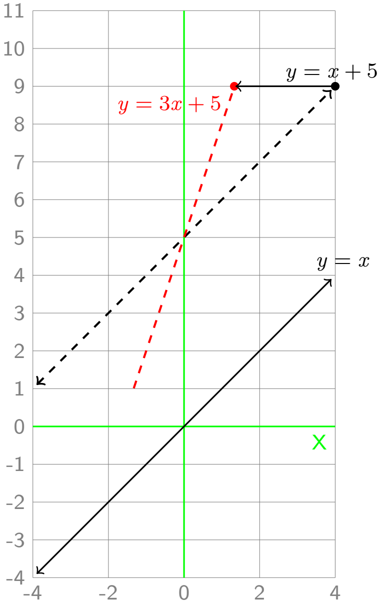
Ex. 2.7.10: What sequence of equation transformations changes $y = x$ to $y = 3x + 5$? What is the effect on the graph at each stage?

- Start by sketching the graph of $y = x$.
- Substitute $x + 5$ for x to get $y = x + 5$. The graph shifts up 5.



Ex. 2.7.10: What sequence of equation transformations changes $y = x$ to $y = 3x + 5$? What is the effect on the graph at each stage?

- Start by sketching the graph of $y = x$.
- Substitute $x + 5$ for x to get $y = x + 5$. The graph shifts up 5.
- Substitute $3x$ for x to get $y = 3x + 5$. The graph H-shrinks by 3. To check this, note that the point $(2, 7)$ moves left to point $(\frac{2}{3}, 7)$ and $(4, 9)$ moves to $(\frac{4}{3}, 9)$.



Ex. 2.7.10: What sequence of equation transformations changes $y = x$ to $y = 3x + 5$? What is the effect on the graph at each stage?

- Start by sketching the graph of $y = x$.
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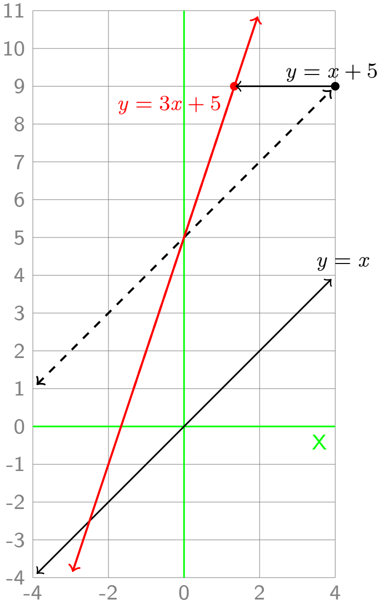
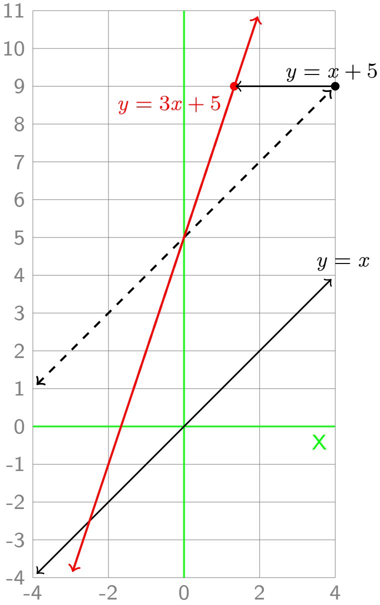
- Extend the straight line $y = 3x + 5$ to the largest possible domain that fits in the grid.



Ex. 2.7.10: What sequence of equation transformations changes $y = x$ to $y = 3x + 5$? What is the effect on the graph at each stage?

- Start by sketching the graph of $y = x$.
- Substitute $x + 5$ for x to get $y = x + 5$. The graph shifts up 5.
- Substitute $3x$ for x to get $y = 3x + 5$. The graph H-shrinks by 3. To check this, note that the point $(2, 7)$ moves left to point $(\frac{2}{3}, 7)$ and $(4, 9)$ moves to $(\frac{4}{3}, 9)$.
- Extend the straight line $y = 3x + 5$ to the largest possible domain that fits in the grid.
- Answer:
  - Shift $y = x$ up 5;
  - H-shrink the result by 3.



Ex. 2.7.11:

Find a sequence of equation transformations that will produce the graph of $y = 3\sin(2x + \pi)$ from the graph of $y = \sin(x)$. What is the effect on the graph at each stage? Don't try to sketch the graph.

Solution:

Ex. 2.7.11:

Find a sequence of equation transformations that will produce the graph of $y = 3\sin(2x + \pi)$ from the graph of $y = \sin(x)$. What is the effect on the graph at each stage? Don't try to sketch the graph.

Solution:

- Start with the equation $y = \sin(x)$.

Ex. 2.7.11:

Find a sequence of equation transformations that will produce the graph of $y = 3\sin(2x + \pi)$ from the graph of $y = \sin(x)$. What is the effect on the graph at each stage? Don't try to sketch the graph.

Solution:

- Start with the equation $y = \sin(x)$.
- Substitute $x + \pi$ for x to get the new equation $y = \sin(x + \pi)$. To graph this equation, shift the graph of $y = \sin(x)$ left π .

Ex. 2.7.11:

Find a sequence of equation transformations that will produce the graph of $y = 3\sin(2x + \pi)$ from the graph of $y = \sin(x)$. What is the effect on the graph at each stage? Don't try to sketch the graph.

Solution:

- Start with the equation $y = \sin(x)$.
- Substitute $x + \pi$ for x to get the new equation $y = \sin(x + \pi)$. To graph this equation, shift the graph of $y = \sin(x)$ left π .
- Substitute $2x$ for x in the equation $y = \sin(x + \pi)$ to get $y = \sin(2x + \pi)$. To graph this equation, H-shrink the graph of $y = \sin(x + \pi)$ by 2.

Ex. 2.7.11:

Find a sequence of equation transformations that will produce the graph of $y = 3\sin(2x + \pi)$ from the graph of $y = \sin(x)$. What is the effect on the graph at each stage? Don't try to sketch the graph.

Solution:

- Start with the equation $y = \sin(x)$.
- Substitute $x + \pi$ for x to get the new equation $y = \sin(x + \pi)$. To graph this equation, shift the graph of $y = \sin(x)$ left π .
- Substitute $2x$ for x in the equation $y = \sin(x + \pi)$ to get $y = \sin(2x + \pi)$. To graph this equation, H-shrink the graph of $y = \sin(x + \pi)$ by 2.
- To graph $y = 3\sin(2x + \pi)$, V-stretch the graph of $y = \sin(2x + \pi)$ by 3.

Ex. 2.7.11:

Find a sequence of equation transformations that will produce the graph of $y = 3\sin(2x + \pi)$ from the graph of $y = \sin(x)$. What is the effect on the graph at each stage? Don't try to sketch the graph.

Solution:

- Start with the equation $y = \sin(x)$.
- Substitute $x + \pi$ for x to get the new equation $y = \sin(x + \pi)$. To graph this equation, shift the graph of $y = \sin(x)$ left π .
- Substitute $2x$ for x in the equation $y = \sin(x + \pi)$ to get $y = \sin(2x + \pi)$. To graph this equation, H-shrink the graph of $y = \sin(x + \pi)$ by 2.
- To graph $y = 3\sin(2x + \pi)$, V-stretch the graph of $y = \sin(2x + \pi)$ by 3.
- Answer:
    - Shift the graph of $y = \sin(x)$ left π;
    - H-shrink the result by 2;
    - V-stretch by 3.

2.7.5 Section 2.7 Quiz

Ex. 2.7.1: Compare the graphs and equations of the circles $x^2 + y^2 = 4$ and $(x-3)^2 + (y+1)^2 = 4$.

Ex. 2.7.2:Compare the graphs and equations of the circles $x^2 + y^2 = 4$ and $(\frac{x}{2})^2 + (3y)^2 = 4$.

Ex. 2.7.3: How do you obtain the graph of $y = -7x^2$ from the graph of $y = x^2$?

Ex. 2.7.4: How do you obtain the graph of $y = \frac{x^2}{4}$ from the graph of $y = x^2$?

Ex. 2.7.5:What sequence of transformations could you apply to the equation $y = x^2$ to obtain the equation $y = 4 - x^2$? What would be the effect on the graph at each stage?

Ex. 2.7.6: What is the equation of the graph obtained by shifting the graph of $y = x^2$ up 4 units, and then reflecting the resulting graph across the y-axis?

Ex. 2.7.7: What sequence of equation transformations changes $y = x^2$ to $y = (x+4)^2 - 5$?

Ex. 2.7.8: What sequence of equation transformations changes the graph of $y = x^2$ to the graph of $y = 4 - (x+2)^2$?

Ex. 2.7.9: What sequence of equation transformations changes the graph of $y = x^2$ to the graph of graph of $y = -2(2x+1)^2$? What is the effect on the graph at each stage?

Ex. 2.7.10: What sequence of equation transformations applied to $y = x$ yields equation $y = 3x + 5$? What is the effect on the graph at each stage?

Ex. 2.7.11: Find a sequence of equation transformations that will produce the graph of $y = 3\sin(2x+5)$ from the graph of $y = \sin(x)$. What is the effect on the graph at each stage?

Chapter 2 Section 8: Inverse functions

- 2.8.1: Formulas and functions, forward and backward
- 2.8.2: Composing a function and its inverse
- 2.8.3: The horizontal line test
- 2.8.4: Restricting the domain
- 2.8.5: Graphing f and f^{-1} on the same grid
- 2.8.6: Section 2.8 Quiz review

Section 2.8 Preview: Definitions and Procedures

- Definition 2.8.1: Composing f and its inverse function f^{-1}
- Definition 2.8.2: A function $y = f(x)$ is *one-to-one* if:
- Definition 2.8.3: *Horizontal line test*
- Definition 2.8.4: A function f is *invertible*] if

- Procedure 2.8.1: How to find the inverse of the function $y = f(x)$
- Procedure 2.8.2: How to test if functions f and g are inverse functions
- Procedure 2.8.3: How to convert the graph of f to the graph of f^{-1}

2.8.1 Formulas and functions, forward and backward

The formula $y = x^2$ shows how y depends on x. The function definition $f(x) = x^2$ shows how $f(x)$ depends on x. What's the difference? None, really. Then why the big deal about functions?

Basically, function notation makes it easy to keep track of different ways that y can depend on x. If you say $y = x^2$ squares an input and $y = x^3$ cubes an input, that's confusing because y means two different things. It's better to say $f(x) = x^2$ squares an input and $g(x) = x^3$ cubes an input.

The name of this section is "inverse functions." Probably it should be called "reverse functions." If a function f converts an input into an output, its inverse function f^{-1} does the reverse and converts an output of f back to an input of f. However, not every function has an inverse.

Warning: Although the inverse of the number 3 is its reciprocal $\frac{1}{3}$, the inverse of function f has nothing to do with the reciprocal function $\frac{1}{f}$.

Example 1: Suppose $y = \frac{3x+2}{5}$. Write a formula that expresses x in terms of y.

Solution: Solve $y = \frac{3x+2}{5}$ for x as follows:

The original formula: $y = \frac{3x+2}{5}$

Multiply both sides by 5: $5y = 3x + 2$

Subtract 2 from both sides: $5y - 2 = 3x$

Divide both sides by 3: $\frac{5y-2}{3} = x$

Write as a formula for x: $\boxed{x = \frac{5y-2}{3}}$

How to find the inverse of the function $y = f(x)$

Solve the equation $y = f(x)$ for x to obtain $x = f^{-1}(y)$.

- If the formula $x = f^{-1}(y)$ does not give exactly one value of y for every x in the domain of f, you will need to restrict that domain. This will be explained later.
- If the inverse function is supposed to be written with input variable x, exchange x and y in $x = f^{-1}(y)$ to obtain the inverse function $y = f^{-1}(x)$

Example 2: Suppose $f(x) = \frac{3x+2}{5}$. Find the inverse function $f^{-1}(x)$.

Solution: This problem uses function language, but the algebra is the same as in Example 1.

- Write y for $f(x)$ to get the formula $y = \frac{3x+2}{5}$.
- Solve $y = \frac{3x+2}{5}$ for x exactly as in Example 1 to get $x = \frac{5y-2}{3}$

  This shows how the input x to function f depends on its output, and we write
- $x = f^{-1}(y) = \frac{5y-2}{3}$. This is a formula for the inverse function f^{-1} using y as the input.

But the question asks for $f^{-1}(x)$: use x as the input. Therefore replace y by x to get

Answer: $\boxed{f^{-1}(x) = \dfrac{5x-2}{3}}$.

In the formulas $y = x + 2$ and $y = \frac{3x+2}{5}$, each output y came from only one input x. This is not always the case.

For example, the two inputs -2 and 2 to $y = x^2$ yield the same output 4. Indeed, for any $y > 0$, solving $y = x^2$ for x gives two solutions: $x = \sqrt{y}$ and $x = -\sqrt{y}$. Thus it seems that the function $f(x) = x^2$ does not have an inverse. Later we will explain how to deal with this issue.

Example 3: Suppose $f(x) = \frac{3x+2}{2x-1}$. Find the inverse function $f^{-1}(x)$.

Solution: The domain of f, not explicitly stated, is all real $x \neq \frac{1}{2}$. Solve $y = \frac{3x+2}{2x-1}$ for x as follows:

The original formula: $y = \frac{3x+2}{2x-1}$

Multiply both sides by $2x-1$: $(2x-1)y = 3x+2$

Multiply out: $2xy - y = 3x + 2$

All terms with x on the left: $2xy - 3x - y = 2$

All other terms on the right: $2xy - 3x = 2 + y$

Factor out x on the left: $x(2y-3) = 2 + y$

Solve for x: If $y \neq \frac{3}{2}$. $x = \frac{2+y}{2y-3} = f^{-1}(y)$

Exchange x and y: $y = \frac{2+x}{2x-3} = f^{-1}(x)$ if $x \neq \frac{3}{2}$

Answer: $\boxed{f^{-1}(x) = \dfrac{2+x}{2x-3}}$

2.8.2: Composing a function and its inverse

f^{-1} uses as its input the output of f and returns as its output the input of f. Thus f and f^{-1} undo each other.

Composing f and its inverse function f^{-1}

- If x is in the domain of f, then $f^{-1}(f(x)) = x$.
- If $y = f(x)$, then $x = f^{-1}(y)$
- If x is in the domain of f^{-1}, then $f(f^{-1}(x)) = x$.

Example 4: In Example 2, we found that the inverse of $f(x) = \dfrac{3x+2}{5}$ is $f^{-1}(x) = \dfrac{5x-2}{3}$

Verify that a) $f^{-1}(f(x)) = x$ and b) $f(f^{-1}(x)) = x$.

Solution to a):

$$f^{-1}(f(x)) = f^{-1}(\tfrac{3x+2}{5}) = \frac{5(\frac{3x+2}{5})-2}{3} = \frac{3x+2-2}{3} = \frac{3x}{3} = x$$

Solution to b):

$$f(f^{-1}(x)) = f(\tfrac{5x-2}{3}) = \frac{3(\frac{5x-2}{3})+2}{5} = \frac{5x-2+2}{5} = \frac{5x}{5} = x$$

Now let's study simple functions with restricted domains.
Suppose $f(x) = x^2$ with domain $\{2, 3, 4\}$.Then
$f(2) = 2^2 = 4$. Therefore $f^{-1}(4) = 2$.
$f(3) = 3^2 = 9$. Therefore $f^{-1}(9) = 3$.
$f(4) = 4^2 = 16$. Therefore $f^{-1}(16) = 4$.

We can represent the functions f and f^{-1} in tabular form:

a	$f(a) = b = x^2$	Point (a, b) on graph of f
2	4	$(2, 4)$
3	9	$(3, 9)$
4	16	$(4, 16)$

b	$f^{-1}(b) = a$	Point (b, a) on graph of f^{-1}
4	2	$(4, 2)$
9	3	$(9, 3)$
16	4	$(16, 4)$

Each red number is an input of f and an output of f^{-1}. Each blue number is an input of f^{-1} and an output of f.

f and f^{-1} exchange inputs and outputs. In other words, $f(a) = b$ if and only if $f^{-1}(b) = a$.

Example 5: If f is invertible and $f(3) = 8$, find $f^{-1}(8)$.
Solution: $f^{-1}(8) = f^{-1}(f(3)) = 3$.

To test if functions f and g are inverse functions, verify all of the following:

- the domain of f is the range of g;
- the domain of g is the range of f;
- for all x in the domain of f: $g(f(x)) = x$; and
- for all x in the domain of g: $f(g(x)) = x$.

Example 6: In Example 3, we found that the inverse of $f(x) = \frac{3x+2}{2x-1}$ is $f^{-1}(x) = \frac{2+x}{2x-3}$.

Verify that

- a) $f^{-1}(f(x)) = x$ and
- b) $f(f^{-1}(x)) = x$.

Solution to a): Start with $f^{-1}(f(x))$

Use the definition of f: $= f^{-1}\left(\frac{3x+2}{2x-1}\right)$

In $f^{-1}(x) = \frac{2+x}{2x-3}$, substitute $\frac{3x+2}{2x-1}$ for x to obtain :

$$\frac{2+\frac{3x+2}{2x-1}}{2\left(\frac{3x+2}{2x-1}\right)-3}$$

Multiply by $\frac{2x-1}{2x-1}$: $$= \frac{(2x-1)\left(2+\frac{3x+2}{2x-1}\right)}{(2x-1)\left(2\left(\frac{3x+2}{2x-1}\right)-3\right)}$$

Distribute $(2x-1)$ and cancel: $$= \frac{(2x-1)\cdot 2 + \cancel{(2x-1)}\left(\frac{3x+2}{\cancel{2x-1}}\right)}{\cancel{(2x-1)}\cdot 2\left(\frac{3x+2}{\cancel{2x-1}}\right) - (2x-1)3}$$

Rewrite and simplify: $$= \frac{(2x-1)2+3x+2}{2(3x+2)-3(2x-1)}$$

Success because the answer is x: $$= \frac{4x-2+3x+2}{6x+4-6x+3} = \frac{7x}{7} \boxed{= x}$$

Solution to b): Try it yourself.

2.8.3: The horizontal line test for one-to-one functions

A function $y = f(x)$ is *one-to-one* means:

for each output, there is only one input.

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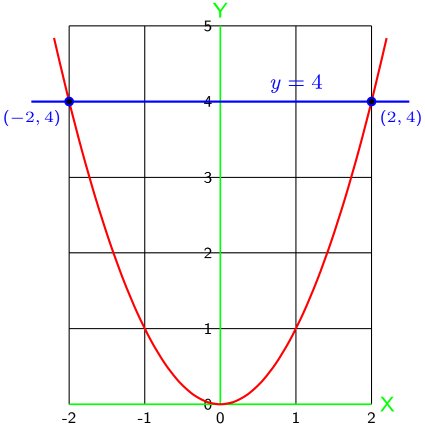
Horizontal line test

- A function is one-to-one provided each horizontal line misses the function's graph or meets it at exactly one point.
- The function is not one-to-one if any horizontal line meets the function's graph more than once.



The above graph of $f(x) = x^2$ with domain $-2 \leq x \leq 2$ fails the horizontal line test because (at least) one horizontal line meets the graph more than once.

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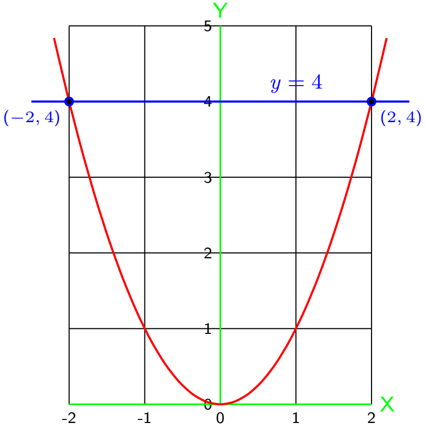
A function f is *invertible*

if it is one-to-one (i.e., it passes the horizontal line test). Then f has an inverse function, called f^{-1}, defined as follows:

- The domain of f^{-1} is the range of f.
- If $f(x) = y$, then $f^{-1}(y) = x$.

If f fails the horizontal line test, it is not invertible.



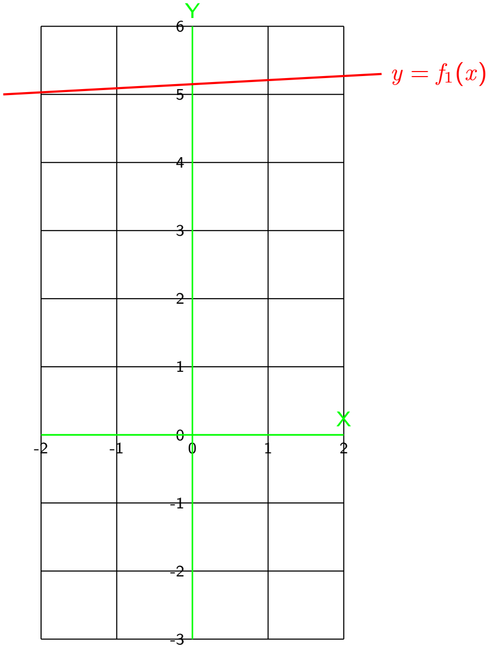
The above graph of $f(x) = x^2$ with domain $-2 \leq x \leq 2$ fails the horizontal line test because (at least) one horizontal line meets the graph more than once. Thus f does not have an inverse.

Example 7: Which graphs are the graphs of one-to-one functions?

Solution: Only the top graph $y = f_1(x)$ passes the horizontal line test. That's because a horizontal line meets a slanted line segment at most once.

For all the other graphs, it's easy to find a horizontal line that meets the graph twice, and so those graphs are not the graphs of one-to-one functions.

It follows that only the top function $y = f_1(x)$ has an inverse. Functions f_2, f_3, f_4, f_5 are not invertible. That's because each of their graphs has an output (a y-coordinate) that comes from more than one input (an x-coordinate).



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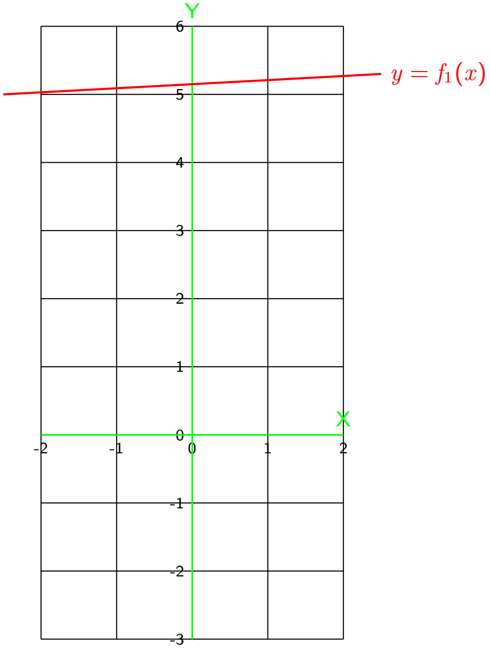

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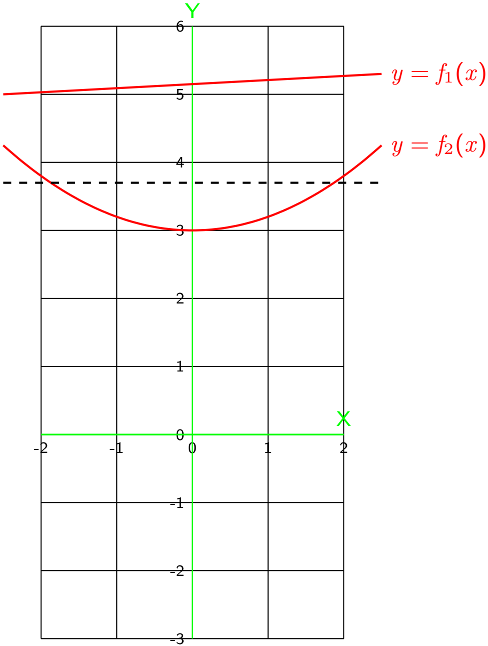

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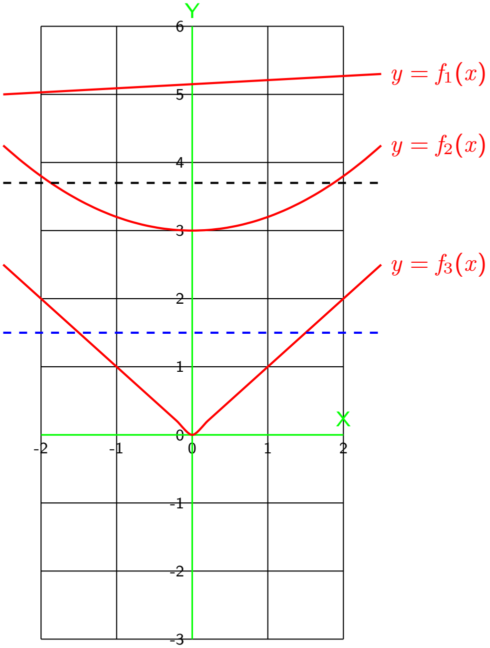

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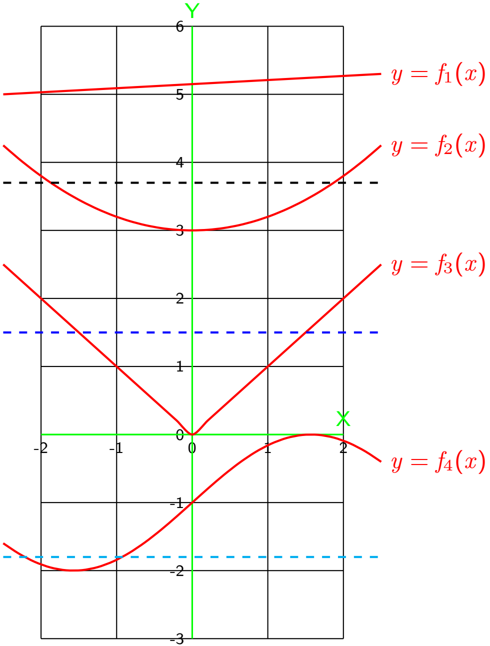

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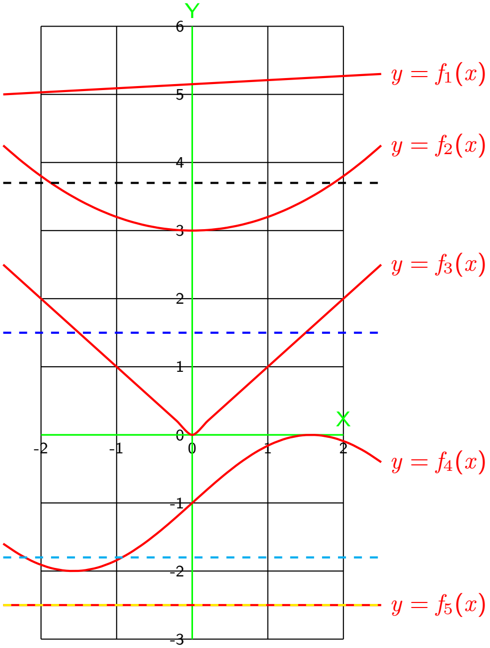
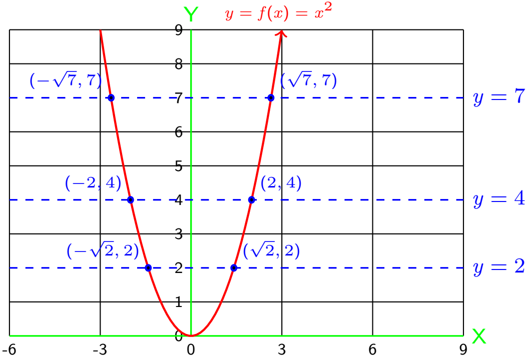


2.8.4: Restricting the domain produces a one-to-one function

- The graph of $y = f(x) = x^2$ with domain all x does not pass the horizontal line test.

  See the graph at the right. Indeed, for any $c > 0$,the horizontal line $y = c$ meets the graph twice, at $x = \pm\sqrt{c}$. As a result, function f does not have an inverse.



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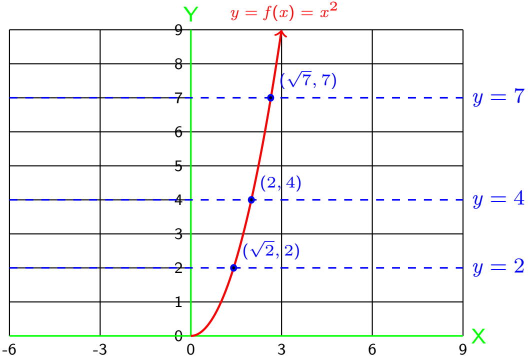

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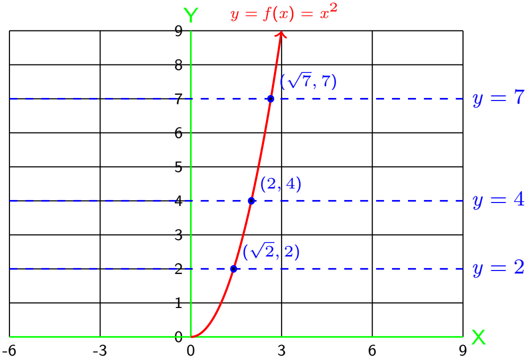

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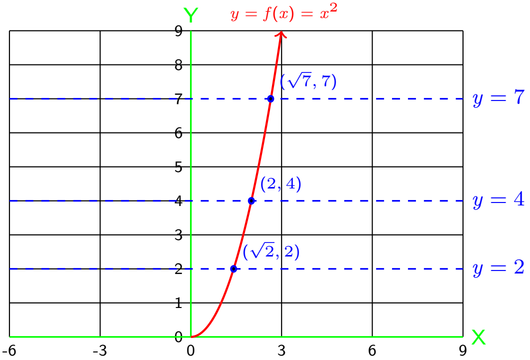
- Now each horizontal line meets the graph just once. Reason: for each $y \geq 0$, equation $y = x^2$ has a single solution with $x \geq 0$, namely $x = \sqrt{y}$.
- Therefore $f(x) = x^2$ with domain $x \geq 0$ does have an inverse function f^{-1}, defined by $f^{-1}(y) = \sqrt{y}$, with domain $y \geq 0$.



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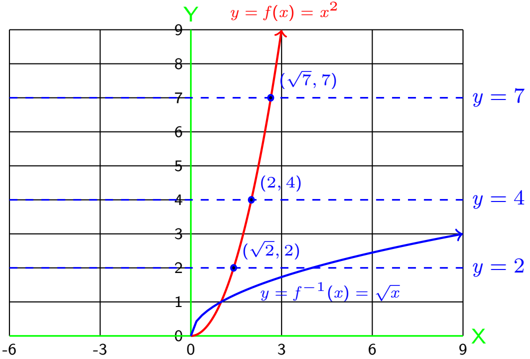
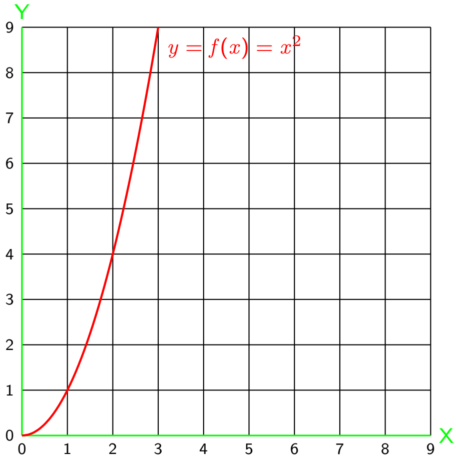
- Therefore $f(x) = x^2$ with domain $x \geq 0$ does have an inverse function f^{-1}, defined by $f^{-1}(y) = \sqrt{y}$, with domain $y \geq 0$.
- To graph f and f^{-1} together, we want x to be the input for both. Therefore we say that the inverse function is $f^{-1}(x) = \sqrt{x}$ with domain $x \geq 0$. See the blue graph at the right.



2.8.5: Graphing f and f^{-1} on the same grid

The next few slides demonstrate the relationship between the graphs of a function and its inverse. First, let's review step by step how input $x = 2$ to the function $f(x) = x^2$ with domain $x \geq 0$ produces output $y = f(2) = 4$.



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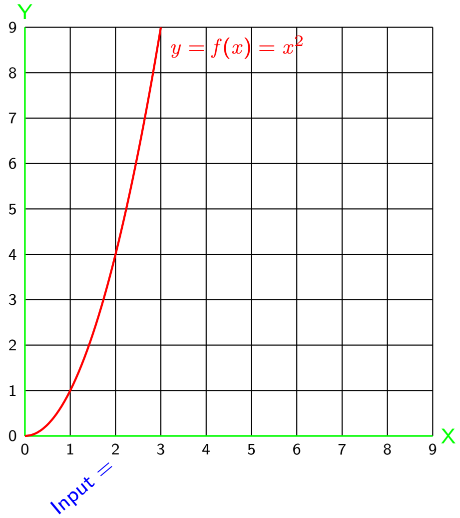
- Start with input $x = 2$ on the x-axis



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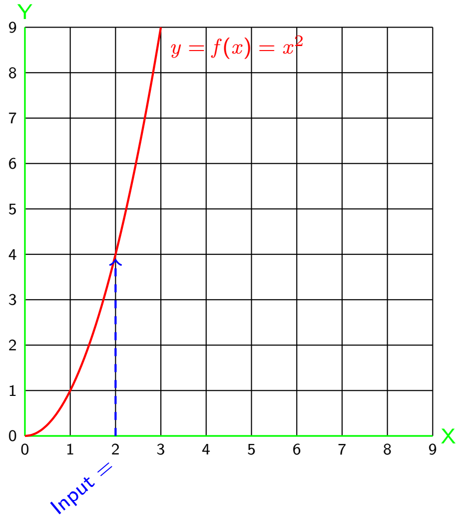
- Start with input $x = 2$ on the x-axis
- Move on the vertical line $x = 2$ up to the graph of $y = f(x) = x^2$.



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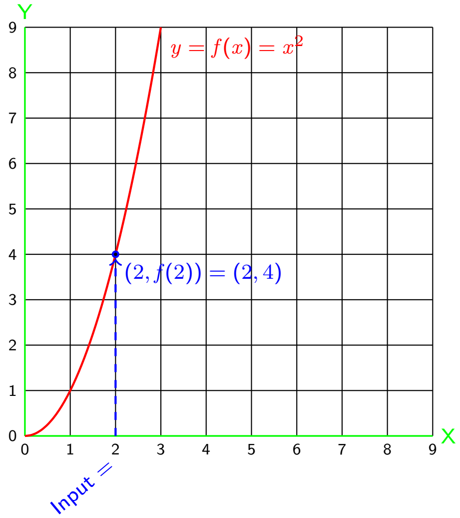
- Start with input $x = 2$ on the x-axis
- Move on the vertical line $x = 2$ up to the graph of $y = f(x) = x^2$.
- Arrive at point $(2, f(2))$ on the graph of f.



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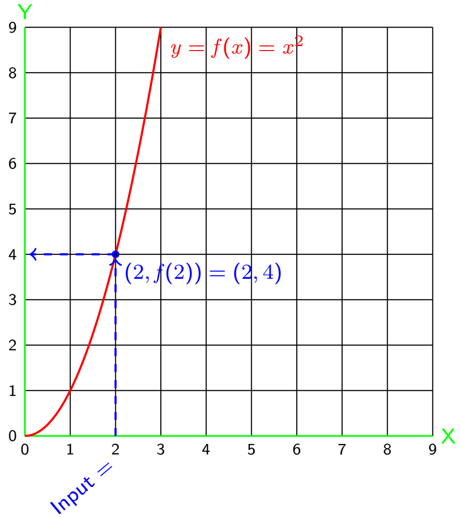
- Arrive at point $(2, f(2))$ on the graph of f.
- Move horizontally to the y-axis.



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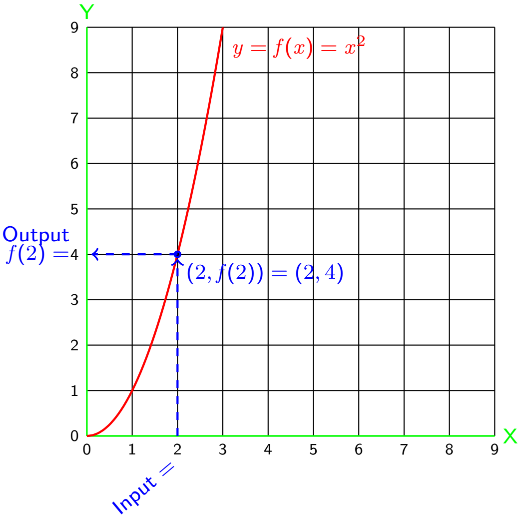
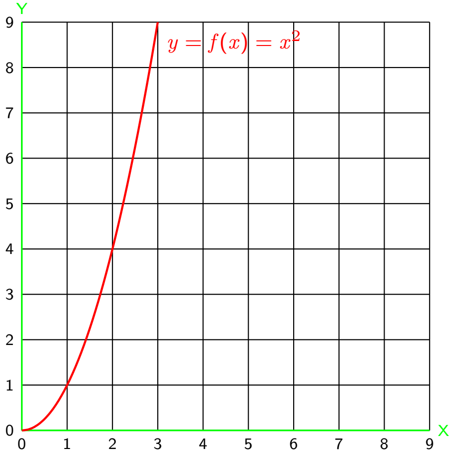
- Arrive at point $(2, f(2))$ on the graph of f.
- Move horizontally to the y-axis.
- Obtain $f(2) = 4$ as the output of the function f when 2 is the input.



To start thinking about the inverse function, start with input 4 on the y-axis.



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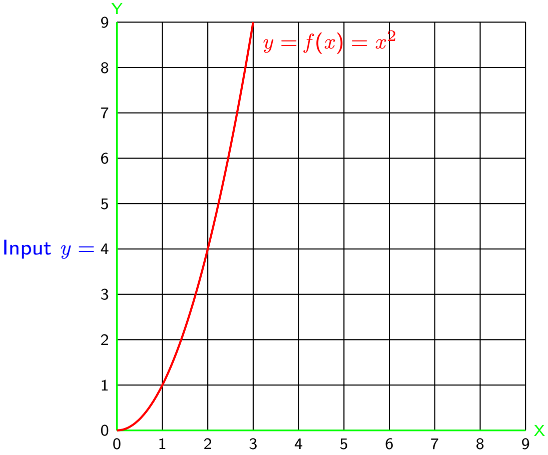

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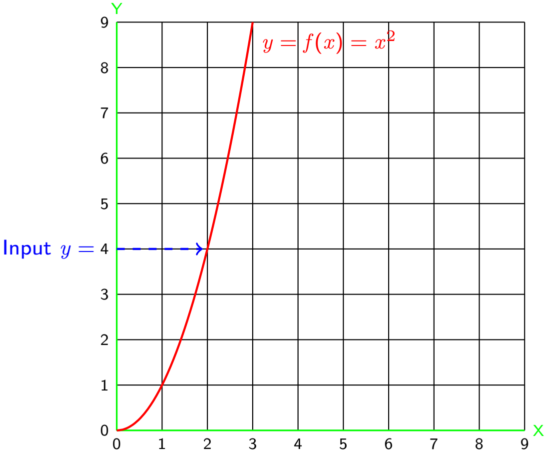
- Move on the horizontal line $y = 4$ to the graph.



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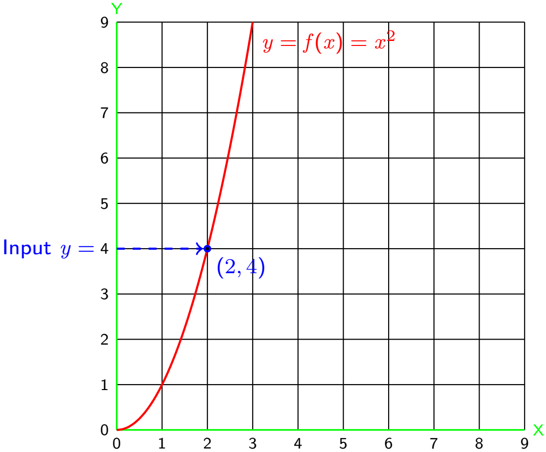
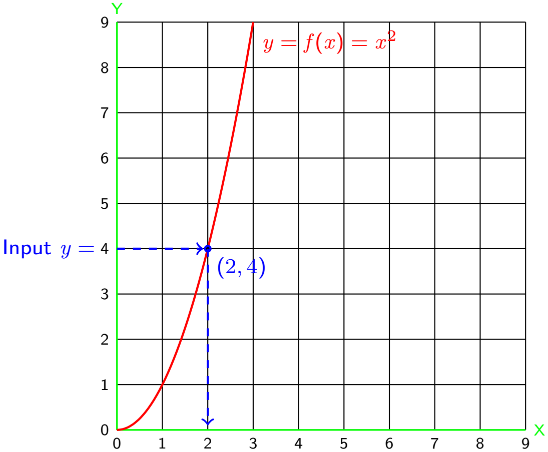
- Move on the horizontal line $y = 4$ to the graph.
- Hit the point $(2, 4)$ on the graph.



To start thinking about the inverse function, start with input 4 on the y-axis.

- Move on the horizontal line $y = 4$ to the graph.
- Hit the point $(2, 4)$ on the graph.
- Move vertically down to the x-axis.



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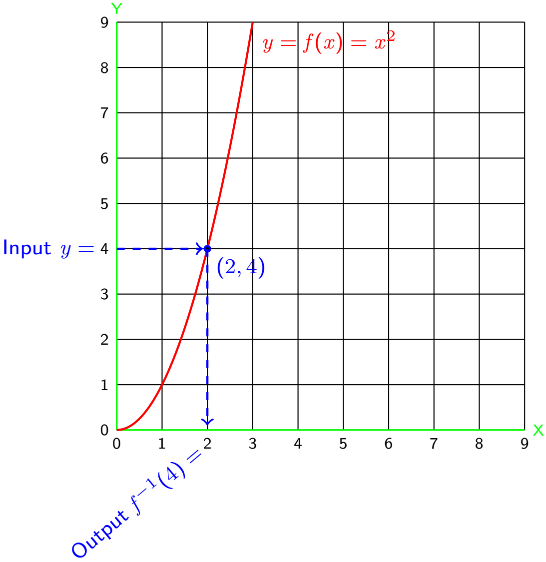
- Move on the horizontal line $y = 4$ to the graph.
- Hit the point $(2, 4)$ on the graph.
- Move vertically down to the x-axis.
- $x = 2 = \sqrt{4}$ is the output of function $f^{-1}$when the input is $y = 4$. Thus $f^{-1}(4) = 2$.



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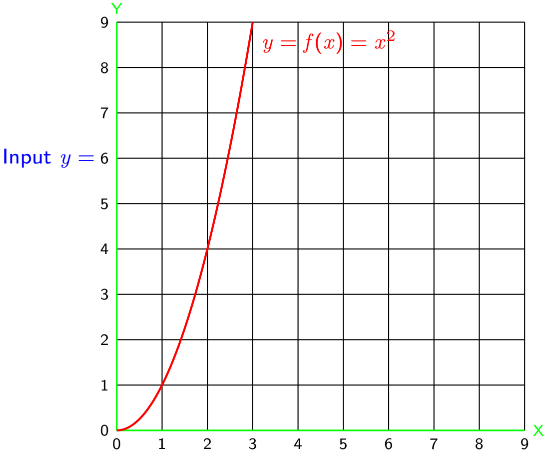
- You can start with any y and go through the same steps.



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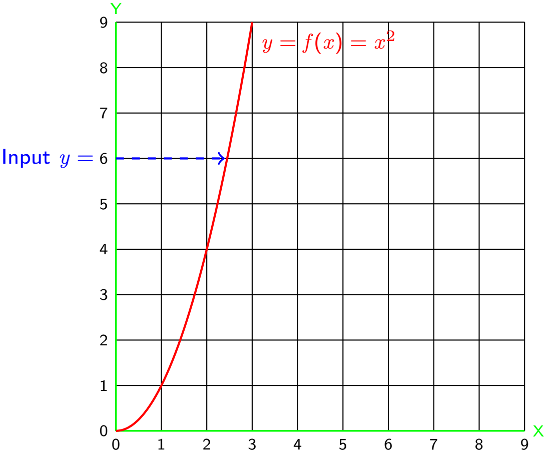
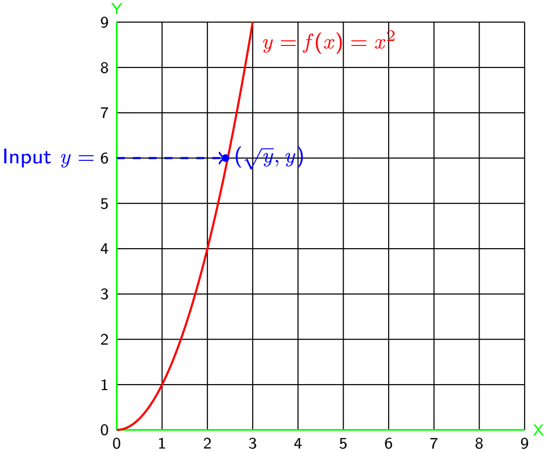
- $x = 2 = \sqrt{4}$ is the output of function $f^{-1}$when the input is $y = 4$. Thus $f^{-1}(4) = 2$.
- You can start with any y and go through the same steps.
- Move horizontally to the graph.



To start thinking about the inverse function, start with input 4 on the y-axis.

- Move on the horizontal line $y = 4$ to the graph.
- Hit the point $(2, 4)$ on the graph.
- Move vertically down to the x-axis.
- $x = 2 = \sqrt{4}$ is the output of function $f^{-1}$when the input is $y = 4$. Thus $f^{-1}(4) = 2$.
- You can start with any y and go through the same steps.
- Move horizontally to the graph.
- Hit the point $(\sqrt{y}, y)$ on the graph



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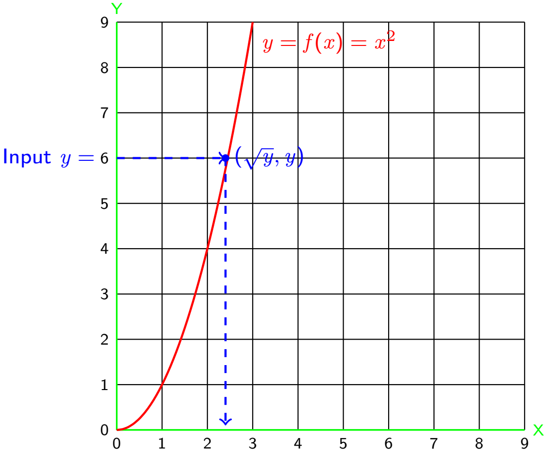

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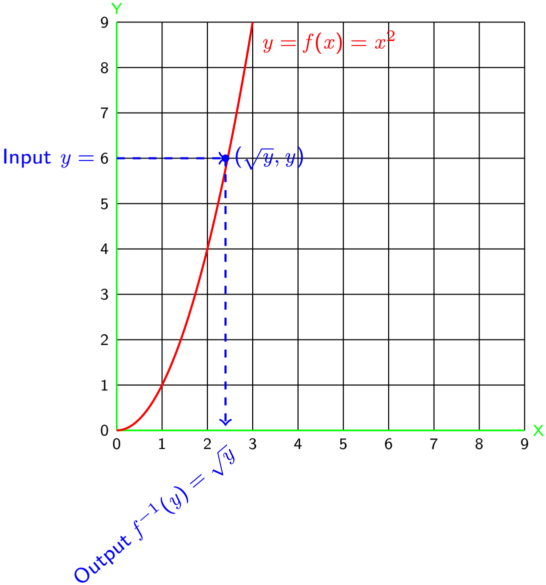
- You can start with any y and go through the same steps.
- Move horizontally to the graph.
- Hit the point $(\sqrt{y}, y)$ on the graph
- Move vertically down to the x-axis.
- Then $f^{-1}(y) = \sqrt{y}$



To start thinking about the inverse function, start with input 4 on the y-axis.

- Move on the horizontal line $y = 4$ to the graph.
- Hit the point $(2, 4)$ on the graph.
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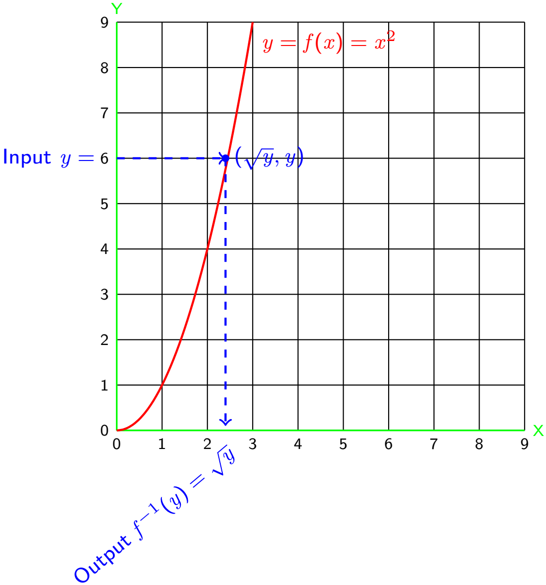
- You can start with any y and go through the same steps.
- Move horizontally to the graph.
- Hit the point $(\sqrt{y}, y)$ on the graph
- Move vertically down to the x-axis.
- Then $f^{-1}(y) = \sqrt{y}$

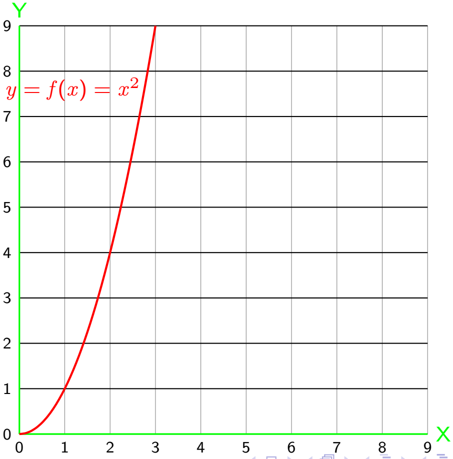
The only problem with this is that we would like the input of f^{-1} to be on the horizontal x-axis. We'll fix this on the next slide.



How to convert the graph of f to the graph of f^{-1}

For every point $(x, f(x))$ on the graph of f, plot the point $(f(x), x)$.

Example: The red graph shows $f(x) = x^2$. with domain $0 \leq x \leq 3$ and range $0 \leq y \leq 9$. Plot the graph of f^{-1}



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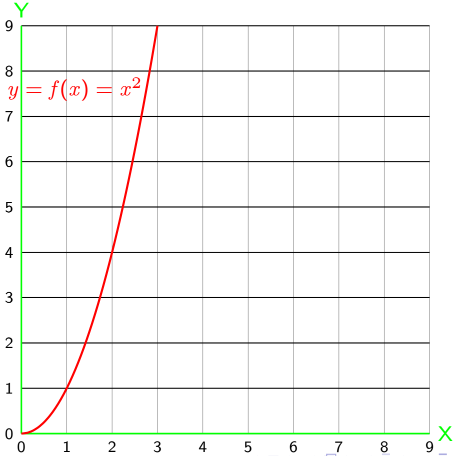

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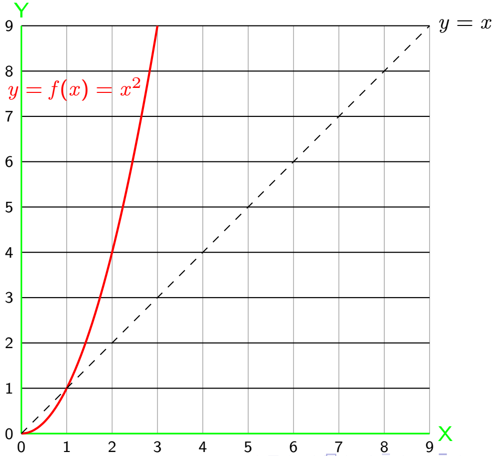
- Reflecting each point on the graph of f through the diagonal line $y = x$ will yield a point on the graph of f^{-1} with coordinates interchanged. Click slowly and watch the grid!



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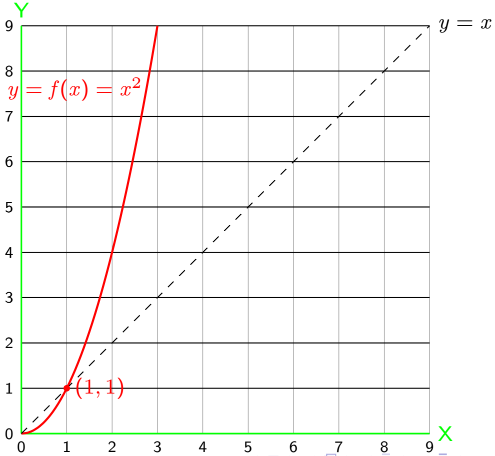
- Reflecting each point on the graph of f through the diagonal line $y = x$ will yield a point on the graph of f^{-1} with coordinates interchanged. Click slowly and watch the grid!
- Interchanging the coordinates of point $(1, 1)$ on the graph of f gives



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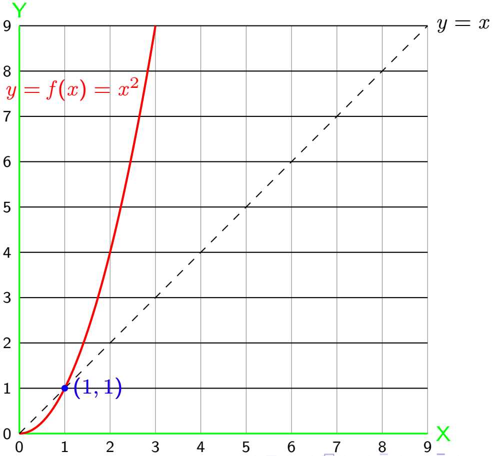
- Reflecting each point on the graph of f through the diagonal line $y = x$ will yield a point on the graph of f^{-1} with coordinates interchanged. Click slowly and watch the grid!
- Interchanging the coordinates of point $(1, 1)$ on the graph of f gives the same point $(1, 1)$ on the graph of f^{-1}.



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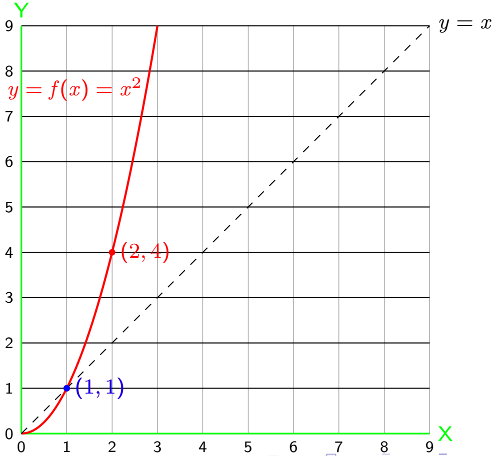
- Reflecting each point on the graph of f through the diagonal line $y = x$ will yield a point on the graph of f^{-1} with coordinates interchanged. Click slowly and watch the grid!
- Interchanging the coordinates of point $(1, 1)$ on the graph of f gives the same point $(1, 1)$ on the graph of f^{-1}.
- Interchanging the coordinates of point $(2, 4)$ on the graph of f



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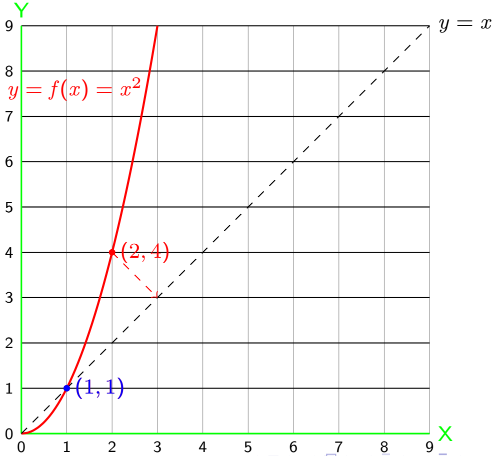
- Reflecting each point on the graph of f through the diagonal line $y = x$ will yield a point on the graph of f^{-1} with coordinates interchanged. Click slowly and watch the grid!
- Interchanging the coordinates of point $(1, 1)$ on the graph of f gives the same point $(1, 1)$ on the graph of f^{-1}.
- Interchanging the coordinates of point $(2, 4)$ on the graph of f by reflecting through $y = x$



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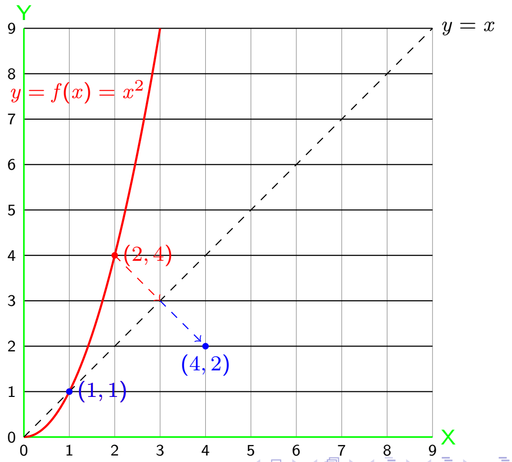
- Reflecting each point on the graph of f through the diagonal line $y = x$ will yield a point on the graph of f^{-1} with coordinates interchanged. Click slowly and watch the grid!
- Interchanging the coordinates of point $(1, 1)$ on the graph of f gives the same point $(1, 1)$ on the graph of f^{-1}.
- Interchanging the coordinates of point $(2, 4)$ on the graph of f by reflecting through $y = x$ gives point $(4, 2)$ on the graph of f^{-1}.



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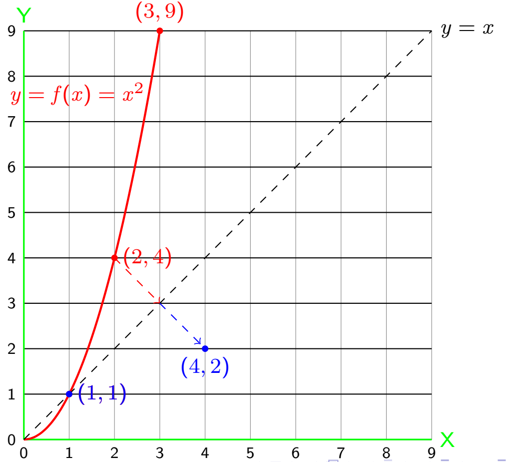
- Interchanging the coordinates of point $(3, 9)$ on the graph of f



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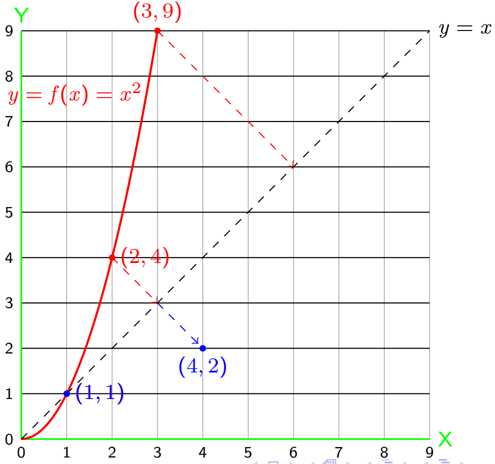

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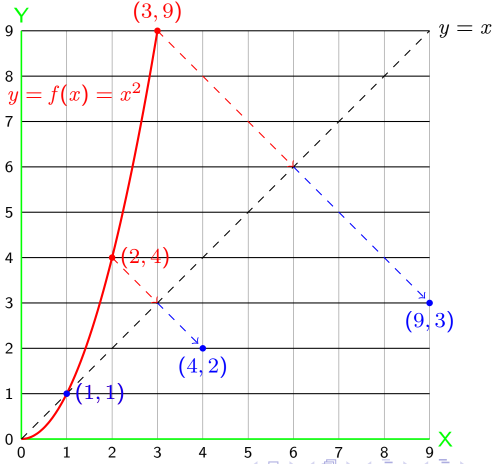

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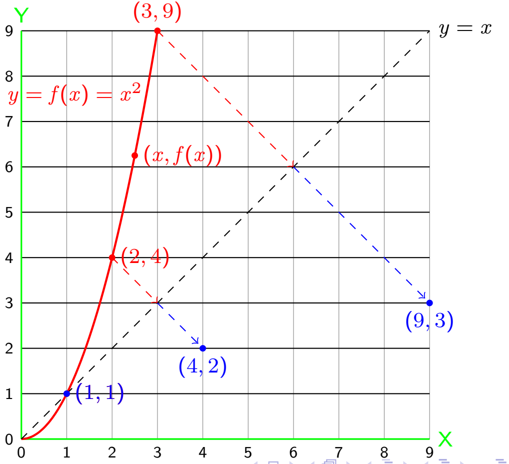
- Interchanging the coordinates of point $(3, 9)$ on the graph of f by reflecting through $y = x$ gives point $(9, 3)$ on the graph of f^{-1}.

- Each point $(x, f(x)) = (x, x^2)$ on the graph of f



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For every point $(x, f(x))$ on the graph of f, plot the point $(f(x), x)$.

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Solution: The graph of $f^{-1}(x) = \sqrt{x}$ will have domain $0 \leq x \leq 9$ and range $0 \leq y \leq 3$.

- Reflecting each point on the graph of f through the diagonal line $y = x$ will yield a point on the graph of f^{-1} with coordinates interchanged. Click slowly and watch the grid!
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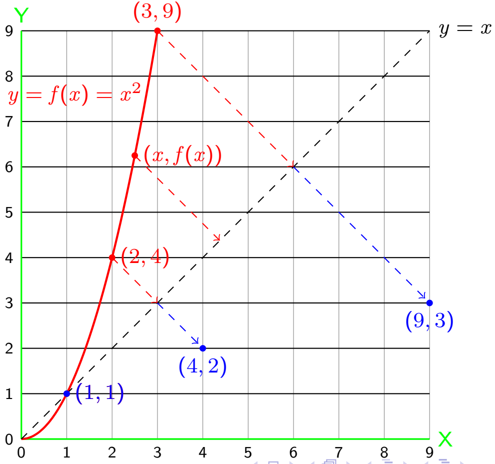
- Interchanging the coordinates of point $(3, 9)$ on the graph of f by reflecting through $y = x$ gives point $(9, 3)$ on the graph of f^{-1}.

- Each point $(x, f(x)) = (x, x^2)$ on the graph of f is reflected across the diagonal $y = x$,



How to convert the graph of f to the graph of f^{-1}

For every point $(x, f(x))$ on the graph of f, plot the point $(f(x), x)$.

Example: The red graph shows $f(x) = x^2$. with domain $0 \le x \le 3$ and range $0 \le y \le 9$. Plot the graph of f^{-1}
Solution: The graph of $f^{-1}(x) = \sqrt{x}$ will have domain $0 \le x \le 9$ and range $0 \le y \le 3$.

- Reflecting each point on the graph of f through the diagonal line $y = x$ will yield a point on the graph of f^{-1} with coordinates interchanged. Click slowly and watch the grid!
- Interchanging the coordinates of point $(1, 1)$ on the graph of f gives the same point $(1, 1)$ on the graph of f^{-1}.
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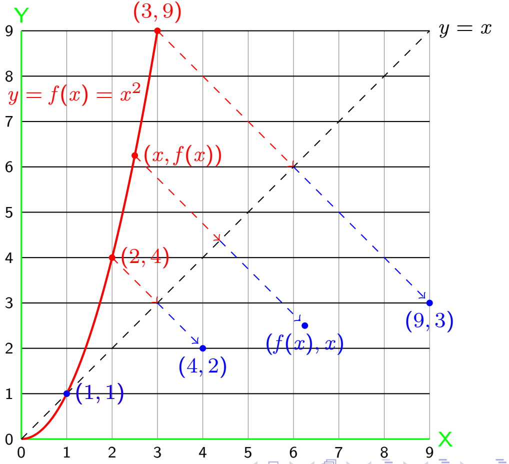
- Interchanging the coordinates of point $(3, 9)$ on the graph of f by reflecting through $y = x$ gives point $(9, 3)$ on the graph of f^{-1}.

- Each point $(x, f(x)) = (x, x^2)$ on the graph of f is reflected across the diagonal $y = x$, to the point $(f(x), x) = (x^2, x) = (x^2, \sqrt{x^2})$ on the graph of f^{-1}. This point is on the graph of f^{-1} because $f^{-1}(f(x)) = x$.



How to convert the graph of f to the graph of f^{-1}

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Solution: The graph of $f^{-1}(x) = \sqrt{x}$ will have domain $0 \leq x \leq 9$ and range $0 \leq y \leq 3$.

- Reflecting each point on the graph of f through the diagonal line $y = x$ will yield a point on the graph of f^{-1} with coordinates interchanged. Click slowly and watch the grid!
- Interchanging the coordinates of point $(1, 1)$ on the graph of f gives the same point $(1, 1)$ on the graph of f^{-1}.
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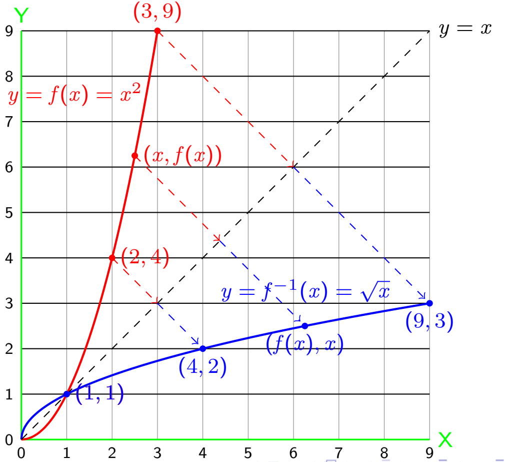
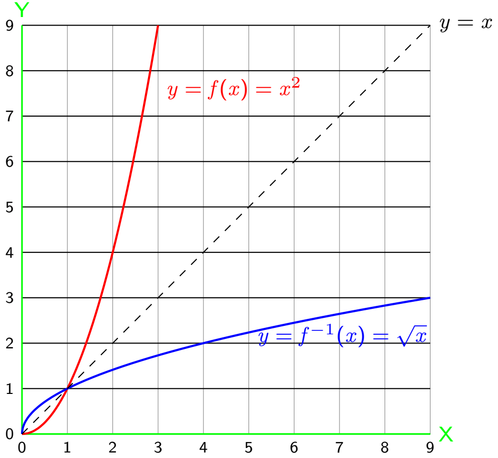
- Interchanging the coordinates of point $(3, 9)$ on the graph of f by reflecting through $y = x$ gives point $(9, 3)$ on the graph of f^{-1}.
- Each point $(x, f(x)) = (x, x^2)$ on the graph of f is reflected across the diagonal $y = x$, to the point $(f(x), x) = (x^2, x) = (x^2, \sqrt{x^2})$ on the graph of f^{-1}. This point is on the graph of f^{-1} because $f^{-1}(f(x)) = x$.
- Draw a smooth curve through the plotted blue points.

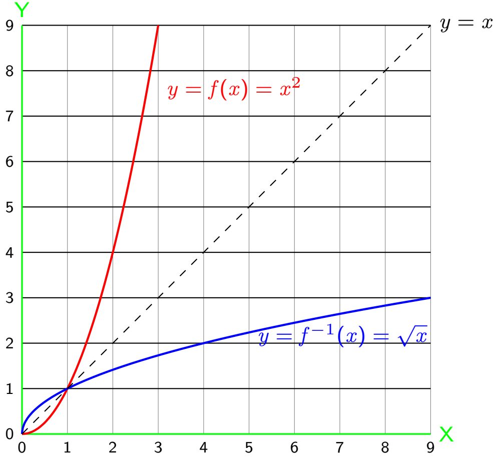


Let's review a bit. On the graph at the right:



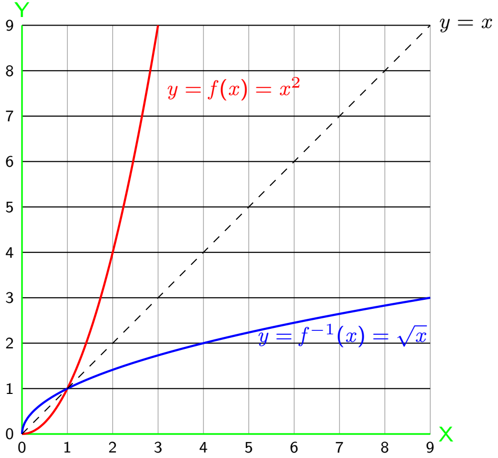
Let's review a bit. On the graph at the right:

- Domain of f = range of $f^{-1} = [0, 3]$.



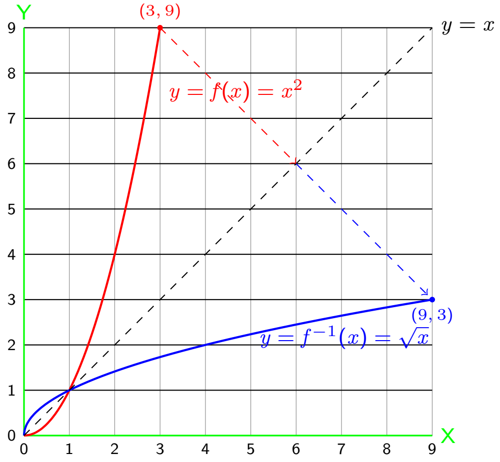
Let's review a bit. On the graph at the right:

- Domain of f = range of $f^{-1} = [0, 3]$.
- Range of f = domain of $f^{-1} = [0, 9]$.



Let's review a bit. On the graph at the right:

- Domain of f = range of $f^{-1} = [0, 3]$.
- Range of f = domain of $f^{-1} = [0, 9]$.
- $(3, 9)$ is on the graph of f and $(9, 3)$ is on the graph of f^{-1}.



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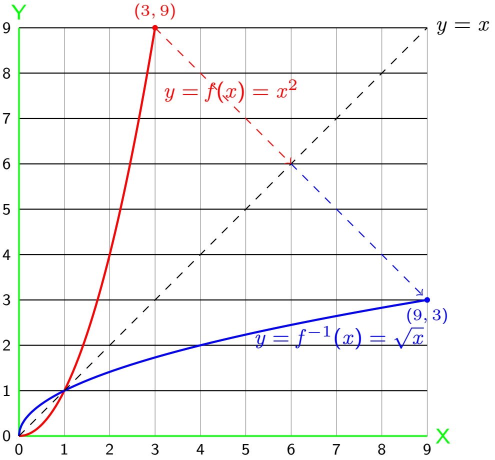
- $9 = f(3)$ and $3 = f^{-1}(9)$.



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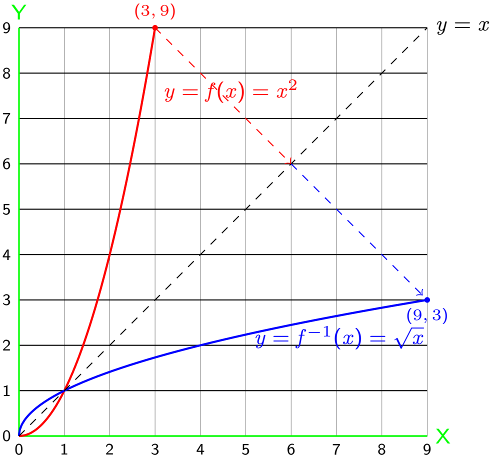
- Domain of f = range of $f^{-1} = [0, 3]$.
- Range of f = domain of $f^{-1} = [0, 9]$.
- $(3, 9)$ is on the graph of f and $(9, 3)$ is on the graph of f^{-1}.
- $9 = f(3)$ and $3 = f^{-1}(9)$.
- $f^{-1}(f(3)) = 3$ and $f(f^{-1}(9)) = 9$.



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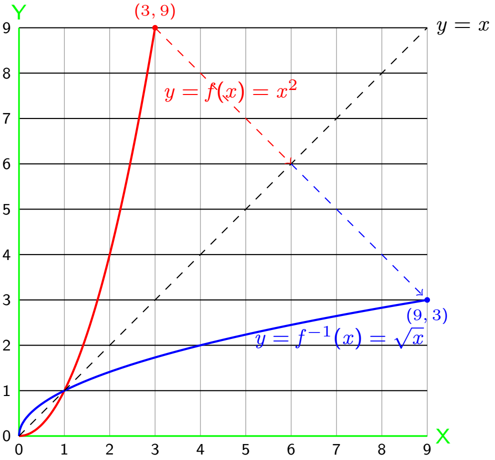
- $f^{-1}(f(3)) = 3$ and $f(f^{-1}(9)) = 9$.

These examples translate to general facts about any one-to-one function f and its inverse f^{-1}. Please remember:



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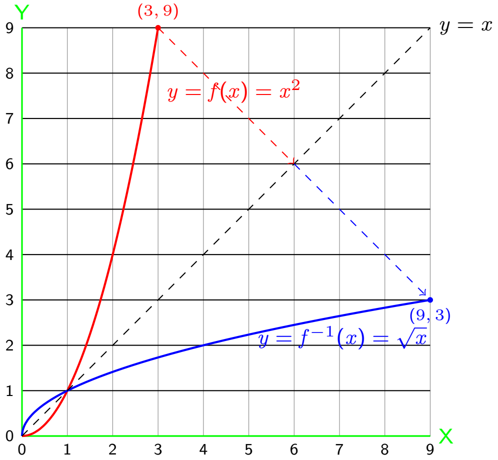
These examples translate to general facts about any one-to-one function f and its inverse f^{-1}. Please remember:

- Domain of f = range of f^{-1}.



Let's review a bit. On the graph at the right:

- Domain of f = range of $f^{-1} = [0, 3]$.
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- $(3, 9)$ is on the graph of f and $(9, 3)$ is on the graph of f^{-1}.
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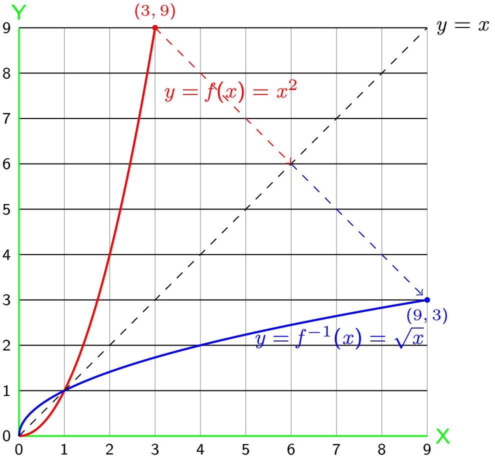

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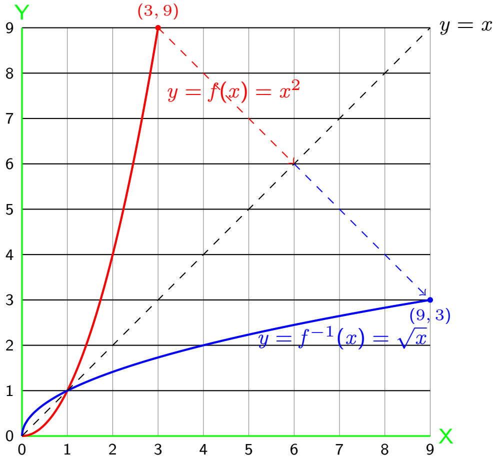

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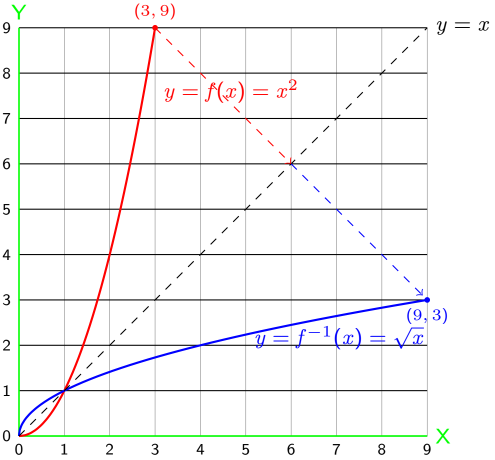

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These examples translate to general facts about any one-to-one function f and its inverse f^{-1}. Please remember:

- Domain of f = range of f^{-1}.
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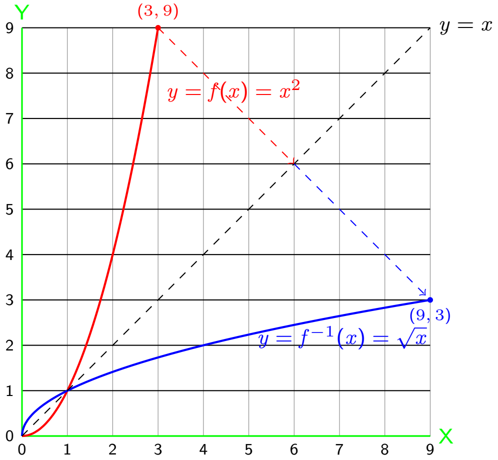
- (x, y) is on the graph of f exactly when (y, x) is on the graph of f^{-1}.
- $y = f(x)$ exactly when $x = f^{-1}(y)$
- For all x in the domain of f, $f^{-1}(f(x)) = x$. For all x in the domain of f^{-1}, $f(f^{-1}(x)) = x$.



Section 2.8 Quiz

Ex. 2.8.1: Suppose $y = \frac{3x+2}{5}$.
Write a formula that expresses x in terms of y.

Ex. 2.8.2: Suppose $f(x) = \frac{3x+2}{5}$.
Find the inverse function $f^{-1}(x)$.

Ex. 2.8.3: Suppose $f(x) = \frac{3x+2}{2x-1}$.
Find the inverse function $f^{-1}(x)$.

Ex. 2.8.4: In Example 2.8.2, verify that
a) $f^{-1}(f(x)) = x$ and b) $f(f^{-1}(x)) = x$.

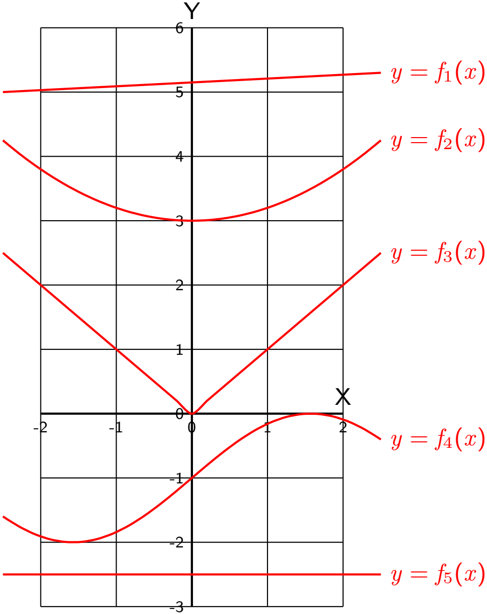
Ex. 2.8.5: If f has inverse function f^{-1} and $f(3) = 8$, find $f^{-1}(8)$.

Ex. 2.8.6: In Example 2.8.3, we found that the inverse

of $f(x) = \frac{3x+2}{2x-1}$ is $f^{-1}(x) = \frac{2+x}{2x-3}$. Verify that

a) $f^{-1}(f(x)) = x$ and b) $f(f^{-1}(x)) = x$.

Ex. 2.8.7: Which graphs at the right are the graphs of one-to-one functions?



Section 2.8 Review: Inverse functions

Ex. 2.8.1: Write a formula that expresses x in terms of y if

- $y = \frac{3x+2}{5} \Rightarrow$
- $y = \frac{4}{5+x} \Rightarrow$
- $y = \frac{6-2x}{5} \Rightarrow$
- $y = \frac{5}{5-3x} \Rightarrow$

Section 2.8 Review: Inverse functions

Ex. 2.8.1: Write a formula that expresses x in terms of y if

- $y = \frac{3x+2}{5} \Rightarrow \ x = \frac{5y-2}{3}$
- $y = \frac{4}{5+x} \Rightarrow \ x = \frac{4-5y}{y}$
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Section 2.8 Review: Inverse functions

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Ex. 2.8.2: Find the inverse function $f^{-1}(x)$. if

- $f(x) = \frac{3x+2}{5} \Rightarrow$
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Section 2.8 Review: Inverse functions

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Ex. 2.8.3: Find the inverse function $f^{-1}(x)$ and state its domain if

- $f(x) = \frac{3x+2}{2x-1} \Rightarrow$
- $f(x) = \frac{3-4x}{2x-1} \Rightarrow$
- $f(x) = \frac{3x}{2x+10} \Rightarrow$
- $f(x) = \frac{2-5x}{2-4x} \Rightarrow$

Section 2.8 Review: Inverse functions

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Section 2.8 Review: Inverse functions

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Section 2.8 Review: Inverse functions

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Section 2.8 Review: Inverse functions

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- $f(x) = \frac{4}{5+x}; f^{-1}(x) = \frac{4-5x}{x}$
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Section 2.8 Review: Inverse functions

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Ex. 2.8.5: If f is invertible and $f(3) = 8, f(-2) = 0, f(14) = 3, f(5) = -2$ find

- $f^{-1}(8) =$ • $f^{-1}(0) =$ • $f^{-1}(f(14)) =$ • $f(f^{-1}(0)) =$.

Section 2.8 Review: Inverse functions

Ex. 2.8.1: Write a formula that expresses x in terms of y if
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- $y = \frac{6-2x}{5} \Rightarrow x = \frac{6-5y}{2}$
- $y = \frac{5}{5-3x} \Rightarrow x = \frac{5y-5}{3y}$

Ex. 2.8.2: Find the inverse function $f^{-1}(x)$. if
- $f(x) = \frac{3x+2}{5} \Rightarrow f^{-1}(x) = \frac{5x-2}{3}$
- $f(x) = \frac{4}{5+x} \Rightarrow f^{-1}(x) = \frac{4-5x}{x}$
- $f(x) = \frac{6-2x}{5} \Rightarrow f^{-1}(x) = \frac{6-5x}{2}$
- $f(x) = \frac{5}{5-3x} \Rightarrow f^{-1}(x) = \frac{5x-5}{3x}$

Ex. 2.8.3: Find the inverse function $f^{-1}(x)$ and state its domain if
- $f(x) = \frac{3x+2}{2x-1} \Rightarrow f^{-1}(x) = \frac{2+x}{2x-3}$ domain $x \neq \frac{3}{2}$
- $f(x) = \frac{3-4x}{2x-1} \Rightarrow f^{-1}(x) = \frac{3+x}{2x+4}$ domain $x \neq -2$
- $f(x) = \frac{3x}{2x+10} \Rightarrow f^{-1}(x) = \frac{10x}{3-2x}$ domain $x \neq \frac{3}{2}$
- $f(x) = \frac{2-5x}{2-4x} \Rightarrow f^{-1}(x) = \frac{2x-2}{4x-5}$ domain $x \neq \frac{5}{4}$

Ex. 2.8.4: For each pair $f(x)$ and $f^{-1}(x)$ in Ex2.8.2 above, verify that a) $f^{-1}(f(x)) = x$ and b) $f(f^{-1}(x)) = x$.

- For $f(x) = \frac{3x+2}{5}$ and $f^{-1}(x) = \frac{5x-2}{3}$ a): $f^{-1}(f(x)) = f^{-1}(\frac{3x+2}{5}) = \frac{5(\frac{3x+2}{5})-2}{3} = \frac{3x+2-2}{3} = \frac{3x}{3} = x$

  b): $f(f^{-1}(x)) = f(\frac{5x-2}{3}) = \frac{3(\frac{5x-2}{3})+2}{5} = \frac{5x-2+2}{5} = \frac{5x}{5} = x$

Do the other pairs f and f^{-1} by yourself!
- $f(x) = \frac{4}{5+x}; f^{-1}(x) = \frac{4-5x}{x}$
- $f(x) = \frac{6-2x}{5}; f^{-1}(x) = \frac{6-5x}{2}$
- $f(x) = \frac{5}{5-3x}; f^{-1}(x) = \frac{5x-5}{3x}$

Ex. 2.8.5: If f is invertible and $f(3) = 8, f(-2) = 0, f(14) = 3, f(5) = -2$ find
- $f^{-1}(8) = 3$
- $f^{-1}(0) = -2.$
- $f^{-1}(f(14)) = 14.$
- $f(f^{-1}(0)) = 0.$

Section 2.8 Review: Inverse functions

Ex. 2.8.1: Write a formula that expresses x in terms of y if

- $y = \frac{3x+2}{5} \Rightarrow x = \frac{5y-2}{3}$ • $y = \frac{4}{5+x} \Rightarrow x = \frac{4-5y}{y}$
- $y = \frac{6-2x}{5} \Rightarrow x = \frac{6-5y}{2}$ • $y = \frac{5}{5-3x} \Rightarrow x = \frac{5y-5}{3y}$

Ex. 2.8.2: Find the inverse function $f^{-1}(x)$. if

- $f(x) = \frac{3x+2}{5} \Rightarrow f^{-1}(x) = \frac{5x-2}{3}$ • $f(x) = \frac{4}{5+x} \Rightarrow f^{-1}(x) = \frac{4-5x}{x}$
- $f(x) = \frac{6-2x}{5} \Rightarrow f^{-1}(x) = \frac{6-5x}{2}$ • $f(x) = \frac{5}{5-3x} \Rightarrow f^{-1}(x) = \frac{5x-5}{3x}$

Ex. 2.8.3: Find the inverse function $f^{-1}(x)$ and state its domain if

- $f(x) = \frac{3x+2}{2x-1} \Rightarrow f^{-1}(x) = \frac{2+x}{2x-3}$ domain $x \neq \frac{3}{2}$ • $f(x) = \frac{3-4x}{2x-1} \Rightarrow f^{-1}(x) = \frac{3+x}{2x+4}$ domain $x \neq -2$
- $f(x) = \frac{3x}{2x+10} \Rightarrow f^{-1}(x) = \frac{10x}{3-2x}$ domain $x \neq \frac{3}{2}$ • $f(x) = \frac{2-5x}{2-4x} \Rightarrow f^{-1}(x) = \frac{2x-2}{4x-5}$ domain $x \neq \frac{5}{4}$

Ex. 2.8.4: For each pair $f(x)$ and $f^{-1}(x)$ in Ex2.8.2 above, verify that a) $f^{-1}(f(x)) = x$ and b) $f(f^{-1}(x)) = x$.

- For $f(x) = \frac{3x+2}{5}$ and $f^{-1}(x) = \frac{5x-2}{3}$ a): $f^{-1}(f(x)) = f^{-1}(\frac{3x+2}{5}) = \frac{5(\frac{3x+2}{5})-2}{3} = \frac{3x+2-2}{3} = \frac{3x}{3} = x$

  b): $f(f^{-1}(x)) = f(\frac{5x-2}{3}) = \frac{3(\frac{5x-2}{3})+2}{5} = \frac{5x-2+2}{5} = \frac{5x}{5} = x$

Do the other pairs f and f^{-1} by yourself!

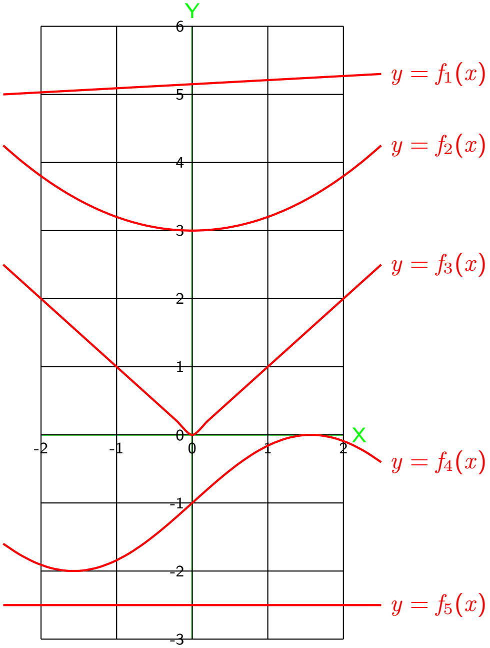
- $f(x) = \frac{4}{5+x}; f^{-1}(x) = \frac{4-5x}{x}$ • $f(x) = \frac{6-2x}{5}; f^{-1}(x) = \frac{6-5x}{2}$ • $f(x) = \frac{5}{5-3x}; f^{-1}(x) = \frac{5x-5}{3x}$

Ex. 2.8.5: If f is invertible and $f(3) = 8, f(-2) = 0, f(14) = 3, f(5) = -2$ find

- $f^{-1}(8) = 3$ • $f^{-1}(0) = -2.$ • $f^{-1}(f(14)) = 14.$ • $f(f^{-1}(0)) = 0.$

Ex. 2.8.6: For each pair $f(x)$ and $f^{-1}(x)$ in Ex2.8.3 above, verify that $a)$ $f^{-1}(f(x)) = x$ and $b)$ $f(f^{-1}(x)) = x$.

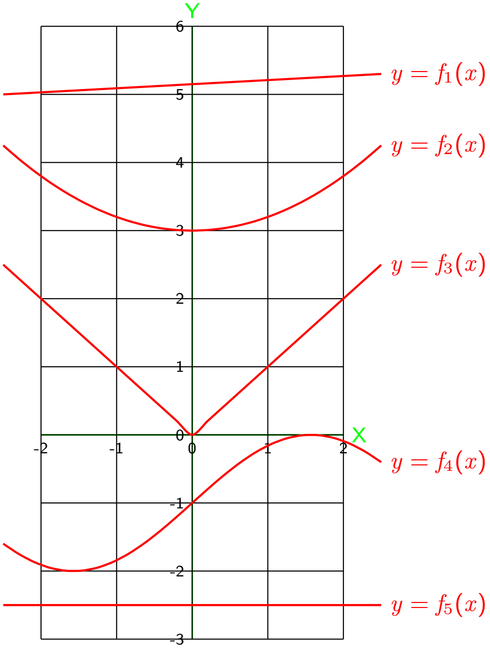
Ex. 2.8.7: Which graphs are the graphs of one-to-one functions?



Ex. 2.8.7: Which graphs are the graphs of one-to-one functions?

Solution: Only the top graph $y = f_1(x)$ passes the horizontal line test. That's because a horizontal line meets a slanted line segment at most once.

For each of the other graphs, draw a horizontal line that meets the graph twice. $\Rightarrow$



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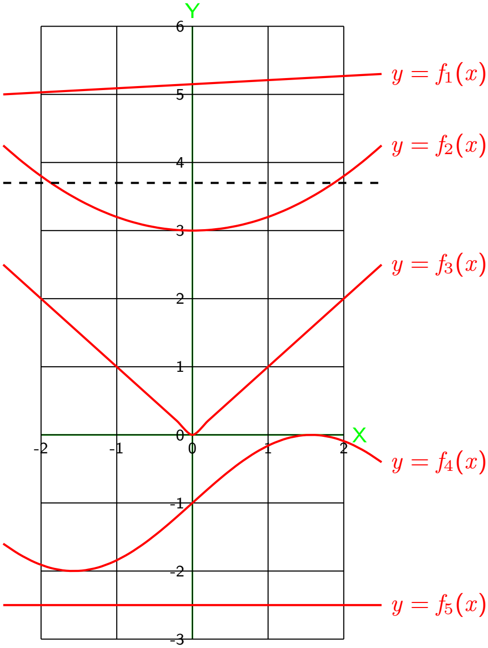

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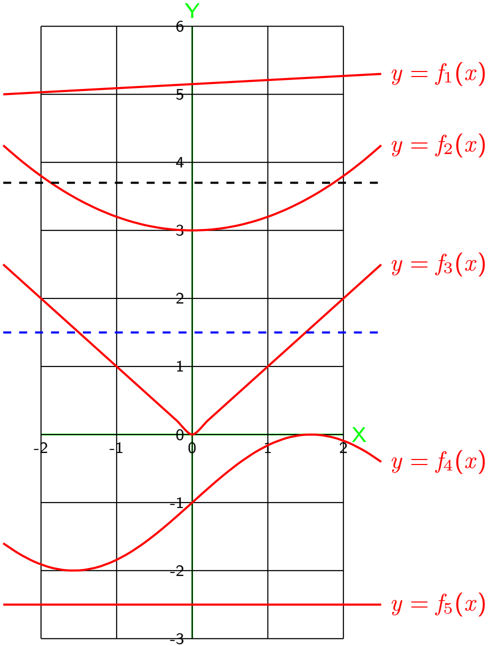

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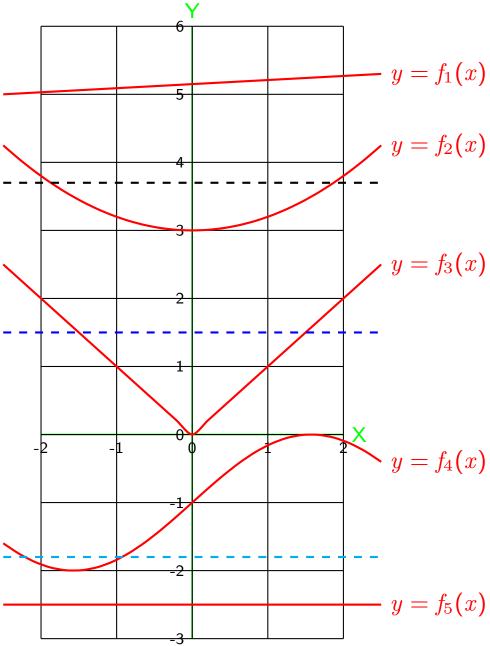
$\Rightarrow$

$\Rightarrow$

$\Rightarrow$



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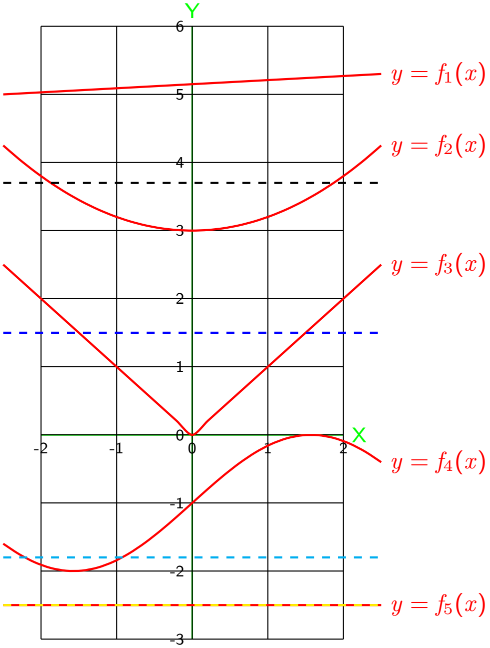
Solution: Only the top graph $y = f_1(x)$ passes the horizontal line test. That's because a horizontal line meets a slanted line segment at most once.
For each of the other graphs, draw a horizontal line that meets the graph twice. ⇒

⇒

⇒

⇒



Ex. 2.8.7: Which graphs are the graphs of one-to-one functions?

Solution: Only the top graph $y = f_1(x)$ passes the horizontal line test. That's because a horizontal line meets a slanted line segment at most once.

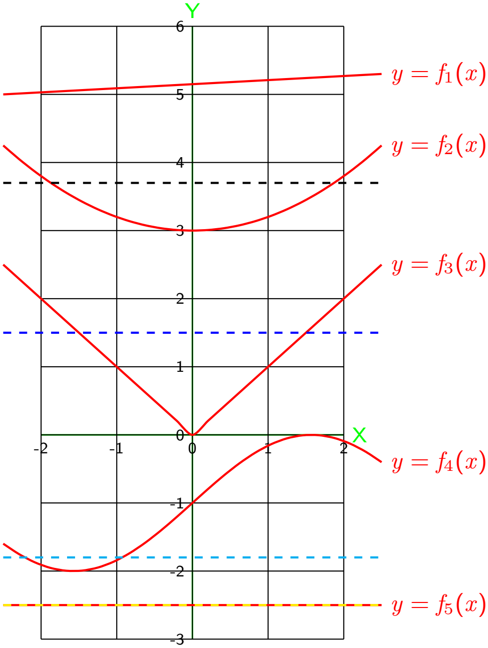
For each of the other graphs, draw a horizontal line that meets the graph twice. $\Rightarrow$

$\Rightarrow$

$\Rightarrow$

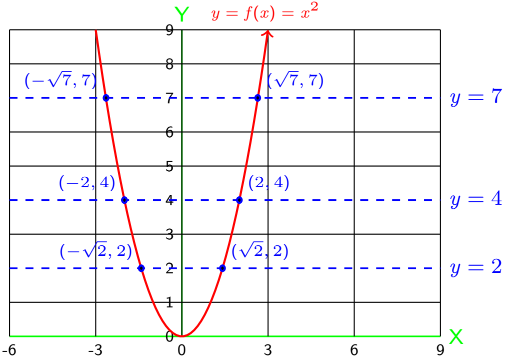
$\Rightarrow$

It follows that only the top function $y = f_1(x)$ has an inverse. Functions f_2, f_3, f_4, f_5 are not invertible. That's because each of their graphs has an output (a y-coordinate) that comes from more than one input (an x-coordinate).



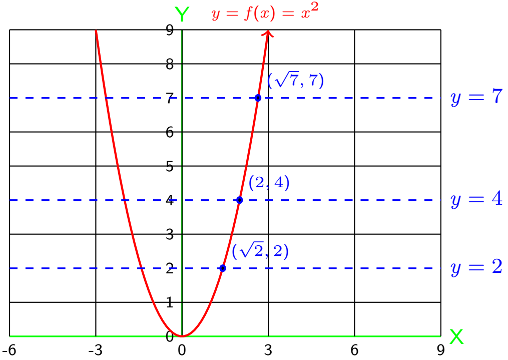
Restricting the domain produces a one-to-one function

- Explain why the graph of $y = f(x) = x^2$ with domain all x does not pass the horizontal line test. $\Rightarrow$



Restricting the domain produces a one-to-one function

- Explain why the graph of $y = f(x) = x^2$ with domain all x does not pass the horizontal line test. $\Rightarrow$ For any $c > 0$,the horizontal line $y = c$ meets the graph twice, at $x = \pm\sqrt{c}$. As a result, function f does not have an inverse.



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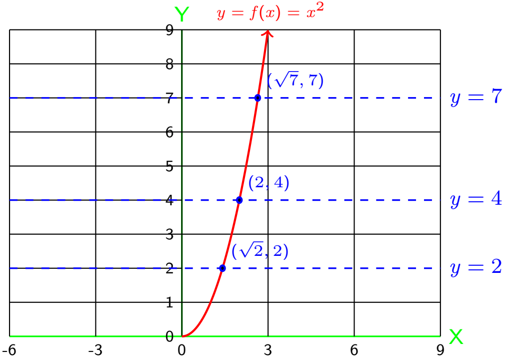
- Why does restricting the domain of $f(x) = x^2$ to positive x-values yield a function with an inverse? $\Rightarrow$



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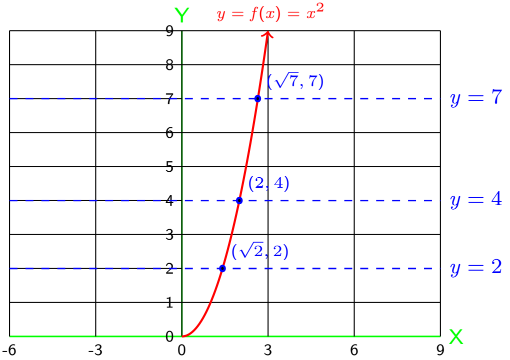
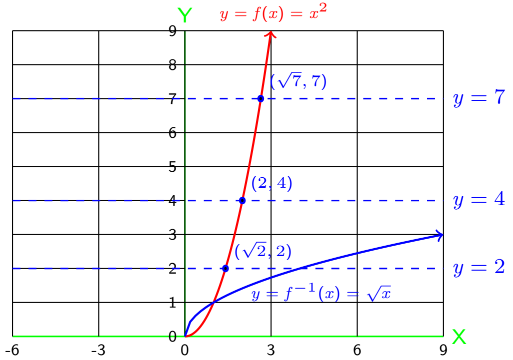
- Why does restricting the domain of $f(x) = x^2$ to positive x-values yield a function with an inverse? $\Rightarrow$ Now each horizontal line meets the graph just once.



Restricting the domain produces a one-to-one function

- Explain why the graph of $y = f(x) = x^2$ with domain all x does not pass the horizontal line test. $\Rightarrow$ For any $c > 0$,the horizontal line $y = c$ meets the graph twice, at $x = \pm\sqrt{c}$. As a result, function f does not have an inverse.
- Why does restricting the domain of $f(x) = x^2$ to positive x-values yield a function with an inverse? $\Rightarrow$ Now each horizontal line meets the graph just once.
- Define the inverse function of $y = f(x)$ with domain $x \geq 0$. $\Rightarrow$



Restricting the domain produces a one-to-one function

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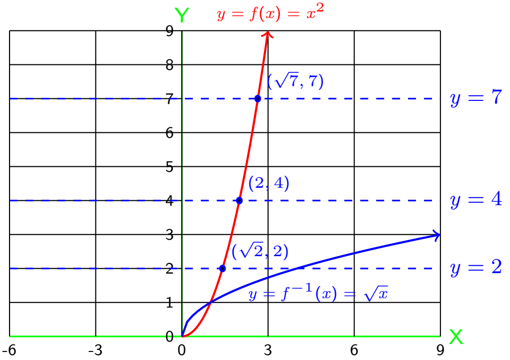
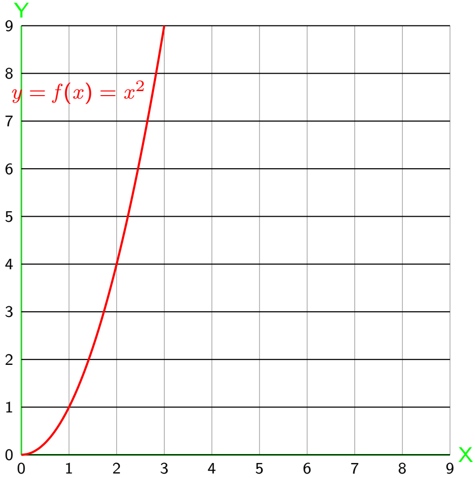
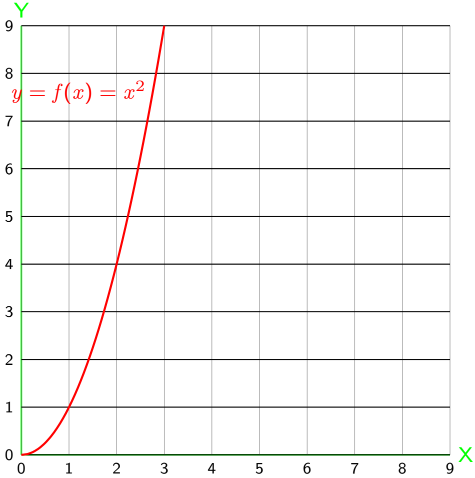
- Why does restricting the domain of $f(x) = x^2$ to positive x-values yield a function with an inverse? $\Rightarrow$ Now each horizontal line meets the graph just once.
- Define the inverse function of $y = f(x)$ with domain $x \geq 0$. $\Rightarrow$ $f^{-1}(x) = \sqrt{x}$ with domain $x \geq 0$.



Example: The red graph shows $f(x) = x^2$. with domain $0 \leq x \leq 3$ and range $0 \leq y \leq 9$. Plot the graph of f^{-1}



Example: The red graph shows $f(x) = x^2$. with domain $0 \leq x \leq 3$ and range $0 \leq y \leq 9$. Plot the graph of f^{-1}
Solution: Click slowly to plot the graph of $f^{-1}(x) = \sqrt{x}$, which will have domain $0 \leq x \leq 9$ and range $0 \leq y \leq 3$.



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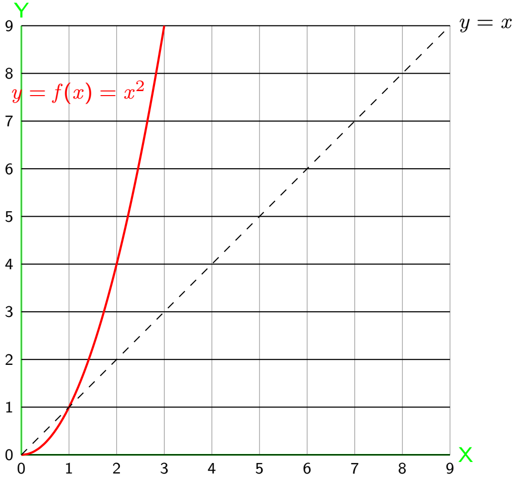

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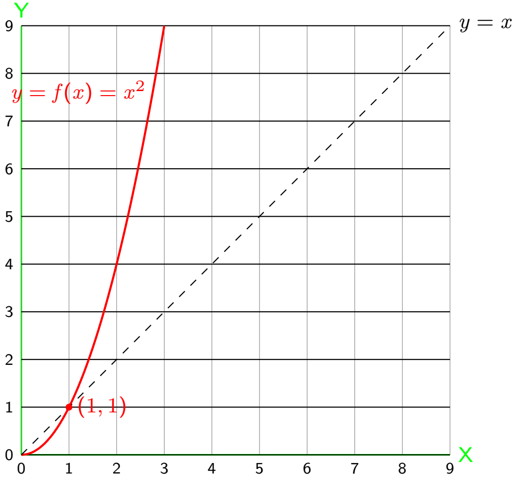

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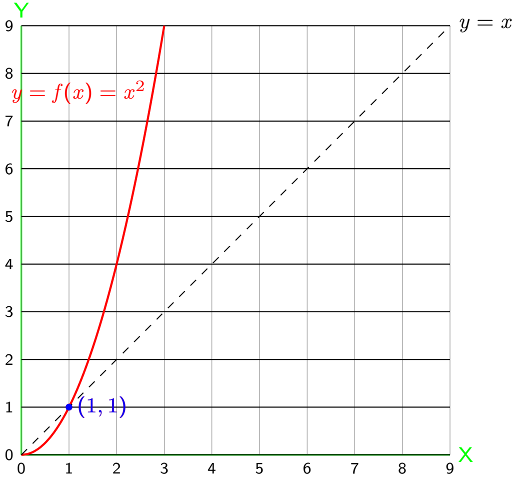

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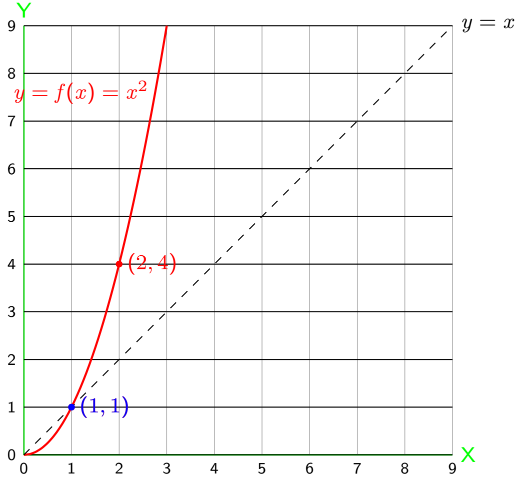

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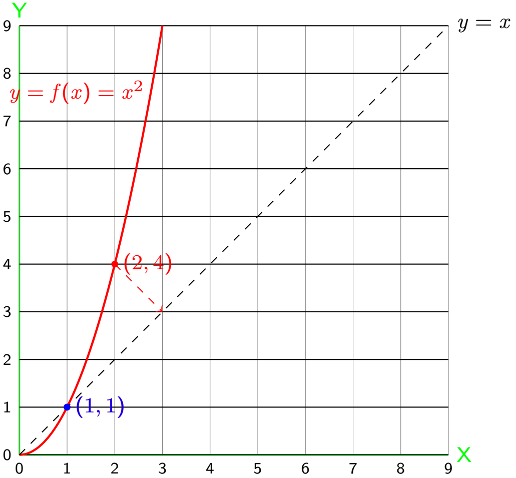

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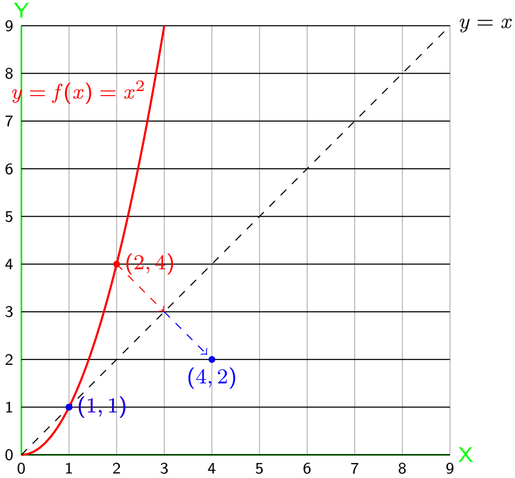

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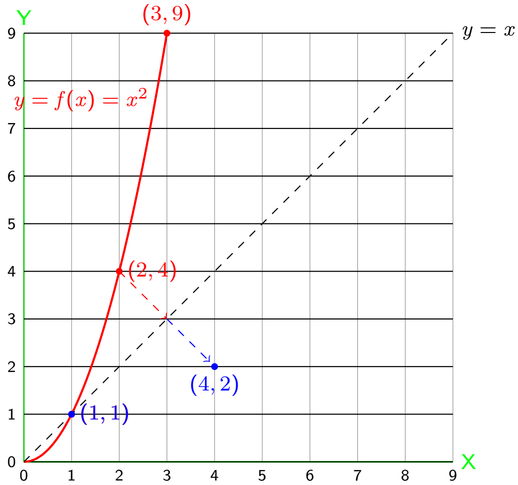

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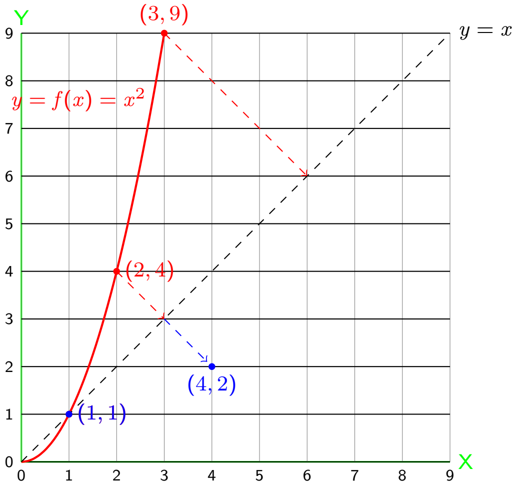

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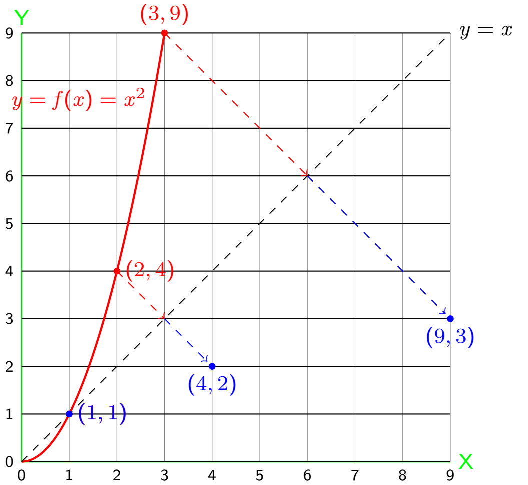

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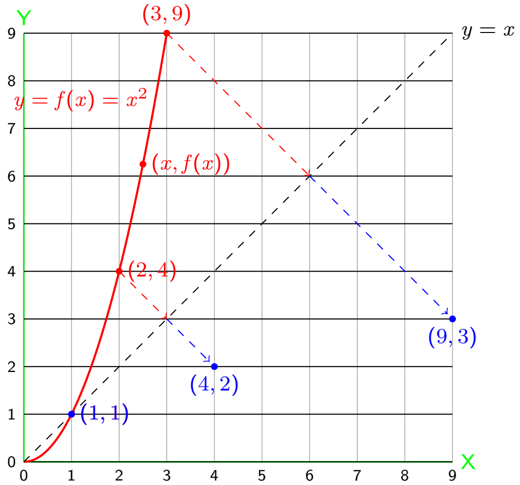

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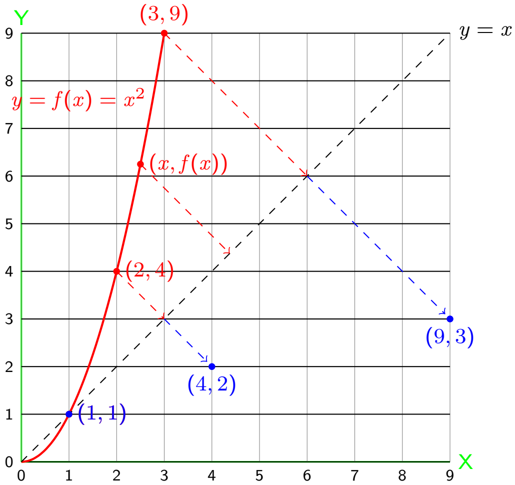

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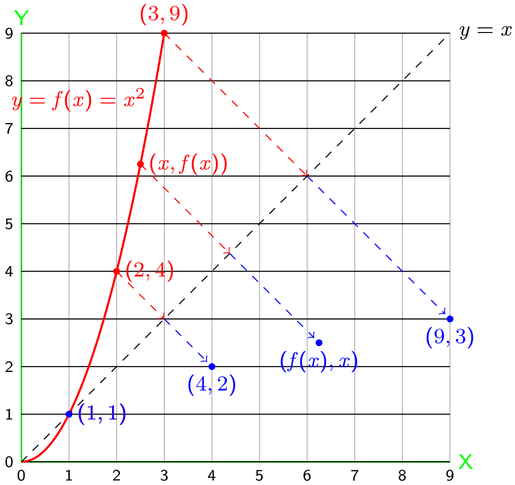

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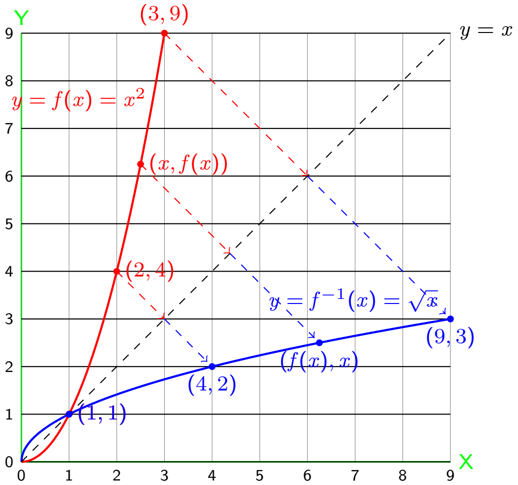


Chapter 2 Review

- Precalculus Section 2.1 Review: Graphs of functions and relations
- Precalculus Section 2.2 Review: Sketching graphs
- Precalculus Section 2.3 Review: Analyzing graphs
- Precalculus Section 2.4 Review: Quadratic functions
- Precalculus Section 2.5 Review: Graphing polynomials
- Precalculus Section 2.6 Review: Average rate of change
- Precalculus Section 2.7 Review: Transformations
- Precalculus Section 2.8 Review: Function inverse

To review a section listed above:

Click on its button to view the first Example in that section as well as three similar questions. Work out the answers, then click again to see if you are correct. If so, keep on clicking.

If you have trouble answering a question, click on the button to its left to access its solution in the text. Then click on the faint ⟲ Adobe control at the bottom right of the text screen to continue your review.

Precalculus Chapter 3: Exponential and log functions

Section 3.1: Functions

- 3.1.1: Exponentiation is repeated multiplication
- 3.1.2: Exp functions with bases 2 and 3
- 3.1.3: Is there a preferred base?
- 3.1.4: Transforming exponential graphs
- 3.1.5: Worked examples
- 3.1 Quiz 3.1 Review

Section 3.2: Natural exponential functions

- 3.2.1: The natural exponential function
- 3.2 Quiz 3.2 Review

Section 3.3: Logarithmic functions

- 3.3.1: Finding values of log functions
- 3.3.2: Log is the inverse function of exp
- 3.3.3: Is there a preferred base for log?
- 3.3.4: Transforming log graphs and equations
- 3.3.5: More examples of transformations
- 3.3 Quiz 3.3 Review

Section 3.4: Laws of logarithms

- 3.4.1: Laws of logarithms
- 3.4.2: Expanding log expressions
- 3.4.3: Combining log expressions
- 3.4 Quiz 3.4 Review

Section 3.5: Exponential and logarithmic equations

- 3.5.1: Exponential equations
- 3.5.2: Logarithmic equations
- 3.5 Quiz 3.5 Review

Section 3.6: Exponential growth and decay

- 3.6.1: Exponential growth and decay
- 3.6.2: The relative growth rate
- 3.6.3: The exponential growth factor
- 3.6.4: Radioactive decay
- 3.5 Quiz 3.6 Review

Chapter 3 Review

Chapter 3 Section 1: Exponential functions

- 3.1.1: Exponentiation is repeated multiplication
- 3.1.2: Exp functions with bases 2 and 3
- 3.1.3: Is there a preferred base?
- 3.1.4: Transforming exponential graphs
- 3.1.5: Worked examples
- 3.1.6: Section 3.1 Quiz

Section 3.1 Preview: Definitions

- Definition 3.1.1: *f is an exponential function with base a*
- Definition 3.1.2: The word *per cent* (or symbol %) means divided by 100
- Definition 3.1.3: Percentage increase principle: Adding 5% to a number multiplies that number by 1.05.
- Definition 3.1.4:Two kinds of 5% annual bank interest: simple vs. compounded
- Definition 3.1.5: The *annual compound interest rate is r* means:
- Definition 3.1.6: Interest compounded n times a year
- Definition 3.1.7: Interest compounded continuously means:
- Definition 3.1.8: *Function $N(t)$ grows (or decays) exponentially* means:

3.1.1 Exponentiation is repeated multiplication

Raising a base a to a whole number exponent is repeated multiplication: $a^3 = a \cdot a \cdot a$. Power rules say: $a^{2.345} = a^{\frac{2345}{1000}} = (\sqrt[1000]{a})^{2345}$. Defining $a^{\sqrt{2}}$ requires calculus.

f *is an exponential function with base* a

if $a > 0$ and $f(x) = a^x$ for all real numbers x.

The amount of money in a bank account is an exponential function of time, determined by the per cent interest rate.

The word *per cent* (or symbol %) means *divided by 100*. *Per cent of* means *per cent times*.

- To divide by 100, multiply by .01: move the decimal point two places left.
- 5% of $10 = 5/100 \cdot 10 = .05 \cdot 10 = .5$
- 235% of 1000 $= 235/100 \cdot 1000 = 2.35(1000) = 2350$

Percentage increase principle: Adding 5% to a number multiplies that number by 1.05.

- $100 + (5\% \text{ of } 100) = 1(100) + .05(100)$ $= (1 + .05)(100) = 1.05(100) = 105$

Two kinds of 5% annual bank interest:

- Simple interest: At the end of each year, the bank *adds to your account* 5% *of your original deposit*.
- Compounded annually: At the end of each year, the bank *multiplies your account by* 1.05. To do this, it adds to your account 5% of the amount that was in the account at the beginning of that year.

Example 1: My bank account starts off at 100 dollars and gets 5 percent interest, compounded annually. How much is in my account after 3 years? After t years?

Solution: Let $N(t)$ be my account in dollars after t years.

	Simple interest	Annual Compounding:
$N(t) =$	$100 + 5t$	$100(1.05)^t$
$N(1) =$	$100 + 5(1) = 105$	$100(1.05) = 105$
$N(2) =$	$100 + 5(2) = 110$	$100(1.05)^2 = 110.25$
$N(3) =$	$100 + 5(3) = 115$	$100(1.05)^3 = 115.7625$

Answer: My account's dollar value is

- $100(1.05)^3 = 115.7625$ dollars after 3 years.
- $100(1.05)^t$ dollars after t years if t is a whole number.

The virtue of compound interest is that it pays interest on prior interest earned. The amount after 3 years is 76.25 cents more than the $115 that would be obtained by adding 5% *of the original deposit* each year. Instead, at the end of each year, the bank adds 5% *of the amount at the beginning of that year*, which is larger than the original deposit because it includes interest earned until then.

The *annual compound interest rate is* r means:

- Each year, the account is multiplied by $1+r$.
- The per cent interest rate is $100r$.
- If the interest rate is $5\%, r = .05$

The bank may decide to split the interest payment into n equal parts. If so, the interest period is $\frac{1}{n}$ of a year, and the interest rate per period is $\frac{r}{n}$. We say that the annual interest rate is r, compounded n times a year.

Let $N(t)$ be the amount in an account with annual interest rate r at the end of t years. If interest is

- compounded annually, then $N(t) = N(0)(1+r)^t$;
- compounded n times a year, then, after k interest periods, $t = \frac{k}{n}$ and
$$N(t) = N(0)\left(1+\frac{r}{n}\right)^k = N(0)\left(1+\frac{r}{n}\right)^{nt}.$$

If the annual interest rate is 5%, compounded 4 times a year, then $r = .05$ and $n = 4$. The interest period is $\frac{1}{4}$ of a year (3 months). The amount after 30 months (ten 3-month interest periods) is $N(0)(1+\frac{.05}{4})^{10}$.

For fixed r, increasing n will raise the account value $N(t)$, but by a small amount. If the interest were compounded a million times a second, the account value will get close to, but is less than, $N(t) = N(0)e^{rt}$, where $e \approx 2.71828183$.

***The annual interest rate on a bank account is* r*, compounded continuously,* means:**

the account value at time t is $N(t) = N(0)e^{rt}$.

Since $N(0)e^{rt} = N(0)K^t$ where $K = e^r$, we say that a bank account with continuously compounded annual interest rate r has exponential growth factor $K = e^r$. More generally:

***Function* $N(t)$ *grows (or decays) exponentially* means:**
***For some number* $K > 0$, $\quad N(t) = N(0)K^t$.**

- If $K > 1$, K is the exponential growth factor.
- If $0 < K < 1$, K is the exponential decay factor.

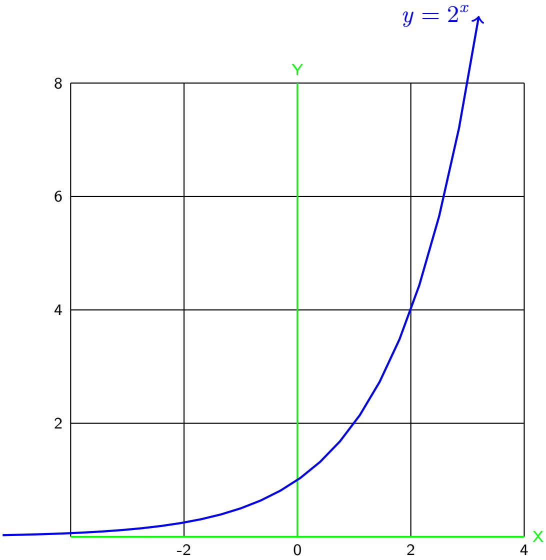
Exponential growth and decay occur in all of the sciences. To a first approximation,

- bacteria population grows exponentially;
- the mass of a radioactive element decays exponentially.

3.1.2 Exponential functions with bases 2 and 3

Let's start with $f(x) = 2^x$, often used to model population growth. When you plot the graph, don't use only integer values for inputs.

x	2^x
-3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$



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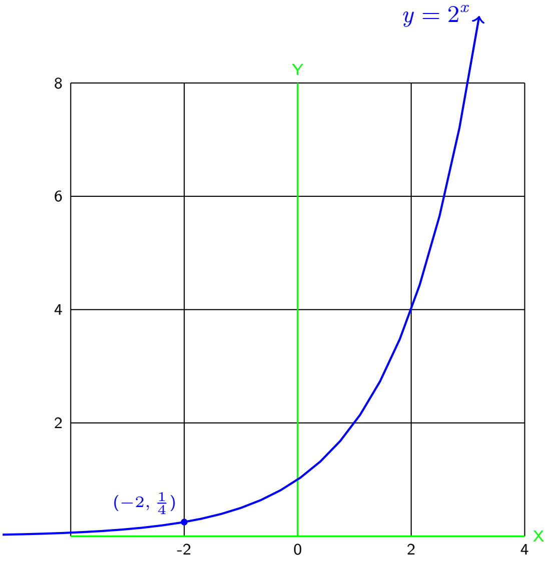

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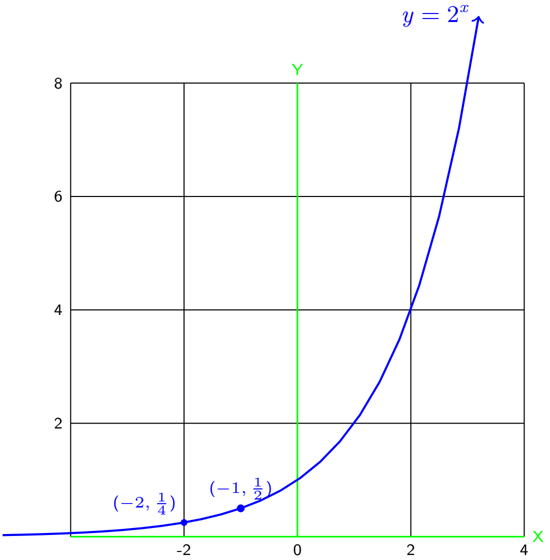

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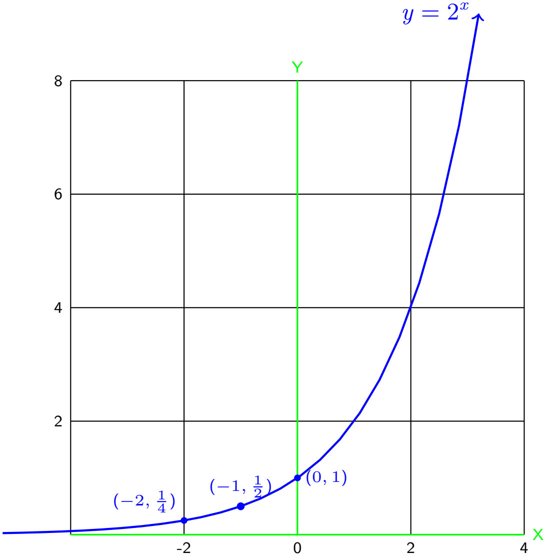
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$
0	1



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0	1
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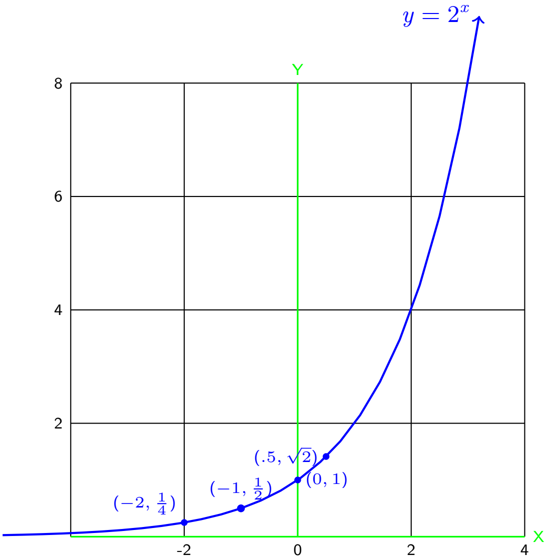

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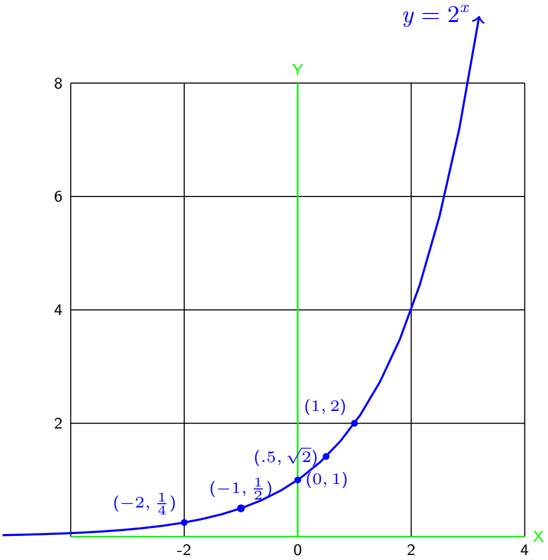
0	1
$.5$	$2^{1/2} = \sqrt{2}$
1	2



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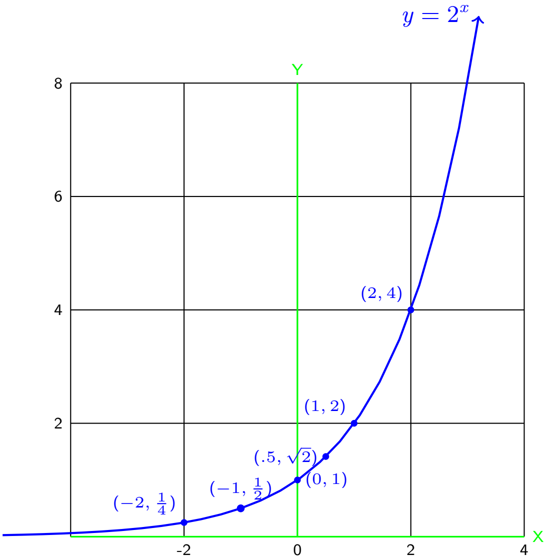
0	1
$.5$	$2^{1/2} = \sqrt{2}$
1	2
2	4



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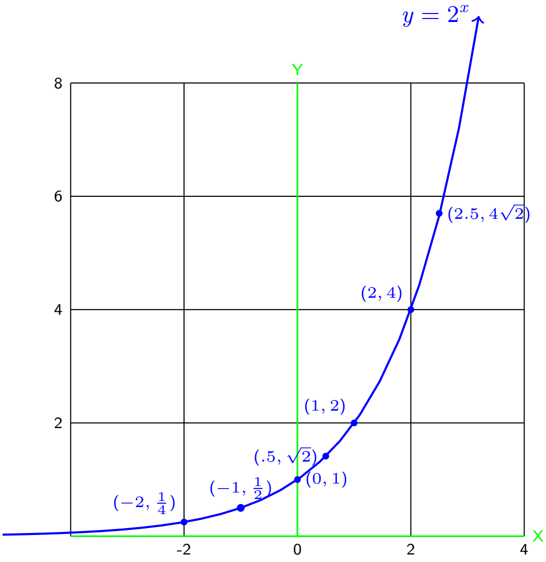
2.5	$2^2 2^{\frac{1}{2}} = 4\sqrt{2}$



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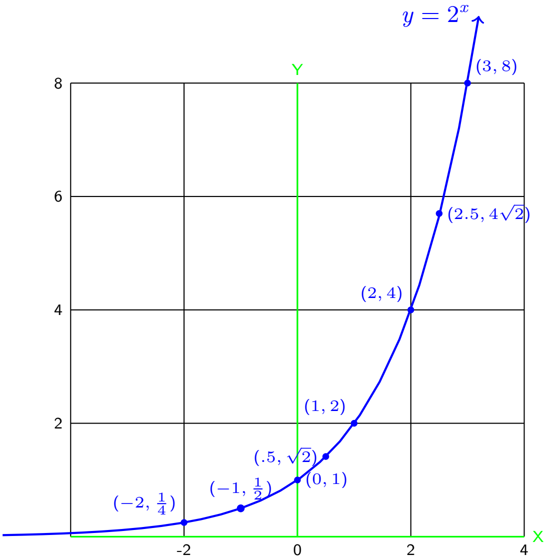
3	8



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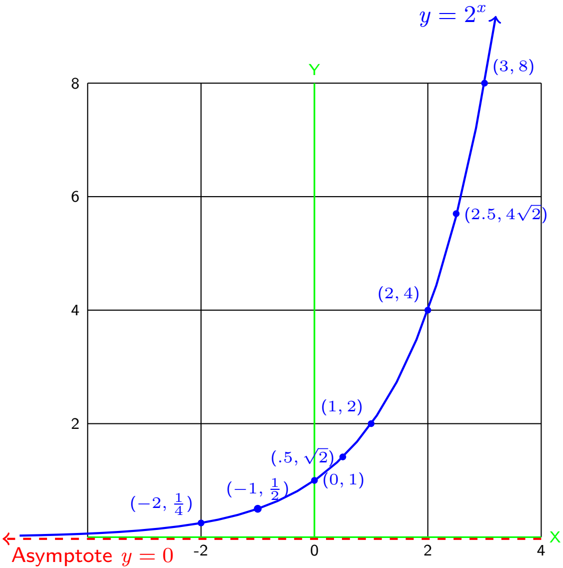
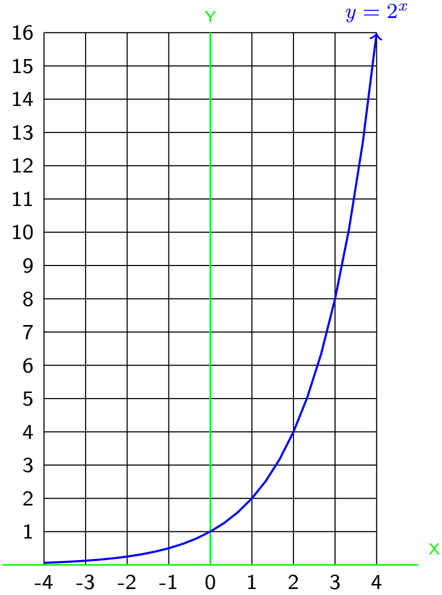


The graph gets close to the x-axis (the line $y = 0$) as x becomes large negative. We say that the line $y = 0$ is a *horizontal asymptote* for the graph $y = 2^x$.

To find the y-intercept, set $x = 0$. Then $f(0) = 2^0 = 1$ and the y-intercept is 1, at point $(0, 1)$.

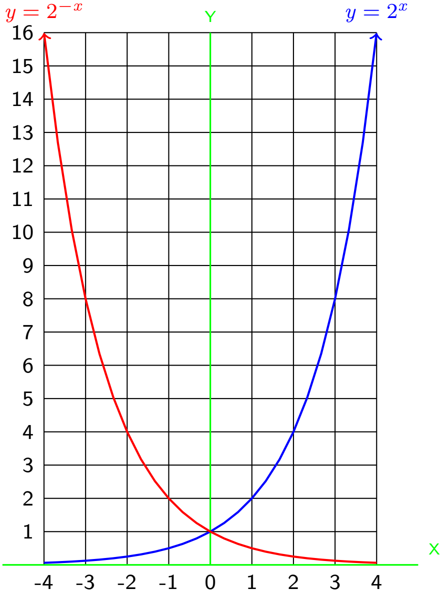
There is no x-intercept.

$y = f(x) = 2^x$ grows very quickly as x increases.



$y = f(x) = 2^x$ grows very quickly as x increases.

- Substituting $-x$ for x reflects the graph of $f(x) = 2^x$ across the y-axis to yield the graph of $g(x) = 2^{-x}$.



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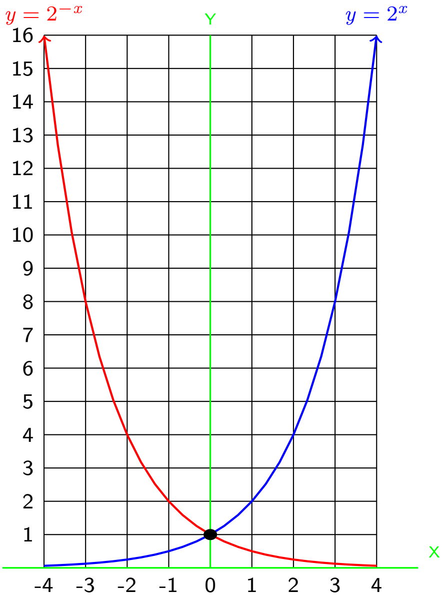
- $2^0 = 2^{-0} = 1$: both graphs cross the y-axis at $(0, 1)$.



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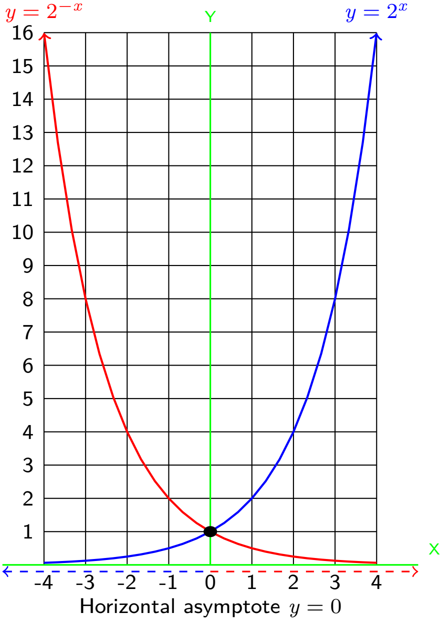
- $2^0 = 2^{-0} = 1$: both graphs cross the y-axis at $(0, 1)$.
- As x gets large positive, the graph of $f(x) = 2^x$ comes arbitrarily close to the x-axis, which we call the *asymptote* of the graph.
- As x gets large negative, the graph of $g(x) = 2^{-x}$ also comes arbitrarily close to the x-axis.



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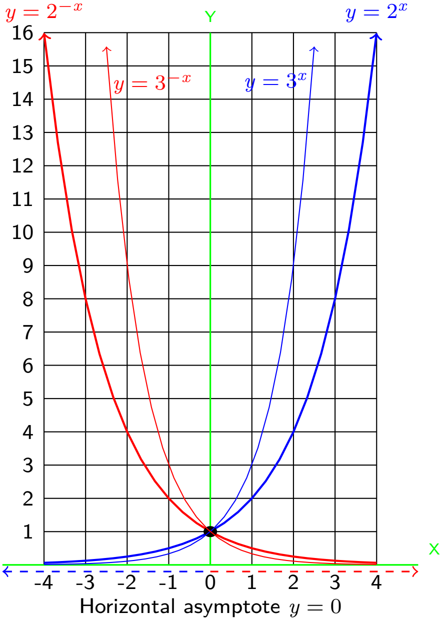
- If we change the base to 3, the graphs of $y = 3^x$ and $y = 3^{-x}$ increase more quickly than the corresponding graphs with base 2, but behave pretty similarly.



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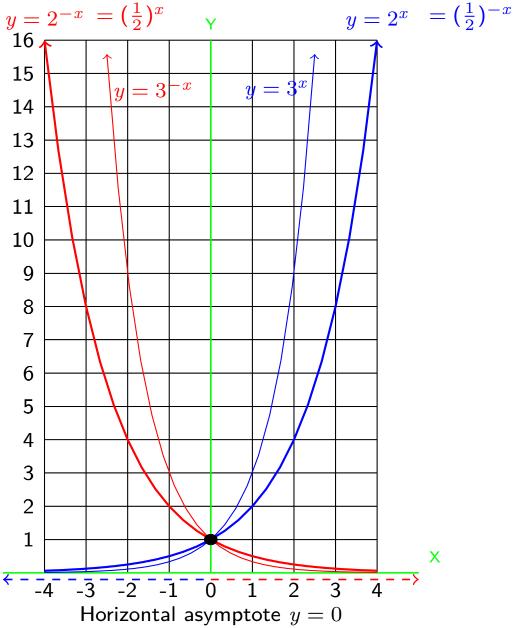
- If we change the base to 3, the graphs of $y = 3^x$ and $y = 3^{-x}$ increase more quickly than the corresponding graphs with base 2, but behave pretty similarly.
- When the base is a fraction $\frac{a}{b}$, recall that $(\frac{a}{b})^x = (\frac{b}{a})^{-x}$
  $(\frac{1}{2})^x = 2^{-x}$: the dark red graph shows $y = (\frac{1}{2})^x$.
  $(\frac{1}{2})^{-x} = 2^x$: the dark blue graph shows $y = (\frac{1}{2})^{-x}$.



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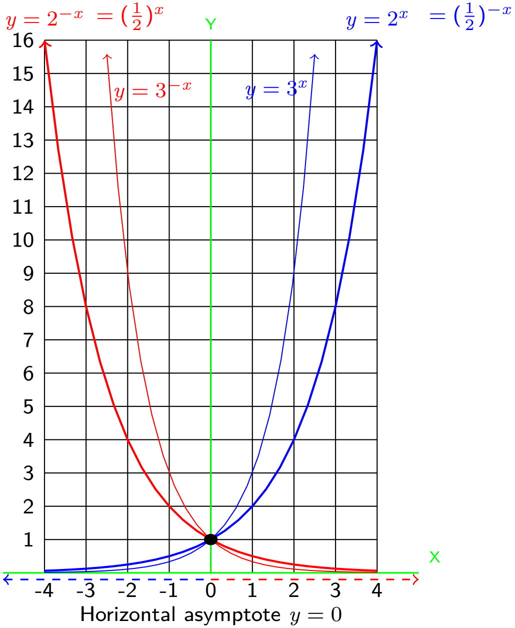
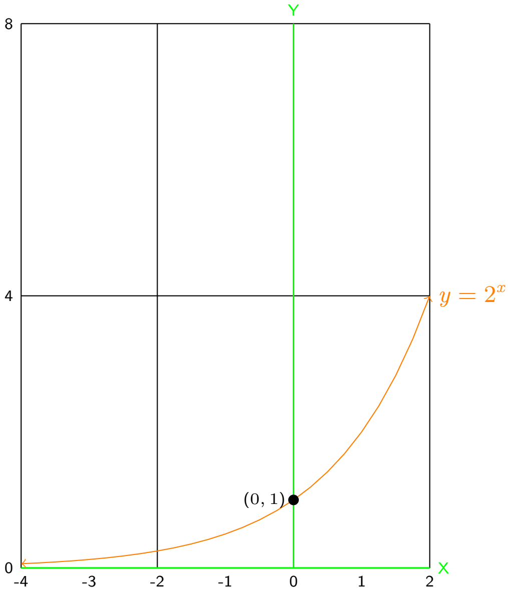
- All of the graphs meet at the y-intercept point $(0, 1)$.
  All have horizontal asymptote $y = 0$.



3.1.3 Is there a preferred base for exponential functions $y = a^x$?

As usual, we use only positive bases $a > 0$. The base $a = 1$ is uninteresting, since $1^x = 1$ for all x.

- It's unusual to use bases between 0 and 1. That's because, for instance, $\frac{1}{2}^x = 2^{-x}$, an easier function to plot and think about.
- We frequently study the doubling function $f(x) = 2^x$ because we like to double our money, or because bacteria split in two every minute.



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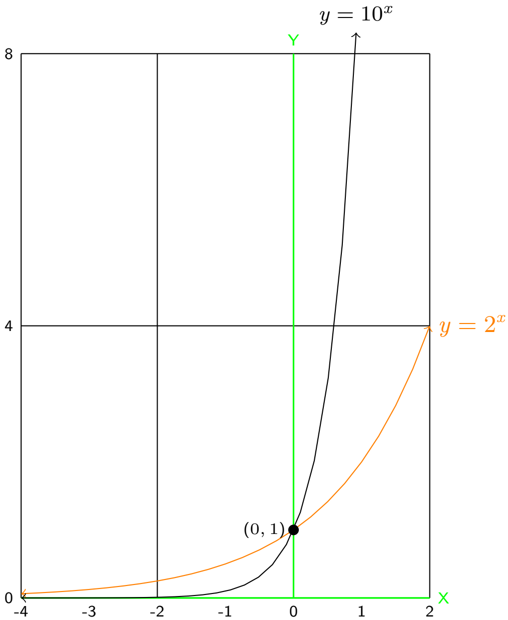
- We frequently study the doubling function $f(x) = 2^x$ because we like to double our money, or because bacteria split in two every minute.
- Our decimal number system is built on the base 10 function $y = 10^x$. For example, $10^3 = 1000$; $10^{-3} = \frac{1}{10^3} = 0.001 =$ one thousandth; and $10^9 = 1,000,000,000 =$ one billion.



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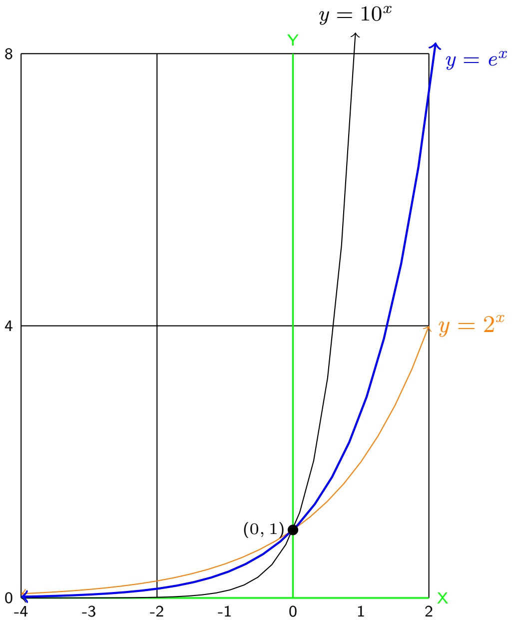
- Calculus focuses on an unusual base, called e (about 2.718) and studies in detail the function $\exp(x) = e^x$.



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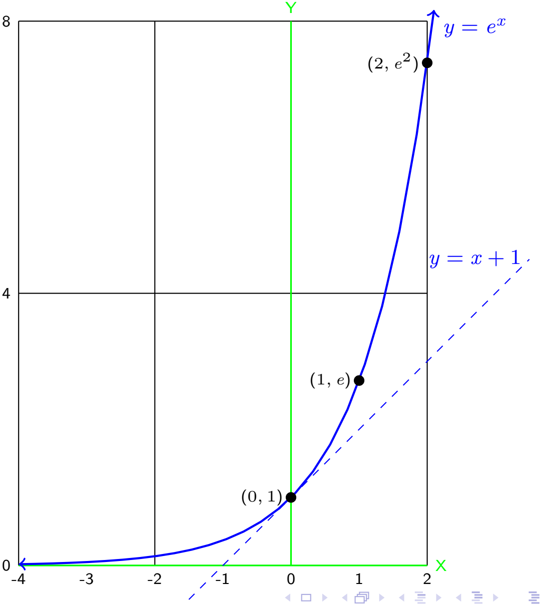
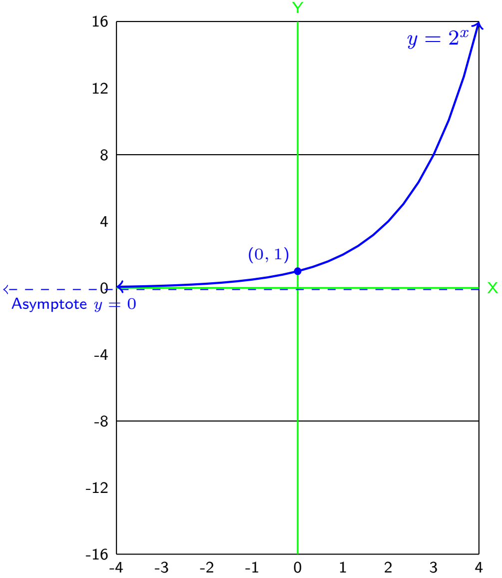
- Our decimal number system is built on the base 10 function $y = 10^x$. For example, $10^3 = 1000$; $10^{-3} = \frac{1}{10^3} = 0.001 =$ one thousandth; and $10^9 = 1,000,000,000 =$ one billion.
- Calculus focuses on an unusual base, called e (about 2.718) and studies in detail the function $\exp(x) = e^x$.
- This base e has an interesting feature: the tangent line to the graph of an exponential function $y = a^x$ at $(0, 1)$ is a *diagonal* line (slope $= 1$) if and only if $a = e$.

  The equation of that tangent line is
  $y - y_0 = m(x - x_0)$,
  $y - 1 = 1(x - 0) \Rightarrow y = x + 1$.



3.1.4 Transforming graphs and equations of exponential functions

First let's show the graph of the base 2 exponential function and its various reflections. Begin with the graph of $y = 2^x$. Its asymptote is the horizontal line $y = 0$, and its y-intercept is $(0, 1)$.



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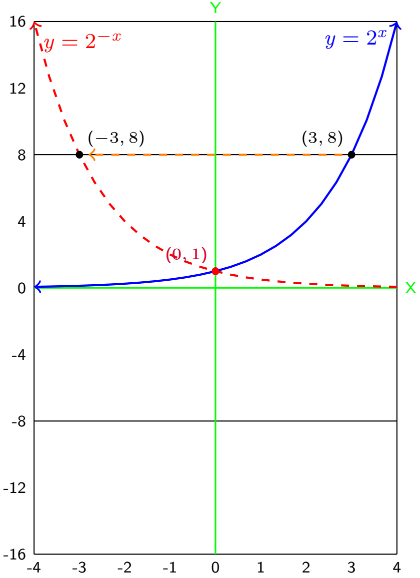

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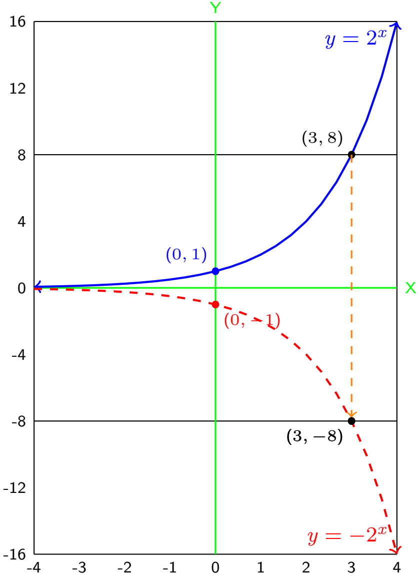

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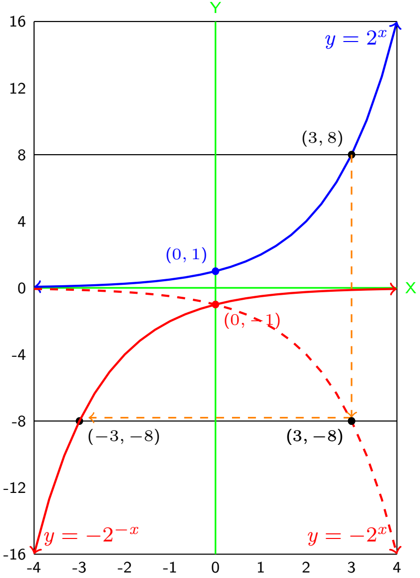

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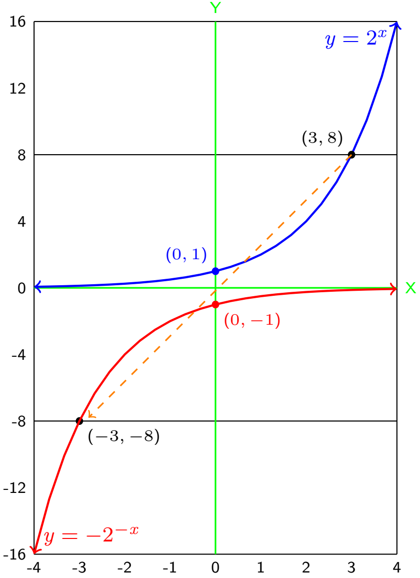

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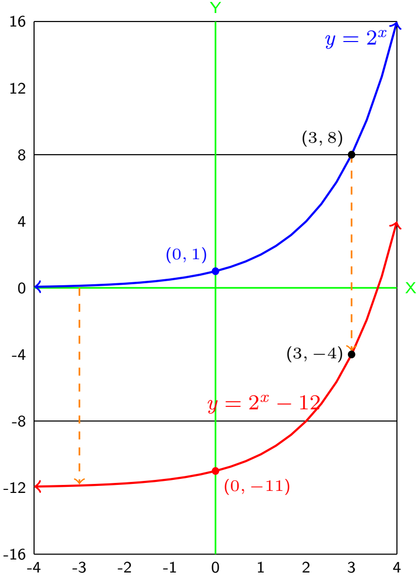
- Now for a shift. Subtract 12 from the RHS of $y = 2^x$ to shift its graph down 12 units to yield the graph of $y = 2^x - 12$.



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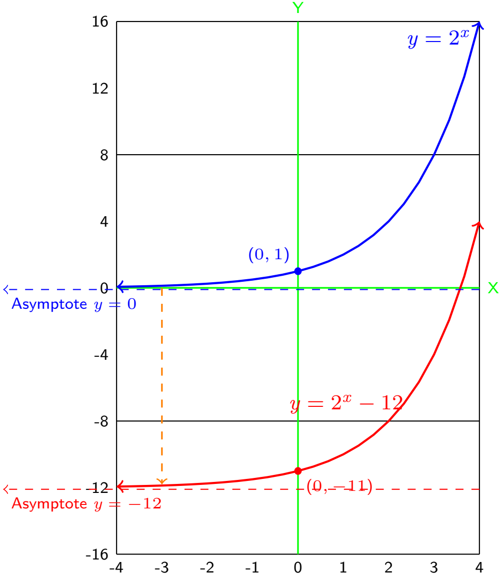
- Now for a shift. Subtract 12 from the RHS of $y = 2^x$ to shift its graph down 12 units to yield the graph of $y = 2^x - 12$.
- When the graph shifts down 12 units, so does its horizontal asymptote, which becomes $y = -12$, and its y-intercept, which becomes $(0, -11)$,



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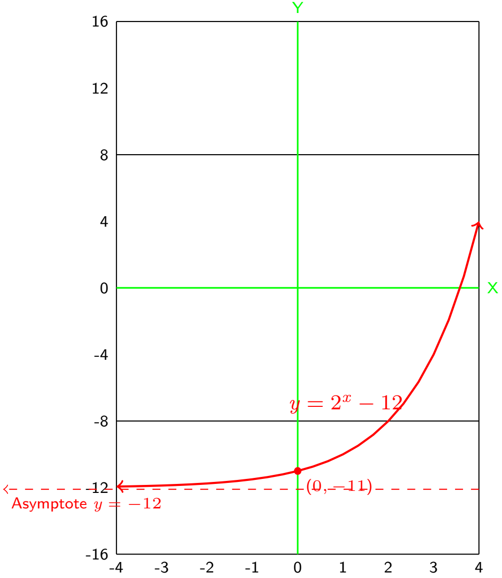
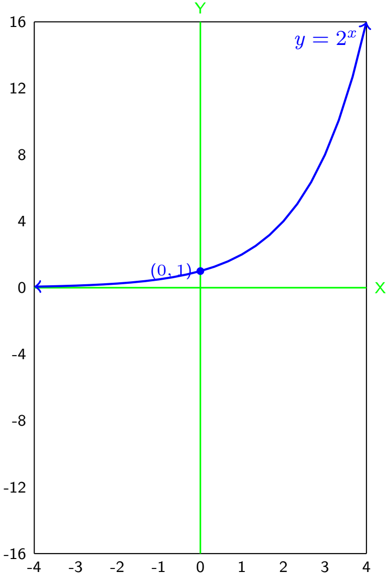


3.1.5 Worked examples

Example 2: Transform the graph of $y = 2^x$ step by step to the graph of $y = 8 - 2^{-x}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^x$.



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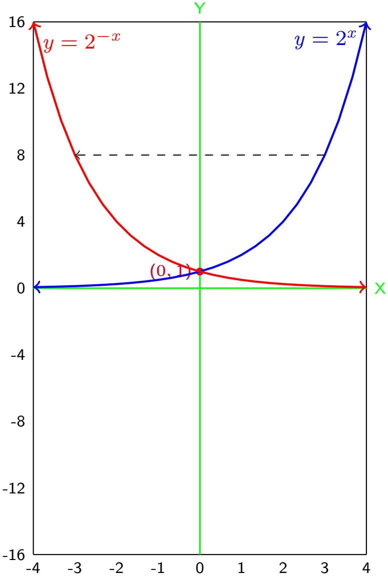

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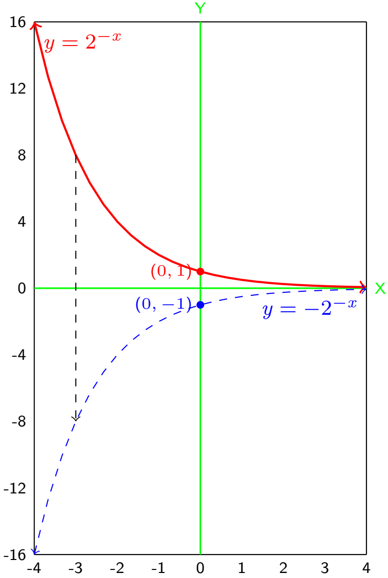

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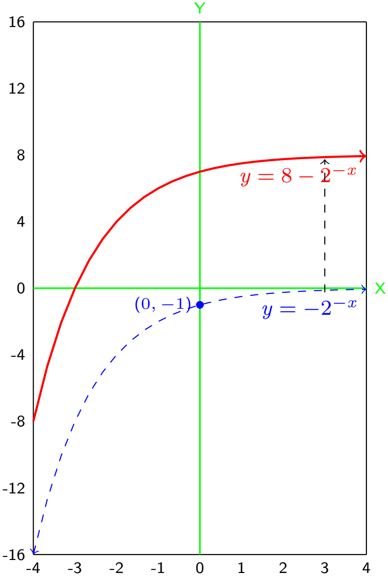

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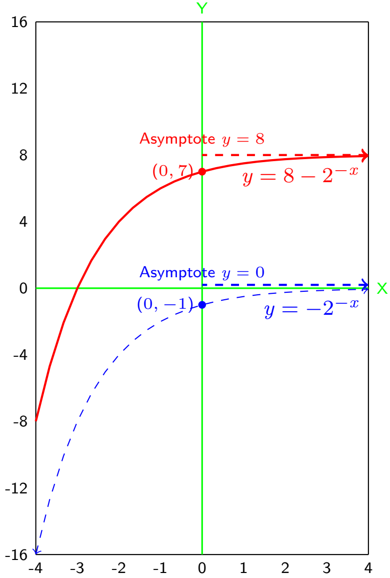

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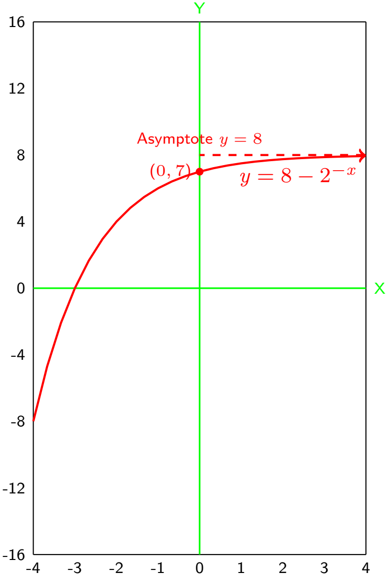
- You could also find the y-intercept by setting $x = 0$ in $y = 8 - 2^{-x}$: $y = 8 - 2^{-0} = 8 - 2^0 = 8 - 1 = 7$, and so the $y-$intercept is 7. The graph meets the y-axis at $(0, 7)$.



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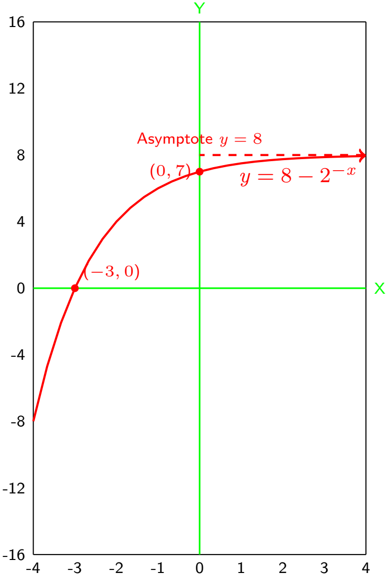
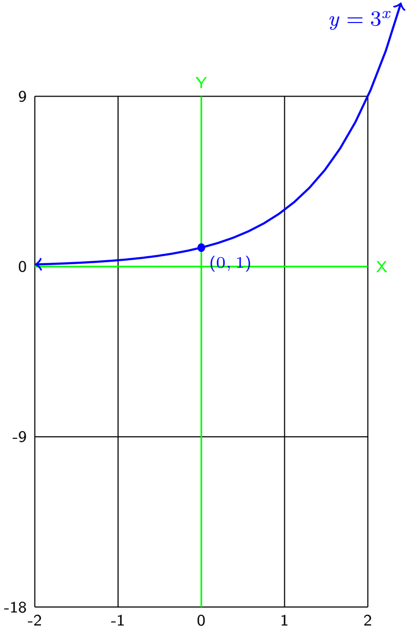
- Set $y = 0$ to find the $x-$intercept: $0 = 8 - 2^{-x}$. $2^{-x} = 2^3 \Rightarrow -x = 3$; $x = -3$. The graph meets the x-axis at $(-3, 0)$.



Example 3: Sketch the graph of $y = 9 - 3^{x+1}$. Show intercepts and asymptote(s).

Solution: Start with the graph of $y = 3^x$, with asymptote the x-axis and y-intercept $(0, 1)$.



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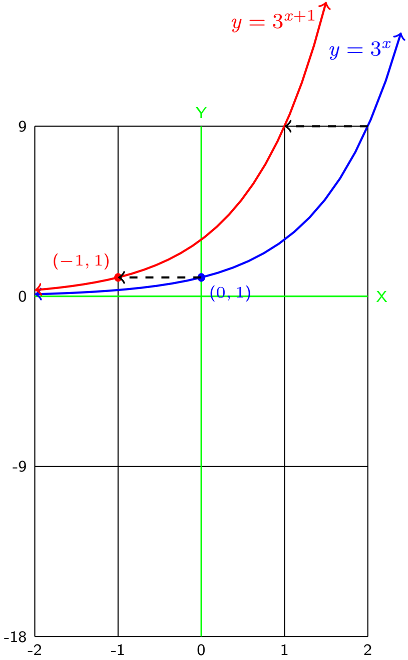

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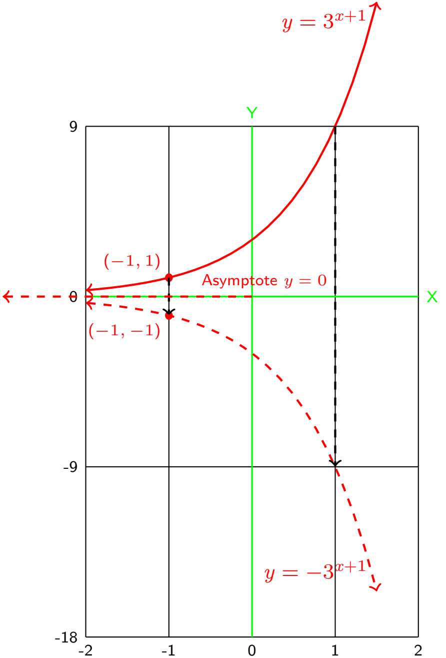

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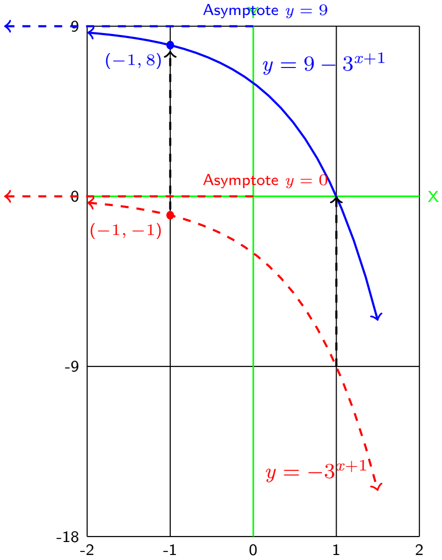
- Add 9 to the RHS of $y = -3^{x+1}$ to shift its graph up 9 units to the requested graph of $y = 9 - 3^{x+1}$. The asymptote (the x-axis $y = 0$) moves up 9 units to the horizontal line $y = 9$. Point $(-1, -1)$ moves up 9 units to point $(-1, 8)$.



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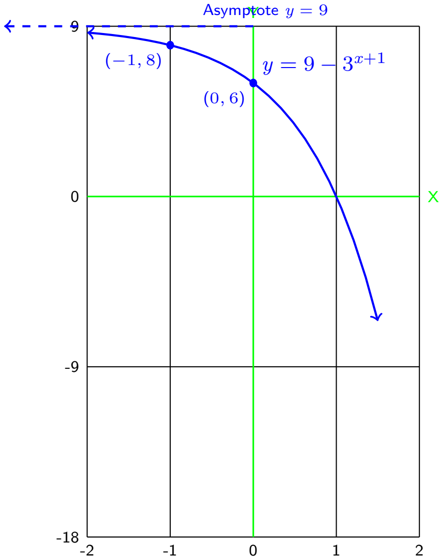

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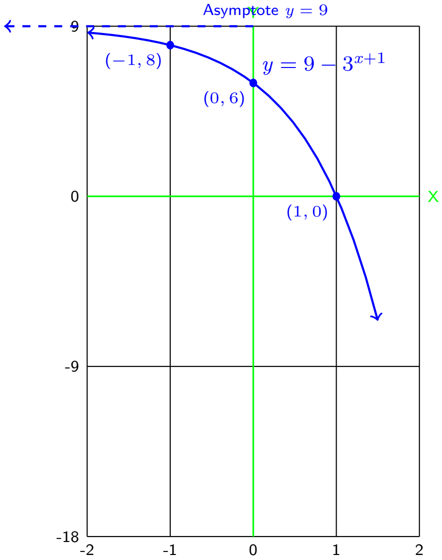
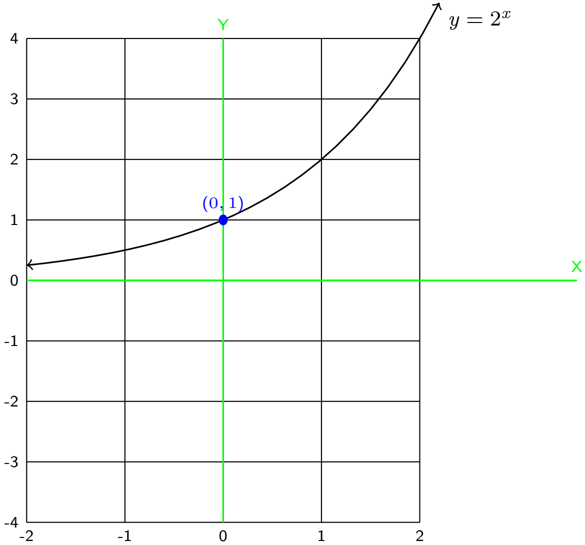
- Set $x = 0$ to find the y-intercept: $y = 9 - 3^{x+1} = 9 - 3^1 = 9 - 3 = 6$, and so the new y-intercept is $(0, 6)$.
- Set $y = 0$ to find the x-intercept: $0 = 9 - 3^{x+1} \Rightarrow 9 = 3^{x+1} \Rightarrow x + 1 = 2 \Rightarrow x = 1$. The graph's x-intercept is $(1, 0)$.



Example 4: Graph the equation $y = 2^{x-2} - 3$. Show intercepts and asymptote(s).

Solution: Start with the graph of $y = 2^x$, with asymptote the x-axis and y-intercept $(0, 1)$.



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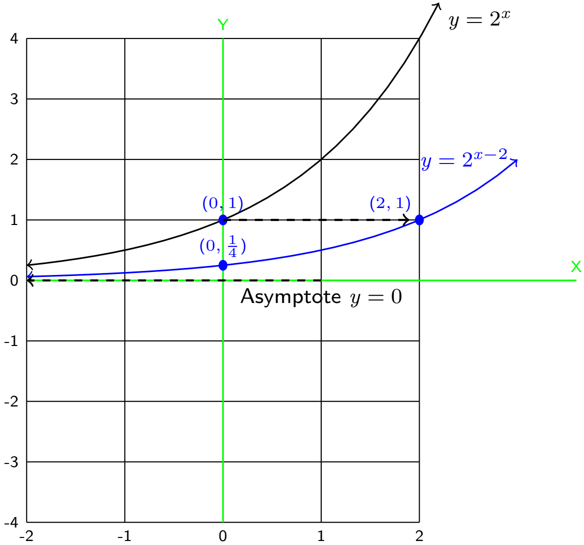
Solution: Start with the graph of $y = 2^x$, with asymptote the x-axis and y-intercept $(0, 1)$.

- Substituting $x - 2$ for x in $y = 2^x$ shifts the graph 2 units *right* to the graph of $y = 2^{x-2}$. The asymptote is still the x-axis. The y-intercept moves to point $(0, \frac{1}{4})$.



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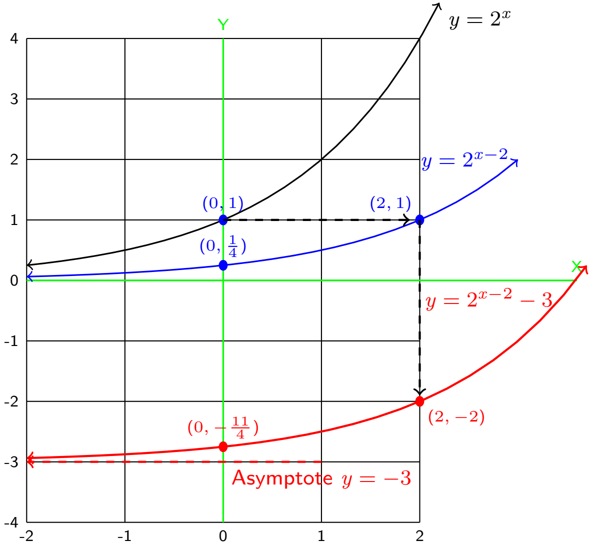
- Subtract 3 from the RHS of the equation $y = 2^{x-2}$ to shift the graph 3 units down to the graph of $y = 2^{x-2} - 3$. The asymptote also shifts down to $y = -3$.



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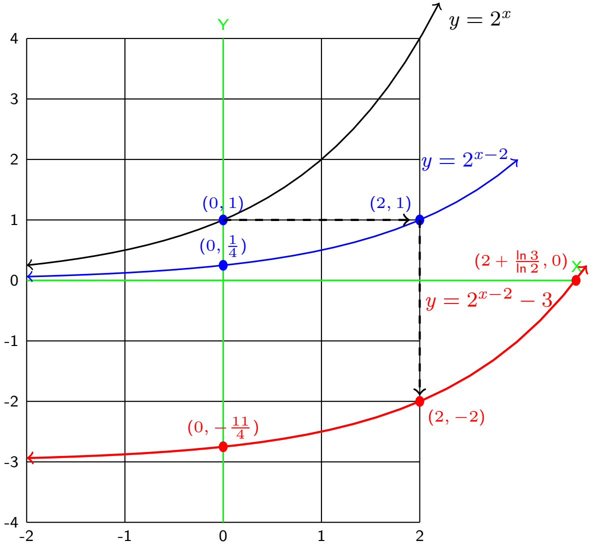

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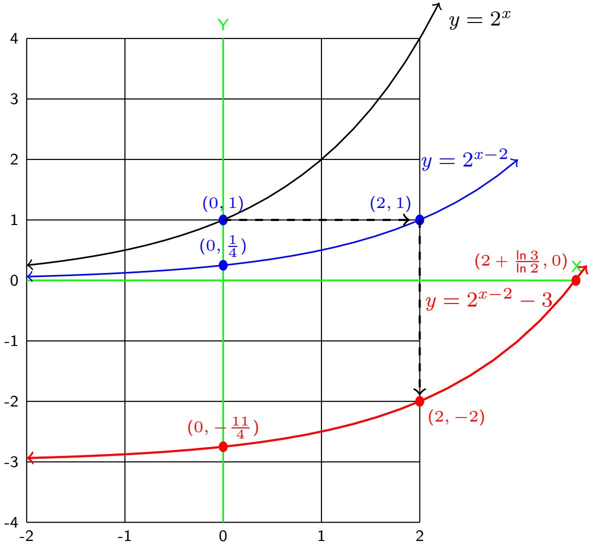
- Set $x = 0$ to find the y-intercept: $2^{x-2} - 3 = 2^{-2} - 3 = -\frac{11}{4}$, and so the new y-intercept is $(0, -\frac{11}{4})$.
- Set $y = 0$ to find the x-intercept: $2^{x-2} - 3 = 0 \Rightarrow 2^{x-2} = 3 \Rightarrow x = 2 + \frac{\ln 3}{\ln 2} \approx 3.59$.



3.1.6 Quiz

Ex. 3.1.1: My bank account starts off at 100 dollars and gets 5 percent interest, compounded annually. How much is in my account after 3 years? After t years?

Ex. 3.1.2: Transform the graph of $y = 2^x$ step by step to the graph of $y = 8 - 2^{-x}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Ex. 3.1.3: Sketch the graph of $y = 9 - 3^{x+1}$. Find and label its intercepts and asymptote.

Ex. 3.1.4: Graph the equation $y = 2^{x-2} - 3$. Find and label its intercepts and asymptote.

Section 3.1 Review: Exponential functions

Ex. 3.1.1: My bank account starts off with \$1 and gets 5 percent interest, compounded
- annually
- every month
- 100 times a year
- 10000 times a year.

How much is in my account after 3 years? After t years? After 100 years? After 1000 years?

	Annually	every month	100 times/year	10,000 times/year
After 3 years	1.05^3	$(1+.05/12)^{36}$	1.0005^{300}	1.000005^{30000}
After t years	1.05^t	$(1+.05/12)^{12t}$	1.0005^{100t}	$1.000005^{10,000t}$
After 100 years	1.05^{100}	$(1+.05/12)^{1200}$	$1.0005^{10,000}$	$1.000005^{1,000,000}$
After 1000 years	1.05^{1000}	$(1+.05/12)^{12000}$	$1.0005^{100,000}$	$1.000005^{10,000,000}$

Use a calculator to find out the dollar value of each answer.

	Annually	every month	100 times/year	10000 times/year
After 3 years	1.16	1.16	1.16	1.16
After t years	1.05^t	$(1+.05/12)^{12t}$	1.0005^{100t}	1.00005^{1000t}
After 100 years	131.50	146.88	148.23	148.41
After 1000 years	$1.5463\cdot 10^{21}$	$4.6732\cdot 10^{21}$	$5.1203\cdot 10^{21}$	$5.1841\cdot 10^{21}$

The solutions to the following begin on the next page.

Ex. 3.1.2: Sketch the graph, labeled with intercepts and asymptotes, obtained by transforming, step by step, the graph of

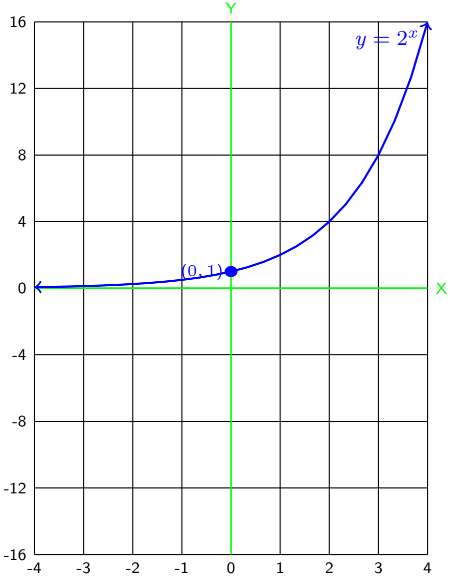
- $y = 2^x$ to the graph of $y = 8 - 2^{-x}$.
- $y = 2^x$ to the graph of $y = 0.5 \cdot 2^{2x}$.
- $y = 2^{-x}$ to the graph of $y = 1 + 2^x$.
- $y = 2^x$ to the graph of $y = -2^{2x+1}$.

Ex. 3.1.3: Sketch the graph, labeled with intercepts and asymptotes, obtained by transforming, step by step, the graph of

- $y = 9 - 3^{x+1}$.
- $y = 2 + 3^{-x}$.
- $y = -3^{2x+1}$.
- $y = 9 - 3^{2x+1}$.

Example 3.1.2a: Transform the graph of $y = 2^x$ step by step to the graph of $y = 8 - 2^{-x}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

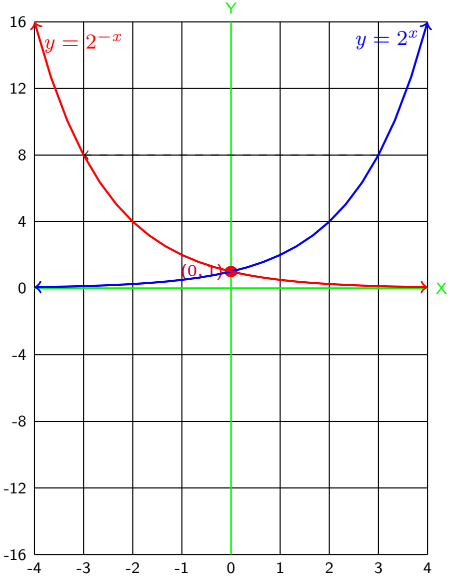
Solution: Start with the graph of $y = 2^x$.



Example 3.1.2a: Transform the graph of $y = 2^x$ step by step to the graph of $y = 8 - 2^{-x}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^x$.

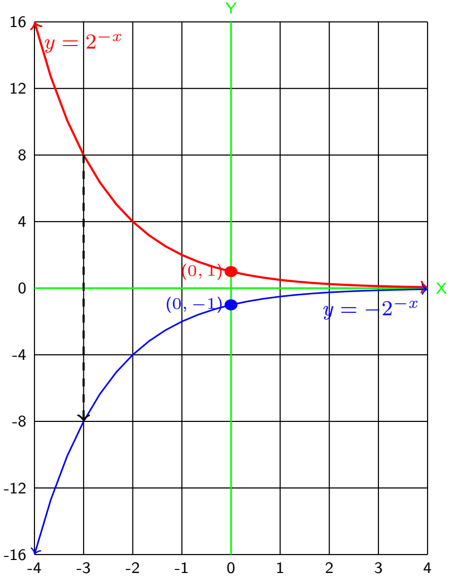
- Substituting $-x$ for x in $y = 2^x$ reflects its graph across the y-axis to the graph of $y = 2^{-x}$.



Example 3.1.2a: Transform the graph of $y = 2^x$ step by step to the graph of $y = 8 - 2^{-x}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^x$.

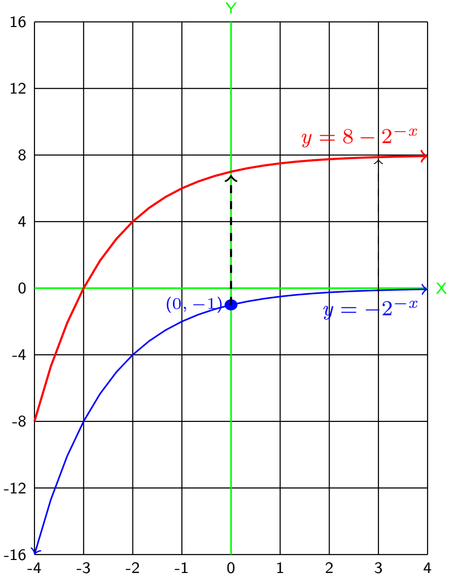
- Substituting $-x$ for x in $y = 2^x$ reflects its graph across the y-axis to the graph of $y = 2^{-x}$.
- Multiplying the RHS of the equation $y = 2^{-x}$ by -1 reflects its graph across the x-axis to the graph of $y = -2^{-x}$.



Example 3.1.2a: Transform the graph of $y = 2^x$ step by step to the graph of $y = 8 - 2^{-x}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^x$.

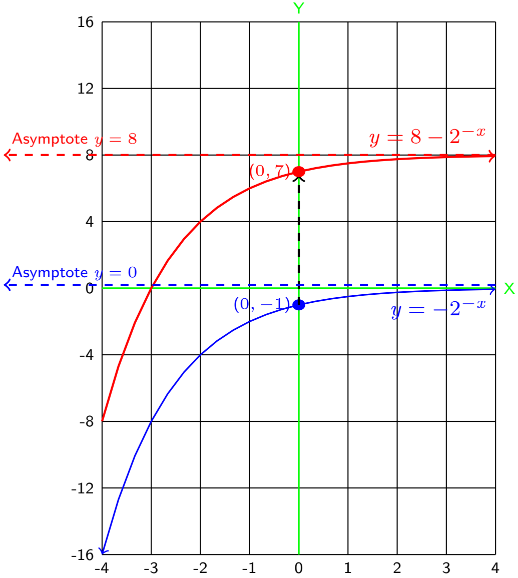
- Substituting $-x$ for x in $y = 2^x$ reflects its graph across the y-axis to the graph of $y = 2^{-x}$.
- Multiplying the RHS of the equation $y = 2^{-x}$ by -1 reflects its graph across the x-axis to the graph of $y = -2^{-x}$.
- Adding 8 to the RHS of $y = -2^{-x}$ shifts its graph up 8 units to give the requested graph of $y = 8 - 2^{-x}$.



Example 3.1.2a: Transform the graph of $y = 2^x$ step by step to the graph of $y = 8 - 2^{-x}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^x$.

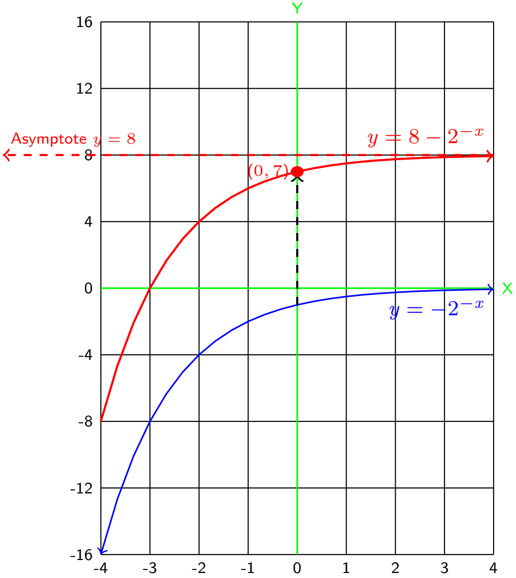
- Substituting $-x$ for x in $y = 2^x$ reflects its graph across the y-axis to the graph of $y = 2^{-x}$.
- Multiplying the RHS of the equation $y = 2^{-x}$ by -1 reflects its graph across the x-axis to the graph of $y = -2^{-x}$.
- Adding 8 to the RHS of $y = -2^{-x}$ shifts its graph up 8 units to give the requested graph of $y = 8 - 2^{-x}$.
- The asymptote $y = 0$ of the original graph also shifts up 8 units to give the new asymptote $y = 8$.
- The intercept point $(0, -1)$ shifts up 8 to $(0, -1 + 8) = (0, 7)$, the new y-intercept point.



Example 3.1.2a: Transform the graph of $y = 2^x$ step by step to the graph of $y = 8 - 2^{-x}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^x$.

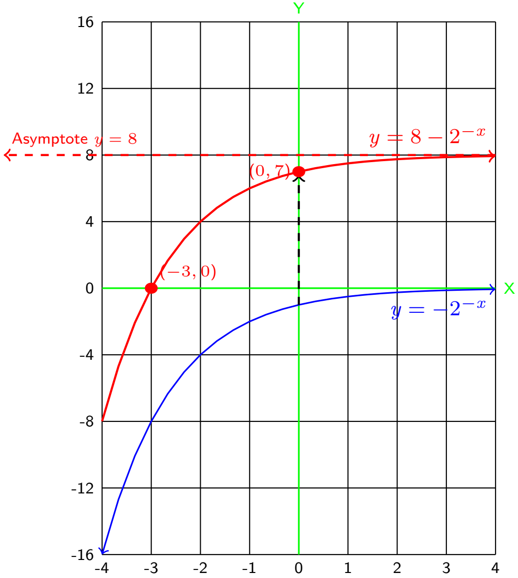
- Substituting $-x$ for x in $y = 2^x$ reflects its graph across the y-axis to the graph of $y = 2^{-x}$.
- Multiplying the RHS of the equation $y = 2^{-x}$ by -1 reflects its graph across the x-axis to the graph of $y = -2^{-x}$.
- Adding 8 to the RHS of $y = -2^{-x}$ shifts its graph up 8 units to give the requested graph of $y = 8 - 2^{-x}$.
- The asymptote $y = 0$ of the original graph also shifts up 8 units to give the new asymptote $y = 8$.
- The intercept point $(0, -1)$ shifts up 8 to $(0, -1 + 8) = (0, 7)$, the new y-intercept point.
- You could also find the y-intercept by setting $x = 0$ in $y = 8 - 2^{-x}$: $y = 8 - 2^{-0} = 8 - 1 = 7$, and so the $y-$intercept is 7. The graph meets the y-axis at $(0, 7)$.



Example 3.1.2a: Transform the graph of $y = 2^x$ step by step to the graph of $y = 8 - 2^{-x}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^x$.

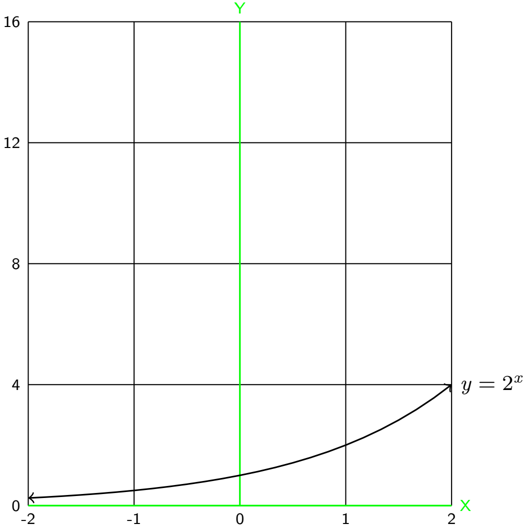
- Substituting $-x$ for x in $y = 2^x$ reflects its graph across the y-axis to the graph of $y = 2^{-x}$.
- Multiplying the RHS of the equation $y = 2^{-x}$ by -1 reflects its graph across the x-axis to the graph of $y = -2^{-x}$.
- Adding 8 to the RHS of $y = -2^{-x}$ shifts its graph up 8 units to give the requested graph of $y = 8 - 2^{-x}$.
- The asymptote $y = 0$ of the original graph also shifts up 8 units to give the new asymptote $y = 8$.
- The intercept point $(0, -1)$ shifts up 8 to $(0, -1 + 8) = (0, 7)$, the new y-intercept point.
- You could also find the y-intercept by setting $x = 0$ in $y = 8 - 2^{-x}$: $y = 8 - 2^{-0} = 8 - 1 = 7$, and so the $y-$intercept is 7. The graph meets the y-axis at $(0, 7)$.
- Set $y = 0$ to find the $x-$intercept: $0 = 8 - 2^{-x}$ so $2^{-x} = 2^3 \Rightarrow -x = 3 \Rightarrow x = 3$. The graph meets the x-axis at $(-3, 0)$.



Solutions to Review Exercises 3.1.2 and 3.1.3.

Example 3.1.2b: Transform the graph of $y = 2^x$ step by step to the graph of $y = 0.5 \cdot 2^{2x}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^x$.

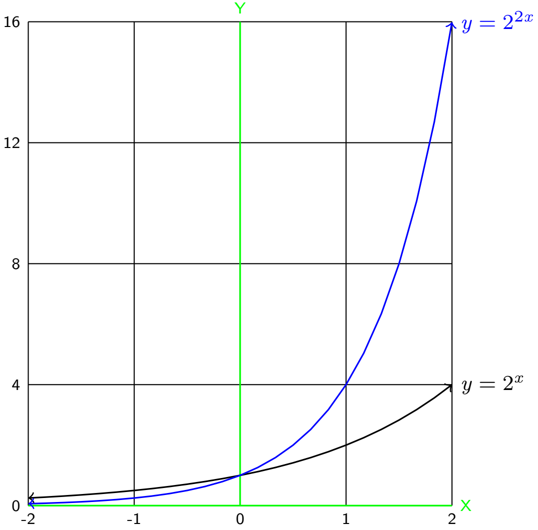


Solutions to Review Exercises 3.1.2 and 3.1.3.

Example 3.1.2b: Transform the graph of $y = 2^x$ step by step to the graph of $y = 0.5 \cdot 2^{2x}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^x$.

- Substituting $2x$ for x in $y = 2^x$ shrinks the graph by a factor of 2 toward the y-axis to the graph of $y = 2^{2x}$.

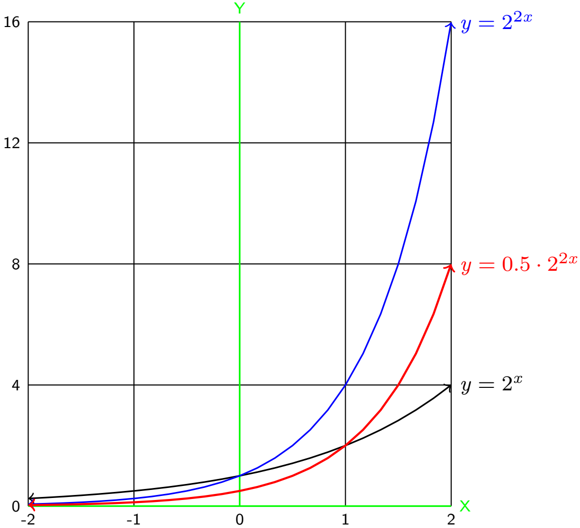


Solutions to Review Exercises 3.1.2 and 3.1.3.

Example 3.1.2b: Transform the graph of $y = 2^x$ step by step to the graph of $y = 0.5 \cdot 2^{2x}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^x$.

- Substituting $2x$ for x in $y = 2^x$ shrinks the graph by a factor of 2 toward the y-axis to the graph of $y = 2^{2x}$.
- Multiplying the RHS of the equation $y = 2^{2x}$ by $\frac{1}{2}$ shrinks its graph by a factor of 2 toward the x-axis to the graph of $y = 0.5 \cdot 2^{2x}$.

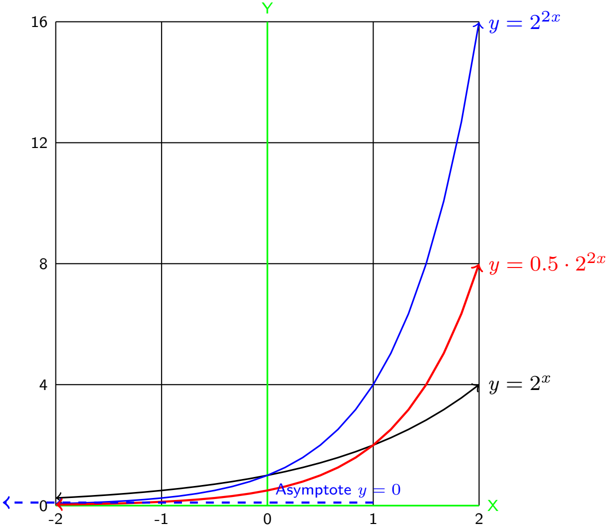


Solutions to Review Exercises 3.1.2 and 3.1.3.

Example 3.1.2b: Transform the graph of $y = 2^x$ step by step to the graph of $y = 0.5 \cdot 2^{2x}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^x$.

- Substituting $2x$ for x in $y = 2^x$ shrinks the graph by a factor of 2 toward the y-axis to the graph of $y = 2^{2x}$.
- Multiplying the RHS of the equation $y = 2^{2x}$ by $\frac{1}{2}$ shrinks its graph by a factor of 2 toward the x-axis to the graph of $y = 0.5 \cdot 2^{2x}$.
- The asymptote $y = 0$ of the original graph does not change.

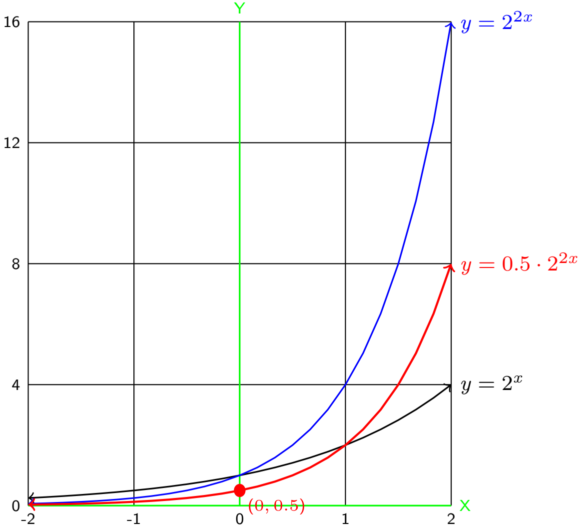


Solutions to Review Exercises 3.1.2 and 3.1.3.

Example 3.1.2b: Transform the graph of $y = 2^x$ step by step to the graph of $y = 0.5 \cdot 2^{2x}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

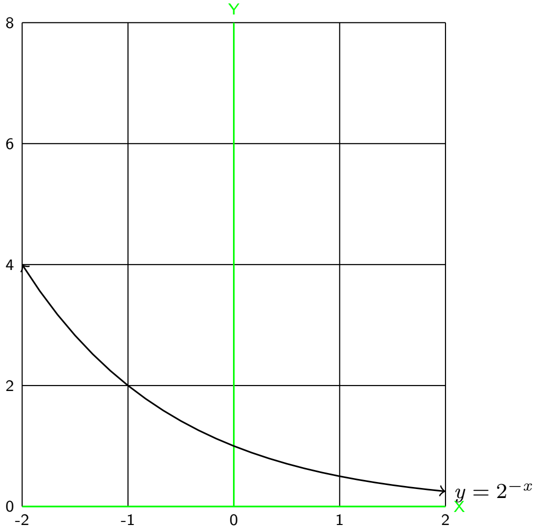
Solution: Start with the graph of $y = 2^x$.

- Substituting $2x$ for x in $y = 2^x$ shrinks the graph by a factor of 2 toward the y-axis to the graph of $y = 2^{2x}$.
- Multiplying the RHS of the equation $y = 2^{2x}$ by $\frac{1}{2}$ shrinks its graph by a factor of 2 toward the x-axis to the graph of $y = 0.5 \cdot 2^{2x}$.
- The asymptote $y = 0$ of the original graph does not change.
- Find the y-intercept by setting $x = 0$ in $y = 0.5 \cdot 2^{2x}$, so the $y-$intercept is 0.5. The graph meets the y-axis at $(0, 0.5)$.



Example 3.1.2c: Transform the graph of $y = 2^{-x}$ to the graph of $y = 1 + 2^{x}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

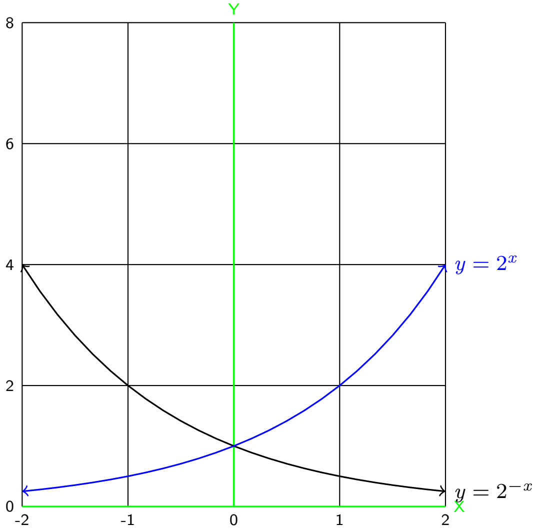
Solution: Start with the graph of $y = 2^{-x}$.



Example 3.1.2c: Transform the graph of $y = 2^{-x}$ to the graph of $y = 1 + 2^x$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^{-x}$.

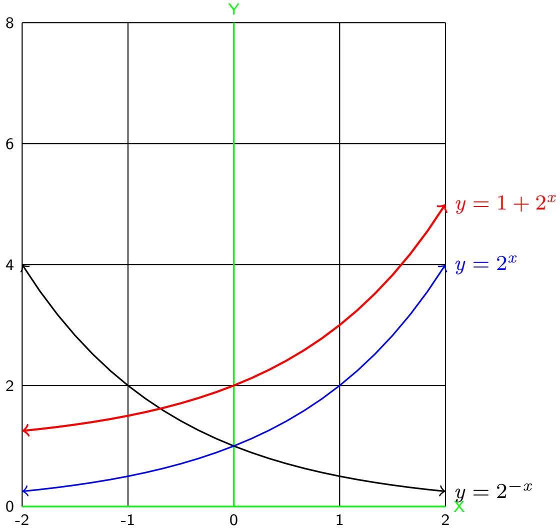
- Substituting $-x$ for x in $y = 2^{-x}$ reflects the graph across the y-axis to the graph of $y = 2^x$.



Example 3.1.2c: Transform the graph of $y = 2^{-x}$ to the graph of $y = 1 + 2^x$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^{-x}$.

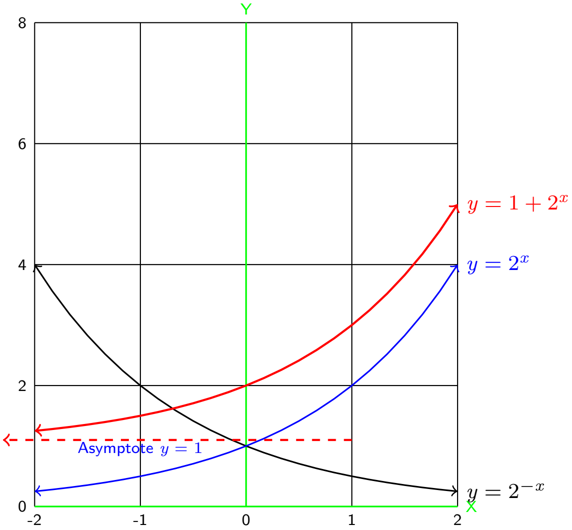
- Substituting $-x$ for x in $y = 2^{-x}$ reflects the graph across the y-axis to the graph of $y = 2^x$.
- Adding 1 the RHS of the equation $y = 2^x$ shifts its graph up 1 to the graph of $y = 1 + 2^x$.



Example 3.1.2c: Transform the graph of $y = 2^{-x}$ to the graph of $y = 1 + 2^x$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^{-x}$.

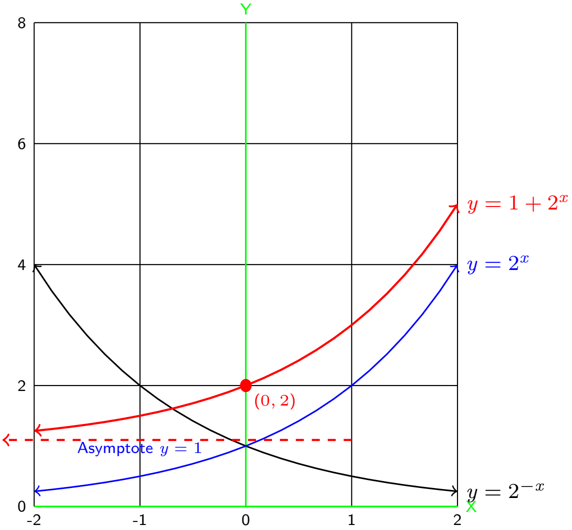
- Substituting $-x$ for x in $y = 2^{-x}$ reflects the graph across the y-axis to the graph of $y = 2^x$.
- Adding 1 the RHS of the equation $y = 2^x$ shifts its graph up 1 to the graph of $y = 1 + 2^x$.
- The asymptote $y = 0$ shifts up 1 to $y = 1$.



Example 3.1.2c: Transform the graph of $y = 2^{-x}$ to the graph of $y = 1 + 2^x$. Sketch the resulting graph. Find and label its intercepts and asymptote.

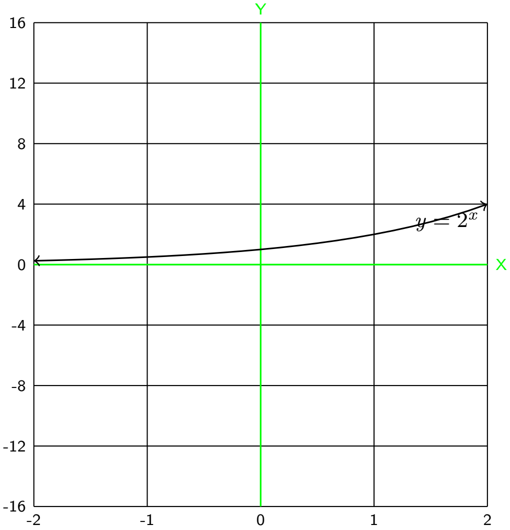
Solution: Start with the graph of $y = 2^{-x}$.

- Substituting $-x$ for x in $y = 2^{-x}$ reflects the graph across the y-axis to the graph of $y = 2^x$.
- Adding 1 the RHS of the equation $y = 2^x$ shifts its graph up 1 to the graph of $y = 1 + 2^x$.
- The asymptote $y = 0$ shifts up 1 to $y = 1$.
- Find the y-intercept by setting $x = 0$ in $y = 1 + 2^x \Rightarrow y = 1 + 2^0 = 2$. The graph meets the y-axis at $(0, 2)$.
- The graph of $y = 1 + 2^x$ has no $x-$intercept since $2^x > 0$ for all x.



Example 3.1.2d: Transform the equation $y = 2^x$ to $y = -2^{2x+1}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

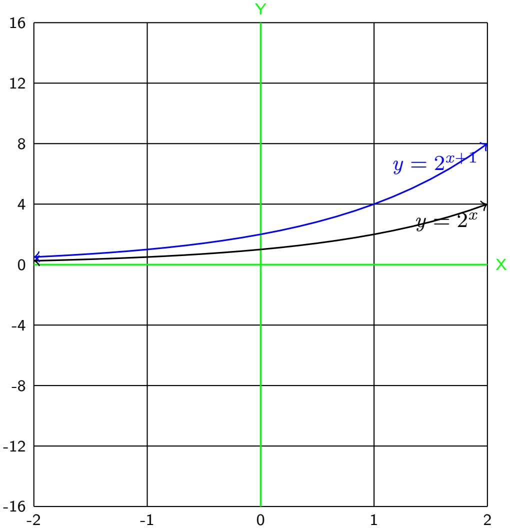
Solution: Start with the graph of $y = 2^x$.



Example 3.1.2d: Transform the equation $y = 2^x$ to $y = -2^{2x+1}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^x$.

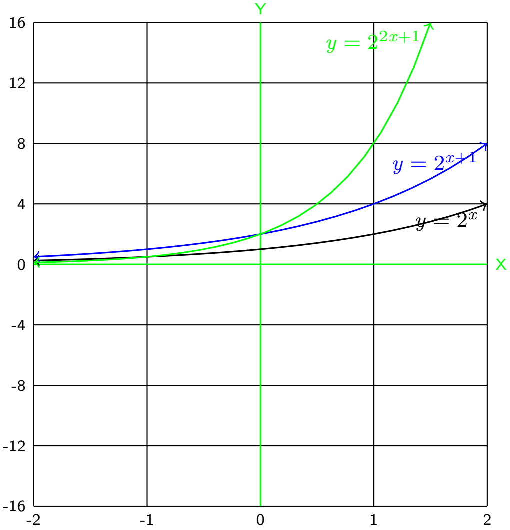
- Substituting $x + 1$ for x in $y = 2^x$ shifts the graph 1 unit left to the graph of $y = 2^{x+1}$.



Example 3.1.2d: Transform the equation $y = 2^x$ to $y = -2^{2x+1}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^x$.

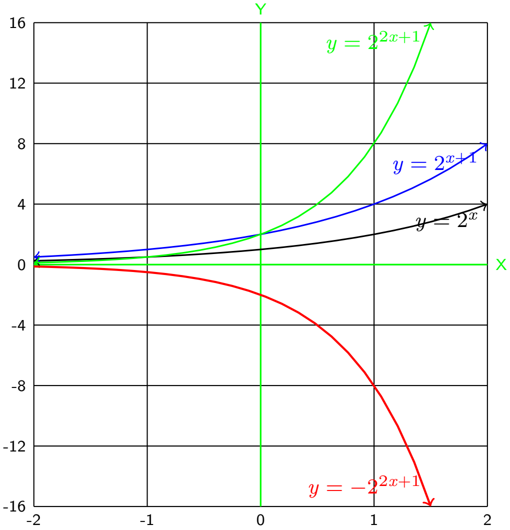
- Substituting $x + 1$ for x in $y = 2^x$ shifts the graph 1 unit left to the graph of $y = 2^{x+1}$.
- Substituting $2x$ for x in $y = 2^{x+1}$ contracts the graph by a factor of 2 toward the x-axis to the graph of $y = 2^{2x+1}$.



Example 3.1.2d: Transform the equation $y = 2^x$ to $y = -2^{2x+1}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^x$.

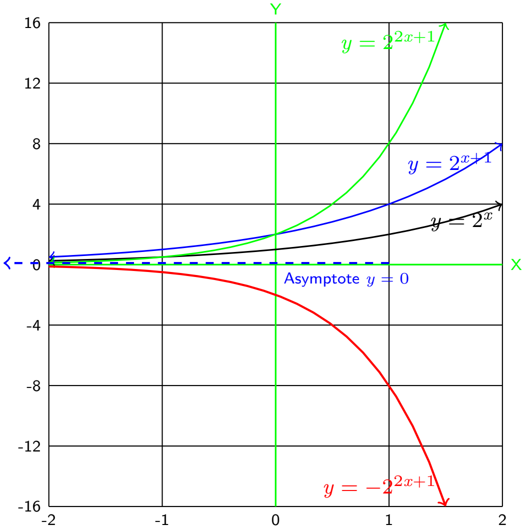
- Substituting $x + 1$ for x in $y = 2^x$ shifts the graph 1 unit left to the graph of $y = 2^{x+1}$.
- Substituting $2x$ for x in $y = 2^{x+1}$ contracts the graph by a factor of 2 toward the x-axis to the graph of $y = 2^{2x+1}$.
- Multiplying the RHS of $y = 2^{2x+1}$ by -1 reflects its graph across the x-axis to give the requested graph of $y = -2^{2x+1}$.



Example 3.1.2d: Transform the equation $y = 2^x$ to $y = -2^{2x+1}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^x$.

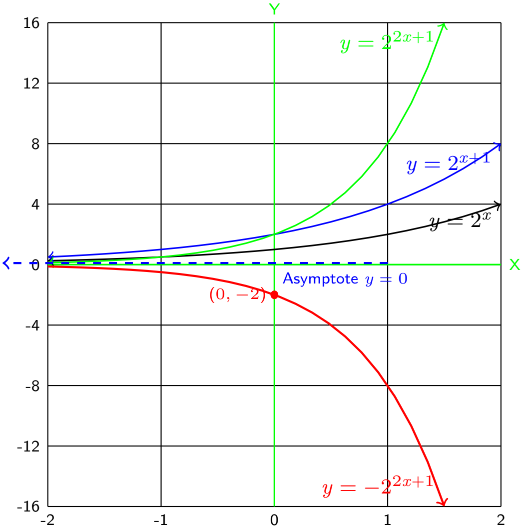
- Substituting $x + 1$ for x in $y = 2^x$ shifts the graph 1 unit left to the graph of $y = 2^{x+1}$.
- Substituting $2x$ for x in $y = 2^{x+1}$ contracts the graph by a factor of 2 toward the x-axis to the graph of $y = 2^{2x+1}$.
- Multiplying the RHS of $y = 2^{2x+1}$ by -1 reflects its graph across the x-axis to give the requested graph of $y = -2^{2x+1}$.
- The asymptote $y = 0$ of the original graph is unchanged. There is no x-intercept.



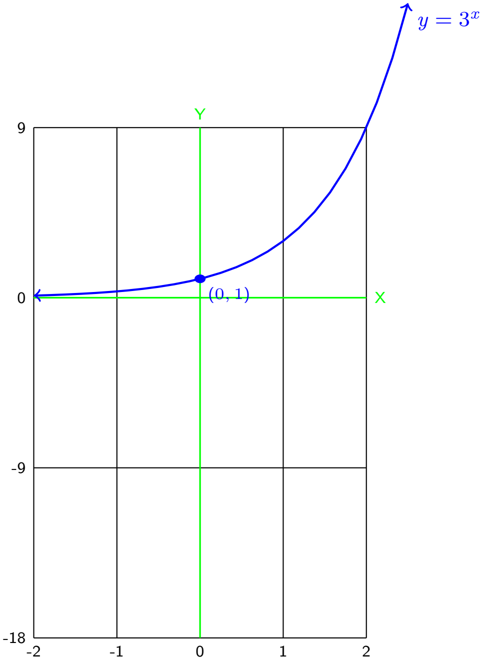
Example 3.1.2d: Transform the equation $y = 2^x$ to $y = -2^{2x+1}$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = 2^x$.

- Substituting $x + 1$ for x in $y = 2^x$ shifts the graph 1 unit left to the graph of $y = 2^{x+1}$.
- Substituting $2x$ for x in $y = 2^{x+1}$ contracts the graph by a factor of 2 toward the x-axis to the graph of $y = 2^{2x+1}$.
- Multiplying the RHS of $y = 2^{2x+1}$ by -1 reflects its graph across the x-axis to give the requested graph of $y = -2^{2x+1}$.
- The asymptote $y = 0$ of the original graph is unchanged. There is no x-intercept.
- Set $x = 0$ to find the $y-$intercept: $-2^{0+1} = -2$. The graph meets the y-axis at $(0, -2)$.



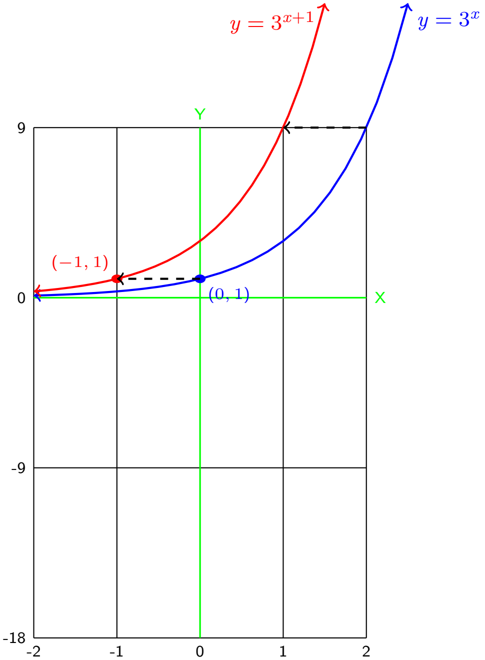
Example 3.1.3: Sketch the graph of $y = 9 - 3^{x+1}$.
Show intercepts and asymptote(s).
Solution: Start with the graph of $y = 3^x$, with asymptote the x-axis and y-intercept $(0, 1)$.



Example 3.1.3: Sketch the graph of $y = 9 - 3^{x+1}$.
Show intercepts and asymptote(s).

Solution: Start with the graph of $y = 3^x$, with asymptote the x-axis and y-intercept $(0, 1)$.

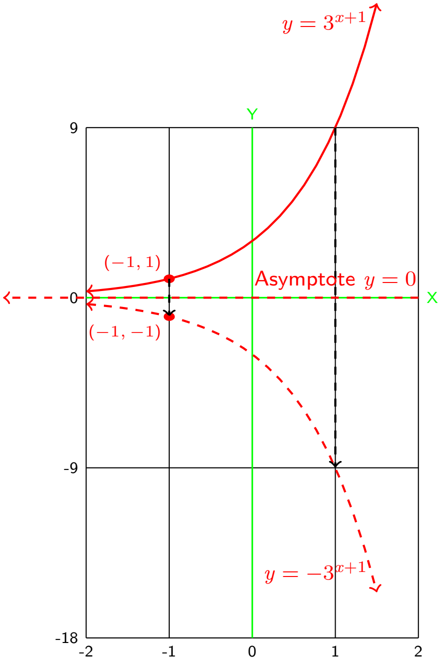
- Substituting $x + 1$ for x in $y = 3^x$ shifts the graph 1 unit *left* to the graph of $y = 3^{x+1}$. The asymptote is still the x-axis. Point $(0, 1)$ moves to $(-1, 1)$.



Example 3.1.3: Sketch the graph of $y = 9 - 3^{x+1}$. Show intercepts and asymptote(s).

Solution: Start with the graph of $y = 3^x$, with asymptote the x-axis and y-intercept $(0, 1)$.

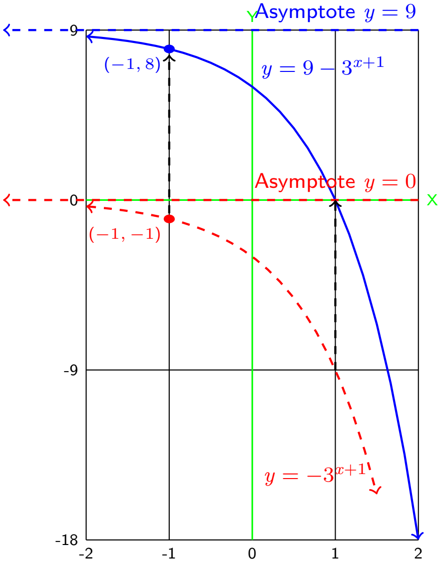
- Substituting $x + 1$ for x in $y = 3^x$ shifts the graph 1 unit *left* to the graph of $y = 3^{x+1}$. The asymptote is still the x-axis. Point $(0, 1)$ moves to $(-1, 1)$.
- Multiplying the RHS of the equation $y = 3^{x+1}$ by -1 reflects its graph across the x-axis to the graph of $y = -3^{x+1}$. The asymptote is still the x-axis. Point $(-1, 1)$ reflects to point $(-1, -1)$.



Example 3.1.3: Sketch the graph of $y = 9 - 3^{x+1}$. Show intercepts and asymptote(s).

Solution: Start with the graph of $y = 3^x$, with asymptote the x-axis and y-intercept $(0, 1)$.

- Substituting $x + 1$ for x in $y = 3^x$ shifts the graph 1 unit *left* to the graph of $y = 3^{x+1}$. The asymptote is still the x-axis. Point $(0, 1)$ moves to $(-1, 1)$.
- Multiplying the RHS of the equation $y = 3^{x+1}$ by -1 reflects its graph across the x-axis to the graph of $y = -3^{x+1}$. The asymptote is still the x-axis. Point $(-1, 1)$ reflects to point $(-1, -1)$.
- Adding 9 to the RHS of $y = -3^{x+1}$ shifts its graph up 9 units to give the requested graph of $y = 9 - 3^{x+1}$. The asymptote (the x-axis $y = 0$) now moves up 9 units to become the horizontal line $y = 9$. Point $(-1, -1)$ moves up 9 units to point $(-1, 8)$.



Example 3.1.3: Sketch the graph of $y = 9 - 3^{x+1}$.
Show intercepts and asymptote(s).

Solution: Start with the graph of $y = 3^x$, with asymptote the x-axis and y-intercept $(0, 1)$.

- Substituting $x + 1$ for x in $y = 3^x$ shifts the graph 1 unit *left* to the graph of $y = 3^{x+1}$. The asymptote is still the x-axis. Point $(0, 1)$ moves to $(-1, 1)$.
- Multiplying the RHS of the equation $y = 3^{x+1}$ by -1 reflects its graph across the x-axis to the graph of $y = -3^{x+1}$. The asymptote is still the x-axis. Point $(-1, 1)$ reflects to point $(-1, -1)$.
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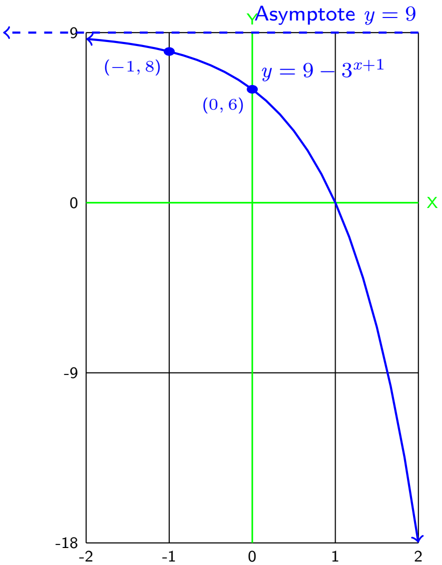
- Set $x = 0$ to find the y-intercept: $y = 9 - 3^{x+1} = 9 - 3^1 = 9 - 3 = 6$, and so the new y-intercept point is $(0, 6)$.



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Solution: Start with the graph of $y = 3^x$, with asymptote the x-axis and y-intercept $(0, 1)$.

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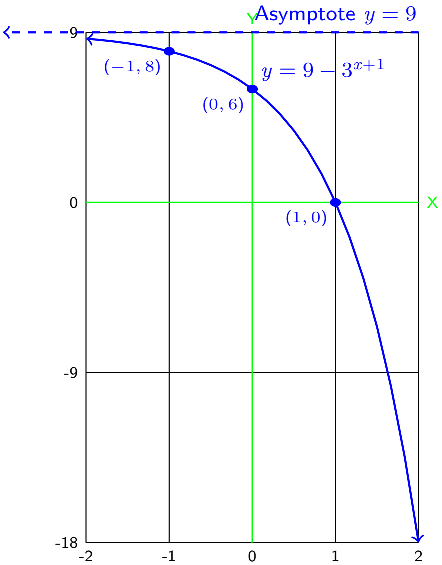
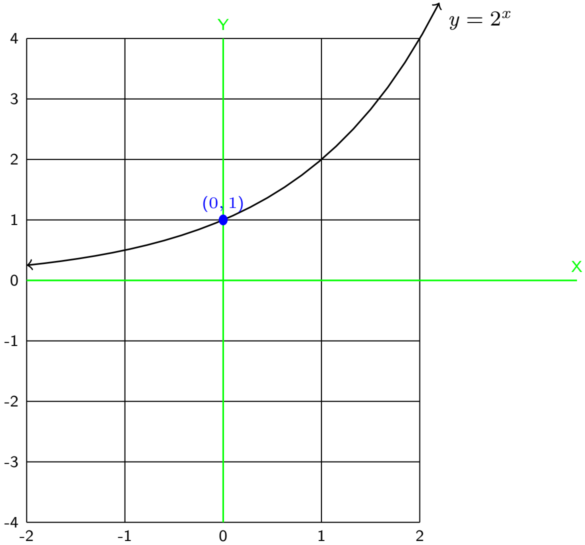
- Set $x = 0$ to find the y-intercept: $y = 9 - 3^{x+1} = 9 - 3^1 = 9 - 3 = 6$, and so the new y-intercept point is $(0, 6)$.
- Set $y = 0$ to find the x-intercept: $0 = 9 - 3^{x+1} \Rightarrow 9 = 3^{x+1} \Rightarrow x + 1 = 2 \Rightarrow x = 1$. The graph's x-intercept point is $(1, 0)$.



Example 3.1.4: Graph the equation $y = 2^{x-2} - 3$. Show intercepts and asymptote(s).

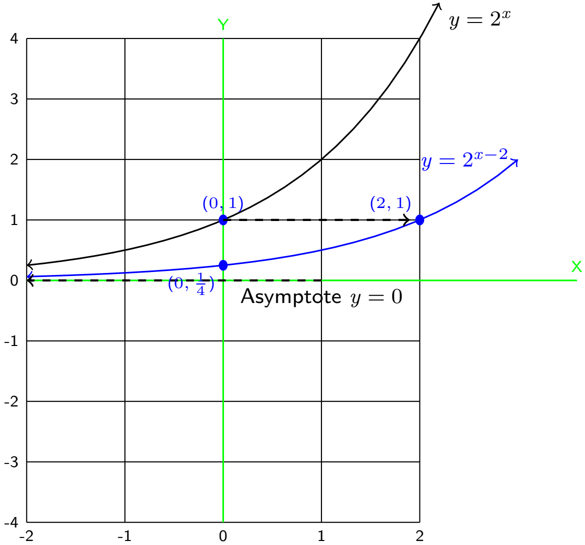
Solution: Start with the graph of $y = 2^x$, with asymptote the x-axis and y-intercept $(0, 1)$?



Example 3.1.4: Graph the equation $y = 2^{x-2} - 3$. Show intercepts and asymptote(s).

Solution: Start with the graph of $y = 2^x$, with asymptote the x-axis and y-intercept $(0, 1)$?

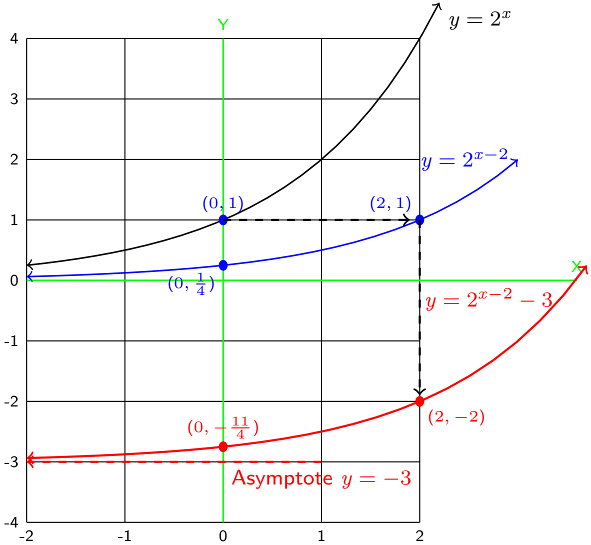
- Substituting $x - 2$ for x in $y = 2^x$ shifts the graph 2 units *right* to the graph of $y = 2^{x-2}$. The asymptote is still the x-axis. The y-intercept moves to point $(0, \frac{1}{4})$.



Example 3.1.4: Graph the equation $y = 2^{x-2} - 3$. Show intercepts and asymptote(s).

Solution: Start with the graph of $y = 2^x$, with asymptote the x-axis and y-intercept $(0, 1)$?

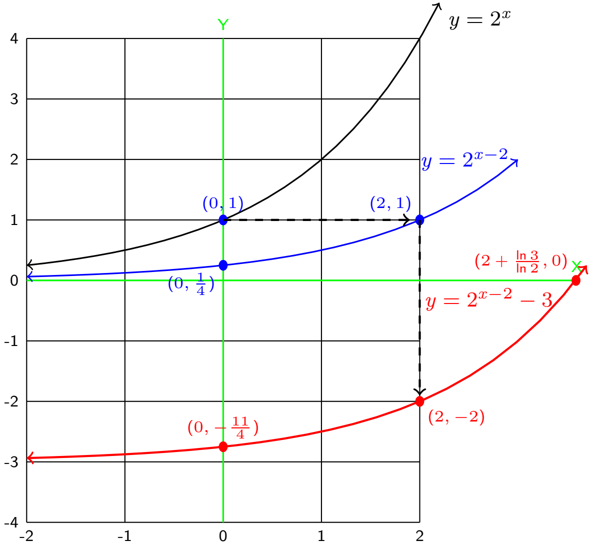
- Substituting $x - 2$ for x in $y = 2^x$ shifts the graph 2 units *right* to the graph of $y = 2^{x-2}$. The asymptote is still the x-axis. The y-intercept moves to point $(0, \frac{1}{4})$.
- Subtract 3 from the RHS of the equation $y = 2^{x-2}$ by -1 to shift the graph 3 units down to the graph of $y = -3^{x+1}$. The asymptote also shifts down to $y = -3$.



Example 3.1.4: Graph the equation $y = 2^{x-2} - 3$. Show intercepts and asymptote(s).

Solution: Start with the graph of $y = 2^x$, with asymptote the x-axis and y-intercept $(0, 1)$?

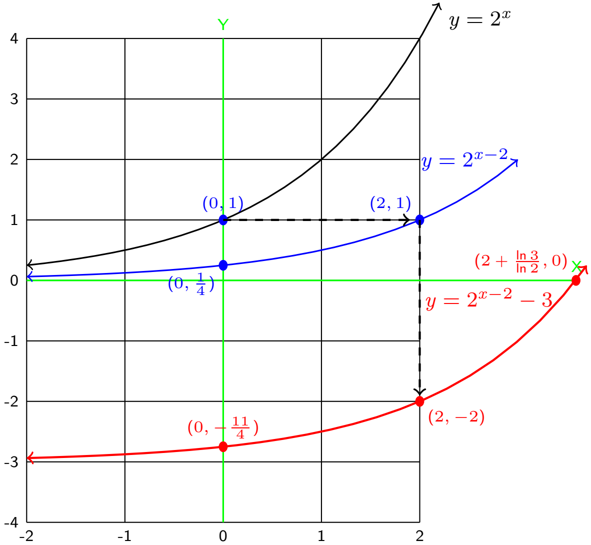
- Substituting $x - 2$ for x in $y = 2^x$ shifts the graph 2 units *right* to the graph of $y = 2^{x-2}$. The asymptote is still the x-axis. The y-intercept moves to point $(0, \frac{1}{4})$.
- Subtract 3 from the RHS of the equation $y = 2^{x-2}$ by -1 to shift the graph 3 units down to the graph of $y = -3^{x+1}$. The asymptote also shifts down to $y = -3$.
- Set $x = 0$ to find the y-intercept: $2^{x-2} - 3 = 2^{-2} - 3 = -\frac{11}{4}$, and so the new y-intercept is $(0, -\frac{11}{4})$.



Example 3.1.4: Graph the equation $y = 2^{x-2} - 3$. Show intercepts and asymptote(s).

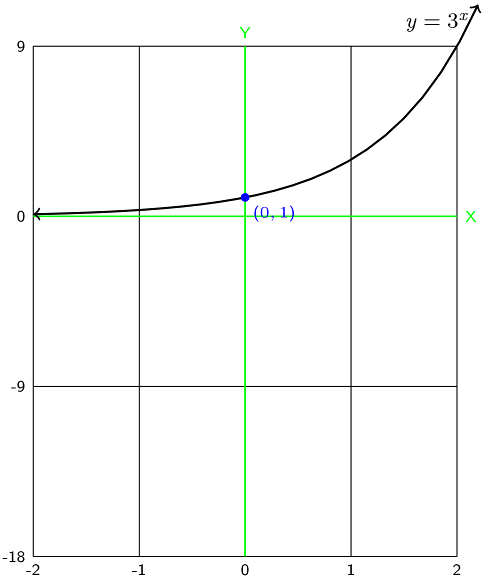
Solution: Start with the graph of $y = 2^x$, with asymptote the x-axis and y-intercept $(0, 1)$?

- Substituting $x - 2$ for x in $y = 2^x$ shifts the graph 2 units *right* to the graph of $y = 2^{x-2}$. The asymptote is still the x-axis. The y-intercept moves to point $(0, \frac{1}{4})$.
- Subtract 3 from the RHS of the equation $y = 2^{x-2}$ by -1 to shift the graph 3 units down to the graph of $y = -3^{x+1}$. The asymptote also shifts down to $y = -3$.
- Set $x = 0$ to find the y-intercept: $2^{x-2} - 3 = 2^{-2} - 3 = -\frac{11}{4}$, and so the new y-intercept is $(0, -\frac{11}{4})$.
- Set $y = 0$ to find the x-intercept: $2^{x-2} - 3 = 0 \Rightarrow 2^{x-2} = 3 \Rightarrow x = 2 + \frac{\ln 3}{\ln 2} \approx 3.59$.



Example 3.1.3c: Sketch the graph of $y = -3^{2x+1}$. Show intercepts and asymptote(s).

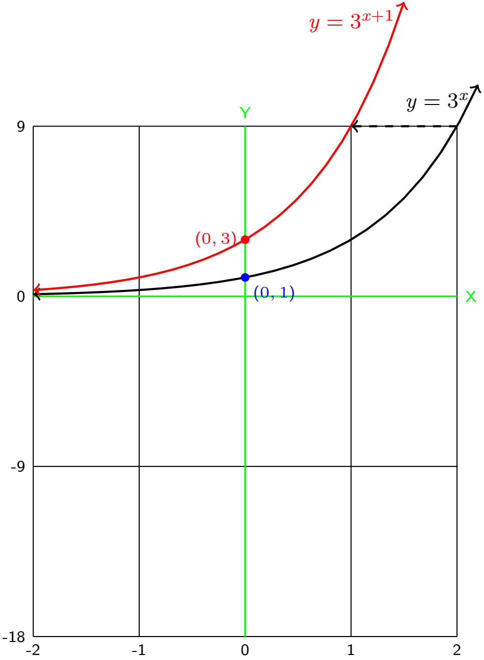
Solution: Start with the graph of $y = 3^x$, with asymptote the x-axis and y-intercept point $(0, 1)$.



Example 3.1.3c: Sketch the graph of $y = -3^{2x+1}$. Show intercepts and asymptote(s).

Solution: Start with the graph of $y = 3^x$, with asymptote the x-axis and y-intercept point $(0, 1)$.

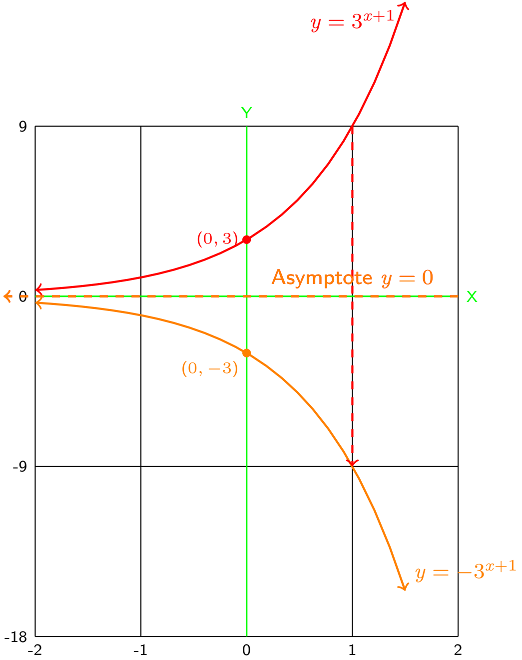
- Substituting $x + 1$ for x in $y = 3^x$ shifts the graph 1 unit *left* to the graph of $y = 3^{x+1}$. The asymptote is still the x-axis. Point $(2, 9)$ moves to $(1, 9)$. The $y-$intercept point is $(0, 3)$.



Example 3.1.3c: Sketch the graph of $y = -3^{2x+1}$. Show intercepts and asymptote(s).

Solution: Start with the graph of $y = 3^x$, with asymptote the x-axis and y-intercept point $(0, 1)$.

- Substituting $x + 1$ for x in $y = 3^x$ shifts the graph 1 unit *left* to the graph of $y = 3^{x+1}$. The asymptote is still the x-axis. Point $(2, 9)$ moves to $(1, 9)$. The $y-$intercept point is $(0, 3)$.
- Multiplying the RHS of the equation $y = 3^{x+1}$ by -1 reflects its graph across the x-axis to the graph of $y = -3^{x+1}$. The asymptote is still the x-axis. $(9, 0)$ reflects to $(-9, 0)$. The new y-intercept point is $(0, -3)$



Example 3.1.3c: Sketch the graph of $y = -3^{2x+1}$.
Show intercepts and asymptote(s).
Solution: Start with the graph of $y = 3^x$, with asymptote the x-axis and y-intercept point $(0, 1)$.

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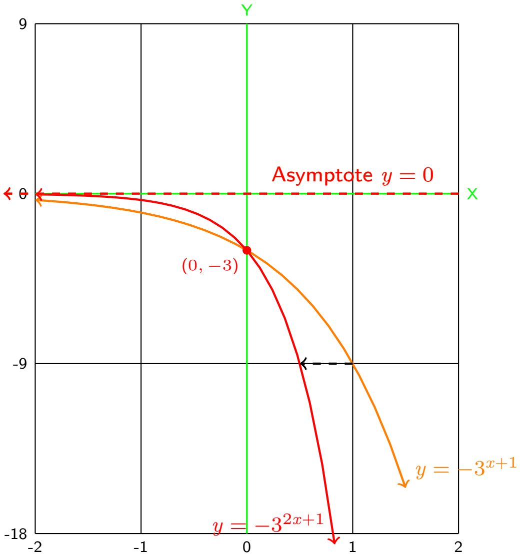
- Multiplying the RHS of the equation $y = 3^{x+1}$ by -1 reflects its graph across the x-axis to the graph of $y = -3^{x+1}$. The asymptote is still the x-axis. $(9, 0)$ reflects to $(-9, 0)$. The new y-intercept point is $(0, -3)$
- Substituting $2x$ for x in $y = -3^{x+1}$ shrinks its graph by a factor of 2 toward the x-axis to give the requested graph of -3^{2x+1}. The asymptote remains $y = 0$. Point $(1, -9)$ moves to $(\frac{1}{2}, -9)$.



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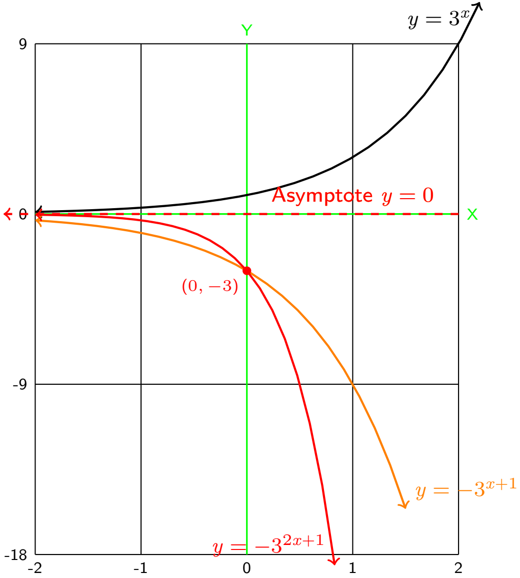
- Substituting $2x$ for x in $y = -3^{x+1}$ shrinks its graph by a factor of 2 toward the x-axis to give the requested graph of -3^{2x+1}. The asymptote remains $y = 0$. Point $(1, -9)$ moves to $(\frac{1}{2}, -9)$.
- Set $x = 0$ to find the y-intercept: $y = -3^{2x+1} = -3$, and so the new y-intercept point is $(0, -3)$.



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Show intercepts and asymptote(s).

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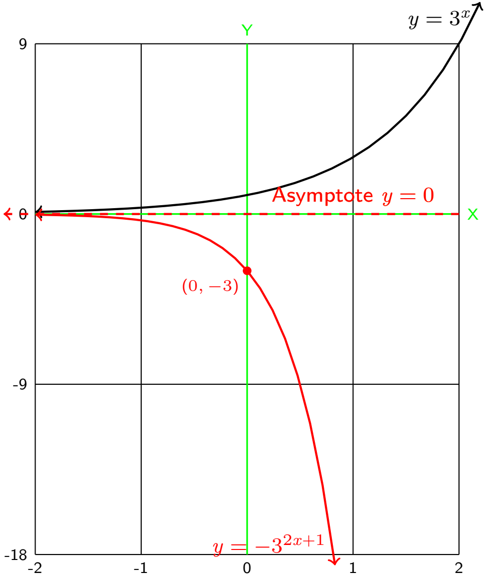
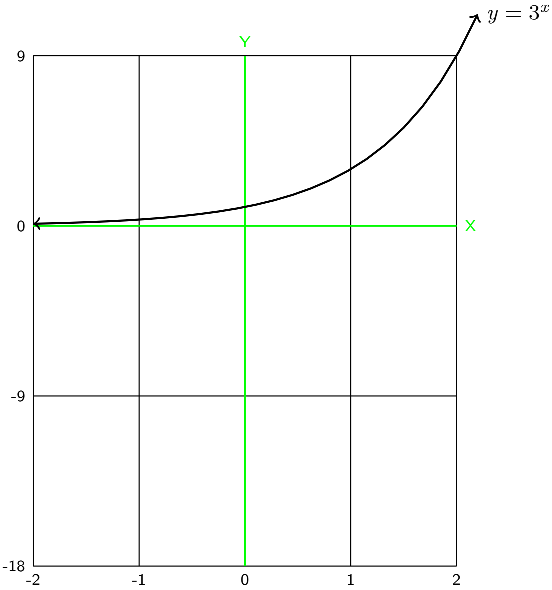
- Substituting $2x$ for x in $y = -3^{x+1}$ shrinks its graph by a factor of 2 toward the x-axis to give the requested graph of -3^{2x+1}. The asymptote remains $y = 0$. Point $(1, -9)$ moves to $(\frac{1}{2}, -9)$.
- Set $x = 0$ to find the y-intercept: $y = -3^{2x+1} = -3$, and so the new y-intercept point is $(0, -3)$.
- Set $y = 0$ to find the x-intercept: $0 = -3^{3x+1}$: No solution, so no $x-$intercept.



Example 3.1.3d: Sketch the graph of $y = 9 - 3^{2x+1}$. Show intercepts and asymptote(s).

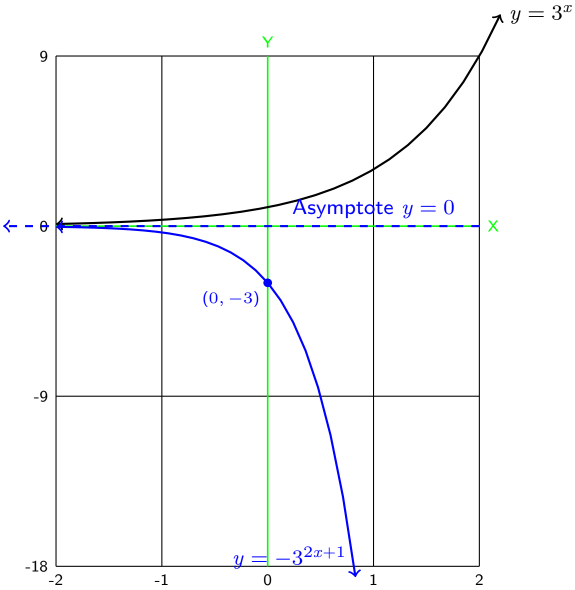
Solution: Start with the graph of $y = 3^x$, with asymptote the x-axis no y-intercept.



Example 3.1.3d: Sketch the graph of $y = 9 - 3^{2x+1}$. Show intercepts and asymptote(s).

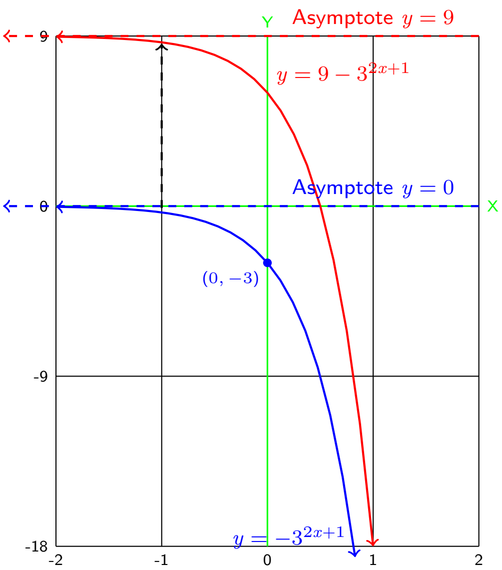
Solution: Start with the graph of $y = 3^x$, with asymptote the x-axis no y-intercept.

- As in the previous example, Substituting $x + 1$ for x in $y = 3^x$, multiplying the RHS of the equation $y = 3^{x+1}$ by -1 , then substituting $2x$ for x in $y = -3^{x+1}$ gives the graph of -3^{2x+1}. The asymptote remains $y = 0$.



Example 3.1.3d: Sketch the graph of $y = 9 - 3^{2x+1}$.
Show intercepts and asymptote(s).
Solution: Start with the graph of $y = 3^x$, with asymptote the x-axis no y-intercept.

- As in the previous example, Substituting $x + 1$ for x in $y = 3^x$, multiplying the RHS of the equation $y = 3^{x+1}$ by -1 , then substituting $2x$ for x in $y = -3^{x+1}$ gives the graph of -3^{2x+1}. The asymptote remains $y = 0$.
- Add 9 to the RHS of $y = -3^{2x+1}$ to shift its graph up 9 to give the requested graph of $y = 9 - 3^{2x+1}$.
  The horizontal asymptote shifts up to $y = 9$.



Example 3.1.3d: Sketch the graph of $y = 9 - 3^{2x+1}$. Show intercepts and asymptote(s).

Solution: Start with the graph of $y = 3^x$, with asymptote the x-axis no y-intercept.

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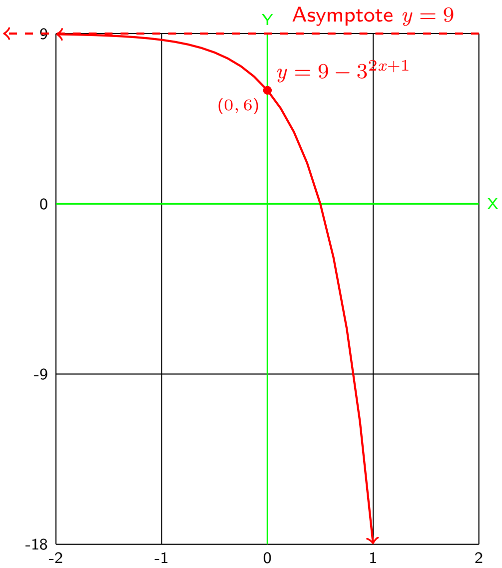
  The horizontal asymptote shifts up to $y = 9$.
- Set $x = 0$ to find the y-intercept: $y = 9 - 3^{2x+1} = 6$, and so the new y-intercept point is $(0, 6)$.



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- Add 9 to the RHS of $y = -3^{2x+1}$ to shift its graph up 9 to give the requested graph of $y = 9 - 3^{2x+1}$.
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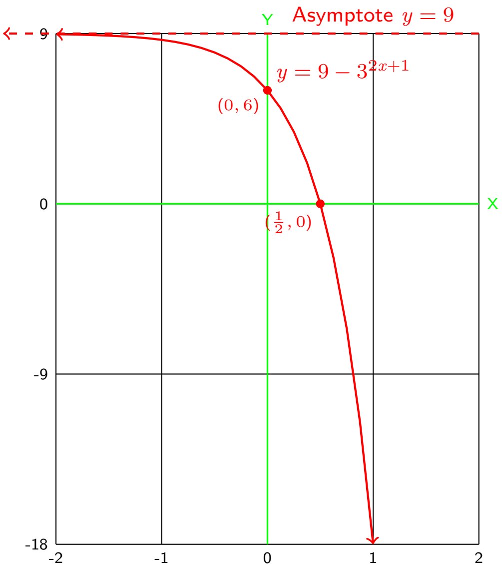
- Set $x = 0$ to find the y-intercept: $y = 9 - 3^{2x+1} = 6$, and so the new y-intercept point is $(0, 6)$.
- Set $y = 0$ to find the x-intercept: $0 = 9 - 3^{2x+1}$ when $2x + 1 = 2 \Rightarrow x = \frac{1}{2}$. The x-intercept point is $(\frac{1}{2}, 0)$.



Chapter 3 Section 2: The natural exponential function

The previous section mentioned that scientific work very often uses an unusual base $e \approx 2.718281828$, and that the *natural exponential function* is $f(x) = e^x$. To see why e is a useful base, you will need to take calculus. But here we will discuss how e arises in everyday life.

The number e is defined in a strange way. It is equal to the limit, as x gets close to zero, of $(1+x)^{\frac{1}{x}}$. This fancy language just means that $(1+x)^{\frac{1}{x}}$ is an excellent approximation to e if x is very,very close to zero.

One everyday problem that uses e is to calculate the risk of engaging very often in behavior that poses a tiny risk. Suppose the risk of catching Covid-19 by going into a store is 1 in 1000. What happens if you go into the store 1000 times?

Each time you go into the store, the chance that you will avoid Covid-19 is $1 - \frac{1}{1000} = \frac{999}{1000} = .999$.

The chance that you will avoid Covid-19 if you go into that store twice is $.999 \cdot .999$, and the chance that you will avoid Covid-19 if you go into that store 1000 times is $.999^{1000} = (1 + -.001)^{1000} = \frac{1}{(1+ \ -.001)^{-1000}}$ This equals $\frac{1}{(1+x)^{\frac{1}{x}}}$ where $x = -.001$, which is close to 0.

So this should be close to $\frac{1}{e}$.

Indeed, $\frac{1}{e} \approx 0.3679$ while $\frac{1}{.999^{1000}} \approx 0.3677$.

In our scenario, the chance of

avoiding Covid-19 is $\approx 36.77\%$ and so the chance of catching Covid-19 is $\approx 63.23\%$

This calculation shows two things. First the risk of catching Covid-19 is not 100%, as you might have guessed. But it's still very high.

Example 1. Suppose that the risk of going into a store once is 1 in 1000, and you go into that store 1000 times a year for 8 years. What is the likelihood that you will catch Covid-19?

Solution. This is the same as the example discussed, except that the exponent 1000 is now 8000. The probability of avoiding Covid-19 is $.999^{8000} \approx .0003$ and so the risk of catching Covid-19 is $1 - .999^{8000} \approx .9997 = 99.97$ %.

The moral: avoid repeated low-risk scenarios.

3.2 Quiz

Ex. 3.2.1: Suppose that the risk of going into a store once is 1 in 1000, and you go into that store 1000 times a year for 8 years. What is the likelihood that you will catch Covid-19?

Section 3.2 Review: Natural exponential function

Ex. 3.2.1: Example 1. if the risk of catching Covid-19 by going into the store once is 0.001, what is the likelihood that you will catch Covid-19 if you go into a store

- 1000 times a year for 8 years ?
- 1000 times a year for 4 years ?
- 100 times a year for 8 years ?
- 100 times a year for 4 years ?

if the risk of catching Covid-19 by going into the store once is 0.01 %, what is the likelihood that at least one person will catch Covid-19 if

- 5000 people go into the store once a month for 8 months ?
- 1000 people go into the store once a month for 4 months?
- 400 people go into the store once a week for 2 weeks ?
- 400 people go into a store once ?

Section 3.2 Review: Natural exponential function

Ex. 3.2.1: Example 1. if the risk of catching Covid-19 by going into the store once is 0.001, what is the likelihood that you will catch Covid-19 if you go into a store

- 1000 times a year for 8 years ? $1 - .999^{8000} \approx .9997 = 99.97$ %.
- 1000 times a year for 4 years ? $1 - .999^{4000} \approx .9817 = 98.17$ %.
- 100 times a year for 8 years ? $1 - .999^{800} \approx .5509 = 55.09$ %.
- 100 times a year for 4 years ? $1 - .999^{400} \approx .3298 = 32.98$ %.

if the risk of catching Covid-19 by going into the store once is 0.01 %, what is the likelihood that at least one person will catch Covid-19 if

- 5000 people go into the store once a month for 8 months ? $1 - .9999^{40000} \approx 98.17$ %.
- 1000 people go into the store once a month for 4 months? $1 - .9999^{4000} \approx 32.97$ %.
- 400 people go into the store once a week for 2 weeks ? $1 - .9999^{800} \approx 7.69$ %.
- 400 people go into a store once ? $1 - .9999^{400} \approx 3.92$ %.

Chapter 3 Section 3: Logarithmic functions

- 3.3.1: Finding values of log functions
- 3.3.2: Log is the inverse function of exp
- 3.3.3: Is there a preferred base for log functions?
- 3.3.4: Transforming logarithm graphs
- 3.3.5: More examples of transformations
- 3.3.6: Section 3.3 Quiz

Section 3.3 Preview: Definitions and Procedures

- Definition 3.3.1:Definition of logarithmic function
- Definition 3.3.2: The inverse function of $y = e^x$ is $y = \ln(x)$.

- Procedure 3.3.1: How to figure out $\log_a(K)$
- Procedure 3.3.2: How to switch between log and exponential functions

3.3.1 Finding values of logarithm functions

In section 3.1 we studied exponential functions $y = a^x$ with various bases $a > 0$. For all such a, the function function a^x has domain $(-\infty, \infty)$ and range $(0, \infty)$. The function is

- increasing if $a > 1$ and
- decreasing if $0 < a < 1$.

Both increasing and decreasing functions are one-to-one. Therefore $y = a^x$ has an inverse function if $0 < a < 1$ or $a > 1$.

Definition of logarithmic functions

Suppose $a > 0$ and $a \neq 1$.

- The function $y = a^x$ has an inverse function $y = log_a(x)$, defined for $x > 0$.
- For every $x > 0$, $\boxed{a^{log_a(x)} = x}$.
- For every real x, $\boxed{\log_a(a^x) = x}$.

  Since $a^0 = 1$ and $a^1 = a$:
- $\boxed{log_a(1) = 0}$ $\boxed{log_a(a) = 1}$.

How to figure out $\log_a(K)$

- Rewrite K as $K = a^x$. Then $\log_a(K) = \log_a(a^x) = x$.
- If you can't rewrite K as $K = a^x$, use a calculator.

Example 1: Find $\log_2(16), \log_2(\frac{1}{8}), \log_2(8\sqrt{2})$, and $\log_2(24)$

Solutions:

- $\log_2(16) = \log_2(2^4) = 4$.
- $\frac{1}{8} = \frac{1}{2^3} = 2^{-3}$, and so $\log_2(\frac{1}{8}) = \log_2(2^{-3}) = -3$.
- $8\sqrt{2} = 2^3 \cdot 2^{1/2} = 2^{3+\frac{1}{2}} = 2^{\frac{7}{2}}$, and so $\log_2(8\sqrt{2}) = \log_2(2^{\frac{7}{2}}) = \frac{7}{2}$.
- There is no obvious way to rewrite 24 as 2^x . Use a calculator.

The inverse function of $y = 10^x$ is $y = \log_{10}(x)$, the *common logarithm of* x, defined by $\boxed{\log_{10}(10^x) = x}$.

Example 2: Find $\log_{10}(1000000)$ and $\log_{10}(.0001)$.

Solutions:

- $\log_{10}(1000000) = \log_{10}(10^6) = 6$.
- $\log_{10}(.0001) = \log_{10}(\frac{1}{10^4}) = \log_{10}(10^{-4}) = -4$.

The inverse function of $y = e^x$ is $y = \ln(x)$.

- $\ln(x)$ is the natural logarithm of x.
- Its defining property is $\ln(e^x) = x$.

Example 3: Find $\ln(1), \ln(e)$, and $\ln(\frac{1}{\sqrt{e}})$.

Solutions:

- $\ln(1) = \ln(e^0) = 0.$
- $\ln(e) = \ln(e^1) = 1.$
- $\ln(\frac{1}{\sqrt{e}}) = \ln\left(\frac{1}{e^{\frac{1}{2}}}\right) = \ln(e^{-\frac{1}{2}}) = -\frac{1}{2}.$

How to switch between log and exponential functions

- $y = a^x$ if and only if $x = \log_a y$.
- $y = 2^x$ if and only if $x = \log_2 y$.
- $y = 10^x$ if and only if $x = \log_{10} y = \log y$.

Example 4: Solve $\log_2(x+2) = 3$

Solution: Rewrite: $2^3 = x + 2$

Solve: $x = 2^3 - 2 = 8 - 2 = 6$ Answer: $x = 6$.

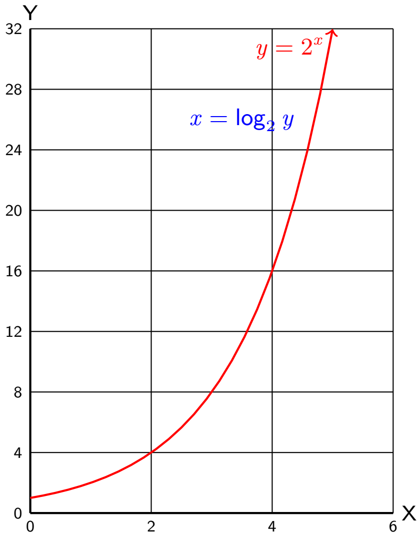
Check: $\log_2(6+2) =?3$

$\log_2(8) =?3$

$\log_2(2^3) =?3$ Yes.

3.3.2 Log is the inverse function of exp

The graphs of $y = 2^x$ with domain $(0, \infty)$ or (using the inverse function) $x = \log_2 y$ with domain $(0, \infty)$ each consist of all points $(x, 2^x)$ with $x > 0$.



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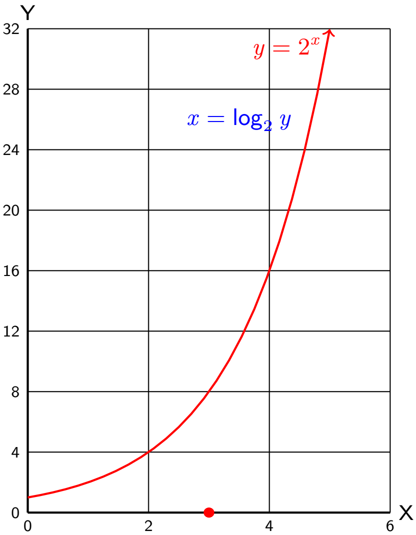

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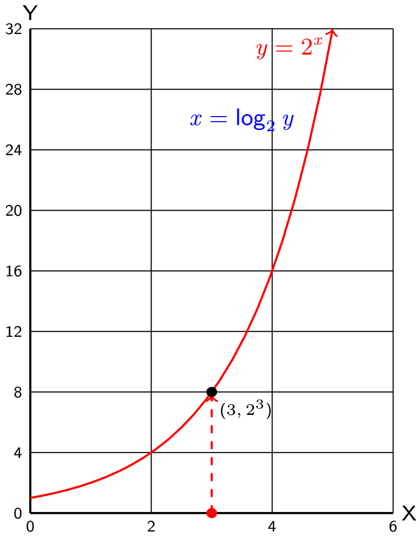

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- To see this, start at 3 on the x-axis.
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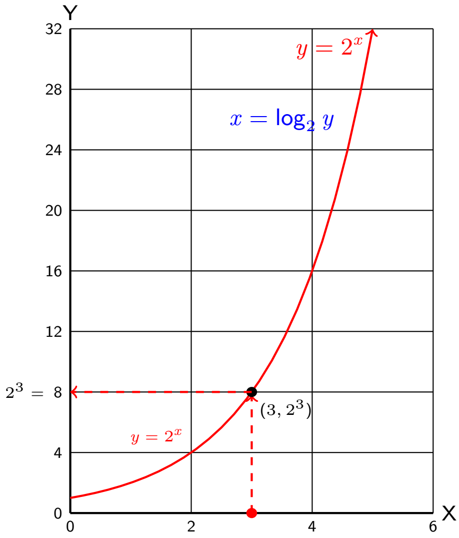

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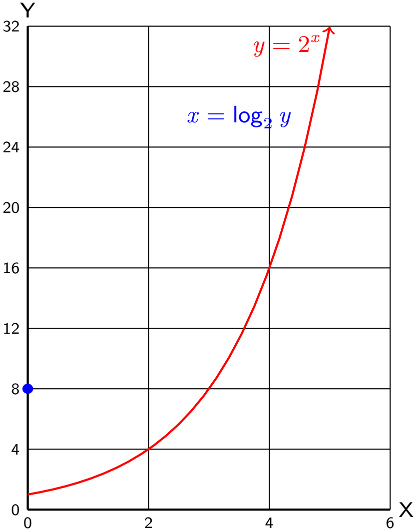
  To figure out $\log_2 8$, proceed in reverse.
- Start at $y = 8$ on the y-axis.



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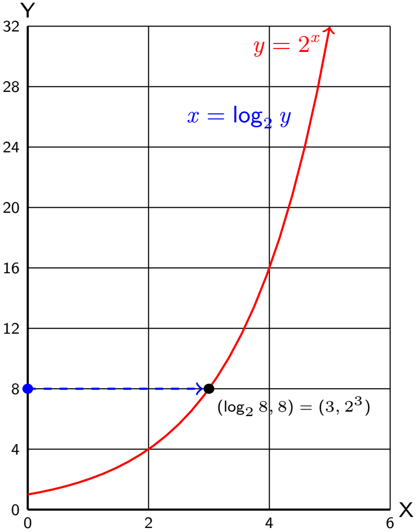

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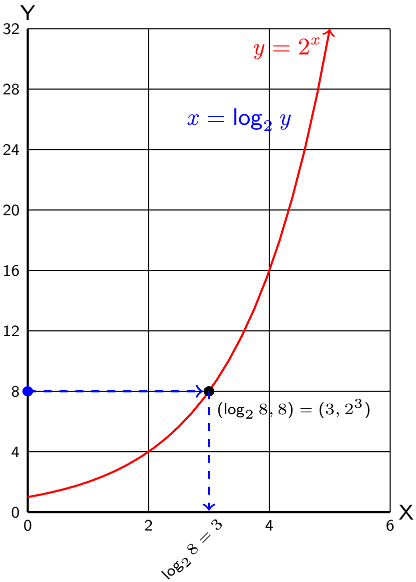
  To figure out $\log_2 8$, proceed in reverse.
- Start at $y = 8$ on the y-axis.
- Move horizontally to the graph, to $(\log_2 8, 8)$.
- Move vertically down to the x-axis.

  Since we arrive at $x = 3$, conclude $\log_2 8 = 3$



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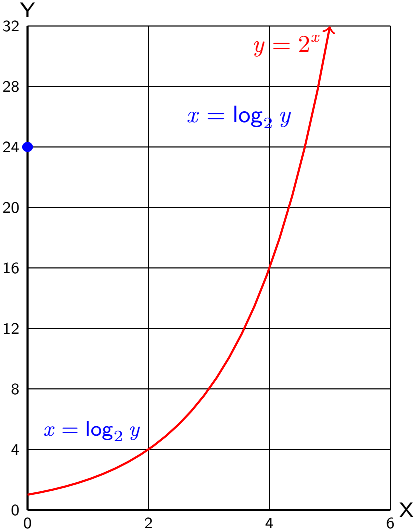
  To figure out $\log_2 8$, proceed in reverse.
- Start at $y = 8$ on the y-axis.
- Move horizontally to the graph, to $(\log_2 8, 8)$.
- Move vertically down to the x-axis.

  Since we arrive at $x = 3$, conclude $\log_2 8 = 3$
- Similarly, to find $\log_2(24)$,

  Start at $y = 24$ on the y-axis.



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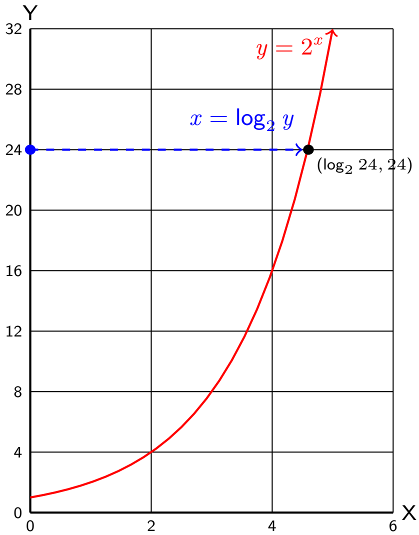
- Move horizontally to the graph, to $(\log_2 8, 8)$.
- Move vertically down to the x-axis.

  Since we arrive at $x = 3$, conclude $\log_2 8 = 3$
- Similarly, to find $\log_2(24)$,

  Start at $y = 24$ on the y-axis.
- Move horizontally to the graph.



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- Move vertically up the graph to point $(3, 2^3)$
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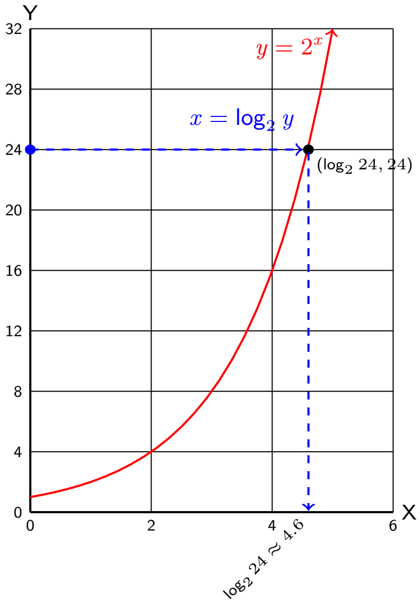
  To figure out $\log_2 8$, proceed in reverse.
- Start at $y = 8$ on the y-axis.
- Move horizontally to the graph, to $(\log_2 8, 8)$.
- Move vertically down to the x-axis.

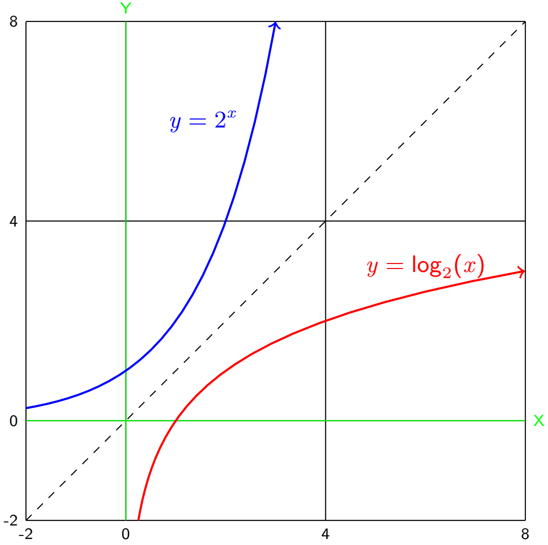
  Since we arrive at $x = 3$, conclude $\log_2 8 = 3$
- Similarly, to find $\log_2(24)$,

  Start at $y = 24$ on the y-axis.
- Move horizontally to the graph.
- Move vertically to the x-axis, where you find x that makes $x = \log_2(24) \approx 4.6$.



To change the graph of $y = 2^x$ to the graph of $y = \log_2 x$, reflect each point $(x, 2^x)$ across the diagonal line $y = x$ to obtain point $(2^x, x) = (2^x, \log_2 2^x)$.

$(x, 2^x)$	$(x, \log_2(x))$



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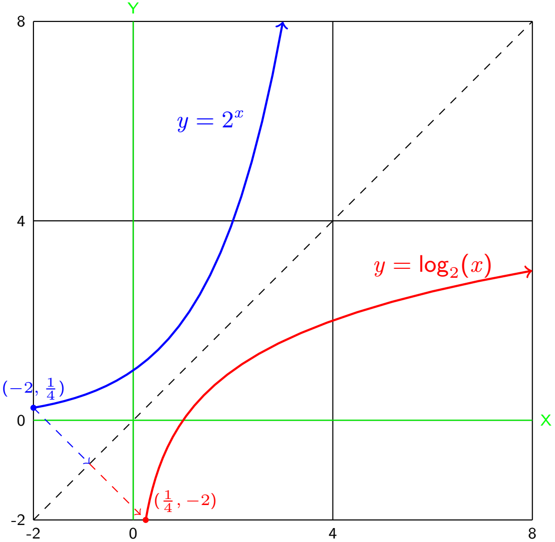

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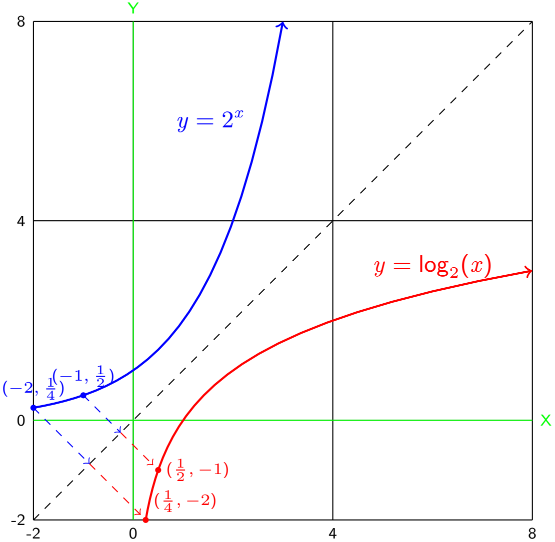

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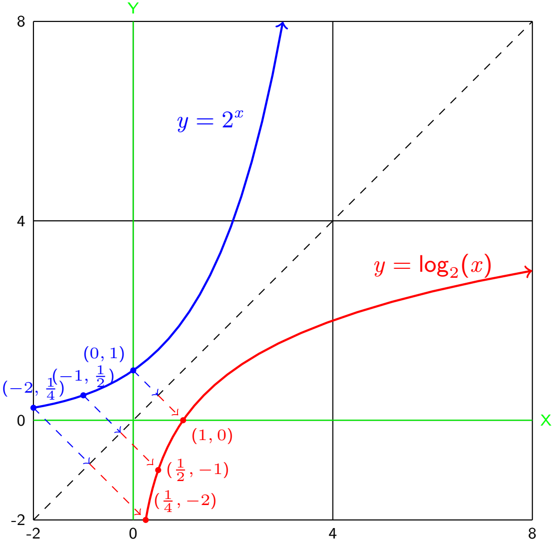

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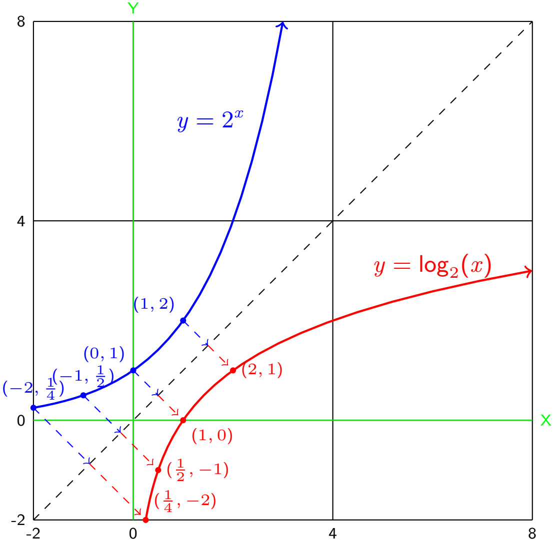

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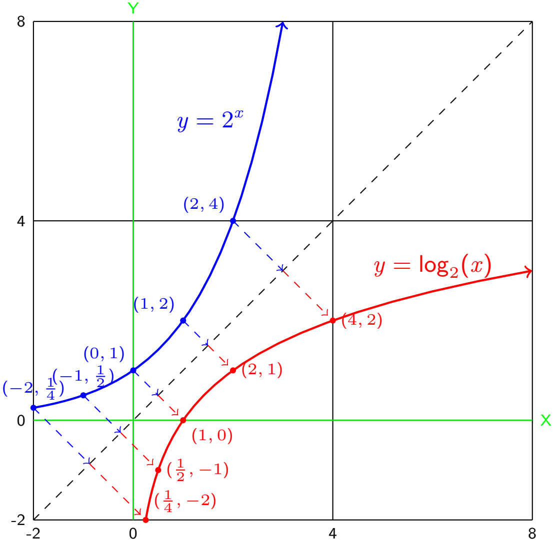
$(2, 4)$	$(4, 2)$



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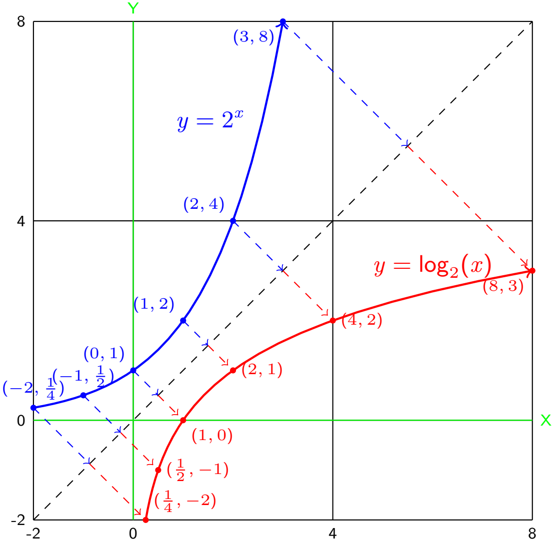
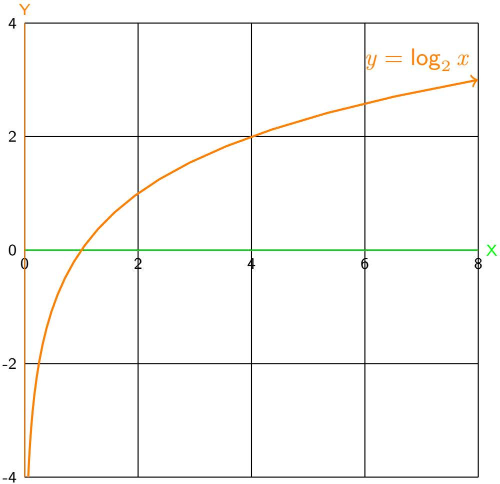
$(3, 8)$	$(8, 3)$



3.3.3 Is there a preferred base for the logarithm functions $y = \log_a x$?

- The decimal system uses the base 10 function $y = 10^x$. Its inverse is $y = \log_{10} x$.This function was always used for scientific computation, and so it is called the *common logarithm of* x. It is usually written without a base: $\log x$ means $\log_{10} x$.
- $10^{-3} = \frac{1}{10^3} = 0.001 =$ one thousandth. Therefore $\log_{10}(.001) = -3$.
- $10^9 = 1,000,000,000 =$ one billion. Therefore $\log_{10}(1,000,000,000) = 9$.



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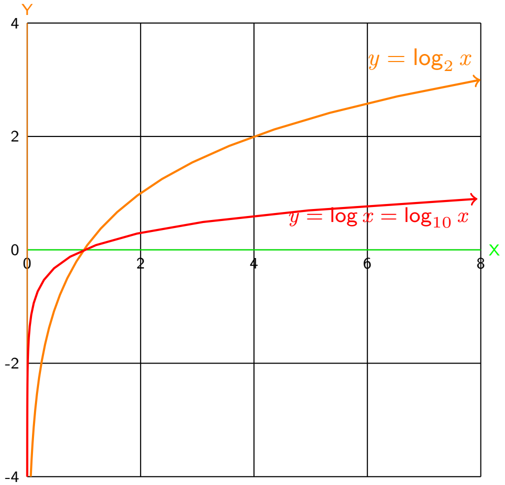

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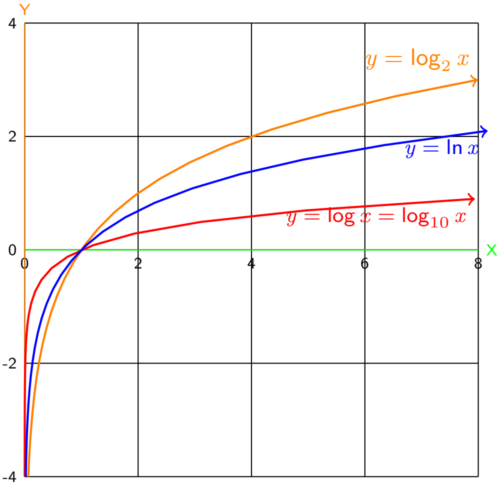

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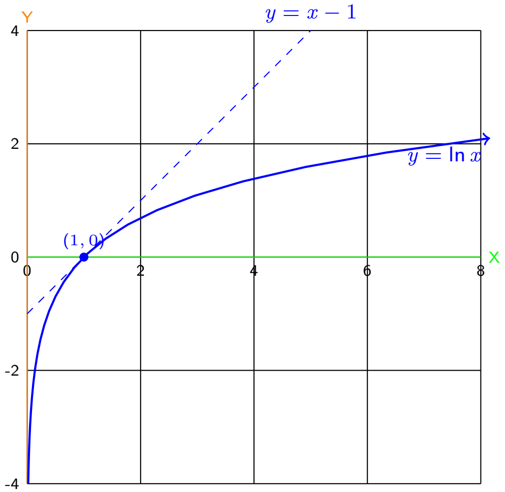
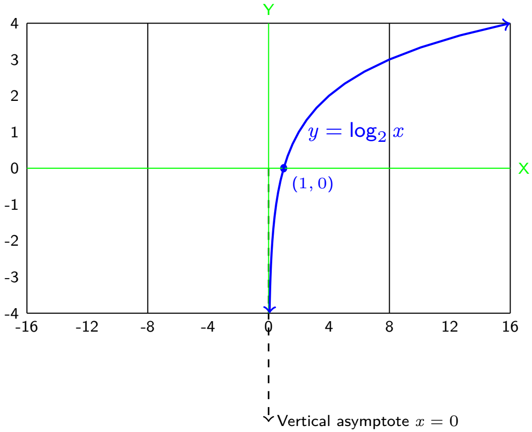
- Calculus uses base $e \approx 2.71828183$. The inverse of the function $\exp(x) = e^x$ is $\log_e x$. It is always written as $\ln x$, the *natural logarithm* of x.
- The tangent line to the graph of $y = \log_a x$ at $(1, 0)$ is a *diagonal* line if and only if $a = e$, that is, when $y = \ln x$.



3.3.4 Transforming graphs and equations of logarithmic functions

- This slide shows the graphs of $y = \log_2 x$ and its various reflections. They share the vertical asymptote line (abbreviated VA) $x = 0$. Start with $y = \log_2 x$, with domain $x > 0$ and x-intercept $(1, 0)$.



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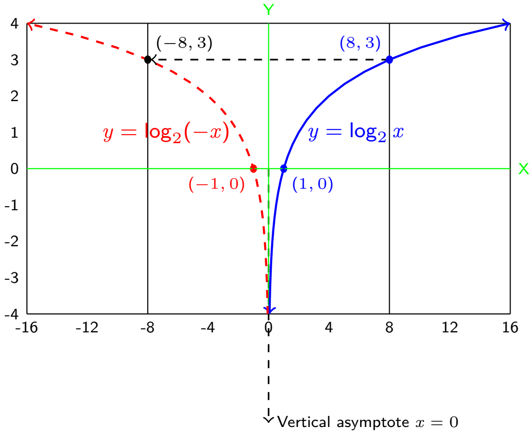
- Substituting $-x$ for x in $y = \log_2 x$ reflects its graph across the y-axis to yield the graph of $y = \log_2(-x)$. This requires $-x > 0$, and so the domain is $x < 0$. The x-intercept is $(-1, 0)$.



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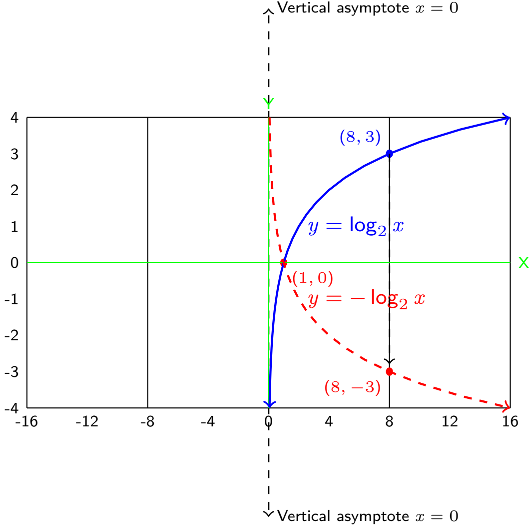
- Multiplying the RHS of $y = \log_2 x$ by -1 reflects its graph across the x-axis to yield the graph of $y = -\log_2 x$ with domain $x > 0$ and x-intercept $(1, 0)$.



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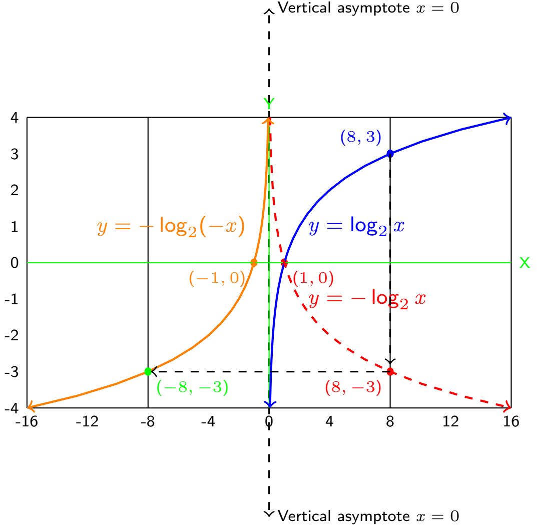
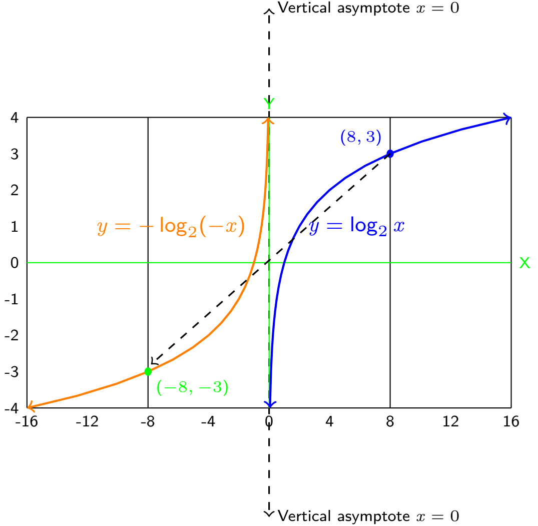
- Substituting $-x$ for x in $y = \log_2 x$ reflects its graph across the y-axis to yield the graph of $y = \log_2(-x)$. This requires $-x > 0$, and so the domain is $x < 0$. The x-intercept is $(-1, 0)$.
- Multiplying the RHS of $y = \log_2 x$ by -1 reflects its graph across the x-axis to yield the graph of $y = -\log_2 x$ with domain $x > 0$ and x-intercept $(1, 0)$.
- Replacing x by $-x$ in equation $y = -\log_2 x$ by -1 reflects its graph across the y-axis to yield the graph of $y = -\log_2(-x)$ with domain $x < 0$ and x-intercept $(-1, 0)$.

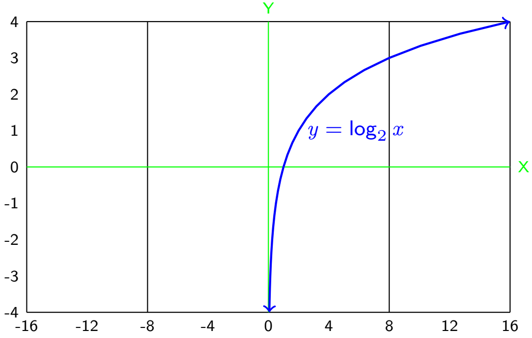


3.3.4 Transforming graphs and equations of logarithmic functions

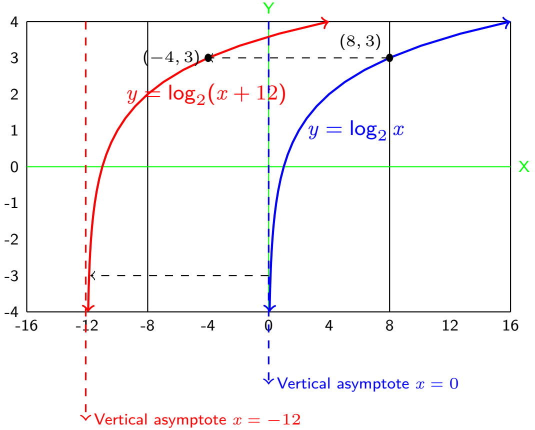
- This slide shows the graphs of $y = \log_2 x$ and its various reflections. They share the vertical asymptote line (abbreviated VA) $x = 0$. Start with $y = \log_2 x$, with domain $x > 0$ and x-intercept $(1, 0)$.
- Substituting $-x$ for x in $y = \log_2 x$ reflects its graph across the y-axis to yield the graph of $y = \log_2(-x)$. This requires $-x > 0$, and so the domain is $x < 0$. The x-intercept is $(-1, 0)$.
- Multiplying the RHS of $y = \log_2 x$ by -1 reflects its graph across the x-axis to yield the graph of $y = -\log_2 x$ with domain $x > 0$ and x-intercept $(1, 0)$.
- Replacing x by $-x$ in equation $y = -\log_2 x$ by -1 reflects its graph across the y-axis to yield the graph of $y = -\log_2(-x)$ with domain $x < 0$ and x-intercept $(-1, 0)$.
- Combine the last two steps: Substitute $-x$ for x in the original $y = \ln x$, then multiply the RHS by -1: the graph reflects through the origin to give $y = -\log_2(-x)$.



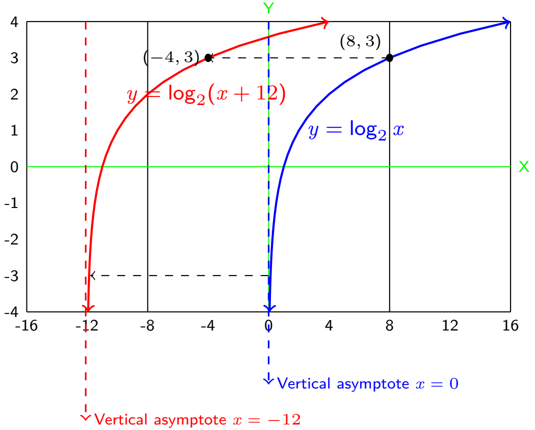
- Start over: Replacing x by $x + 12$ in the equation $y = \log_2 x$ shifts its graph *left* 12 units to yield the graph of $y = \log_2(x + 12)$, with domain $x + 12 > 0$, i.e., $x > -12$.



- Start over: Replacing x by $x+12$ in the equation $y = \log_2 x$ shifts its graph *left* 12 units to yield the graph of $y = \log_2(x+12)$, with domain $x+12>0$, i.e., $x > -12$.
- When the graph shifts left 12 units, so does its VA $x=0$, which becomes $x=-12$.



- Start over: Replacing x by $x + 12$ in the equation $y = \log_2 x$ shifts its graph *left* 12 units to yield the graph of $y = \log_2(x + 12)$, with domain $x + 12 > 0$, i.e., $x > -12$.
- When the graph shifts left 12 units, so does its VA $x = 0$, which becomes $x = -12$.



It's easy to find the vertical asymptote of a log graph just by looking at the function definition. The idea is that the VA of $y = \log x$ is $x = 0$. If you have a more complicated log function, the vertical asymptote is obtained by setting the input of the log function to zero.

Example: Find an equation of the vertical asymptote of the graph of $y = 4 + 3\log(2x + 5)$.

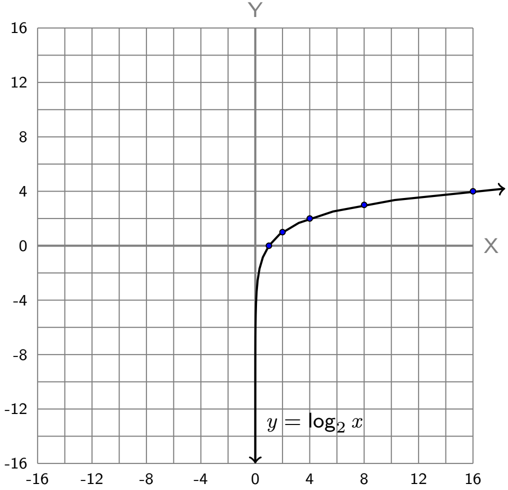
Solution: The input to the log function is $2x + 5$. Therefore the VA is given by setting $2x + 5 = 0$. Solve for x to obtain the

Answer: The vertical asymptote is the line $x = -5/2$.

3.3.5 More examples of transforming log and exp graphs

Example 5: Transform the graph of $y = \log_2 x$ step by step to the graph of $y = 4 - \log_2(x + 4)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_2 x$, with asymptote the y-axis. It passes through points $(2^x, x)$, including $(1, 0), (2, 1), (4, 2), (8, 2), (16, 4)$.



3.3.5 More examples of transforming log and exp graphs

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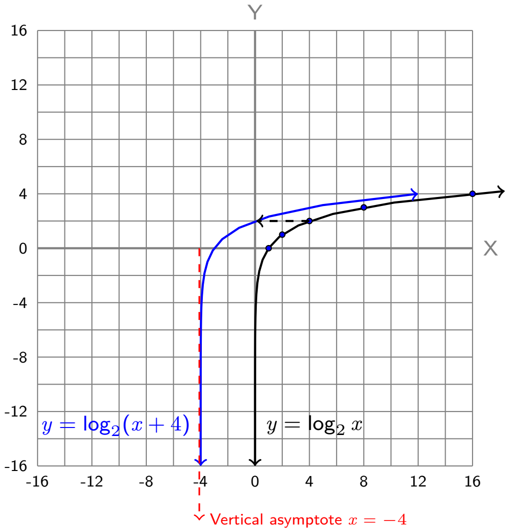
Solution: Start with the graph of $y = \log_2 x$, with asymptote the y-axis. It passes through points $(2^x, x)$, including $(1, 0), (2, 1), (4, 2), (8, 2), (16, 4)$.

- First substitute $x + 4$ for x in $y = \log_2 x$. The graph shifts 4 units left to yield the graph of $y = \log_2(x + 4)$. The asymptote shifts 4 left to $x = -4$.



3.3.5 More examples of transforming log and exp graphs

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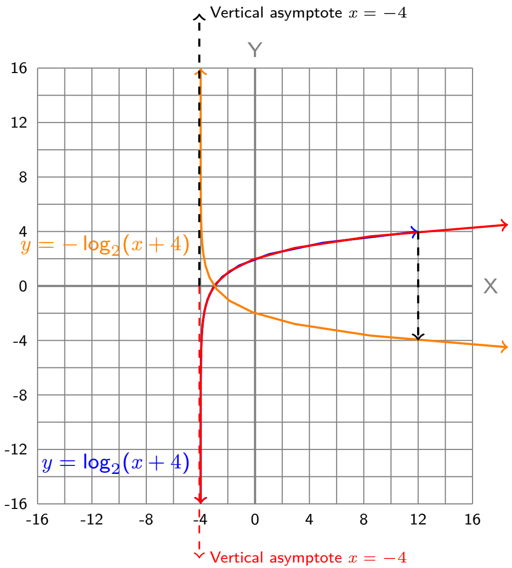
Solution: Start with the graph of $y = \log_2 x$, with asymptote the y-axis. It passes through points $(2^x, x)$, including $(1, 0), (2, 1), (4, 2), (8, 2), (16, 4)$.

- First substitute $x + 4$ for x in $y = \log_2 x$. The graph shifts 4 units left to yield the graph of $y = \log_2(x + 4)$. The asymptote shifts 4 left to $x = -4$.
- Multiply the RHS of the equation $y = \log_2(x + 4)$ by -1. Its graph reflects across the x-axis to yield the graph of $y = -\log_2(x + 4)$. The asymptote remains $x = -4$.



3.3.5 More examples of transforming log and exp graphs

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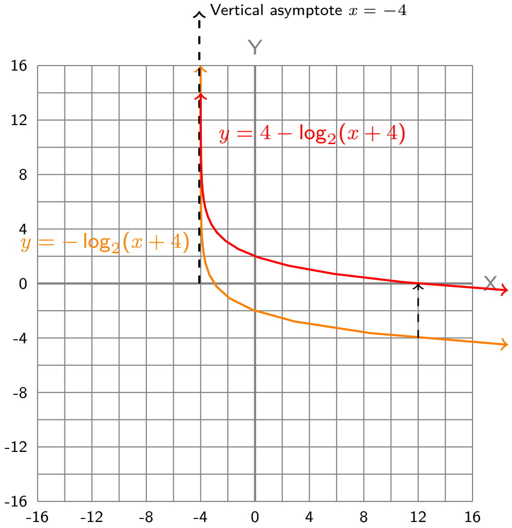
Solution: Start with the graph of $y = \log_2 x$, with asymptote the y-axis. It passes through points $(2^x, x)$, including $(1,0), (2,1), (4,2), (8,2), (16,4)$.

- First substitute $x+4$ for x in $y = \log_2 x$. The graph shifts 4 units left to yield the graph of $y = \log_2(x+4)$. The asymptote shifts 4 left to $x = -4$.
- Multiply the RHS of the equation $y = \log_2(x+4)$ by -1. Its graph reflects across the x-axis to yield the graph of $y = -\log_2(x+4)$. The asymptote remains $x = -4$.
- Adding 4 to the RHS of $y = -\log_2(x+4)$ shifts its graph up 4 units to give the requested graph of $y = 4 - \log_2(x+4)$. The asymptote remains $x = -4$.

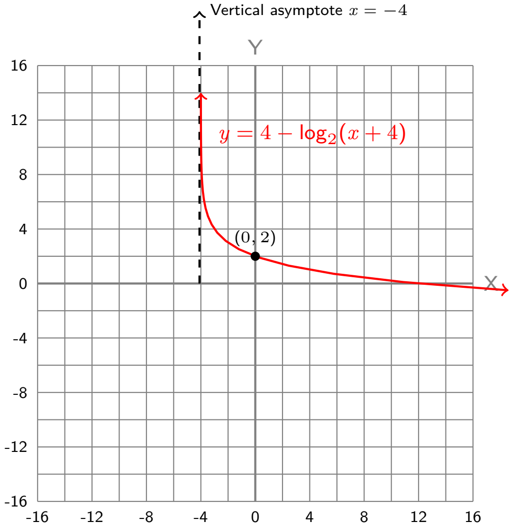


3.3.5 More examples of transforming log and exp graphs

Example 5: Transform the graph of $y = \log_2 x$ step by step to the graph of $y = 4 - \log_2(x + 4)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_2 x$, with asymptote the y-axis. It passes through points $(2^x, x)$, including $(1, 0), (2, 1), (4, 2), (8, 2), (16, 4)$.

- First substitute $x + 4$ for x in $y = \log_2 x$. The graph shifts 4 units left to yield the graph of $y = \log_2(x + 4)$. The asymptote shifts 4 left to $x = -4$.
- Multiply the RHS of the equation $y = \log_2(x + 4)$ by -1. Its graph reflects across the x-axis to yield the graph of $y = -\log_2(x + 4)$. The asymptote remains $x = -4$.
- Adding 4 to the RHS of $y = -\log_2(x + 4)$ shifts its graph up 4 units to give the requested graph of $y = 4 - \log_2(x + 4)$. The asymptote remains $x = -4$.
- Set $x = 0$ in $y = 4\log_2(x + 4)$ to find the y-intercept: $y = 4 - \log_2(4) = 4 - \log_2(2^2) = 4 - 2 = 2$, and so the y-intercept is 2. The y-intercept point is $(0, 2)$.

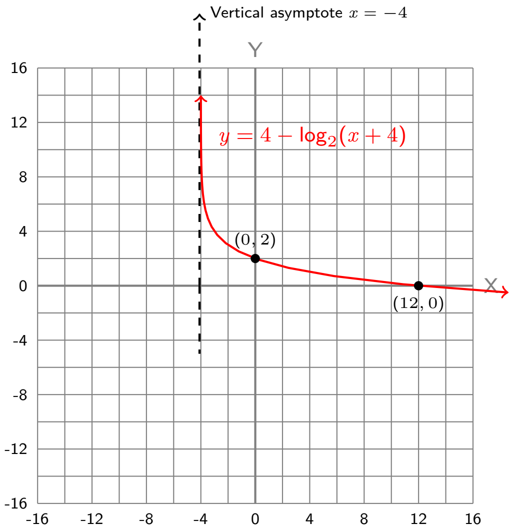


3.3.5 More examples of transforming log and exp graphs

Example 5: Transform the graph of $y = \log_2 x$ step by step to the graph of $y = 4 - \log_2(x+4)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_2 x$, with asymptote the y-axis. It passes through points $(2^x, x)$, including $(1,0), (2,1), (4,2), (8,2), (16,4)$.

- First substitute $x+4$ for x in $y = \log_2 x$. The graph shifts 4 units left to yield the graph of $y = \log_2(x+4)$. The asymptote shifts 4 left to $x = -4$.
- Multiply the RHS of the equation $y = \log_2(x+4)$ by -1. Its graph reflects across the x-axis to yield the graph of $y = -\log_2(x+4)$. The asymptote remains $x = -4$.
- Adding 4 to the RHS of $y = -\log_2(x+4)$ shifts its graph up 4 units to give the requested graph of $y = 4 - \log_2(x+4)$. The asymptote remains $x = -4$.
- Set $x = 0$ in $y = 4\log_2(x+4)$ to find the y-intercept: $y = 4 - \log_2(4) = 4 - \log_2(2^2) = 4 - 2 = 2$, and so the y-intercept is 2. The y-intercept point is $(0,2)$.

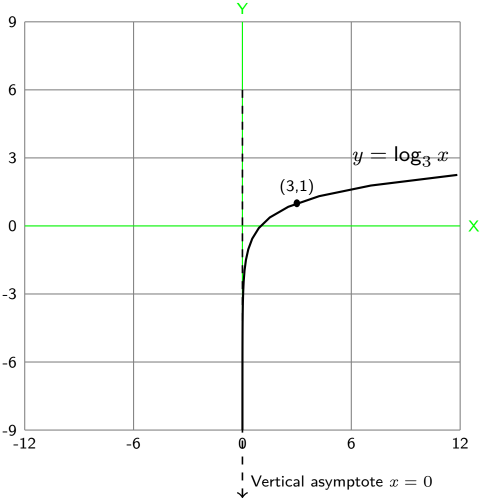


- Set $y = 0$ to find the x-intercept: $y = 0 = 4 - \log_2(x+4) \Rightarrow 4 = \log_2(x+4) \Rightarrow 2^4 = x+4 \Rightarrow x = 12$.
- The x-intercept point is $(12,0)$. The vertical asymptote is the line $x = -4$.

Transforming logarithmic graphs and equations

Example 6: Transform the graph of $y = \log_3 x$ step by step to obtain the graph of $y = \log_3(-x) - 2$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_3 x$, with asymptote the y-axis (the line $x = 0$.).

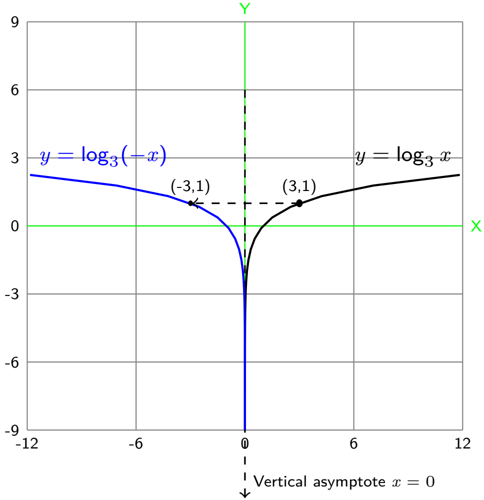


Transforming logarithmic graphs and equations

Example 6: Transform the graph of $y = \log_3 x$ step by step to obtain the graph of $y = \log_3(-x) - 2$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_3 x$, with asymptote the y-axis (the line $x = 0$.).

- Substituting $-x$ for x in $y = \log_3 x$ reflects its graph across the y-axis to yield the graph of $y = \log_3(-x)$. The asymptote is still $x = 0$.

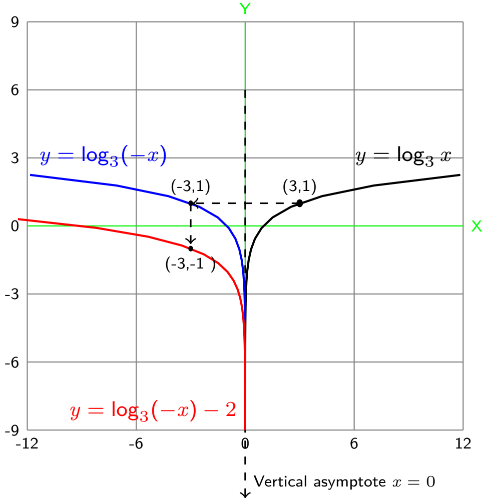


Transforming logarithmic graphs and equations

Example 6: Transform the graph of $y = \log_3 x$ step by step to obtain the graph of $y = \log_3(-x) - 2$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_3 x$, with asymptote the y-axis (the line $x = 0$.).

- Substituting $-x$ for x in $y = \log_3 x$ reflects its graph across the y-axis to yield the graph of $y = \log_3(-x)$. The asymptote is still $x = 0$.
- Subtracting 2 from RHS of the equation $y = \log_3(-x)$ shifts its graph down 2 units to yield the requested graph of $y = \log_3(-x) - 2$. The asymptote shifts down and is still $x = 0$.

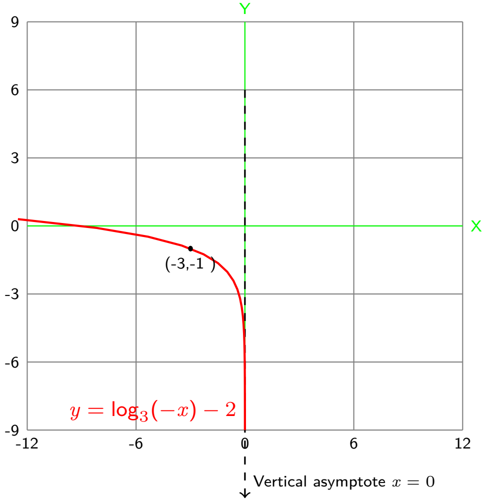


Transforming logarithmic graphs and equations

Example 6: Transform the graph of $y = \log_3 x$ step by step to obtain the graph of $y = \log_3(-x) - 2$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_3 x$, with asymptote the y-axis (the line $x = 0$.).

- Substituting $-x$ for x in $y = \log_3 x$ reflects its graph across the y-axis to yield the graph of $y = \log_3(-x)$. The asymptote is still $x = 0$.
- Subtracting 2 from RHS of the equation $y = \log_3(-x)$ shifts its graph down 2 units to yield the requested graph of $y = \log_3(-x) - 2$. The asymptote shifts down and is still $x = 0$.
- Setting $x = 0$ in $y = \log_3(-x) - 2$ makes the RHS undefined. There is no y-intercept.

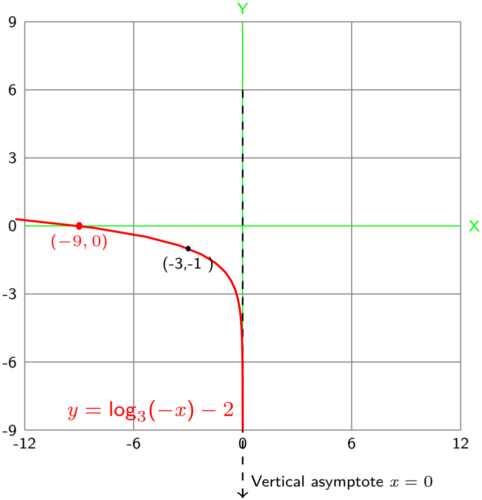


Transforming logarithmic graphs and equations

Example 6: Transform the graph of $y = \log_3 x$ step by step to obtain the graph of $y = \log_3(-x) - 2$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_3 x$, with asymptote the y-axis (the line $x = 0$.).

- Substituting $-x$ for x in $y = \log_3 x$ reflects its graph across the y-axis to yield the graph of $y = \log_3(-x)$. The asymptote is still $x = 0$.
- Subtracting 2 from RHS of the equation $y = \log_3(-x)$ shifts its graph down 2 units to yield the requested graph of $y = \log_3(-x) - 2$. The asymptote shifts down and is still $x = 0$.
- Setting $x = 0$ in $y = \log_3(-x) - 2$ makes the RHS undefined. There is no y-intercept.
- Set $y = 0$ in that equation to get $0 = \log_3(-x) - 2 \Rightarrow \log_3(-x) = 2 \Rightarrow -x = 3^2 = 9$ and so $x = -9$ is the x-intercept, at point $(-9, 0)$. The vertical asymptote is $x = 0$ (the y-axis).



3.3.6 Quiz

Example 3.3.1: Find $\log_2(16), \log_2(\frac{1}{8}), \log_2(8\sqrt{2})$, and $\log_2(24)$.

Example 3.3.2: Find $\log_{10}(1000000)$ and $\log_{10}(.0001)$.

Example 3.3.3: Find $\ln(1), \ln(e)$, and $\ln(\frac{1}{\sqrt{e}})$.

Example 3.3.4: Solve $\log_2(x+2) = 3$

Example 3.3.5: Transform the graph of $y = \log_2 x$ step by step to the graph of $y = 4 - \log_2(x+4)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Example 3.3.6: Transform the graph of $y = \log_3 x$ step by step to the graph of $y = \log_3(-x) - 2$. Sketch the resulting graph. Find and label its intercepts and asymptote.

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) =$
- $\log_2(\frac{1}{8}) =$
- $\log_2(8\sqrt{2}) =$
- $\log_2(24)=$

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_2(8\sqrt{2}) = \frac{7}{2}$
- $\log_2(24) =$ Does not simplify

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_2(8\sqrt{2}) = \frac{7}{2}$
- $\log_2(24) =$ Does not simplify
- $\log_3(9) =$
- $\log_3 \frac{1}{9} =$
- $\log_3(27\sqrt{3}) =$
- $\log_3(9\sqrt[3]{3}) =$

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_2(8\sqrt{2}) = \frac{7}{2}$
- $\log_2(24)=$ Does not simplify
- $\log_3(9) = 2$
- $\log_3 \frac{1}{9} = -2$
- $\log_3(27\sqrt{3}) = \frac{7}{2}$,
- $\log_3(9\sqrt[3]{3}) = \frac{7}{3}$

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) = 4$
- $\log_3(9) = 2$
- $\log_4(2) =$
- $\log_2(\frac{1}{8}) = -3$
- $\log_3 \frac{1}{9} = -2$
- $\log_4(\frac{1}{4}) =$
- $\log_2(8\sqrt{2}) = \frac{7}{2}$
- $\log_3(27\sqrt{3}) = \frac{7}{2},$
- $\log_4(64) =$
- $\log_2(24)=$ Does not simplify
- $\log_3(9\sqrt[3]{3}) = \frac{7}{3}$
- $\log_4(64\sqrt{8}) =$

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) = 4$
- $\log_3(9) = 2$
- $\log_4(2) = \frac{1}{2}$
- $\log_2(\frac{1}{8}) = -3$
- $\log_3 \frac{1}{9} = -2$
- $\log_4(\frac{1}{4}) = -1$
- $\log_2(8\sqrt{2}) = \frac{7}{2}$
- $\log_3(27\sqrt{3}) = \frac{7}{2}$,
- $\log_4(64) = 3$
- $\log_2(24)=$ Does not simplify
- $\log_3(9\sqrt[3]{3}) = \frac{7}{3}$
- $\log_4(64\sqrt{8}) = \frac{15}{4}$

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) = 4$
- $\log_3(9) = 2$
- $\log_4(2) = \frac{1}{2}$
- $\log_5(\frac{1}{25}) =$
- $\log_2(\frac{1}{8}) = -3$
- $\log_3 \frac{1}{9} = -2$
- $\log_4(\frac{1}{4}) = -1$
- $\log_5(\frac{\sqrt{5}}{25}) =$
- $\log_2(8\sqrt{2}) = \frac{7}{2}$
- $\log_3(27\sqrt{3}) = \frac{7}{2},$
- $\log_4(64) = 3$
- $\log_5(\frac{1}{125}) =$
- $\log_2(24) =$ Does not simplify
- $\log_3(9\sqrt[3]{3}) = \frac{7}{3}$
- $\log_4(64\sqrt{8}) = \frac{15}{4}$
- $\log_5(1) =$

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_2(8\sqrt{2}) = \frac{7}{2}$
- $\log_2(24)=$ Does not simplify
- $\log_3(9) = 2$
- $\log_3 \frac{1}{9} = -2$
- $\log_3(27\sqrt{3}) = \frac{7}{2}$,
- $\log_3(9\sqrt[3]{3}) = \frac{7}{3}$
- $\log_4(2) = \frac{1}{2}$
- $\log_4(\frac{1}{4}) = -1$
- $\log_4(64) = 3$
- $\log_4(64\sqrt{8}) = \frac{15}{4}$
- $\log_5(\frac{1}{25}) = -2$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$
- $\log_5(\frac{1}{125}) = -3$
- $\log_5(1) = 0$

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_2(8\sqrt{2}) = \frac{7}{2}$
- $\log_2(24) =$ Does not simplify
- $\log_3(9) = 2$
- $\log_3 \frac{1}{9} = -2$
- $\log_3(27\sqrt{3}) = \frac{7}{2}$,
- $\log_3(9\sqrt[3]{3}) = \frac{7}{3}$
- $\log_4(2) = \frac{1}{2}$
- $\log_4(\frac{1}{4}) = -1$
- $\log_4(64) = 3$
- $\log_4(64\sqrt{8}) = \frac{15}{4}$
- $\log_5(\frac{1}{25}) = -2$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$
- $\log_5(\frac{1}{125}) = -3$
- $\log_5(1) = 0$

Example 3.3.2: Find

- $\log_{10}(1000000) =$
- $\log_{10}(.0001) =$
- $\log_{10}(10) =$
- $\log_{10}(100) =$

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_2(8\sqrt{2}) = \frac{7}{2}$
- $\log_2(24) =$ Does not simplify
- $\log_3(9) = 2$
- $\log_3 \frac{1}{9} = -2$
- $\log_3(27\sqrt{3}) = \frac{7}{2}$,
- $\log_3(9\sqrt[3]{3}) = \frac{7}{3}$
- $\log_4(2) = \frac{1}{2}$
- $\log_4(\frac{1}{4}) = -1$
- $\log_4(64) = 3$
- $\log_4(64\sqrt{8}) = \frac{15}{4}$
- $\log_5(\frac{1}{25}) = -2$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$
- $\log_5(\frac{1}{125}) = -3$
- $\log_5(1) = 0$

Example 3.3.2: Find

- $\log_{10}(1000000) = 6$
- $\log_{10}(.0001) = -4$
- $\log_{10}(10) = 1$
- $\log_{10}(100) = 2$

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_2(8\sqrt{2}) = \frac{7}{2}$
- $\log_2(24)=$ Does not simplify
- $\log_3(9) = 2$
- $\log_3 \frac{1}{9} = -2$
- $\log_3(27\sqrt{3}) = \frac{7}{2}$,
- $\log_3(9\sqrt[3]{3}) = \frac{7}{3}$
- $\log_4(2) = \frac{1}{2}$
- $\log_4(\frac{1}{4}) = -1$
- $\log_4(64) = 3$
- $\log_4(64\sqrt{8}) = \frac{15}{4}$
- $\log_5(\frac{1}{25}) = -2$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$
- $\log_5(\frac{1}{125}) = -3$
- $\log_5(1) = 0$

Example 3.3.2: Find

- $\log_{10}(1000000) = 6$
- $\log_{10}(.0001) = -4$
- $\log_{10}(10) = 1$
- $\log_{10}(100) = 2$
- $\log_{10}(100\sqrt{10}) =$
- $\log_{10}(\frac{10}{\sqrt{1000}}) =$
- $\log_{10}(\frac{1}{1000}) =$
- $\log_{10}(100\sqrt[3]{10}) =$

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_2(8\sqrt{2}) = \frac{7}{2}$
- $\log_2(24)=$ Does not simplify
- $\log_3(9) = 2$
- $\log_3 \frac{1}{9} = -2$
- $\log_3(27\sqrt{3}) = \frac{7}{2}$,
- $\log_3(9\sqrt[3]{3}) = \frac{7}{3}$
- $\log_4(2) = \frac{1}{2}$
- $\log_4(\frac{1}{4}) = -1$
- $\log_4(64) = 3$
- $\log_4(64\sqrt{8}) = \frac{15}{4}$
- $\log_5(\frac{1}{25}) = -2$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$
- $\log_5(\frac{1}{125}) = -3$
- $\log_5(1) = 0$

Example 3.3.2: Find

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- $\log_{10}(.0001) = -4$
- $\log_{10}(10) = 1$
- $\log_{10}(100) = 2$
- $\log_{10}(100\sqrt{10}) = \frac{5}{2}$
- $\log_{10}(\frac{10}{\sqrt{1000}}) = -\frac{1}{2}$
- $\log_{10}(\frac{1}{1000}) = -3$
- $\log_{10}(100\sqrt[3]{10}) = \frac{7}{3}$

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

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- $\log_2(\frac{1}{8}) = -3$
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- $\log_2(24)=$ Does not simplify
- $\log_3(9) = 2$
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- $\log_3(27\sqrt{3}) = \frac{7}{2}$,
- $\log_3(9\sqrt[3]{3}) = \frac{7}{3}$
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- $\log_4(\frac{1}{4}) = -1$
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- $\log_4(64\sqrt{8}) = \frac{15}{4}$
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- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$
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Example 3.3.2: Find

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- $\log_{10}(\frac{1}{1000}) = -3$
- $\log_{10}(100\sqrt[3]{10}) = \frac{7}{3}$

Example 3.3.3: Find

- $\ln(1) =$
- $\ln(e) =$
- $\ln(\frac{1}{\sqrt{e}}) =$

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) = 4$
- $\log_2(\frac{1}{8}) = -3$
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- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$
- $\log_5(\frac{1}{125}) = -3$
- $\log_5(1) = 0$

Example 3.3.2: Find

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- $\log_{10}(.0001) = -4$
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- $\log_{10}(100) = 2$
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- $\log_{10}(\frac{10}{\sqrt{1000}}) = -\frac{1}{2}$
- $\log_{10}(\frac{1}{1000}) = -3$
- $\log_{10}(100\sqrt[3]{10}) = \frac{7}{3}$

Example 3.3.3: Find

- $\ln(1) = 0$
- $\ln(e) = 1$
- $\ln(\frac{1}{\sqrt{e}}) = -\frac{1}{2}$

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_2(8\sqrt{2}) = \frac{7}{2}$
- $\log_2(24)=$ Does not simplify
- $\log_3(9) = 2$
- $\log_3 \frac{1}{9} = -2$
- $\log_3(27\sqrt{3}) = \frac{7}{2}$,
- $\log_3(9\sqrt[3]{3}) = \frac{7}{3}$
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- $\log_4(\frac{1}{4}) = -1$
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- $\log_4(64\sqrt{8}) = \frac{15}{4}$
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- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$
- $\log_5(\frac{1}{125}) = -3$
- $\log_5(1) = 0$

Example 3.3.2: Find

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- $\log_{10}(.0001) = -4$
- $\log_{10}(10) = 1$
- $\log_{10}(100) = 2$
- $\log_{10}(100\sqrt{10}) = \frac{5}{2}$
- $\log_{10}(\frac{10}{\sqrt{1000}}) = -\frac{1}{2}$
- $\log_{10}(\frac{1}{1000}) = -3$
- $\log_{10}(100\sqrt[3]{10}) = \frac{7}{3}$

Example 3.3.3: Find

- $\ln(1) = 0$
- $\ln(e) = 1$
- $\ln(\frac{1}{\sqrt{e}}) = -\frac{1}{2}$
- $\ln(e^3) =$
- $\ln(e^2 e^5) =$
- $\ln(-\frac{1}{\sqrt{e^3}}) =$

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_2(8\sqrt{2}) = \frac{7}{2}$
- $\log_2(24)=$ Does not simplify
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- $\log_4(2) = \frac{1}{2}$
- $\log_4(\frac{1}{4}) = -1$
- $\log_4(64) = 3$
- $\log_4(64\sqrt{8}) = \frac{15}{4}$
- $\log_5(\frac{1}{25}) = -2$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$
- $\log_5(\frac{1}{125}) = -3$
- $\log_5(1) = 0$

Example 3.3.2: Find

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- $\log_{10}(.0001) = -4$
- $\log_{10}(10) = 1$
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- $\log_{10}(\frac{10}{\sqrt{1000}}) = -\frac{1}{2}$
- $\log_{10}(\frac{1}{1000}) = -3$
- $\log_{10}(100\sqrt[3]{10}) = \frac{7}{3}$

Example 3.3.3: Find

- $\ln(1) = 0$
- $\ln(e) = 1$
- $\ln(\frac{1}{\sqrt{e}}) = -\frac{1}{2}$
- $\ln(e^3) = 3$
- $\ln(e^2 e^5) = 7$
- $\ln(-\frac{1}{\sqrt{e^3}}) =$ not defined

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_2(8\sqrt{2}) = \frac{7}{2}$
- $\log_2(24)=$ Does not simplify
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- $\log_3(9\sqrt[3]{3}) = \frac{7}{3}$
- $\log_4(2) = \frac{1}{2}$
- $\log_4(\frac{1}{4}) = -1$
- $\log_4(64) = 3$
- $\log_4(64\sqrt{8}) = \frac{15}{4}$
- $\log_5(\frac{1}{25}) = -2$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$
- $\log_5(\frac{1}{125}) = -3$
- $\log_5(1) = 0$

Example 3.3.2: Find

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- $\log_{10}(\frac{10}{\sqrt{1000}}) = -\frac{1}{2}$
- $\log_{10}(\frac{1}{1000}) = -3$
- $\log_{10}(100\sqrt[3]{10}) = \frac{7}{3}$

Example 3.3.3: Find

- $\ln(1) = 0$
- $\ln(e) = 1$
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- $\ln(e^3) = 3$
- $\ln(e^2 e^5) = 7$
- $\ln(-\frac{1}{\sqrt{e^3}}) =$ not defined
- $\ln(\frac{1}{e}) =$
- $\ln(100e^3) =$
- $\ln(10e) =$

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) = 4$
- $\log_2(\frac{1}{8}) = -3$
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Example 3.3.3: Find

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- $\ln(e) = 1$
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- $\ln(e^3) = 3$
- $\ln(e^2 e^5) = 7$
- $\ln(-\frac{1}{\sqrt{e^3}}) =$ not defined
- $\ln(\frac{1}{e}) = -1$
- $\ln(100e^3) = 3 + \ln 100$
- $\ln(10e) = 1 + \ln 10$

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_2(8\sqrt{2}) = \frac{7}{2}$
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- $\log_4(64) = 3$
- $\log_4(64\sqrt{8}) = \frac{15}{4}$
- $\log_5(\frac{1}{25}) = -2$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$
- $\log_5(\frac{1}{125}) = -3$
- $\log_5(1) = 0$

Example 3.3.2: Find

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- $\log_{10}(.0001) = -4$
- $\log_{10}(10) = 1$
- $\log_{10}(100) = 2$
- $\log_{10}(100\sqrt{10}) = \frac{5}{2}$
- $\log_{10}(\frac{10}{\sqrt{1000}}) = -\frac{1}{2}$
- $\log_{10}(\frac{1}{1000}) = -3$
- $\log_{10}(100\sqrt[3]{10}) = \frac{7}{3}$

Example 3.3.3: Find

- $\ln(1) = 0$
- $\ln(e) = 1$
- $\ln(\frac{1}{\sqrt{e}}) = -\frac{1}{2}$
- $\ln(e^3) = 3$
- $\ln(e^2 e^5) = 7$
- $\ln(-\frac{1}{\sqrt{e^3}}) =$ not defined
- $\ln(\frac{1}{e}) = -1$
- $\ln(100e^3) = 3 + \ln 100$
- $\ln(10e) = 1 + \ln 10$
- $\ln(-1) =$
- $\ln(e^{-5}) =$
- $\ln(\frac{e^3}{\sqrt{e}}) =$

3.3 Section 3.3 Review: logarithmic functions

Example 3.3.1: Find

- $\log_2(16) = 4$
- $\log_2(\frac{1}{8}) = -3$
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- $\log_4(64\sqrt{8}) = \frac{15}{4}$
- $\log_5(\frac{1}{25}) = -2$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$
- $\log_5(\frac{1}{125}) = -3$
- $\log_5(1) = 0$

Example 3.3.2: Find

- $\log_{10}(1000000) = 6$
- $\log_{10}(.0001) = -4$
- $\log_{10}(10) = 1$
- $\log_{10}(100) = 2$
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- $\log_{10}(\frac{10}{\sqrt{1000}}) = -\frac{1}{2}$
- $\log_{10}(\frac{1}{1000}) = -3$
- $\log_{10}(100\sqrt[3]{10}) = \frac{7}{3}$

Example 3.3.3: Find

- $\ln(1) = 0$
- $\ln(e) = 1$
- $\ln(\frac{1}{\sqrt{e}}) = -\frac{1}{2}$
- $\ln(e^3) = 3$
- $\ln(e^2 e^5) = 7$
- $\ln(-\frac{1}{\sqrt{e^3}}) =$ not defined
- $\ln(\frac{1}{e}) = -1$
- $\ln(100e^3) = 3 + \ln 100$
- $\ln(10e) = 1 + \ln 10$
- $\ln(-1) =$ not defined
- $\ln(e^{-5}) = -5$
- $\ln(\frac{e^3}{\sqrt{e}}) = \frac{5}{2}$

Example 3.3.4: Solve

- $\log_2(x+2) = 3 \quad \Rightarrow$
- $\log_2(3x+2) = 3 \ \Rightarrow$
- $\log_5(x-1) = 2 \ \Rightarrow$
- $\log_5(x+1) = 3 \ \Rightarrow$

Example 3.3.5: Sketch the graph, labeled with intercepts and asymptotes, obtained by transforming, step by step, the graph of

- $y = \log_2 x$ to the graph of $y = 4 - \log_2(x+4)$.
- $y = \log_3(x)$ to the graph of $y = 4 + 2\log_3(x)$.
- $y = \log_2 x$ to the graph of $y = -2\log_2(x-4)$.
- $y = \log_2 x$ to the graph of $\log_2(x^2)$.

Example 3.3.4: Solve

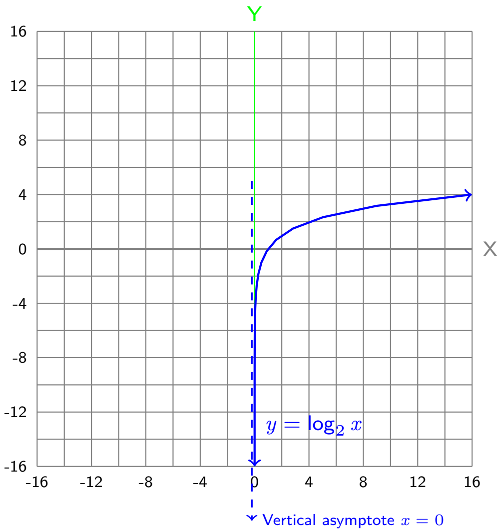
- $\log_2(x+2) = 3 \Rightarrow x = 6$
- $\log_2(3x+2) = 3 \Rightarrow x = 2$
- $\log_5(x-1) = 2 \Rightarrow x = 26$
- $\log_5(x+1) = 3 \Rightarrow x = 124$

Example 3.3.5: Sketch the graph, labeled with intercepts and asymptotes, obtained by transforming, step by step, the graph of

- $y = \log_2 x$ to the graph of $y = 4 - \log_2(x+4)$.
- $y = \log_3(x)$ to the graph of $y = 4 + 2\log_3(x)$.
- $y = \log_2 x$ to the graph of $y = -2\log_2(x-4)$.
- $y = \log_2 x$ to the graph of $\log_2(x^2)$.

Example 3.3.5a: Transform the graph of $y = \log_2 x$ step by step to the graph of $y = 4 - \log_2(x + 4)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

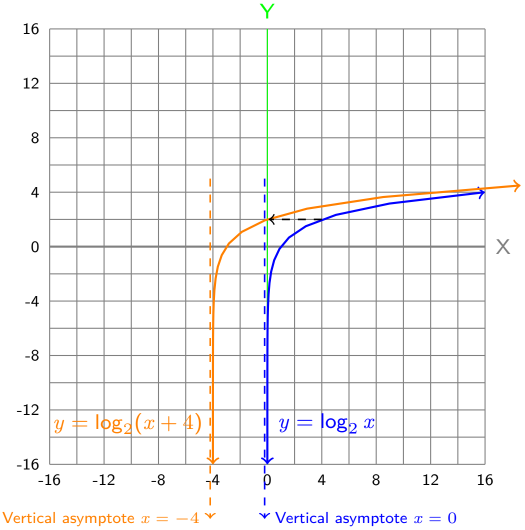
Solution: Start with the graph of $y = \log_2 x$, with asymptote the y-axis.



Example 3.3.5a: Transform the graph of $y = \log_2 x$ step by step to the graph of $y = 4 - \log_2(x + 4)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_2 x$, with asymptote the y-axis.

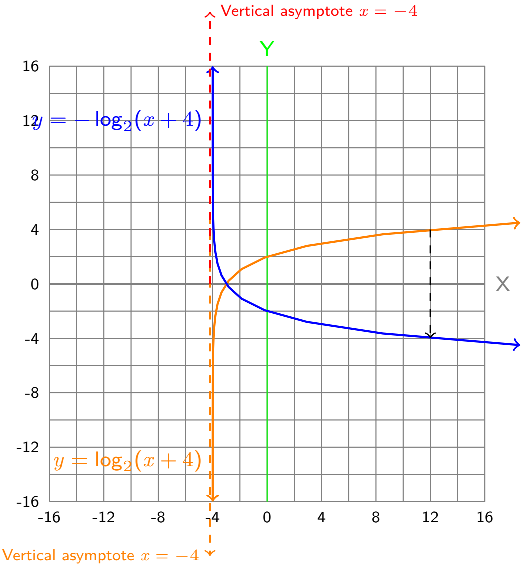
- First substitute $x + 4$ for x in $y = \log_2 x$. The graph shifts 4 units left to yield the graph of $y = \log_2(x + 4)$. The asymptote shifts 4 left to $x = -4$.



Example 3.3.5a: Transform the graph of $y = \log_2 x$ step by step to the graph of $y = 4 - \log_2(x + 4)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_2 x$, with asymptote the y-axis.

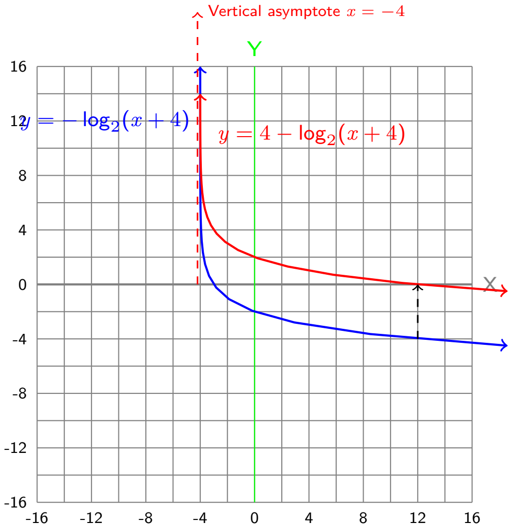
- First substitute $x + 4$ for x in $y = \log_2 x$. The graph shifts 4 units left to yield the graph of $y = \log_2(x + 4)$. The asymptote shifts 4 left to $x = -4$.
- Multiply the RHS of the equation $y = \log_2(x + 4)$ by -1. Its graph reflects across the x-axis to yield the graph of $y = -\log_2(x + 4)$. The asymptote remains $x = -4$.



Example 3.3.5a: Transform the graph of $y = \log_2 x$ step by step to the graph of $y = 4 - \log_2(x + 4)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_2 x$, with asymptote the y-axis.

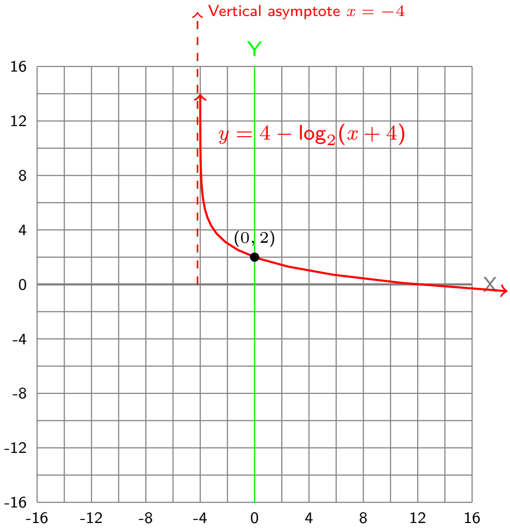
- First substitute $x + 4$ for x in $y = \log_2 x$. The graph shifts 4 units left to yield the graph of $y = \log_2(x + 4)$. The asymptote shifts 4 left to $x = -4$.
- Multiply the RHS of the equation $y = \log_2(x + 4)$ by -1. Its graph reflects across the x-axis to yield the graph of $y = -\log_2(x + 4)$. The asymptote remains $x = -4$.
- Adding 4 to the RHS of $y = -\log_2(x + 4)$ shifts its graph up 4 units to give the requested graph of $y = 4 - \log_2(x + 4)$. The asymptote remains $x = -4$.



Example 3.3.5a: Transform the graph of $y = \log_2 x$ step by step to the graph of $y = 4 - \log_2(x + 4)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_2 x$, with asymptote the y-axis.

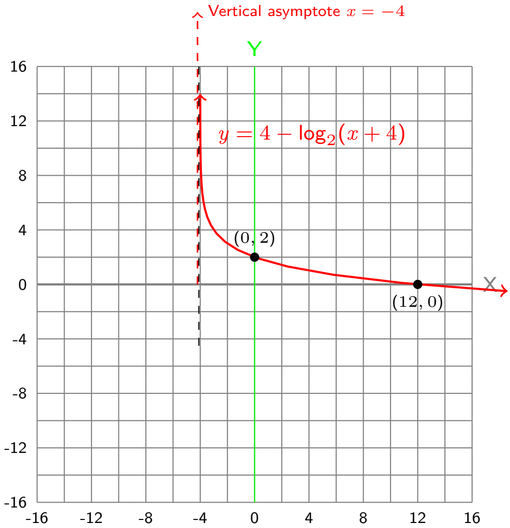
- First substitute $x + 4$ for x in $y = \log_2 x$. The graph shifts 4 units left to yield the graph of $y = \log_2(x + 4)$. The asymptote shifts 4 left to $x = -4$.
- Multiply the RHS of the equation $y = \log_2(x + 4)$ by -1. Its graph reflects across the x-axis to yield the graph of $y = -\log_2(x + 4)$. The asymptote remains $x = -4$.
- Adding 4 to the RHS of $y = -\log_2(x + 4)$ shifts its graph up 4 units to give the requested graph of $y = 4 - \log_2(x + 4)$. The asymptote remains $x = -4$.
- Set $x = 0$ in $y = 4\log_2(x + 4)$.to find the y-intercept: $y = 4 - \log_2(4) = 4 - \log_2(2^2) = 4 - 2 = 2$, and so the y-intercept is 2. The y-intercept point is $(0, 2)$.



Example 3.3.5a: Transform the graph of $y = \log_2 x$ step by step to the graph of $y = 4 - \log_2(x + 4)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_2 x$, with asymptote the y-axis.

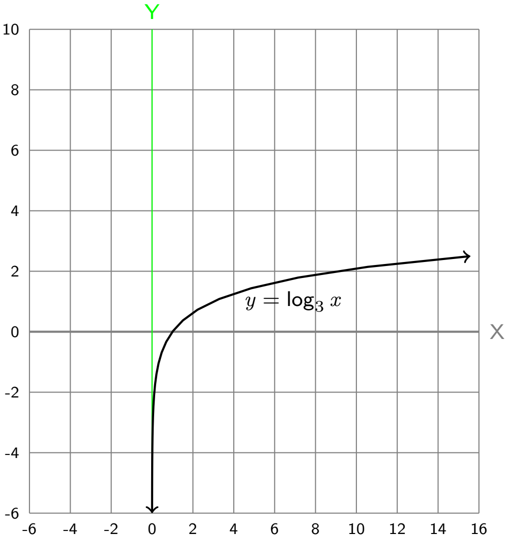
- First substitute $x + 4$ for x in $y = \log_2 x$. The graph shifts 4 units left to yield the graph of $y = \log_2(x + 4)$. The asymptote shifts 4 left to $x = -4$.
- Multiply the RHS of the equation $y = \log_2(x + 4)$ by -1. Its graph reflects across the x-axis to yield the graph of $y = -\log_2(x + 4)$. The asymptote remains $x = -4$.
- Adding 4 to the RHS of $y = -\log_2(x + 4)$ shifts its graph up 4 units to give the requested graph of $y = 4 - \log_2(x + 4)$. The asymptote remains $x = -4$.
- Set $x = 0$ in $y = 4\log_2(x + 4)$.to find the y-intercept: $y = 4 - \log_2(4) = 4 - \log_2(2^2) = 4 - 2 = 2$, and so the y-intercept is 2. The y-intercept point is $(0, 2)$.



- Set $y = 0$ to find the x-intercept: $y = 0 = 4 - \log_2(x + 4) \Rightarrow 4 = \log_2(x + 4) \Rightarrow 2^4 = x + 4 \Rightarrow x = 12$.
- The x-intercept point is $(12, 0)$.
- The asymptote is the vertical line $x = -4$.

Example 3.3.5b: Transform the graph of $y = \log_3 x$ step by step to the graph of $y = 4 + 2\log_3(x)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

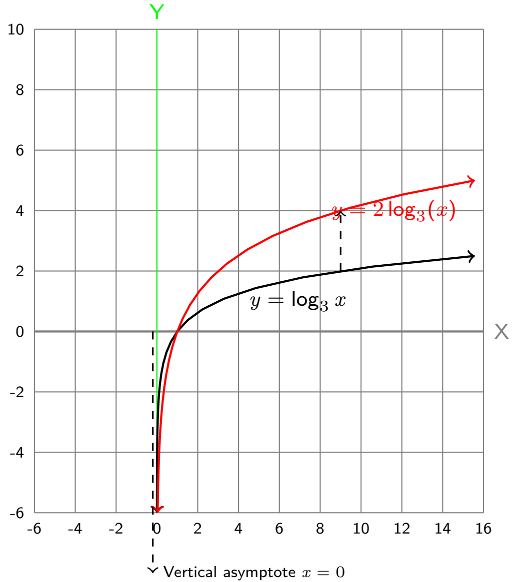
Solution: Start with the graph of $y = \log_3 x$, with vertical asymptote the y-axis.



Example 3.3.5b: Transform the graph of $y = \log_3 x$ step by step to the graph of $y = 4 + 2\log_3(x)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_3 x$, with vertical asymptote the y-axis.

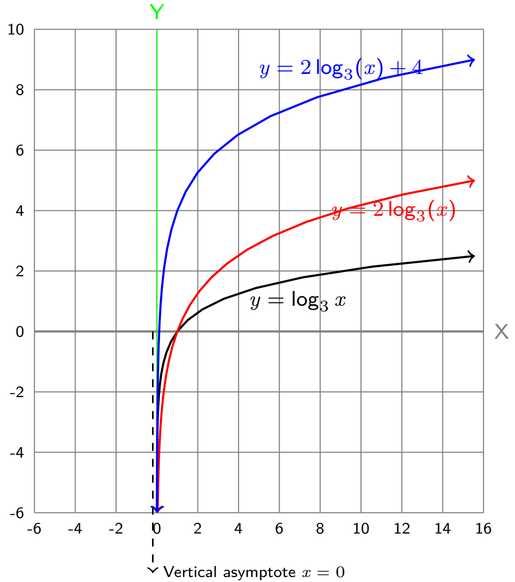
- Multiply the RHS of the equation $y = \log_3(x)$ by 2. This multiplies each points y-coordinate by 2: point $(9, 2)$ goes to $(9, 4)$. The graph stretches away by a factor of 2 from the x-axis to yield the graph of $y = 2\log_3(x)$..



Example 3.3.5b: Transform the graph of $y = \log_3 x$ step by step to the graph of $y = 4 + 2\log_3(x)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_3 x$, with vertical asymptote the y-axis.

- Multiply the RHS of the equation $y = \log_3(x)$ by 2. This multiplies each points y-coordinate by 2: point $(9, 2)$ goes to $(9, 4)$. The graph stretches away by a factor of 2 from the x-axis to yield the graph of $y = 2\log_3(x)$..
- Adding 4 to the RHS of $y = 2\log_3(x)$ shifts its graph up 4 units to give the requested graph of $y = 4 + 2\log_3(x)$.



Example 3.3.5b: Transform the graph of $y = \log_3 x$ step by step to the graph of $y = 4 + 2\log_3(x)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_3 x$, with vertical asymptote the y-axis.

- Multiply the RHS of the equation $y = \log_3(x)$ by 2. This multiplies each points y-coordinate by 2: point $(9, 2)$ goes to $(9, 4)$. The graph stretches away by a factor of 2 from the x-axis to yield the graph of $y = 2\log_3(x)$..
- Adding 4 to the RHS of $y = 2\log_3(x)$ shifts its graph up 4 units to give the requested graph of $y = 4 + 2\log_3(x)$.
- There is no $y-$intercept since setting $y = 4 + 2\log_3(0)$ is undefined.

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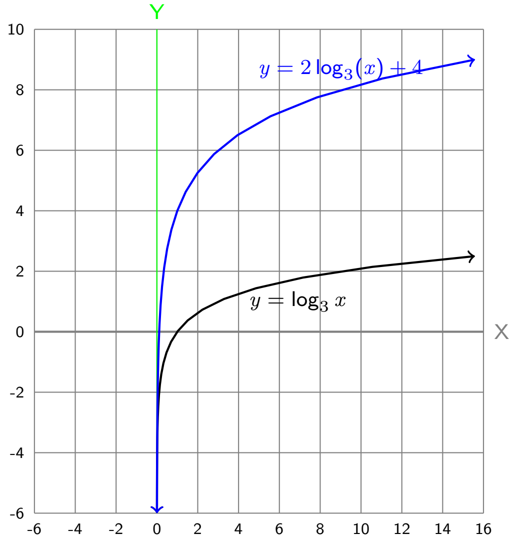
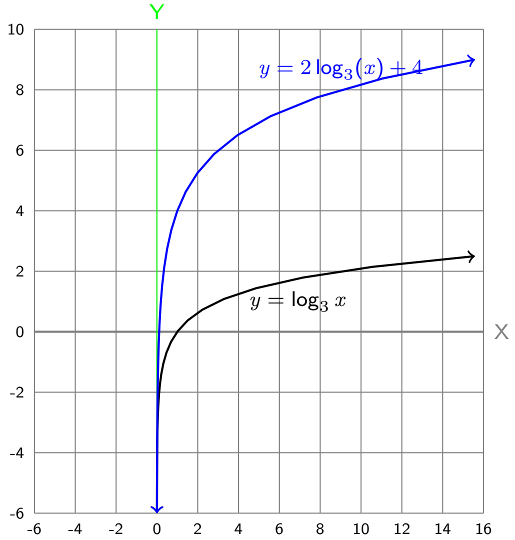


Example 3.3.5b: Transform the graph of $y = \log_3 x$ step by step to the graph of $y = 4 + 2\log_3(x)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_3 x$, with vertical asymptote the y-axis.

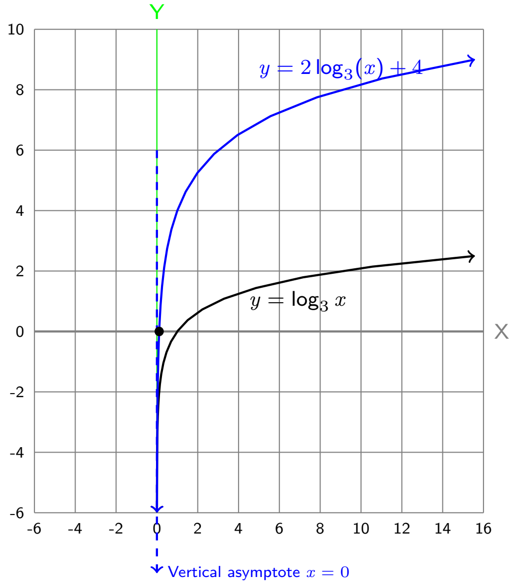
- Multiply the RHS of the equation $y = \log_3(x)$ by 2. This multiplies each points y-coordinate by 2: point $(9, 2)$ goes to $(9, 4)$. The graph stretches away by a factor of 2 from the x-axis to yield the graph of $y = 2\log_3(x)$..
- Adding 4 to the RHS of $y = 2\log_3(x)$ shifts its graph up 4 units to give the requested graph of $y = 4 + 2\log_3(x)$.
- There is no $y-$intercept since setting $y = 4 + 2\log_3(0)$ is undefined.
- Set $y = 0$ to find the x-intercept: $y = 0 = 4+2\log_3(x) \Rightarrow \log_3(x) = -2 \Rightarrow x = 3^{-2} = \frac{1}{9}$ The x-intercept point is $(\frac{1}{9}, 0)$.



Example 3.3.5b: Transform the graph of $y = \log_3 x$ step by step to the graph of $y = 4 + 2\log_3(x)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

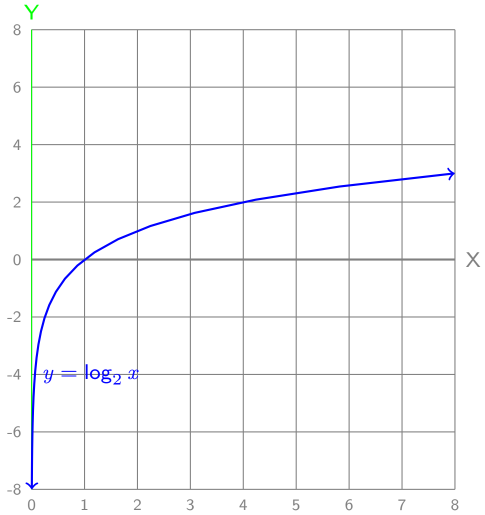
Solution: Start with the graph of $y = \log_3 x$, with vertical asymptote the y-axis.

- Multiply the RHS of the equation $y = \log_3(x)$ by 2. This multiplies each points y-coordinate by 2: point $(9, 2)$ goes to $(9, 4)$. The graph stretches away by a factor of 2 from the x-axis to yield the graph of $y = 2\log_3(x)$..
- Adding 4 to the RHS of $y = 2\log_3(x)$ shifts its graph up 4 units to give the requested graph of $y = 4 + 2\log_3(x)$.
- There is no $y-$intercept since setting $y = 4 + 2\log_3(0)$ is undefined.
- Set $y = 0$ to find the x-intercept: $y = 0 = 4+2\log_3(x) \Rightarrow \log_3(x) = -2 \Rightarrow x = 3^{-2} = \frac{1}{9}$ The x-intercept point is $(\frac{1}{9}, 0)$.
- The vertical asymptote remains $x = 0$.



Example 3.3.5c: Transform the graph of $y = \log_2 x$ step by step to the graph of $y = -2\log_2(x-4)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

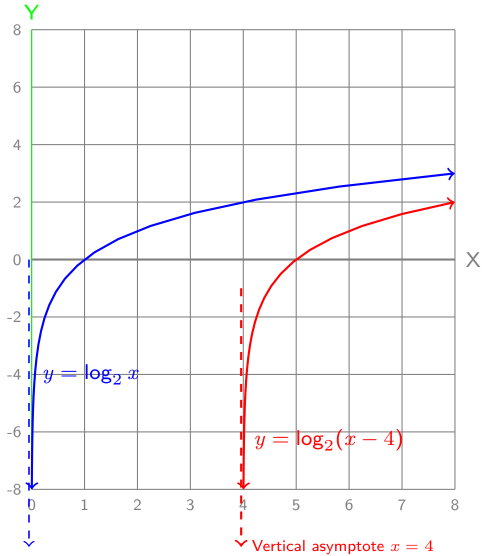
Solution: Start with the graph of $y = \log_2 x$, with asymptote the y-axis and x-intercept $(1, 0)$



Example 3.3.5c: Transform the graph of $y = \log_2 x$ step by step to the graph of $y = -2\log_2(x-4)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

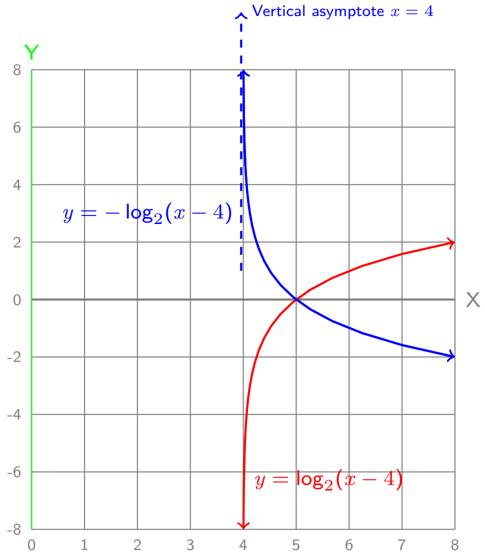
Solution: Start with the graph of $y = \log_2 x$, with asymptote the y-axis and x-intercept $(1, 0)$

- First substitute $x - 4$ for x in $y = \log_2 x$. The graph shifts 4 units right to yield the graph of $y = \log_2(x-4)$. The vertical asymptote and the x-intercept shift 4 right to $x = 4$ and $(5, 0)$, respectively.



Example 3.3.5c: Transform the graph of $y = \log_2 x$ step by step to the graph of $y = -2\log_2(x-4)$. Sketch the resulting graph. Find and label its intercepts and asymptote.
Solution: Start with the graph of $y = \log_2 x$, with asymptote the y-axis and x-intercept $(1, 0)$

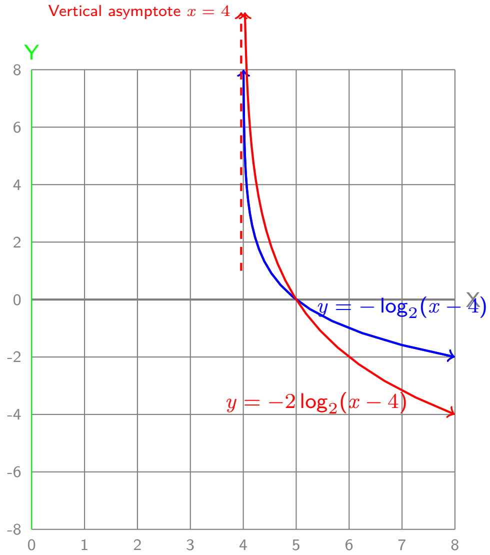
- First substitute $x - 4$ for x in $y = \log_2 x$. The graph shifts 4 units right to yield the graph of $y = \log_2(x-4)$. The vertical asymptote and the x-intercept shift 4 right to $x = 4$ and $(5, 0)$, respectively.
- Multiply the RHS of the equation $y = \log_2(x-4)$ by -1. Its graph reflects across the x-axis to yield the graph of $y = -\log_2(x-4)$. The asymptote remains $x = 4$ but now points up instead of down.



Example 3.3.5c: Transform the graph of $y = \log_2 x$ step by step to the graph of $y = -2\log_2(x-4)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_2 x$, with asymptote the y-axis and x-intercept $(1, 0)$

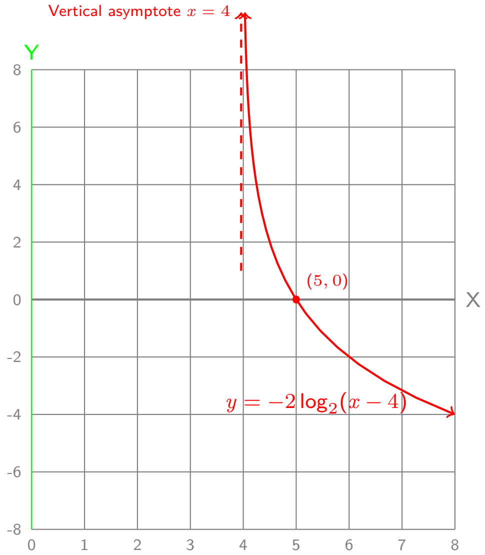
- First substitute $x - 4$ for x in $y = \log_2 x$. The graph shifts 4 units right to yield the graph of $y = \log_2(x-4)$. The vertical asymptote and the x-intercept shift 4 right to $x = 4$ and $(5, 0)$, respectively.
- Multiply the RHS of the equation $y = \log_2(x-4)$ by -1. Its graph reflects across the x-axis to yield the graph of $y = -\log_2(x-4)$. The asymptote remains $x = 4$ but now points up instead of down.
- Multiply the RHS of $y = -\log_2(x-4)$ by 2. Every point's $y-$coordinate is multiplied by 2: the graph stretches away from the x-axis by a factor of 2. This yields the requested graph of $y = -2\log_2(x-4)$. The vertical asymptote is $x = 4$. and the x-intercept is $(5, 0)$



Example 3.3.5c: Transform the graph of $y = \log_2 x$ step by step to the graph of $y = -2\log_2(x-4)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_2 x$, with asymptote the y-axis and x-intercept $(1, 0)$

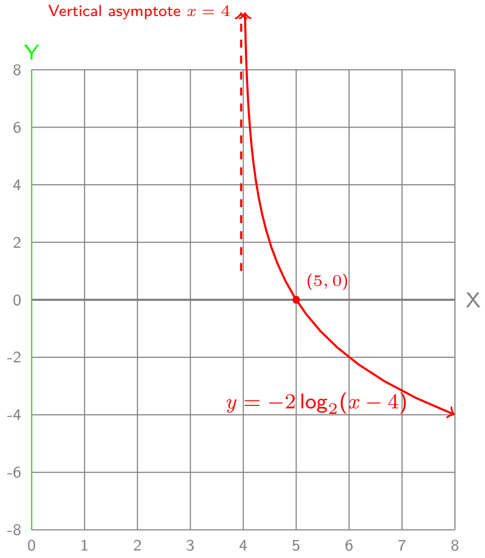
- First substitute $x - 4$ for x in $y = \log_2 x$. The graph shifts 4 units right to yield the graph of $y = \log_2(x-4)$. The vertical asymptote and the x-intercept shift 4 right to $x = 4$ and $(5, 0)$, respectively.
- Multiply the RHS of the equation $y = \log_2(x-4)$ by -1. Its graph reflects across the x-axis to yield the graph of $y = -\log_2(x-4)$. The asymptote remains $x = 4$ but now points up instead of down.
- Multiply the RHS of $y = -\log_2(x-4)$ by 2. Every point's $y-$coordinate is multiplied by 2: the graph stretches away from the x-axis by a factor of 2. This yields the requested graph of $y = -2\log_2(x-4)$. The vertical asymptote is $x = 4$. and the x-intercept is $(5, 0)$
- Another way to find the x-intercept: set $y = 0 = \log_2(x-4) \Rightarrow x - 4 = 1 \Rightarrow x = 5$.. The $x-$intercept point is $(5, 0)$.



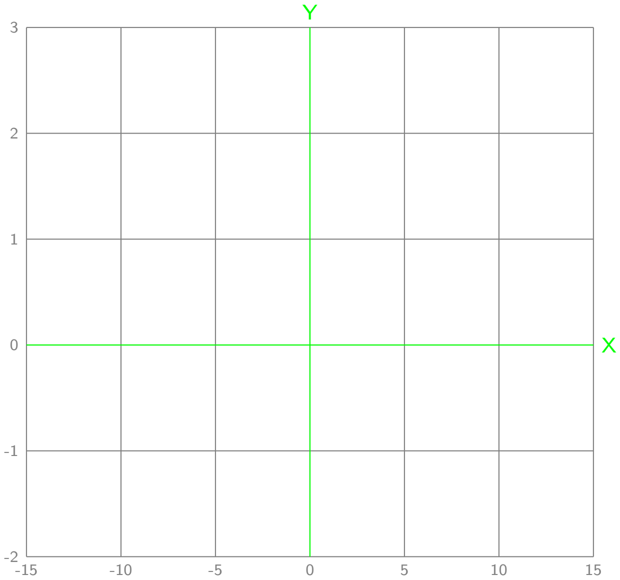
Example 3.3.5c: Transform the graph of $y = \log_2 x$ step by step to the graph of $y = -2\log_2(x-4)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_2 x$, with asymptote the y-axis and x-intercept $(1, 0)$

- First substitute $x - 4$ for x in $y = \log_2 x$. The graph shifts 4 units right to yield the graph of $y = \log_2(x-4)$. The vertical asymptote and the x-intercept shift 4 right to $x = 4$ and $(5, 0)$, respectively.
- Multiply the RHS of the equation $y = \log_2(x-4)$ by -1. Its graph reflects across the x-axis to yield the graph of $y = -\log_2(x-4)$. The asymptote remains $x = 4$ but now points up instead of down.
- Multiply the RHS of $y = -\log_2(x-4)$ by 2. Every point's $y-$coordinate is multiplied by 2: the graph stretches away from the x-axis by a factor of 2. This yields the requested graph of $y = -2\log_2(x-4)$. The vertical asymptote is $x = 4$. and the x-intercept is $(5, 0)$
- Another way to find the x-intercept: set $y = 0 = \log_2(x-4) \Rightarrow x - 4 = 1 \Rightarrow x = 5$.. The $x-$intercept point is $(5, 0)$.
- There is no y-intercept, since $\log_2(0-4)$ is undefined.

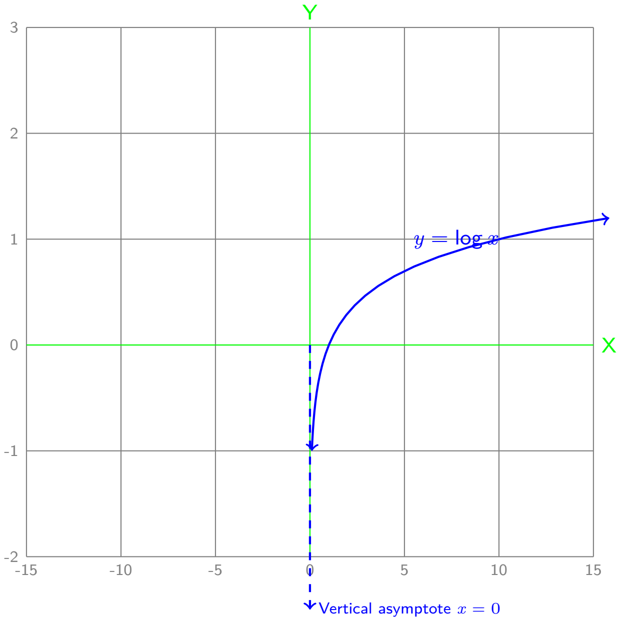


Example 3.3.5d: Transform the graph of $y = \log x = \log_{10} x$ step by step to the graph of $y = \log(x^2)$. Sketch the resulting graph. Find and label its intercepts and asymptote.



Example 3.3.5d: Transform the graph of $y = \log x = \log_{10} x$ step by step to the graph of $y = \log(x^2)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

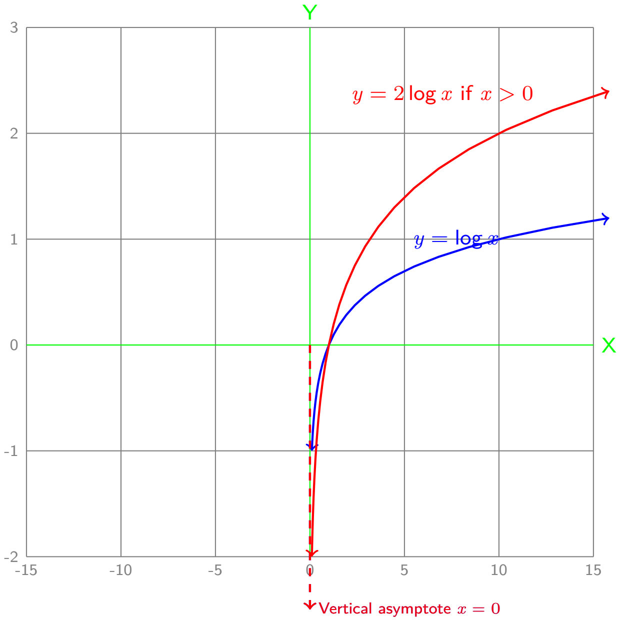
This is a trick question. There is no way to know in general what happens to the graph of $y = f(x)$ if you substitute x^2 for x. In this example, however, we know that $\log(x^2) = 2\log x$ if $x > 0$.



Example 3.3.5d: Transform the graph of $y = \log x = \log_{10} x$ step by step to the graph of $y = \log(x^2)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

This is a trick question. There is no way to know in general what happens to the graph of $y = f(x)$ if you substitute x^2 for x. In this example, however, we know that $\log(x^2) = 2\log x$ if $x > 0$.

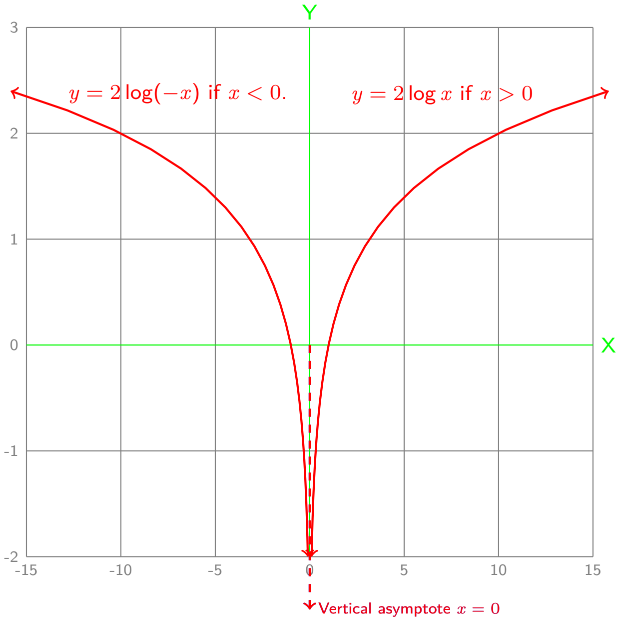
- Draw the graph of $y = \log x$ and then multiply the RHS by 2 to get the graph of $y = 2\log x$ by doubling each point's $y-$coordinate. The domain is $x > 0$.



Example 3.3.5d: Transform the graph of $y = \log x = \log_{10} x$ step by step to the graph of $y = \log(x^2)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

This is a trick question. There is no way to know in general what happens to the graph of $y = f(x)$ if you substitute x^2 for x. In this example, however, we know that $\log(x^2) = 2 \log x$ if $x > 0$.

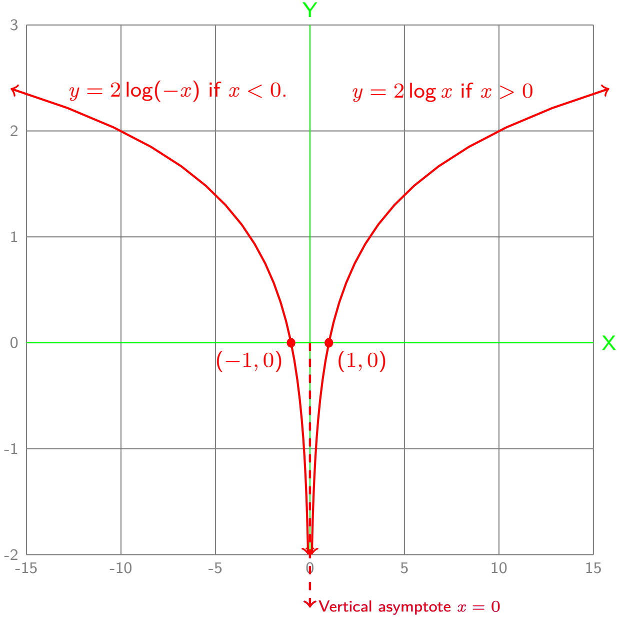
- Draw the graph of $y = \log x$ and then multiply the RHS by 2 to get the graph of $y = 2 \log x$ by doubling each point's $y-$coordinate. The domain is $x > 0$.
- To graph $y = \log(x^2)$ for $x < 0$ note that $x^2 = |x|^2$ for all x. Thus $y = \log(x^2) = \log(|x|^2) = 2 \log |x|$ for *all* $x \neq 0$. In particular, for $x < 0, |x| = -x$ and so $\log(x^2) = 2 \log |x| = 2 \log(-x)$. To graph $y = 2 \log(-x)$, reflect the graph of $2 \log x$ across the y-axis.



Example 3.3.5d: Transform the graph of $y = \log x = \log_{10} x$ step by step to the graph of $y = \log(x^2)$. Sketch the resulting graph. Find and label its intercepts and asymptote.

This is a trick question. There is no way to know in general what happens to the graph of $y = f(x)$ if you substitute x^2 for x. In this example, however, we know that $\log(x^2) = 2\log x$ if $x > 0$.

- Draw the graph of $y = \log x$ and then multiply the RHS by 2 to get the graph of $y = 2\log x$ by doubling each point's $y-$coordinate. The domain is $x > 0$.
- To graph $y = \log(x^2)$ for $x < 0$ note that $x^2 = |x|^2$ for all x. Thus $y = \log(x^2) = \log(|x|^2) = 2\log|x|$ for *all* $x \neq 0$. In particular, for $x < 0, |x| = -x$ and so $\log(x^2) = 2\log|x| = 2\log(-x)$. To graph $y = 2\log(-x)$, reflect the graph of $2\log x$ across the y-axis.
- The requested graph of $y = \log(x^2)$ has domain all $x \neq 0$. There is one vertical asymptote $x = 0$ and two x-intercepts $(\pm 1, 0)$.

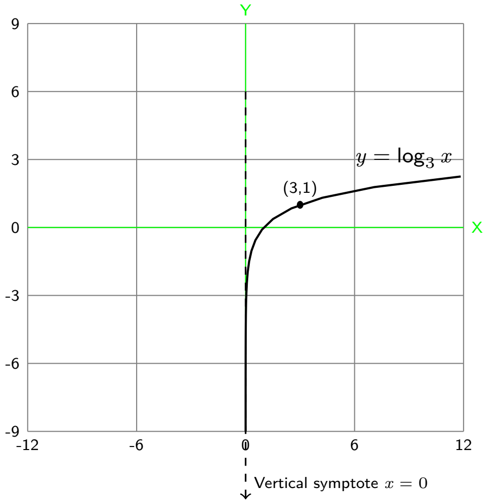


Example 3.3.6: Sketch the graph, labeled with intercepts and asymptotes, obtained by transforming, step by step, the graph of

- $y = \log_3 x$ to the graph of $y = \log_3(-x) - 2$.
- $y = \ln x$ to the graph of $y = 2 - \ln(2x + 1)$
- $y = \ln x$ to the graph of $y = 2\ln(2x - 1)$.
- $y = \ln 3x$ to the graph of $y = \ln(4 - 2x)$

Example 3.3.6a: Transform the graph of $y = \log_3 x$ step by step to obtain the graph of $y = \log_3(-x) - 2$. Sketch the resulting graph. Find and label its intercepts and asymptote.

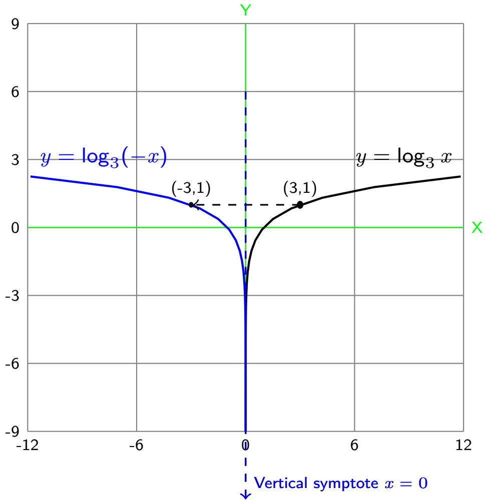
Solution: Start with the graph of $y = \log_3 x$, with asymptote the y-axis (the line $x = 0$.).



Example 3.3.6a: Transform the graph of $y = \log_3 x$ step by step to obtain the graph of $y = \log_3(-x) - 2$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_3 x$, with asymptote the y-axis (the line $x = 0$.).

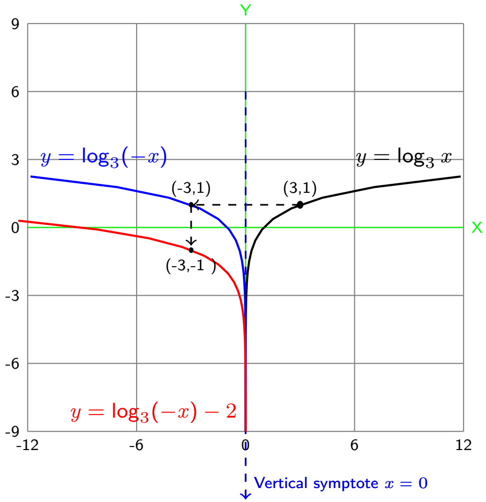
- Substituting $-x$ for x in $y = \log_3 x$ reflects its graph across the y-axis to yield the graph of $y = \log_3(-x)$. The asymptote is still $x = 0$.



Example 3.3.6a: Transform the graph of $y = \log_3 x$ step by step to obtain the graph of $y = \log_3(-x) - 2$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_3 x$, with asymptote the y-axis (the line $x = 0$.).

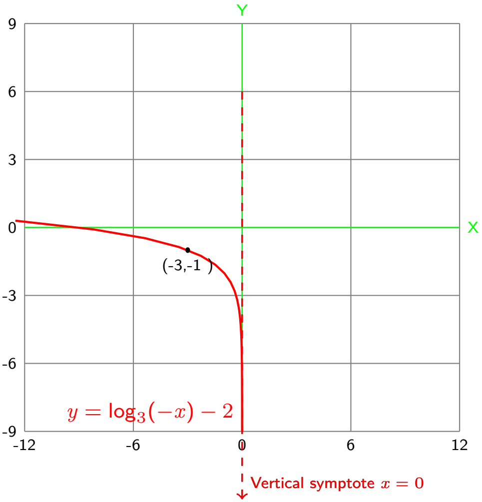
- Substituting $-x$ for x in $y = \log_3 x$ reflects its graph across the y-axis to yield the graph of $y = \log_3(-x)$. The asymptote is still $x = 0$.
- Subtracting 2 from RHS of the equation $y = \log_3(-x)$ shifts its graph down 2 units to yield the requested graph of $y = \log_3(-x) - 2$. The asymptote shifts down and is still $x = 0$.



Example 3.3.6a: Transform the graph of $y = \log_3 x$ step by step to obtain the graph of $y = \log_3(-x) - 2$. Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \log_3 x$, with asymptote the y-axis (the line $x = 0$.).

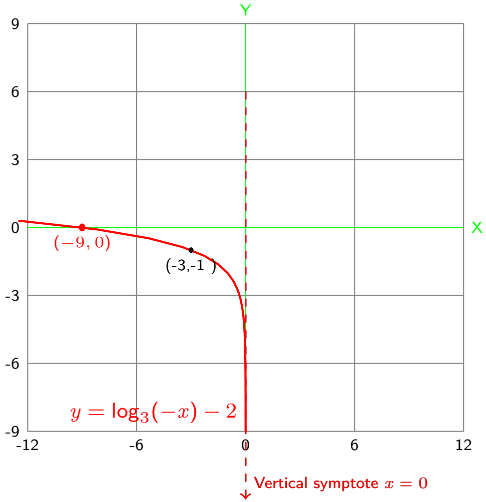
- Substituting $-x$ for x in $y = \log_3 x$ reflects its graph across the y-axis to yield the graph of $y = \log_3(-x)$. The asymptote is still $x = 0$.
- Subtracting 2 from RHS of the equation $y = \log_3(-x)$ shifts its graph down 2 units to yield the requested graph of $y = \log_3(-x) - 2$. The asymptote shifts down and is still $x = 0$.
- Setting $x = 0$ in $y = \log_3(-x) - 2$ makes the RHS undefined. There is no y-intercept.



Example 3.3.6a: Transform the graph of $y = \log_3 x$ step by step to obtain the graph of $y = \log_3(-x) - 2$. Sketch the resulting graph. Find and label its intercepts and asymptote.

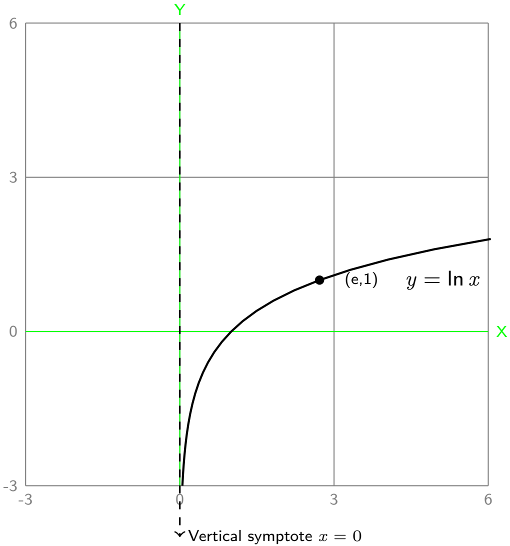
Solution: Start with the graph of $y = \log_3 x$, with asymptote the y-axis (the line $x = 0$.).

- Substituting $-x$ for x in $y = \log_3 x$ reflects its graph across the y-axis to yield the graph of $y = \log_3(-x)$. The asymptote is still $x = 0$.
- Subtracting 2 from RHS of the equation $y = \log_3(-x)$ shifts its graph down 2 units to yield the requested graph of $y = \log_3(-x) - 2$. The asymptote shifts down and is still $x = 0$.
- Setting $x = 0$ in $y = \log_3(-x) - 2$ makes the RHS undefined. There is no y-intercept.
- Set $y = 0$ in that equation to get $0 = \log_3(-x) - 2 \Rightarrow \log_3(-x) = 2 \Rightarrow -x = 3^2 = 9$ and so $x = -9$ is the x-intercept, at point $(-9, 0)$.
- The vertical asymptote is the line $x = 0$ (the y-axis).



Example 3.3.6b: Transform the graph of $y = \ln x$ to the graph of $y = 2 + \ln(2x + 1)$
Sketch the resulting graph. Find and label its intercepts and asymptote.

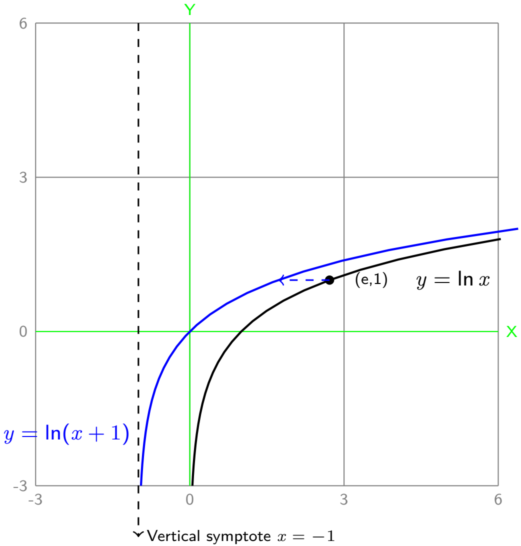
Solution: Start with the graph of $y = \ln x$, with asymptote $x = 0$.



Example 3.3.6b: Transform the graph of $y = \ln x$ to the graph of $y = 2 + \ln(2x + 1)$
Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \ln x$, with asymptote $x = 0$.

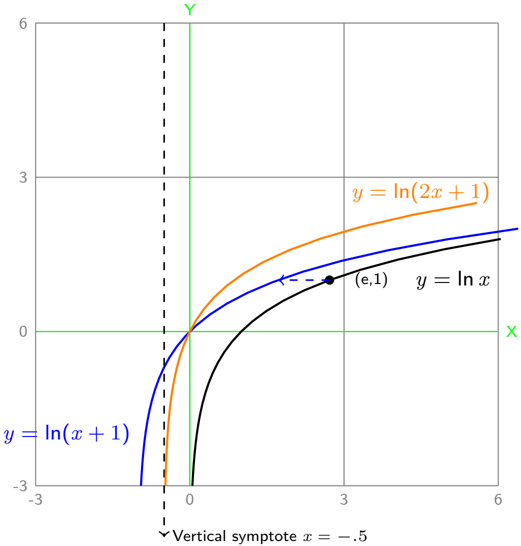
- Substituting $x + 1$ for x in $y = \ln x$ shifts its graph left 1 unit to yield the graph of $y = \ln(x + 1)$. The asymptote shifts to $x = -1$.



Example 3.3.6b: Transform the graph of $y = \ln x$ to the graph of $y = 2 + \ln(2x + 1)$
Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \ln x$, with asymptote $x = 0$.

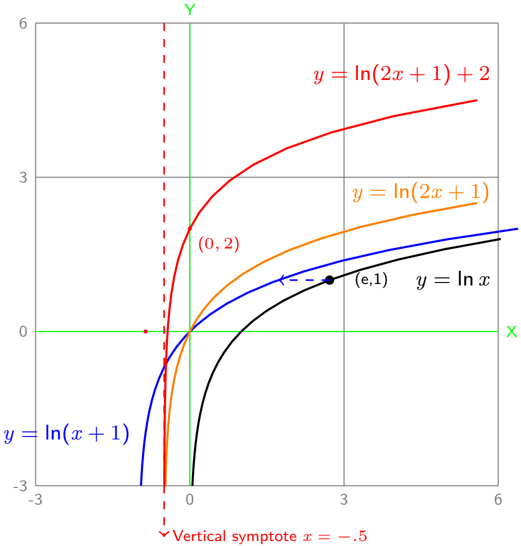
- Substituting $x + 1$ for x in $y = \ln x$ shifts its graph left 1 unit to yield the graph of $y = \ln(x + 1)$. The asymptote shifts to $x = -1$.
- Substituting $2x$ for x in $y = \ln(x + 1)$ contracts its graph by a factor of 2 toward the x-axis to yield the graph of $y = \ln(2x + 1)$ Every point's x-coordinate is divided by 2. The asymptote is now $x = -\frac{1}{2}$.



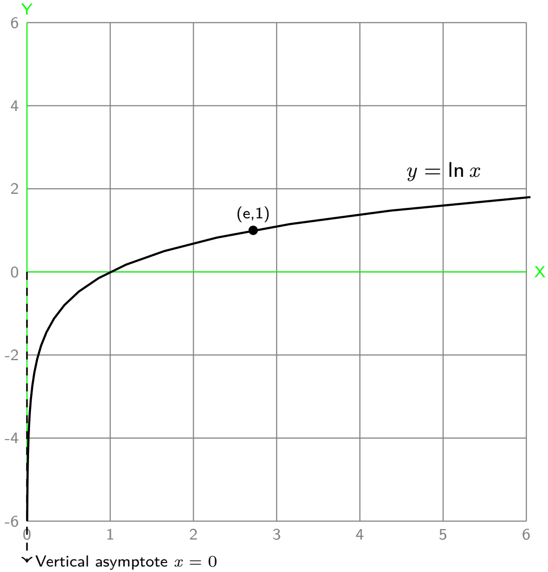
Example 3.3.6b: Transform the graph of $y = \ln x$ to the graph of $y = 2 + \ln(2x + 1)$
Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \ln x$, with asymptote $x = 0$.

- Substituting $x + 1$ for x in $y = \ln x$ shifts its graph left 1 unit to yield the graph of $y = \ln(x + 1)$. The asymptote shifts to $x = -1$.
- Substituting $2x$ for x in $y = \ln(x + 1)$ contracts its graph by a factor of 2 toward the x-axis to yield the graph of $y = \ln(2x + 1)$ Every point's x-coordinate is divided by 2. The asymptote is now $x = -\frac{1}{2}$.
- Adding 2 to the RHS of $y = \ln(2x + 1)$ shifts the graph up to the desired graph of $y = 2 + \ln(2x + 1)$. Setting $x = 0$ in $y = 2 + \ln(2x + 1)$ gives $y-$intercept $2 + 2\ln(1) = 2$.
- Set $y = 0$ to get $\ln(2x + 1) = -2$ $\Rightarrow 2x+1 = e^{-2} \Rightarrow x = \frac{e^-2-1}{2} = \frac{1-e^2}{2e^2} \approx -0.86$, the x-intercept. The vertical asymptote is $x = -\frac{1}{2}$.

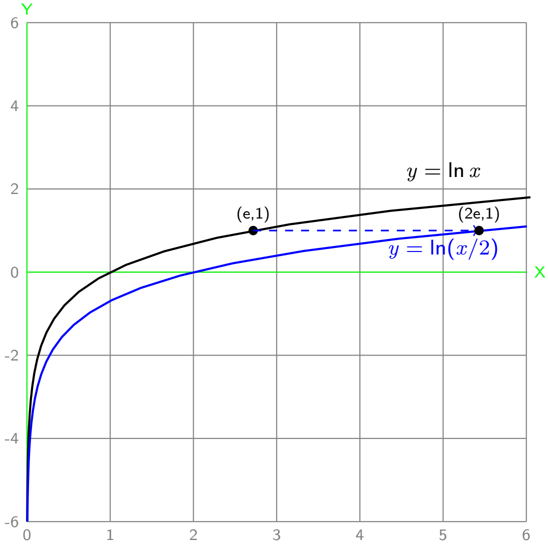


Example 3.3.6c: Transform the graph of $y = \ln x$ to the graph of $y = 2 - \ln(x/2)$.
Sketch the resulting graph. Find and label its intercepts and asymptote.
Solution: Start with the graph of $y = \ln x$, with asymptote $x = 0$.



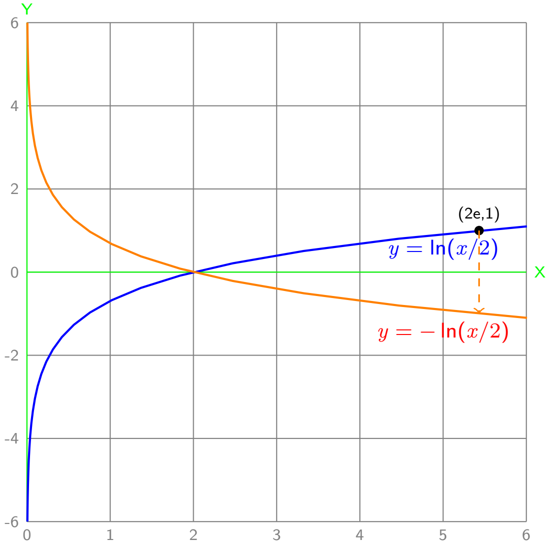
Example 3.3.6c: Transform the graph of $y = \ln x$ to the graph of $y = 2 - \ln(x/2)$.
Sketch the resulting graph. Find and label its intercepts and asymptote.
Solution: Start with the graph of $y = \ln x$, with asymptote $x = 0$.

- Substituting $x/2$ for x in $y = \ln x$ stretches its graph by a factor of 2 from the x-axis to the graph of $y = \ln(x/2)$. Every point's x-coordinate is multiplied by 2. Point $(e, 1)$ goes to $(2e, 1)$



Example 3.3.6c: Transform the graph of $y = \ln x$ to the graph of $y = 2 - \ln(x/2)$.
Sketch the resulting graph. Find and label its intercepts and asymptote.

- Substituting $x/2$ for x in $y = \ln x$ stretches its graph by a factor of 2 from the x-axis to the graph of $y = \ln(x/2)$. Every point's x-coordinate is multiplied by 2. Point $(e, 1)$ goes to $(2e, 1)$
- Multiplying RHS by -1 reflects the graph of $y = \ln x/2$ across the x-axis to yield the graph $y = -\ln x/2$ of Every point's y-coordinate is multiplied by -1. Point $(2e, 1)$ goes to $(2e, -1)$.



Example 3.3.6c: Transform the graph of $y = \ln x$ to the graph of $y = 2 - \ln(x/2)$.
Sketch the resulting graph. Find and label its intercepts and asymptote.

- Substituting $x/2$ for x in $y = \ln x$ stretches its graph by a factor of 2 from the x-axis to the graph of $y = \ln(x/2)$. Every point's x-coordinate is multiplied by 2. Point $(e, 1)$ goes to $(2e, 1)$
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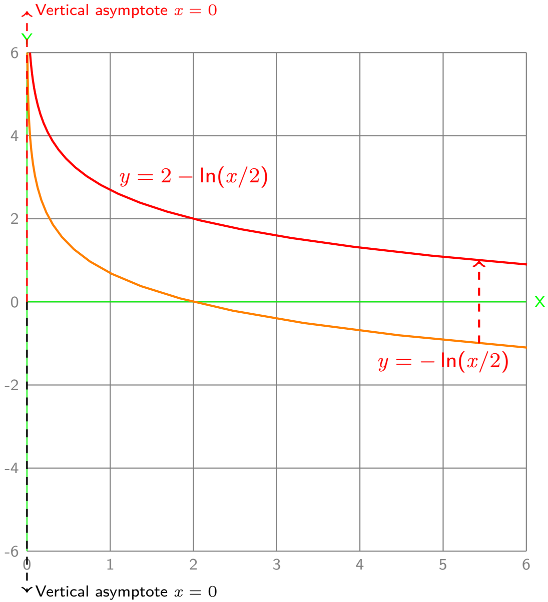
- Adding 2 to the RHS of $y = -\ln x/2$ shifts the graph up to the desired graph of $y = 2 - \ln(x/2)$.The asymptote remains $x = 0$.



Example 3.3.6c: Transform the graph of $y = \ln x$ to the graph of $y = 2 - \ln(x/2)$.
Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \ln x$, with asymptote $x = 0$.

- Substituting $x/2$ for x in $y = \ln x$ stretches its graph by a factor of 2 from the x-axis to the graph of $y = \ln(x/2)$. Every point's x-coordinate is multiplied by 2. Point $(e, 1)$ goes to $(2e, 1)$
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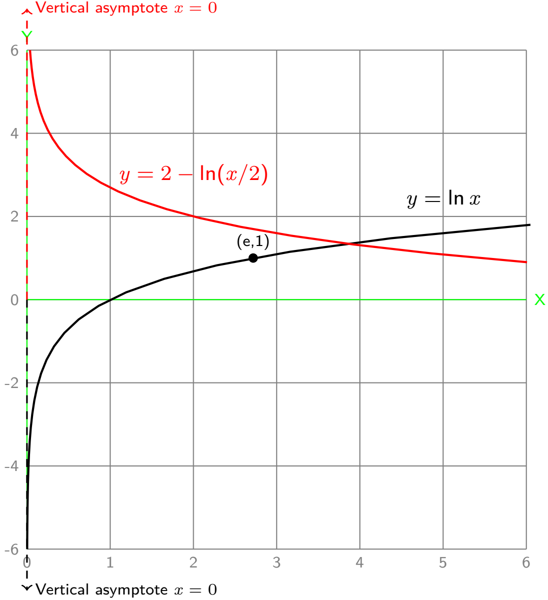
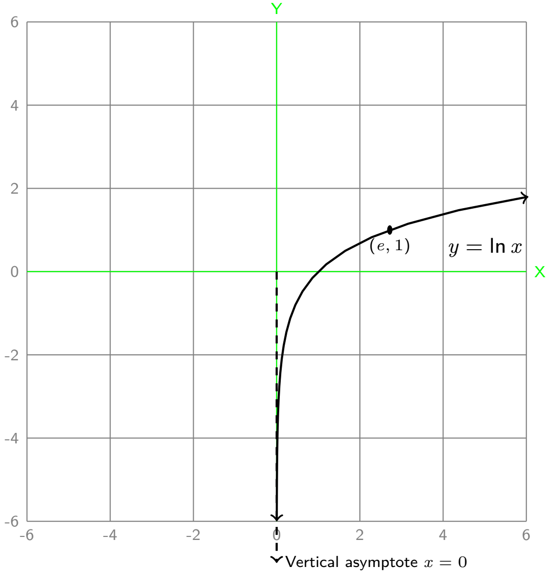
- Adding 2 to the RHS of $y = -\ln x/2$ shifts the graph up to the desired graph of $y = 2 - \ln(x/2)$.The asymptote remains $x = 0$. Setting $x = 0$ in $y = 2 - \ln(x/2)$ yields undefined: There is no $y-$intercept.
- Set $y = 0$ to get $\ln(x/2) = 2 \Rightarrow x/2 = e^2 \Rightarrow x = 2e^2 \approx 14.8$, the x-intercept, not shown on the graph. The vertical asymptote is $x = 0$.



Example 3.3.6d: Transform the graph of $y = \ln x$ to the graph of $y = \ln(4 - 2x)$ Sketch the resulting graph. Find and label its intercepts and asymptote.

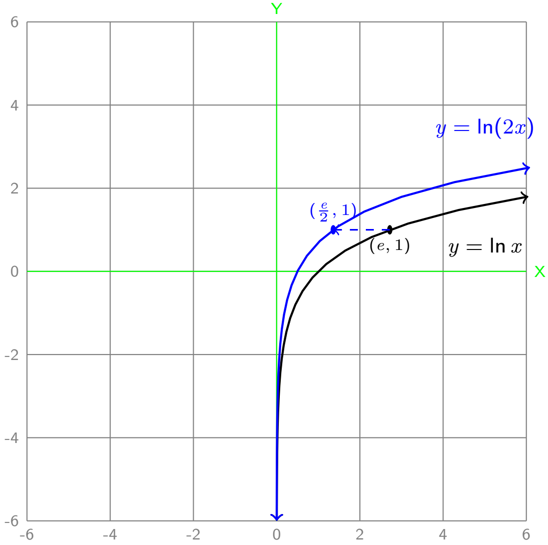
Solution: Start with the graph of $y = \ln x$, with asymptote $x = 0$.



Example 3.3.6d: Transform the graph of $y = \ln x$ to the graph of $y = \ln(4 - 2x)$ Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \ln x$, with asymptote $x = 0$.

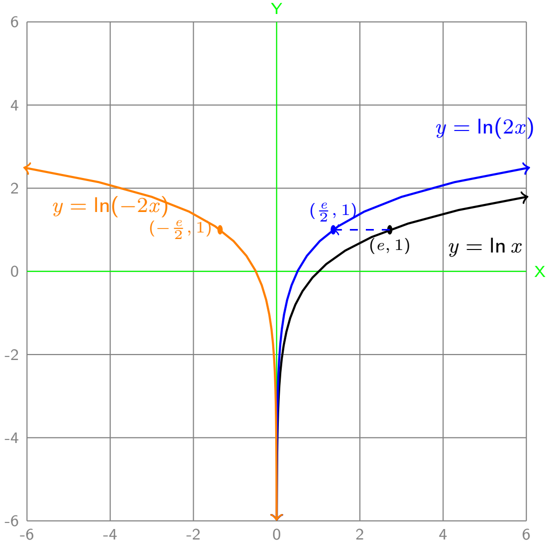
- Substituting $2x$ for x in $y = \ln x$ shrinks its graph by a factor of 2 toward the x-axis to yield the graph of $y = \ln(2x)$. Every point's x-coordinate is divided by 2. Point $(e, 1)$ goes to $(e/2, 1)$



Example 3.3.6d: Transform the graph of $y = \ln x$ to the graph of $y = \ln(4 - 2x)$ Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \ln x$, with asymptote $x = 0$.

- Substituting $2x$ for x in $y = \ln x$ shrinks its graph by a factor of 2 toward the x-axis to yield the graph of $y = \ln(2x)$. Every point's x-coordinate is divided by 2. Point $(e, 1)$ goes to $(e/2, 1)$
- Substituting $-x$ for x in $y = \ln(2x)$ reflects the graph of $y = \ln 2x$ across the y-axis to yield the graph $y = \ln(-2x)$. Every point's x-coordinate is multiplied by -1. Point $(e/2, 1)$ goes to $(-e/2, 1)$.



Example 3.3.6d: Transform the graph of $y = \ln x$ to the graph of $y = \ln(4 - 2x)$ Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \ln x$, with asymptote $x = 0$.

- Substituting $2x$ for x in $y = \ln x$ shrinks its graph by a factor of 2 toward the x-axis to yield the graph of $y = \ln(2x)$. Every point's x-coordinate is divided by 2. Point $(e, 1)$ goes to $(e/2, 1)$
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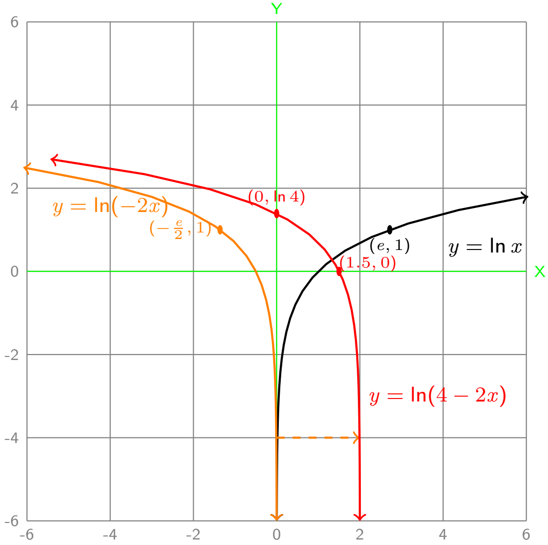
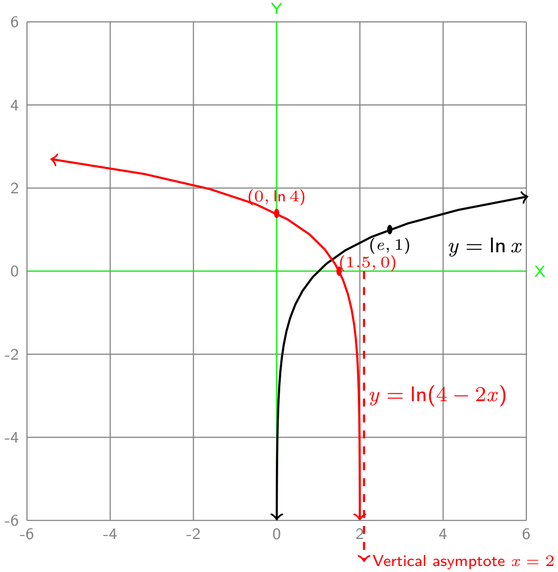
- Substituting $x - 2$ for x in $y = \ln(-2x)$ moves the graph right 2 units to the desired graph $y = \ln(-2(x - 2)) = \ln(4 - 2x)$.



Example 3.3.6d: Transform the graph of $y = \ln x$ to the graph of $y = \ln(4 - 2x)$ Sketch the resulting graph. Find and label its intercepts and asymptote.

Solution: Start with the graph of $y = \ln x$, with asymptote $x = 0$.

- Substituting $2x$ for x in $y = \ln x$ shrinks its graph by a factor of 2 toward the x-axis to yield the graph of $y = \ln(2x)$. Every point's x-coordinate is divided by 2. Point $(e, 1)$ goes to $(e/2, 1)$
- Substituting $-x$ for x in $y = \ln(2x)$ reflects the graph of $y = \ln 2x$ across the y-axis to yield the graph $y = \ln(-2x)$. Every point's x-coordinate is multiplied by -1. Point $(e/2, 1)$ goes to $(-e/2, 1)$.
- Substituting $x - 2$ for x in $y = \ln(-2x)$ moves the graph right 2 units to the desired graph $y = \ln(-2(x - 2)) = \ln(4 - 2x)$.
- Setting $x = 0$ in $y = \ln(4 - 2x)$ yields y-intercept $\ln 4 \approx 1.39$. Set $y = 0$ to get $\ln(4 - 2x) \Rightarrow 4 - 2x = 1 \Rightarrow x = 1.5$, the x-intercept. The vertical asymptote is $x = 0$.



Chapter 3 Section 4: Laws of logarithms

- 3.4.1: Laws of logarithms
- 3.4.2: Expanding log expressions
- 3.4.3: Combining log expressions
- 3.4.4: Section 3.4 Quiz

Section 3.4 Preview: Definitions

- Definition 3.4.1: Log and exp are inverse functions.
- Definition 3.4.2: To find $\log_a x$, express x as $x = a^K$.
- Definition 3.4.3: Laws of logarithms
- Definition 3.4.4: The input to a log function is not always enclosed in parentheses.

3.4.1 Laws of logarithms

We have seen that the function $y = a^x$ has inverse function $y = \log_a(x)$, defined for $x > 0$. In the following, suppose $a > 0, u > 0, v > 0$, so that $\log_a u$ and $\log_a v$ are defined.

Log and exp are inverse functions

- $\log_a(a^x) = x$
- $\ln e^x = \log_e(e^x) = x$
- $a^{\log_a x} = x$
- $e^{\ln x} = x$

To find $\log_a x$, express x as $x = a^K$.

- $\log_a(1) = \log_a(a^0) = 0$
- $\log_a(a) = \log_a(a^1) = 1$
- $\log_a x = \log_a(a^K) = K$

Laws of logarithms

- $\log_a(uv) = \log_a u + \log_a v$: Log of product = sum of logs.
- $\log_a\left(\frac{u}{v}\right) = \log_a u - \log v$: Log of quotient = difference of logs.
- $\log_a\left(\frac{1}{v}\right) = -\log_a v$: Log of reciprocal = negative of log.
- Let n be any real number. Then $\log_a(u^n) = n\log_a u$: Log of power = exponent times log of base.

The input to a log function is not always enclosed in parentheses.

- $\log a = \log(a)$
- $\log uv = \log(uv)$
- $\log abcd^3 = \log(abcd^3)$
- $\ln \frac{a\sqrt{b}}{c+e} = \ln(\frac{a\sqrt{b}}{c+e})$

Example 1: Find $\log_3(81), \log_2(\frac{1}{9})$, and $\log_2(9\sqrt{3})$.

Solutions:

- $\log_3(81) = \log_3(3^4) = 4$.
- $\frac{1}{9} = \frac{1}{3^2} = 3^{-2}$, so $\log_3(\frac{1}{9}) = \log_3(3^{-2}) = -2$.
- $9\sqrt{3} = 3^2 \cdot 3^{1/2} = 3^{2+1/2} = 3^{\frac{5}{2}}$, and so $\log_3(9\sqrt{3}) = \log_3(3^{\frac{5}{2}}) = \boxed{\frac{5}{2}}$

Example 2: Find $\log_{10}(\sqrt[3]{100})$ and $\log_{10}\left(\frac{1}{100\sqrt{10}}\right)$

Solutions:

- Rewrite $\sqrt[3]{100} = 100^{\frac{1}{3}} = (10^2)^{1/3} = 10^{\frac{2}{3}}$

  Then $\log_{10}(\sqrt[3]{100}) = \log_{10}(10^{\frac{2}{3}}) = \boxed{\frac{2}{3}}$.

- Rewrite

$$\frac{1}{100\sqrt{10}} = \frac{1}{10^2 \cdot 10^{1/2}} = \frac{1}{10^{2+1/2}} = \frac{1}{10^{5/2}} = 10^{-5/2}.$$

Then $\log_{10}\left(\frac{1}{100\sqrt{10}}\right) = \log_{10} 10^{-5/2} = \boxed{-\frac{5}{2}}$.

3.4.2 Expanding logs of products or quotients

Example 3: Find $\ln\left(\frac{e^9}{\sqrt{e}}\right)$. Solution: $\frac{e^9}{\sqrt{e}} = \frac{e^9}{e^{1/2}} = e^{9-1/2} = e^{17/2}$ and so $\ln\left(\frac{e^9}{\sqrt{e}}\right) = \ln\left(e^{\frac{17}{2}}\right) = \boxed{\frac{17}{2}}$.

Example 4: Use $\log uv = \log u + \log v$ to find $\log\left[(x^2+1)^4 \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right]$ if $x > 0$.

Solution: To find: $\log\left[(x^2+1)^4 \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right]$

Apply $\log uv = \log u + \log v$: $= \log\,(x^2+1)^4 + \log\,(x^2+1)^{\frac{1}{2}} + \log\, x^{\frac{1}{3}}$

Bring down powers: $= 4\log(x^2+1) + \frac{1}{2}\log(x^2+1) + \frac{1}{3}\log x$

Combine like terms: $= \boxed{\frac{9}{2}\log(x^2+1) + \frac{1}{3}\log x}$.

Example 5: Use $\log \frac{u}{v} = \log u - \log v$ to find $\log \frac{x^3}{\sqrt{x^2+1}\cdot(x+2)^3}$ if $x > 0$.

Solution: To find: $\log \frac{x^3}{\sqrt{x^2+1}\cdot(x+2)^3}$

Apply $\log \frac{u}{v} = \log u - \log v$ $= \log x^3 - \log\left(\sqrt{x^2+1}\cdot(x+2)^3\right)$

Apply $\log uv = \log u + \log v$

Make sure to use parentheses! $= \log x^3 - \left(\log\,(x^2+1)^{\frac{1}{2}} + \log(x+2)^3\right)$

Bring down powers: $= 3\log x - \left(\frac{1}{2}\log(x^2+1) + 3\log(x+2)\right)$

Distribute minus sign $= \boxed{3\log x - \frac{1}{2}\log(x^2+1) - 3\log(x+2)}$.

Example 6: Use laws of logarithms to expand and simplify $\log \sqrt[3]{a^5 bc^3}$ if a, b, c are > 0.

Solution:

To simplify:	$\log \sqrt[3]{a^5 bc^3}$
Rewrite the cube root as the power $\frac{1}{3}$	$= \log \left(a^5 bc^3\right)^{\frac{1}{3}}$
Apply $\log u^n = n \log u$	$= \frac{1}{3} \log(a^5 bc^3)$
Apply $\log uvw = \log u + \log v + \log w$	$= \frac{1}{3}(\log a^5 + \log b + \log c^3)$
Apply $\log u^n = n \log u$	$= \frac{1}{3}(5 \log a + \log b + 3 \log c)$
Multiply out	$= \boxed{\frac{5}{3} \log a + \frac{1}{3} \log b + \log c}$

Example 7: Use laws of logarithms to expand and simplify $\log \dfrac{\sqrt[3]{b}}{abc^3}$ if a, b, c are > 0.

Solution:

To simplify:	$\log \dfrac{\sqrt[3]{b}}{abc^3}$ Note: $\sqrt[3]{b} = b^{\frac{1}{3}}$
Apply $\log \frac{u}{v} = \log u - \log v$:	$= \log b^{\frac{1}{3}} - \log abc^3$
Apply $\log u^n = n \log u$	$= \frac{1}{3} \log b - \log abc^3$
Apply $\log uvw = \log u + \log v + \log w$	$= \frac{1}{3} \log b - (\log a + \log b + \log c^3)$
Apply $\log u^3 = 3 \log u$ and	
Distribute minus sign	$= \frac{1}{3} \log b - \log a - \log b - 3 \log c$
Combine terms	$= \boxed{-\frac{2}{3} \log b - \log a - 3 \log c}$

3.4.3 Combining log expressions

The previous examples rewrote the log of a complicated expression as the sum or difference of logs of simpler expressions. The following examples illustrate the reverse process:

Example 8: Simplify: $\log_8 2 + \log_8 32$.

Easy solution: Combine terms by using the logarithm product formula:
$\log_8 u + \log_8 v = \log_8 uv$ with $u = 2$ and $v = 32$.
$\log_8 2 + \log_8 32 = \log_8(2 \cdot 32) = \log_8 64 = \log_8 8^2 = \boxed{2}$.

Hard solution:

- Since $2^3 = 8$, then $2 = 8^{1/3}$ and so $\log_8 2 = \log_8(8^{1/3}) = \frac{1}{3}$.
- Since $32 = 2^5$, then $32 = (8^{\frac{1}{3}})^5 = 8^{\frac{5}{3}}$ and so $\log_8(32) = \log_8(8^{5/3}) = \frac{5}{3}$.

Therefore $\log_8 2 + \log_8 32 = \frac{1}{3} + \frac{5}{3} = \frac{6}{3} = \boxed{2}$.

The following examples use log as an abbreviation for $\log_k$ for any positive base k, not just base $k = 10$.

Rewrite each of the following as a single logarithm: Assume a, b, c are > 0.

Example 9: $\log a + \log b$

Solution: $\log a + \log b = \boxed{\log\ ab}$

Example 10: $\log a + \frac{1}{2}\log b$

Solution:

$\log a + \frac{1}{2}\log b = \log a + \log\ b^{\frac{1}{2}} = \log ab^{\frac{1}{2}} = \boxed{\log\ a\sqrt{b}}$

Example 11: $\log a - \log b$

Solution: $\log a - \log b = \boxed{\log \frac{a}{b}}$

Example 12: $\log a - \log b - \log c$

Solution: $\log a - \log b - \log c = \log a - (\log b + \log c)$
$= \log a - \log bc = \boxed{\log \frac{a}{bc}}$

Example 13: Use log laws to combine $3\log a + 2\log b + 4\log c$ into a single log.

Solution:

To combine: $3\log a + 2\log b + 4\log c$

Apply $n \ln u = \ln u^n$: $= \log a^3 + \log b^2 + \log c^4$

Apply $\ln u + \ln v + \ln w = \ln uvw$: $= \boxed{\log a^3 b^2 c^4}$

Example 14: Assume x, y are > 0. Use laws of logarithms to combine $3\ln x + \frac{1}{3}\ln y - 4\ln(t^2+1)$ into a single ln.

Solution:

To combine: $3\ln x + \frac{1}{3}\ln y - 4\ln(t^2+1)$

Apply $n \log u = \log u^n$: $= \ln x^3 + \ln y^{\frac{1}{3}} - \ln\left((t^2+1)^4\right)$

Apply $\log u + \log v = \log uv$ $= \ln(x^3 y^{\frac{1}{3}}) - \ln\left((t^2+1)^4\right)$

Apply $\log u - \log v = \log \frac{u}{v}$ $= \boxed{\ln \dfrac{x^3 y^{\frac{1}{3}}}{(t^2+1)^4}}$

3.4.4 Quiz

Example 3.4.1: Find $\log_3(81), \log_2(\frac{1}{9})$, and $\log_2(9\sqrt{3})$

Example 3.4.2: Find $\log_{10}(\frac{1}{100\sqrt{10}})$ and $\log_{10}(\sqrt[3]{100})$

Example 3.4.3: Find $\ln(\frac{e^9}{\sqrt{e}})$.

Example 3.4.4: Use laws of logarithms to expand and simplify $\log \frac{\sqrt[3]{b}}{abc^3}$.

Example 3.4.5: Use $\log uv = \log u + \log v$ to find $\log \left[(x^2+1)^4 \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right]$.

Example 3.4.6: Use $\log \frac{u}{v} = \log u - \log v$ to find $\log \frac{x^3}{\sqrt{x^2+1} \cdot (x+2)^3}$.

Example 3.4.7: Use laws of logarithms to expand and simplify $\log \sqrt[3]{a^5 bc^3}$.

Example 3.4.8: Find $\log_8 2 + \log_8 32$.

Example 3.4.9: Use log laws to combine $\log a + \log b$ into a single expression.

Example 3.4.10: Use log laws to combine $\log a + \frac{1}{2} \log b$ into a single expression.

Example 3.4.11: Use log laws to combine $\log a - \log b$ into a single expression.

Example 3.4.12: Use log laws to combine $\log a - \log b - \log c$ into a single expression.

Example 3.4.13: Use log laws to combine $3 \log a + 2 \log b + 4 \log c$ into a single expression.

Example 3.4.14: Use laws of logarithms to combine $3 \ln x + \frac{1}{3} \ln y - 4 \ln(t^2+1)$ into a single expression.

3.4 Review: Laws of logarithms

Example 3.4.1: Find

- $\log_3(81) =$
- $\log_2(\frac{1}{8})=$
- $\log_3(9\sqrt{3})=$

3.4 Review: Laws of logarithms

Example 3.4.1: Find

- $\log_3(81) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_3(9\sqrt{3}) = \frac{5}{2}$

3.4 Review: Laws of logarithms

Example 3.4.1: Find

- $\log_3(81) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_3(9\sqrt{3}) = \frac{5}{2}$
- $\log_2(16) =$
- $\log_2(\frac{1}{4}) =$
- $\log_5(25\sqrt{5}) =$

3.4 Review: Laws of logarithms

Example 3.4.1: Find

- $\log_3(81) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_3(9\sqrt{3}) = \frac{5}{2}$
- $\log_2(16) = 4$
- $\log_2(\frac{1}{4}) = -2$
- $\log_5(25\sqrt{5}) = \frac{5}{2}$

3.4 Review: Laws of logarithms

Example 3.4.1: Find

- $\log_3(81) = 4$
- $\log_2(16) = 4$
- $\log_9(81) = 2$
- $\log_2(\frac{1}{8}) = -3$
- $\log_2(\frac{1}{4}) = -2$
- $\log_9 \sqrt{9} = \frac{1}{2}$
- $\log_3(9\sqrt{3}) = \frac{5}{2}$
- $\log_5(25\sqrt{5}) = \frac{5}{2}$
- $\log_9(9\sqrt{3}) = \frac{5}{4}$

3.4 Review: Laws of logarithms

Example 3.4.1: Find

- $\log_3(81) = 4$
- $\log_2(16) = 4$
- $\log_9(81) = 2$
- $\log_2(\frac{1}{8}) = -3$
- $\log_2(\frac{1}{4}) = -2$
- $\log_9 \sqrt{9} = \frac{1}{2}$
- $\log_3(9\sqrt{3}) = \frac{5}{2}$
- $\log_5(25\sqrt{5}) = \frac{5}{2}$
- $\log_9(9\sqrt{3}) = \frac{5}{4}$

3.4 Review: Laws of logarithms

Example 3.4.1: Find

- $\log_3(81) = 4$
- $\log_2(16) = 4$
- $\log_9(81) = 2$
- $\log_5(\frac{1}{5}) = -1$
- $\log_2(\frac{1}{8}) = -3$
- $\log_2(\frac{1}{4}) = -2$
- $\log_9 \sqrt{9} = \frac{1}{2}$
- $\log_5(\frac{1}{25}) = -2$
- $\log_3(9\sqrt{3}) = \frac{5}{2}$
- $\log_5(25\sqrt{5}) = \frac{5}{2}$
- $\log_9(9\sqrt{3}) = \frac{5}{4}$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$

3.4 Review: Laws of logarithms

Example 3.4.1: Find

- $\log_3(81) = 4$
- $\log_2(16) = 4$
- $\log_9(81) = 2$
- $\log_5(\frac{1}{5}) = -1$
- $\log_2(\frac{1}{8}) = -3$
- $\log_2(\frac{1}{4}) = -2$
- $\log_9 \sqrt{9} = \frac{1}{2}$
- $\log_5(\frac{1}{25}) = -2$
- $\log_3(9\sqrt{3}) = \frac{5}{2}$
- $\log_5(25\sqrt{5}) = \frac{5}{2}$
- $\log_9(9\sqrt{3}) = \frac{5}{4}$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$

3.4 Review: Laws of logarithms

Example 3.4.1: Find

- $\log_3(81) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_3(9\sqrt{3}) = \frac{5}{2}$
- $\log_2(16) = 4$
- $\log_2(\frac{1}{4}) = -2$
- $\log_5(25\sqrt{5}) = \frac{5}{2}$
- $\log_9(81) = 2$
- $\log_9\sqrt{9} = \frac{1}{2}$
- $\log_9(9\sqrt{3}) = \frac{5}{4}$
- $\log_5(\frac{1}{5}) = -1$
- $\log_5(\frac{1}{25}) = -2$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$

Example 3.4.2: Find

- $\log_{10}(\frac{1}{100\sqrt{10}}) =$
- $\log_{10}(\sqrt[3]{100}) =$
- $\log_{10}(1000\sqrt{10}) =$
- $\log_{10}(\frac{1}{100000}) =$

3.4 Review: Laws of logarithms

Example 3.4.1: Find

- $\log_3(81) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_3(9\sqrt{3}) = \frac{5}{2}$
- $\log_2(16) = 4$
- $\log_2(\frac{1}{4}) = -2$
- $\log_5(25\sqrt{5}) = \frac{5}{2}$
- $\log_9(81) = 2$
- $\log_9 \sqrt{9} = \frac{1}{2}$
- $\log_9(9\sqrt{3}) = \frac{5}{4}$
- $\log_5(\frac{1}{5}) = -1$
- $\log_5(\frac{1}{25}) = -2$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$

Example 3.4.2: Find

- $\log_{10}(\frac{1}{100\sqrt{10}}) = -\frac{5}{2}$
- $\log_{10}(\sqrt[3]{100}) = \frac{2}{3}$
- $\log_{10}(1000\sqrt{10}) = \frac{7}{2}$
- $\log_{10}(\frac{1}{100000}) = -5$

3.4 Review: Laws of logarithms

Example 3.4.1: Find

- $\log_3(81) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_3(9\sqrt{3}) = \frac{5}{2}$
- $\log_2(16) = 4$
- $\log_2(\frac{1}{4}) = -2$
- $\log_5(25\sqrt{5}) = \frac{5}{2}$
- $\log_9(81) = 2$
- $\log_9 \sqrt{9} = \frac{1}{2}$
- $\log_9(9\sqrt{3}) = \frac{5}{4}$
- $\log_5(\frac{1}{5}) = -1$
- $\log_5(\frac{1}{25}) = -2$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$

Example 3.4.2: Find

- $\log_{10}(\frac{1}{100\sqrt{10}}) = -\frac{5}{2}$
- $\log_{10}(\sqrt[3]{100}) = \frac{2}{3}$
- $\log_{10}(1000\sqrt{10}) = \frac{7}{2}$
- $\log_{10}(\frac{1}{100000}) = -5$
- $\log_{10}(\frac{10}{\sqrt[4]{10}}) =$
- $\log_{10}(\sqrt[5]{10}) =$
- $\log_{10}(.00001) =$
- $\log_{10}(1) =$

3.4 Review: Laws of logarithms

Example 3.4.1: Find

- $\log_3(81) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_3(9\sqrt{3}) = \frac{5}{2}$
- $\log_2(16) = 4$
- $\log_2(\frac{1}{4}) = -2$
- $\log_5(25\sqrt{5}) = \frac{5}{2}$
- $\log_9(81) = 2$
- $\log_9\sqrt{9} = \frac{1}{2}$
- $\log_9(9\sqrt{3}) = \frac{5}{4}$
- $\log_5(\frac{1}{5}) = -1$
- $\log_5(\frac{1}{25}) = -2$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$

Example 3.4.2: Find

- $\log_{10}(\frac{1}{100\sqrt{10}}) = -\frac{5}{2}$
- $\log_{10}(\sqrt[3]{100}) = \frac{2}{3}$
- $\log_{10}(1000\sqrt{10}) = \frac{7}{2}$
- $\log_{10}(\frac{1}{100000}) = -5$
- $\log_{10}(\frac{10}{\sqrt[4]{10}}) = \frac{3}{4}$
- $\log_{10}(\sqrt[5]{10}) = \frac{1}{5}$
- $\log_{10}(.00001) = -5$
- $\log_{10}(1) = 0$

3.4 Review: Laws of logarithms

Example 3.4.1: Find

- $\log_3(81) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_3(9\sqrt{3}) = \frac{5}{2}$
- $\log_2(16) = 4$
- $\log_2(\frac{1}{4}) = -2$
- $\log_5(25\sqrt{5}) = \frac{5}{2}$
- $\log_9(81) = 2$
- $\log_9 \sqrt{9} = \frac{1}{2}$
- $\log_9(9\sqrt{3}) = \frac{5}{4}$
- $\log_5(\frac{1}{5}) = -1$
- $\log_5(\frac{1}{25}) = -2$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$

Example 3.4.2: Find

- $\log_{10}(\frac{1}{100\sqrt{10}}) = -\frac{5}{2}$
- $\log_{10}(\sqrt[3]{100}) = \frac{2}{3}$
- $\log_{10}(1000\sqrt{10}) = \frac{7}{2}$
- $\log_{10}(\frac{1}{100000}) = -5$
- $\log_{10}(\frac{10}{\sqrt[4]{10}}) = \frac{3}{4}$
- $\log_{10}(\sqrt[5]{10}) = \frac{1}{5}$
- $\log_{10}(.00001) = -5$
- $\log_{10}(1) = 0$

Example 3.4.3: Find

- $\ln(\frac{e^9}{\sqrt{e}}) =$
- $\ln(\frac{\sqrt{e}}{e^9}) =$
- $\ln(\frac{e}{\sqrt{e}}) =$
- $\ln(\frac{\sqrt[5]{e}}{\sqrt{e}}) =$

3.4 Review: Laws of logarithms

Example 3.4.1: Find

- $\log_3(81) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_3(9\sqrt{3}) = \frac{5}{2}$
- $\log_2(16) = 4$
- $\log_2(\frac{1}{4}) = -2$
- $\log_5(25\sqrt{5}) = \frac{5}{2}$
- $\log_9(81) = 2$
- $\log_9 \sqrt{9} = \frac{1}{2}$
- $\log_9(9\sqrt{3}) = \frac{5}{4}$
- $\log_5(\frac{1}{5}) = -1$
- $\log_5(\frac{1}{25}) = -2$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$

Example 3.4.2: Find

- $\log_{10}(\frac{1}{100\sqrt{10}}) = -\frac{5}{2}$
- $\log_{10}(\sqrt[3]{100}) = \frac{2}{3}$
- $\log_{10}(1000\sqrt{10}) = \frac{7}{2}$
- $\log_{10}(\frac{1}{100000}) = -5$
- $\log_{10}(\frac{10}{\sqrt[4]{10}}) = \frac{3}{4}$
- $\log_{10}(\sqrt[5]{10}) = \frac{1}{5}$
- $\log_{10}(.00001) = -5$
- $\log_{10}(1) = 0$

Example 3.4.3: Find

- $\ln(\frac{e^9}{\sqrt{e}}) = \frac{17}{2}$
- $\ln(\frac{\sqrt{e}}{e^9}) = -\frac{17}{2}$
- $\ln(\frac{e}{\sqrt{e}}) = \frac{1}{2}$
- $\ln(\frac{\sqrt[5]{e}}{\sqrt{e}}) = -\frac{3}{10}$

3.4 Review: Laws of logarithms

Example 3.4.1: Find

- $\log_3(81) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_3(9\sqrt{3}) = \frac{5}{2}$
- $\log_2(16) = 4$
- $\log_2(\frac{1}{4}) = -2$
- $\log_5(25\sqrt{5}) = \frac{5}{2}$
- $\log_9(81) = 2$
- $\log_9 \sqrt{9} = \frac{1}{2}$
- $\log_9(9\sqrt{3}) = \frac{5}{4}$
- $\log_5(\frac{1}{5}) = -1$
- $\log_5(\frac{1}{25}) = -2$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$

Example 3.4.2: Find

- $\log_{10}(\frac{1}{100\sqrt{10}}) = -\frac{5}{2}$
- $\log_{10}(\sqrt[3]{100}) = \frac{2}{3}$
- $\log_{10}(1000\sqrt{10}) = \frac{7}{2}$
- $\log_{10}(\frac{1}{100000}) = -5$
- $\log_{10}(\frac{10}{\sqrt[4]{10}}) = \frac{3}{4}$
- $\log_{10}(\sqrt[5]{10}) = \frac{1}{5}$
- $\log_{10}(.00001) = -5$
- $\log_{10}(1) = 0$

Example 3.4.3: Find

- $\ln(\frac{e^9}{\sqrt{e}}) = \frac{17}{2}$
- $\ln(\frac{\sqrt{e}}{e^9}) = -\frac{17}{2}$
- $\ln(\frac{e}{\sqrt{e}}) = \frac{1}{2}$
- $\ln(\frac{\sqrt[5]{e}}{\sqrt{e}}) = -\frac{3}{10}$

Example 3.4.4: Assume a, b, c are all > 0. Use laws of logarithms to expand and simplify

- $\log \frac{\sqrt[3]{b}}{abc^3} =$
- $\log \frac{\sqrt[3]{abc}}{ac^3} =$
- $\log \frac{\sqrt{b}}{\sqrt{abc}} =$
- $\log \frac{\sqrt[3]{b^6 c^9}}{a^3} =$

3.4 Review: Laws of logarithms

Example 3.4.1: Find

- $\log_3(81) = 4$
- $\log_2(\frac{1}{8}) = -3$
- $\log_3(9\sqrt{3}) = \frac{5}{2}$
- $\log_2(16) = 4$
- $\log_2(\frac{1}{4}) = -2$
- $\log_5(25\sqrt{5}) = \frac{5}{2}$
- $\log_9(81) = 2$
- $\log_9 \sqrt{9} = \frac{1}{2}$
- $\log_9(9\sqrt{3}) = \frac{5}{4}$
- $\log_5(\frac{1}{5}) = -1$
- $\log_5(\frac{1}{25}) = -2$
- $\log_5(\frac{\sqrt{5}}{25}) = -\frac{3}{2}$

Example 3.4.2: Find

- $\log_{10}(\frac{1}{100\sqrt{10}}) = -\frac{5}{2}$
- $\log_{10}(\sqrt[3]{100}) = \frac{2}{3}$
- $\log_{10}(1000\sqrt{10}) = \frac{7}{2}$
- $\log_{10}(\frac{1}{100000}) = -5$
- $\log_{10}(\frac{10}{\sqrt[4]{10}}) = \frac{3}{4}$
- $\log_{10}(\sqrt[5]{10}) = \frac{1}{5}$
- $\log_{10}(.00001) = -5$
- $\log_{10}(1) = 0$

Example 3.4.3: Find

- $\ln(\frac{e^9}{\sqrt{e}}) = \frac{17}{2}$
- $\ln(\frac{\sqrt{e}}{e^9}) = -\frac{17}{2}$
- $\ln(\frac{e}{\sqrt{e}}) = \frac{1}{2}$
- $\ln(\frac{\sqrt[5]{e}}{\sqrt{e}}) = -\frac{3}{10}$

Example 3.4.4: Assume a, b, c are all > 0. Use laws of logarithms to expand and simplify

- $\log \frac{\sqrt[3]{b}}{abc^3} = -\log a - \frac{2}{3}\log b - 3\log c$
- $\log \frac{\sqrt[3]{abc}}{ac^3} = -\frac{2}{3}\log a + \frac{1}{3}\log b - \frac{8}{3}\log c$
- $\log \frac{\sqrt{b}}{\sqrt{abc}} = -\frac{1}{2}\log a - \frac{1}{2}\log c$
- $\log \frac{\sqrt[3]{b^6 c^9}}{a^3} = -3\log a + 2\log b + 3\log c$

Example 3.4.5: Assume $x > 0$ Use $\log uv = \log u + \log v$ to find

- $\log\left((x^2+1)^4 \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right) =$
- $\log\left((x^2+1)^{-5} \cdot \sqrt[3]{x^2} \cdot \sqrt[5]{x}\right) =$
- $\log\left(10(x^2+1)^2 \cdot \sqrt{x^8} \cdot \sqrt[3]{x^5}\right) =$
- $\log\left(\sqrt{x^3} \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right) =$

Example 3.4.5: Assume $x > 0$ Use $\log uv = \log u + \log v$ to find

- $\log\left((x^2+1)^4 \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right) = 4\log(x^2+1) + \frac{1}{2}\log(x^2+1) + \frac{1}{3}\log x = \frac{9}{2}\log(x^2+1) + \frac{1}{3}\log x$
- $\log\left((x^2+1)^{-5} \cdot \sqrt[3]{x^2} \cdot \sqrt[5]{x}\right) = -5\log(x^2+1) + \frac{2}{3}\log x + \frac{1}{5}\log x = -5\log(x^2+1) + \frac{13}{15}\log x$
- $\log\left(10(x^2+1)^2 \cdot \sqrt{x^8} \cdot \sqrt[3]{x^5}\right) = \log 10 + 2\log(x^2+1) + \frac{17}{3}\log x$
- $\log\left(\sqrt{x^3} \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right) = \frac{11}{6}\log(x) + \frac{1}{2}\log(x^2+1)$

Example 3.4.5: Assume $x > 0$ Use $\log uv = \log u + \log v$ to find

- $\log\left((x^2+1)^4 \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right) = 4\log(x^2+1) + \frac{1}{2}\log(x^2+1) + \frac{1}{3}\log x = \frac{9}{2}\log(x^2+1) + \frac{1}{3}\log x$
- $\log\left((x^2+1)^{-5} \cdot \sqrt[3]{x^2} \cdot \sqrt[5]{x}\right) = -5\log(x^2+1) + \frac{2}{3}\log x + \frac{1}{5}\log x = -5\log(x^2+1) + \frac{13}{15}\log x$
- $\log\left(10(x^2+1)^2 \cdot \sqrt{x^8} \cdot \sqrt[3]{x^5}\right) = \log 10 + 2\log(x^2+1) + \frac{17}{3}\log x$
- $\log\left(\sqrt{x^3} \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right) = \frac{11}{6}\log(x) + \frac{1}{2}\log(x^2+1)$

Example 3.4.6: Using $\log \frac{u}{v} = \log u - \log v$, state the domain for x and find

- $\log \frac{x^3}{(x+2)^3\sqrt{x^2+1}} =$
- $\log \frac{x^2(x-2)^5}{x^2+1} =$
- $\log \frac{1}{(2x-2)^3\sqrt{x^2+1}} =$
- $\log \frac{|x-4|^3}{\sqrt{x-2}} =$

Example 3.4.5: Assume $x > 0$ Use $\log uv = \log u + \log v$ to find

- $\log\left((x^2+1)^4 \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right) = 4\log(x^2+1) + \frac{1}{2}\log(x^2+1) + \frac{1}{3}\log x = \frac{9}{2}\log(x^2+1) + \frac{1}{3}\log x$
- $\log\left((x^2+1)^{-5} \cdot \sqrt[3]{x^2} \cdot \sqrt[5]{x}\right) = -5\log(x^2+1) + \frac{2}{3}\log x + \frac{1}{5}\log x = -5\log(x^2+1) + \frac{13}{15}\log x$
- $\log\left(10(x^2+1)^2 \cdot \sqrt{x^8} \cdot \sqrt[3]{x^5}\right) = \log 10 + 2\log(x^2+1) + \frac{17}{3}\log x$
- $\log\left(\sqrt{x^3} \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right) = \frac{11}{6}\log(x) + \frac{1}{2}\log(x^2+1)$

Example 3.4.6: Using $\log \frac{u}{v} = \log u - \log v$, state the domain for x and find

- $\log \frac{x^3}{(x+2)^3\sqrt{x^2+1}} = 3\log x - 3\log(x+2) - \frac{1}{2}\log(x^2+1)$; domain $x > 0$
- $\log \frac{x^2(x-2)^5}{x^2+1} = 2\log x + 5\log(x-2) - \log(x^2+1)$; domain $x > 2$
- $\log \frac{1}{(2x-2)^3\sqrt{x^2+1}} = -3\log(2x-2) - \frac{1}{2}\log(x^2+1)$; domain $x > 1$
- $\log \frac{|x-4|^3}{\sqrt{x-2}} = 3\log|x-4| - \frac{1}{2}\log(x-2)$; domain $(2,4) \cup (4,\infty)$

Example 3.4.5: Assume $x > 0$ Use $\log uv = \log u + \log v$ to find

- $\log\left((x^2+1)^4 \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right) = 4\log(x^2+1) + \frac{1}{2}\log(x^2+1) + \frac{1}{3}\log x = \frac{9}{2}\log(x^2+1) + \frac{1}{3}\log x$
- $\log\left((x^2+1)^{-5} \cdot \sqrt[3]{x^2} \cdot \sqrt[5]{x}\right) = -5\log(x^2+1) + \frac{2}{3}\log x + \frac{1}{5}\log x = -5\log(x^2+1) + \frac{13}{15}\log x$
- $\log\left(10(x^2+1)^2 \cdot \sqrt{x^8} \cdot \sqrt[3]{x^5}\right) = \log 10 + 2\log(x^2+1) + \frac{17}{3}\log x$
- $\log\left(\sqrt{x^3} \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right) = \frac{11}{6}\log(x) + \frac{1}{2}\log(x^2+1)$

Example 3.4.6: Using $\log\frac{u}{v} = \log u - \log v$, state the domain for x and find

- $\log\frac{x^3}{(x+2)^3\sqrt{x^2+1}} = 3\log x - 3\log(x+2) - \frac{1}{2}\log(x^2+1)$; domain $x > 0$
- $\log\frac{x^2(x-2)^5}{x^2+1} = 2\log x + 5\log(x-2) - \log(x^2+1)$; domain $x > 2$
- $\log\frac{1}{(2x-2)^3\sqrt{x^2+1}} = -3\log(2x-2) - \frac{1}{2}\log(x^2+1)$; domain $x > 1$
- $\log\frac{|x-4|^3}{\sqrt{x-2}} = 3\log|x-4| - \frac{1}{2}\log(x-2)$; domain $(2,4)\cup(4,\infty)$

Example 3.4.7: Assume $a, b, c > 0$ and use laws of logarithms to expand and simplify

- $\log\sqrt[3]{a^5bc^3} =$
- $\log\sqrt{a^5bc^3} =$
- $\log\sqrt[3]{\frac{a^5}{b^3}} =$
- $\log\frac{(a^2+1)^5bc^3}{a^5b^7} =$

Example 3.4.5: Assume $x > 0$ Use $\log uv = \log u + \log v$ to find

- $\log\left((x^2+1)^4 \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right) = 4\log(x^2+1) + \frac{1}{2}\log(x^2+1) + \frac{1}{3}\log x = \frac{9}{2}\log(x^2+1) + \frac{1}{3}\log x$
- $\log\left((x^2+1)^{-5} \cdot \sqrt[3]{x^2} \cdot \sqrt[5]{x}\right) = -5\log(x^2+1) + \frac{2}{3}\log x + \frac{1}{5}\log x = -5\log(x^2+1) + \frac{13}{15}\log x$
- $\log\left(10(x^2+1)^2 \cdot \sqrt{x^8} \cdot \sqrt[3]{x^5}\right) = \log 10 + 2\log(x^2+1) + \frac{17}{3}\log x$
- $\log\left(\sqrt{x^3} \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right) = \frac{11}{6}\log(x) + \frac{1}{2}\log(x^2+1)$

Example 3.4.6: Using $\log\frac{u}{v} = \log u - \log v$, state the domain for x and find

- $\log\frac{x^3}{(x+2)^3\sqrt{x^2+1}} = 3\log x - 3\log(x+2) - \frac{1}{2}\log(x^2+1)$; domain $x > 0$
- $\log\frac{x^2(x-2)^5}{x^2+1} = 2\log x + 5\log(x-2) - \log(x^2+1)$; domain $x > 2$
- $\log\frac{1}{(2x-2)^3\sqrt{x^2+1}} = -3\log(2x-2) - \frac{1}{2}\log(x^2+1)$; domain $x > 1$
- $\log\frac{|x-4|^3}{\sqrt{x-2}} = 3\log|x-4| - \frac{1}{2}\log(x-2)$; domain $(2,4)\cup(4,\infty)$

Example 3.4.7: Assume $a, b, c > 0$ and use laws of logarithms to expand and simplify

- $\log\sqrt[3]{a^5bc^3} = \frac{1}{3}(5\log a + \log b + 3\log c)$
- $\log\sqrt{a^5bc^3} = \frac{1}{2}(5\log a + \log b + 3\log c)$
- $\log\sqrt[3]{\frac{a^5}{b^3}} = \frac{5}{3}\log a - \log b$
- $\log\frac{(a^2+1)^5bc^3}{a^5b^7} = 5\log(a^2+1) - 5\log a - 6\log b + 3\log c$

Example 3.4.5: Assume $x > 0$ Use $\log uv = \log u + \log v$ to find

- $\log\left((x^2+1)^4 \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right) = 4\log(x^2+1) + \frac{1}{2}\log(x^2+1) + \frac{1}{3}\log x = \frac{9}{2}\log(x^2+1) + \frac{1}{3}\log x$
- $\log\left((x^2+1)^{-5} \cdot \sqrt[3]{x^2} \cdot \sqrt[5]{x}\right) = -5\log(x^2+1) + \frac{2}{3}\log x + \frac{1}{5}\log x = -5\log(x^2+1) + \frac{13}{15}\log x$
- $\log\left(10(x^2+1)^2 \cdot \sqrt{x^8} \cdot \sqrt[3]{x^5}\right) = \log 10 + 2\log(x^2+1) + \frac{17}{3}\log x$
- $\log\left(\sqrt{x^3} \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right) = \frac{11}{6}\log(x) + \frac{1}{2}\log(x^2+1)$

Example 3.4.6: Using $\log \frac{u}{v} = \log u - \log v$, state the domain for x and find

- $\log \frac{x^3}{(x+2)^3\sqrt{x^2+1}} = 3\log x - 3\log(x+2) - \frac{1}{2}\log(x^2+1)$; domain $x > 0$
- $\log \frac{x^2(x-2)^5}{x^2+1} = 2\log x + 5\log(x-2) - \log(x^2+1)$; domain $x > 2$
- $\log \frac{1}{(2x-2)^3\sqrt{x^2+1}} = -3\log(2x-2) - \frac{1}{2}\log(x^2+1)$; domain $x > 1$
- $\log \frac{|x-4|^3}{\sqrt{x-2}} = 3\log|x-4| - \frac{1}{2}\log(x-2)$; domain $(2,4)\cup(4,\infty)$

Example 3.4.7: Assume $a, b, c > 0$ and use laws of logarithms to expand and simplify

- $\log \sqrt[3]{a^5bc^3} = \frac{1}{3}(5\log a + \log b + 3\log c)$
- $\log \sqrt{a^5bc^3} = \frac{1}{2}(5\log a + \log b + 3\log c)$
- $\log \sqrt[3]{\frac{a^5}{b^3}} = \frac{5}{3}\log a - \log b$
- $\log \frac{(a^2+1)^5bc^3}{a^5b^7} = 5\log(a^2+1) - 5\log a - 6\log b + 3\log c$

Example 3.4.8: Find

- $\log_8 2 + \log_8 32 =$
- $\log_3 27 + \log_2 32 =$
- $\log_5 125 + \log_5 25\sqrt{5} =$
- $\log_2(2\sqrt{2}) + \log_8 64 =$

Example 3.4.5: Assume $x > 0$ Use $\log uv = \log u + \log v$ to find

- $\log\left((x^2+1)^4 \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right) = 4\log(x^2+1) + \frac{1}{2}\log(x^2+1) + \frac{1}{3}\log x = \frac{9}{2}\log(x^2+1) + \frac{1}{3}\log x$
- $\log\left((x^2+1)^{-5} \cdot \sqrt[3]{x^2} \cdot \sqrt[5]{x}\right) = -5\log(x^2+1) + \frac{2}{3}\log x + \frac{1}{5}\log x = -5\log(x^2+1) + \frac{13}{15}\log x$
- $\log\left(10(x^2+1)^2 \cdot \sqrt{x^8} \cdot \sqrt[3]{x^5}\right) = \log 10 + 2\log(x^2+1) + \frac{17}{3}\log x$
- $\log\left(\sqrt{x^3} \cdot \sqrt{x^2+1} \cdot \sqrt[3]{x}\right) = \frac{11}{6}\log(x) + \frac{1}{2}\log(x^2+1)$

Example 3.4.6: Using $\log \frac{u}{v} = \log u - \log v$, state the domain for x and find

- $\log \frac{x^3}{(x+2)^3\sqrt{x^2+1}} = 3\log x - 3\log(x+2) - \frac{1}{2}\log(x^2+1)$; domain $x > 0$
- $\log \frac{x^2(x-2)^5}{x^2+1} = 2\log x + 5\log(x-2) - \log(x^2+1)$; domain $x > 2$
- $\log \frac{1}{(2x-2)^3\sqrt{x^2+1}} = -3\log(2x-2) - \frac{1}{2}\log(x^2+1)$; domain $x > 1$
- $\log \frac{|x-4|^3}{\sqrt{x-2}} = 3\log|x-4| - \frac{1}{2}\log(x-2)$; domain $(2,4) \cup (4,\infty)$

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- $\log \sqrt[3]{a^5bc^3} = \frac{1}{3}(5\log a + \log b + 3\log c)$
- $\log \sqrt{a^5bc^3} = \frac{1}{2}(5\log a + \log b + 3\log c)$
- $\log \sqrt[3]{\frac{a^5}{b^3}} = \frac{5}{3}\log a - \log b$
- $\log \frac{(a^2+1)^5bc^3}{a^5b^7} = 5\log(a^2+1) - 5\log a - 6\log b + 3\log c$

Example 3.4.8: Find

- $\log_8 2 + \log_8 32 = 2$
- $\log_3 27 + \log_2 32 = 8$
- $\log_5 125 + \log_5 25\sqrt{5} = \frac{11}{2}$
- $\log_2(2\sqrt{2}) + \log_8 64 = \frac{7}{2}$

In the remaining problems, assume that a, b, c, x, t are all > 0

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Example 3.4.9: Rewrite as a single logarithm:

- $\log a + \log b =$
- $\log 2 + \log 3 - \log 4 =$
- $\log x\sqrt{x} + \log x^5 - \log x\sqrt[3]{x} =$
- $\log x^3 - 2\log(b^2+1) =$

In the remaining problems, assume that a, b, c, x, t are all > 0 so that the given expression is defined.

Example 3.4.9: Rewrite as a single logarithm:

- $\log a + \log b = \log ab$
- $\log x\sqrt{x} + \log x^5 - \log x\sqrt[3]{x} = \frac{31}{6}\log x$
- $\log 2 + \log 3 - \log 4 = \log \frac{3}{2}$
- $\log x^3 - 2\log(b^2+1) = \log \frac{x^3}{(b^2+1)^2}$

In the remaining problems, assume that a, b, c, x, t are all > 0 so that the given expression is defined.

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Example 3.4.10: Use log laws to combine into a single simplified expression

- $\log a + \frac{1}{2}\log b =$
- $5\log a + \frac{1}{2}\log b =$
- $\frac{1}{6}\log a + \frac{1}{7}\log b =$
- $8\log a + \frac{2}{5}\log b =$

In the remaining problems, assume that a, b, c, x, t are all > 0 so that the given expression is defined.

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- $\log a + \frac{1}{2}\log b = \log(a\sqrt{b})$
- $5\log a + \frac{1}{2}\log b = \log(a^5\sqrt{b})$
- $\frac{1}{6}\log a + \frac{1}{7}\log b = \log(\sqrt[6]{a}\sqrt[7]{b})$
- $8\log a + \frac{2}{5}\log b = \log(a^8\sqrt[5]{b^2})$

In the remaining problems, assume that a, b, c, x, t are all > 0 so that the given expression is defined.

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- $8\log a + \frac{2}{5}\log b = \log(a^8\sqrt[5]{b^2})$

Example 3.4.11: Use log laws to combine into a single expression.

- $\log a - \log b =$
- $\log(x^2+1) - 2\log(x^5) =$
- $\log(x^2+1) - \log(x^2+2) =$
- $4\log a - 5\log b =$

In the remaining problems, assume that a, b, c, x, t are all > 0 so that the given expression is defined.

Example 3.4.9: Rewrite as a single logarithm:

- $\log a + \log b = \log ab$
- $\log x\sqrt{x} + \log x^5 - \log x\sqrt[3]{x} = \frac{31}{6}\log x$
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- $5\log a + \frac{1}{2}\log b = \log(a^5\sqrt{b})$
- $\frac{1}{6}\log a + \frac{1}{7}\log b = \log(\sqrt[6]{a}\sqrt[7]{b})$
- $8\log a + \frac{2}{5}\log b = \log(a^8\sqrt[5]{b^2})$

Example 3.4.11: Use log laws to combine into a single expression.

- $\log a - \log b = \log \frac{a}{b}$
- $\log(x^2+1) - 2\log(x^5) = \log \frac{x^2+1}{x^{10}}$
- $\log(x^2+1) - \log(x^2+2) = \log \frac{x^2+1}{x^2+2}$
- $4\log a - 5\log b = \log \frac{a^4}{b^5}$

In the remaining problems, assume that a, b, c, x, t are all > 0 so that the given expression is defined.

Example 3.4.9: Rewrite as a single logarithm:

- $\log a + \log b = \log ab$
- $\log x\sqrt{x} + \log x^5 - \log x\sqrt[3]{x} = \frac{31}{6}\log x$
- $\log 2 + \log 3 - \log 4 = \log \frac{3}{2}$
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- $\log a + \frac{1}{2}\log b = \log(a\sqrt{b})$
- $5\log a + \frac{1}{2}\log b = \log(a^5\sqrt{b})$
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- $8\log a + \frac{2}{5}\log b = \log(a^8\sqrt[5]{b^2})$

Example 3.4.11: Use log laws to combine into a single expression.

- $\log a - \log b = \log \frac{a}{b}$
- $\log(x^2+1) - 2\log(x^5) = \log \frac{x^2+1}{x^{10}}$
- $\log(x^2+1) - \log(x^2+2) = \log \frac{x^2+1}{x^2+2}$
- $4\log a - 5\log b = \log \frac{a^4}{b^5}$

Example 3.4.12: Rewrite as a single log :

- $\log a - \log b - \log c =$
- $\log\sqrt{a} - \log b - \log c - \log d =$
- $\log ab + \log b - 5\log bc =$
- $\log a + \log b - \log c - \log d - \log e =$

In the remaining problems, assume that a, b, c, x, t are all > 0 so that the given expression is defined.

Example 3.4.9: Rewrite as a single logarithm:

- $\log a + \log b = \log ab$
- $\log x\sqrt{x} + \log x^5 - \log x\sqrt[3]{x} = \frac{31}{6}\log x$
- $\log 2 + \log 3 - \log 4 = \log \frac{3}{2}$
- $\log x^3 - 2\log(b^2+1) = \log \frac{x^3}{(b^2+1)^2}$

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- $5\log a + \frac{1}{2}\log b = \log(a^5\sqrt{b})$
- $\frac{1}{6}\log a + \frac{1}{7}\log b = \log(\sqrt[6]{a}\sqrt[7]{b})$
- $8\log a + \frac{2}{5}\log b = \log(a^8\sqrt[5]{b^2})$

Example 3.4.11: Use log laws to combine into a single expression.

- $\log a - \log b = \log \frac{a}{b}$
- $\log(x^2+1) - 2\log(x^5) = \log \frac{x^2+1}{x^{10}}$
- $\log(x^2+1) - \log(x^2+2) = \log \frac{x^2+1}{x^2+2}$
- $4\log a - 5\log b = \log \frac{a^4}{b^5}$

Example 3.4.12: Rewrite as a single log :

- $\log a - \log b - \log c = \log \frac{a}{bc}$
- $\log \sqrt{a} - \log b - \log c - \log d = \log \frac{\sqrt{a}}{bcd}$
- $\log ab + \log b - 5\log bc = \log \frac{a}{b^3c^5}$
- $\log a + \log b - \log c - \log d - \log e = \log \frac{ab}{cde}$

In the remaining problems, assume that a, b, c, x, t are all > 0 so that the given expression is defined.

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- $\log a + \log b = \log ab$
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- $8\log a + \frac{2}{5}\log b = \log(a^8\sqrt[5]{b^2})$

Example 3.4.11: Use log laws to combine into a single expression.

- $\log a - \log b = \log \frac{a}{b}$
- $\log(x^2+1) - 2\log(x^5) = \log \frac{x^2+1}{x^{10}}$
- $\log(x^2+1) - \log(x^2+2) = \log \frac{x^2+1}{x^2+2}$
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- $\log a - \log b - \log c = \log \frac{a}{bc}$
- $\log \sqrt{a} - \log b - \log c - \log d = \log \frac{\sqrt{a}}{bcd}$
- $\log ab + \log b - 5\log bc = \log \frac{a}{b^3 c^5}$
- $\log a + \log b - \log c - \log d - \log e = \log \frac{ab}{cde}$

Example 3.4.13: Rewrite as a single log. Don't use radicals.

- $3\log a + 2\log b + 4\log c =$
- $3\log a - 2\log b - 4\log c =$
- $\frac{1}{2}\log a - 2\log b + \frac{1}{3}\log c =$
- $-3\log a + \frac{1}{2}\log b + \frac{1}{3}\log b =$

In the remaining problems, assume that a, b, c, x, t are all > 0 so that the given expression is defined.

Example 3.4.9: Rewrite as a single logarithm:

- $\log a + \log b = \log ab$
- $\log x\sqrt{x} + \log x^5 - \log x\sqrt[3]{x} = \frac{31}{6}\log x$
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Example 3.4.10: Use log laws to combine into a single simplified expression

- $\log a + \frac{1}{2}\log b = \log(a\sqrt{b})$
- $5\log a + \frac{1}{2}\log b = \log(a^5\sqrt{b})$
- $\frac{1}{6}\log a + \frac{1}{7}\log b = \log(\sqrt[6]{a}\sqrt[7]{b})$
- $8\log a + \frac{2}{5}\log b = \log(a^8\sqrt[5]{b^2})$

Example 3.4.11: Use log laws to combine into a single expression.

- $\log a - \log b = \log \frac{a}{b}$
- $\log(x^2+1) - 2\log(x^5) = \log \frac{x^2+1}{x^{10}}$
- $\log(x^2+1) - \log(x^2+2) = \log \frac{x^2+1}{x^2+2}$
- $4\log a - 5\log b = \log \frac{a^4}{b^5}$

Example 3.4.12: Rewrite as a single log :

- $\log a - \log b - \log c = \log \frac{a}{bc}$
- $\log \sqrt{a} - \log b - \log c - \log d = \log \frac{\sqrt{a}}{bcd}$
- $\log ab + \log b - 5\log bc = \log \frac{a}{b^3c^5}$
- $\log a + \log b - \log c - \log d - \log e = \log \frac{ab}{cde}$

Example 3.4.13: Rewrite as a single log. Don't use radicals.

- $3\log a + 2\log b + 4\log c = \log a^3b^2c^4$
- $3\log a - 2\log b - 4\log c = \log \frac{a^3}{b^2c^4}$
- $\frac{1}{2}\log a - 2\log b + \frac{1}{3}\log c = \log \frac{a^{\frac{1}{2}}c^{\frac{1}{3}}}{b^2}$
- $-3\log a + \frac{1}{2}\log b + \frac{1}{3}\log b = \log \frac{b^{\frac{5}{6}}}{a^3}$

In the remaining problems, assume that a, b, c, x, t are all > 0 so that the given expression is defined.

Example 3.4.9: Rewrite as a single logarithm:

- $\log a + \log b = \log ab$
- $\log x\sqrt{x} + \log x^5 - \log x\sqrt[3]{x} = \frac{31}{6}\log x$
- $\log 2 + \log 3 - \log 4 = \log \frac{3}{2}$
- $\log x^3 - 2\log(b^2+1) = \log \frac{x^3}{(b^2+1)^2}$

Example 3.4.10: Use log laws to combine into a single simplified expression

- $\log a + \frac{1}{2}\log b = \log(a\sqrt{b})$
- $5\log a + \frac{1}{2}\log b = \log(a^5\sqrt{b})$
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- $8\log a + \frac{2}{5}\log b = \log(a^8\sqrt[5]{b^2})$

Example 3.4.11: Use log laws to combine into a single expression.

- $\log a - \log b = \log \frac{a}{b}$
- $\log(x^2+1) - 2\log(x^5) = \log \frac{x^2+1}{x^{10}}$
- $\log(x^2+1) - \log(x^2+2) = \log \frac{x^2+1}{x^2+2}$
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Example 3.4.12: Rewrite as a single log :

- $\log a - \log b - \log c = \log \frac{a}{bc}$
- $\log \sqrt{a} - \log b - \log c - \log d = \log \frac{\sqrt{a}}{bcd}$
- $\log ab + \log b - 5\log bc = \log \frac{a}{b^3c^5}$
- $\log a + \log b - \log c - \log d - \log e = \log \frac{ab}{cde}$

Example 3.4.13: Rewrite as a single log. Don't use radicals.

- $3\log a + 2\log b + 4\log c = \log a^3b^2c^4$
- $3\log a - 2\log b - 4\log c = \log \frac{a^3}{b^2c^4}$
- $\frac{1}{2}\log a - 2\log b + \frac{1}{3}\log c = \log \frac{a^{\frac{1}{2}}c^{\frac{1}{3}}}{b^2}$
- $-3\log a + \frac{1}{2}\log b + \frac{1}{3}\log b = \log \frac{b^{\frac{5}{6}}}{a^3}$

Example 3.4.14: Rewrite as a single log : Don't use fractional exponents.

- $3\ln x + \frac{1}{4}\ln y - 4\ln(t^2+1) =$
- $-3\ln x + \frac{1}{4}\ln y - \frac{2}{3}\ln(\sqrt{t^2+1}) =$
- $3\ln x - \frac{1}{3}\ln y + \frac{1}{2}\ln(t^2+1) =$
- $-3\ln x - \frac{1}{3}\ln y - 4\ln(t^2+1) =$

In the remaining problems, assume that a, b, c, x, t are all > 0 so that the given expression is defined.

Example 3.4.9: Rewrite as a single logarithm:

- $\log a + \log b = \log ab$
- $\log x\sqrt{x} + \log x^5 - \log x\sqrt[3]{x} = \frac{31}{6}\log x$
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Example 3.4.10: Use log laws to combine into a single simplified expression

- $\log a + \frac{1}{2}\log b = \log(a\sqrt{b})$
- $5\log a + \frac{1}{2}\log b = \log(a^5\sqrt{b})$
- $\frac{1}{6}\log a + \frac{1}{7}\log b = \log(\sqrt[6]{a}\sqrt[7]{b})$
- $8\log a + \frac{2}{5}\log b = \log(a^8\sqrt[5]{b^2})$

Example 3.4.11: Use log laws to combine into a single expression.

- $\log a - \log b = \log \frac{a}{b}$
- $\log(x^2+1) - 2\log(x^5) = \log \frac{x^2+1}{x^{10}}$
- $\log(x^2+1) - \log(x^2+2) = \log \frac{x^2+1}{x^2+2}$
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Example 3.4.12: Rewrite as a single log :

- $\log a - \log b - \log c = \log \frac{a}{bc}$
- $\log \sqrt{a} - \log b - \log c - \log d = \log \frac{\sqrt{a}}{bcd}$
- $\log ab + \log b - 5\log bc = \log \frac{a}{b^3c^5}$
- $\log a + \log b - \log c - \log d - \log e = \log \frac{ab}{cde}$

Example 3.4.13: Rewrite as a single log. Don't use radicals.

- $3\log a + 2\log b + 4\log c = \log a^3b^2c^4$
- $3\log a - 2\log b - 4\log c = \log \frac{a^3}{b^2c^4}$
- $\frac{1}{2}\log a - 2\log b + \frac{1}{3}\log c = \log \frac{a^{\frac{1}{2}}c^{\frac{1}{3}}}{b^2}$
- $-3\log a + \frac{1}{2}\log b + \frac{1}{3}\log b = \log \frac{b^{\frac{5}{6}}}{a^3}$

Example 3.4.14: Rewrite as a single log : Don't use fractional exponents.

- $3\ln x + \frac{1}{4}\ln y - 4\ln(t^2+1) = \ln \frac{x^3\sqrt[4]{y}}{(t^2+1)^4}$
- $-3\ln x + \frac{1}{4}\ln y - \frac{2}{3}\ln(\sqrt{t^2+1}) = \ln \frac{\sqrt[4]{y}}{x^3\sqrt[3]{(t^2+1)^2}}$
- $3\ln x - \frac{1}{3}\ln y + \frac{1}{2}\ln(t^2+1) = \ln \frac{x^3\sqrt{t^2+1}}{\sqrt[3]{y}}$
- $-3\ln x - \frac{1}{3}\ln y - 4\ln(t^2+1) = \ln \frac{1}{x^3\sqrt[3]{y}(t^2+1)^4}$

Chapter 3 Section 5: Exponential and logarithmic equations

- 3.5.1: Exponential equations
- 3.5.2: Logarithmic equations
- 3.5.3: Section 3.5 Quiz

Section 3.5 Preview: Definitions

- Definition 3.5.1: Inverse Principle for exponential and log functions
- Definition 3.5.2: If a and b are > 0 the solution of $a = b^x$ is $x = log_b(a)$.

3.5.1 Exponential equations

The laws of exponents and logarithms allow us to solve

- exponential equations such as $7 \cdot 10^{2x+3} = 9$ and
- logarithmic equations such as $\log x + \log(x + 90) = 3$.

The basic tool is the

Inverse Principle for exponential and log functions
$log_b(x)$ and b^x are inverse functions.

- If $b > 0$ then $log_b(b^x) = x$ for all real x.
- If $b > 0$ then $b^{log_b(x)} = x$ for all real $x > 0$.

To solve an exponential equation, take the appropriate log of both sides and then solve the resulting simpler equation.

If a and b are > 0 the solution of $a = b^x$ is $x = log_b(a)$.

- To see this, take log_b of both sides of $a = b^x$ to obtain $log_b(a) = log_b(b^x) = x$
- There is no solution if if $a < 0$.

Example 1: Solve a) $2^x = 35$ b)$2^x = -35$.

Solution to a) $x = \log_2 2^x = \boxed{\log_2 35}$.

Taking ln or log of both sides yields correct but complicated answers.

Alternate solutions:
Take log or ln of both sides to solve the equation $2^x = 35$:

$\log 2^x = \log 35 \Rightarrow x \log 2 = \log 35$ $\boxed{x = \frac{\log 35}{\log 2}}$

$\ln 2^x = \ln 35 \Rightarrow x \ln 2 = \ln 35$ $\boxed{x = \frac{\ln 35}{\ln 2}}$

Solution to b)
Taking $\log_2$ of $2^x = -35$ gives $x = \log_2(-35)$, which is undefined, because log functions have domain $(0, \infty)$. $\boxed{\text{No solution}}$

Example 2: Solve $10^{2x+2} = 100\sqrt[3]{10}$.
Solution: Rewrite right side as a power of 10.
Then apply log = $\log_{10}$ to both sides.

Original equation is:	$10^{2x+2} = 100\sqrt[3]{10}$.
Rewrite right side:	$10^{2x+2} = 10^2 \cdot 10^{1/3}$
	$10^{2x+2} = 10^{2+1/3}$.
	$10^{2x+2} = 10^{\frac{7}{3}}$
Take log = $\log_{10}$:	$2x + 2 = \frac{7}{3}$
Solve for x	$2x = \frac{7}{3} - 2 = \frac{1}{3}$
	$x = \frac{1}{2}(\frac{1}{3})$
Answer:	$x = \boxed{\frac{1}{6}}$.

Example 3: Solve $7e^{x+3} = 12$.

Solution: Divide by 7:	$e^{x+3} = \frac{12}{7}$
Take ln of both sides:	$\ln e^{x+3} = \ln \frac{12}{7}$
$\ln e^u = u$ and so:	$x + 3 = \ln \frac{12}{7}$
	$\boxed{x = \ln \frac{12}{7} - 3}$

Example 4: Solve $2^{2x} - 2^x - 6 = 0$.
Solution: This is a quadratic equation in disguise:
recognize that $2^{2x} = (2^x)^2$.

Original equation is:	$2^{2x} - 2^x - 6 = 0$.
Rewrite the first term:	$(2^x)^2 - 2^x + -6 = 0$.
Substitute u for 2^x:	$u^2 - u - 6 = 0$.
Solve by factoring:	$(u + 2)(u - 3) = 0$ so
	$u = -2$ or $u = 3$
$u = 2^x \Rightarrow x = \log_2(u)$:	If $u = -2,\ \ x = \log_2(-2)$ is undefined.
	If $u = 3,\ \ x = \log_2 3$
	$\boxed{x = \log_2 3}$

Example 5: Solve $4xe^x + x^2e^x = 0$.

Solution: Factor:	$xe^x(4 + x) = 0$
e^x can't be 0, divide by it:	$x(4 + x) = 0$
Set each factor to zero.	$\boxed{x = 0; \quad x = -4}$

3.5.2 Logarithmic equations

The strategy for solving logarithmic equations is: combine the logs of expressions into the log of a single expression. Then apply the Inverse Principle.

If $b > 0$, the solution of $a = \log_b(x)$ for x is $x = b^a$.

- $a = \log_b(x)$ Raise base b to both sides.
- $b^a = b^{\log_b(x)} = x$.

Example 6: Solve $5\log(3x+2) = 7$.
Solution: Remember that log means $\log_{10}$.

To solve:	$5\log_{10}(3x+2) = 7$
Divide by 5.	$\log_{10}(3x+2) = \frac{7}{5}$
Inverse principle:	$3x + 2 = 10^{\frac{7}{5}}$
Solve for x:	$3x = 10^{\frac{7}{5}} - 2$
	$\boxed{x = \frac{10^{\frac{7}{5}} - 2}{3}}$

Example 7: Solve $\log_2 x = 3 - \log_2(x+2)$

Solution: To solve:	$\log_2 x = 3 - \log_2(x+2)$
Get log terms on left side:	$\log_2 x + \log_2(x+2) = 3$
$\log u + \log v = \log uv$:	$\log_2(x(x+2)) = 3$
Inverse principle:	$x(x+2) = 2^3 = 8$
Rewrite	$x^2 + 2x - 8 = 0$
Factor	$(x+4)(x-2) = 0$
Set each factor to 0	$x = -4; \quad x = 2$

Because the equation involves log functions, which are not defined for negative inputs, you must check your answers.

Check the solution $x = 2$

$\log_2(x) =?\ 3 - \log_2(x+2)$
$\log_2(2) =?\ 3 - \log_2(2+2)$
$\log_2(2^1) =?\ 3 - \log_2(2^2)$
$1 =?\ 3 - 2$ Yes! $x = 2$ is a solution.

Check the solution $x = -4$:

$\log_2 x =?\ 3 - \log_2(x+2)$
$\log_2(-4) =?\ 3 - \log_2(-4+2)$
Reject $x = -4$ since $\log_2(-4)$ is undefined.

Answer: $\boxed{x = 2}$

Example 8: Solve $\log_3(x^2) + 1 = \log_3(x^2 + 18)$

Solution: To solve: $\log_3(x^2) + 1 = \log_3(x^2 + 18)$

Get all log terms on left side: $\log_3(x^2) - \log_3(x^2 + 18) = -1$

$\log u - \log v = \log \frac{u}{v}$ $\quad \log_3\left(\frac{x^2}{x^2+18}\right) = -1$

Inverse principle: $\quad \frac{x^2}{x^2+18} = 3^{-1} = \frac{1}{3}$

Multiply by$(x^2 + 18)(3)$ $\quad 3x^2 = x^2 + 18$

$$2x^2 = 18$$

$$x^2 = 9$$

$$x = \pm\sqrt{9} = \pm 3$$

Check the answers $x = \pm 3$. For both values, $x^2 = 9$.

$\log_3(x^2) + 1 \ =?\ \log_3(x^2 + 18)$

$\log_3(9) + 1 \ =?\ \log_3(9 + 18)$

$\log_3(3^2) + 1 \ =?\ \log_3(3^3)$

$2 + 1 \ =?\ 3$ Yes! Answer: $\boxed{x = -3; \quad x = 3}$

Section 3.5 Quiz

- Example 3.5.1: Solve a) $2^x = 35$ b) $2^x = -35$
- Example 3.5.2: Solve $10^{2x+2} = 100\sqrt[3]{10}$.
- Example 3.5.3: Solve $7e^{x+3} = 12$.
- Example 3.5.4: Solve $2^{2x} + 5 \cdot 2^{2x} + 6 = 0$.
- Example 3.5.5: Solve $4xe^x + x^2 e^x = 0$.
- Example 3.5.6: Solve $5\log(3x+2) = 7$.
- Example 3.5.7: Solve $\log_2 x = 3 - \log_2(x+2)$
- Example 3.5.8: Solve $\log_3(x^2) + 1 = \log_3(x^2 + 18)$

Section 3.5 Review: Exponential and logarithmic equations

"Solve" in the following means "Find all real solutions."

Example 3.5.1: Solve

- $2^x = 35 \Rightarrow$
- $2^x = -35 \Rightarrow$
- $5^x = 35 \Rightarrow$
- $5^x = \frac{1}{5} \Rightarrow$
- $2^x = 32 \Rightarrow$
- $4^x = 17 \Rightarrow$
- $7^x = 1 \Rightarrow$
- $7^x = -1 \Rightarrow$

Section 3.5 Review: Exponential and logarithmic equations

"Solve" in the following means "Find all real solutions."

Example 3.5.1: Solve

- $2^x = 35 \Rightarrow x = \log_2 35$
- $2^x = 32 \Rightarrow x = 5$
- $2^x = -35 \Rightarrow$ No solution
- $4^x = 17 \Rightarrow x = \log_4 17$
- $5^x = 35 \Rightarrow x = \log_5 35$
- $7^x = 1 \Rightarrow x = 0$
- $5^x = \frac{1}{5} \Rightarrow x = -1$
- $7^x = -1 \Rightarrow$ No solution

Section 3.5 Review: Exponential and logarithmic equations

"Solve" in the following means "Find all real solutions."

Example 3.5.1: Solve

- $2^x = 35 \Rightarrow x = \log_2 35$
- $2^x = -35 \Rightarrow$ No solution
- $5^x = 35 \Rightarrow x = \log_5 35$
- $5^x = \frac{1}{5} \Rightarrow x = -1$
- $2^x = 32 \Rightarrow x = 5$
- $4^x = 17 \Rightarrow x = \log_4 17$
- $7^x = 1 \Rightarrow x = 0$
- $7^x = -1 \Rightarrow$ No solution

Example 3.5.2: Solve

- $10^{2x+2} = 100\sqrt[3]{10} \Rightarrow$
- $10^{3x-1} = 10\sqrt[3]{1000} \Rightarrow$
- $10^{x^2} = 100 \Rightarrow$
- $10^{5x^2} = 100\sqrt[3]{10} \Rightarrow$

Section 3.5 Review: Exponential and logarithmic equations

"Solve" in the following means "Find all real solutions."

Example 3.5.1: Solve

- $2^x = 35 \Rightarrow x = \log_2 35$
- $2^x = -35 \Rightarrow$ No solution
- $5^x = 35 \Rightarrow x = \log_5 35$
- $5^x = \frac{1}{5} \Rightarrow x = -1$
- $2^x = 32 \Rightarrow x = 5$
- $4^x = 17 \Rightarrow x = \log_4 17$
- $7^x = 1 \Rightarrow x = 0$
- $7^x = -1 \Rightarrow$ No solution

Example 3.5.2: Solve

- $10^{2x+2} = 100\sqrt[3]{10} \Rightarrow x = \frac{1}{6}$
- $10^{3x-1} = 10\sqrt[3]{1000} \Rightarrow x = 1$
- $10^{x^2} = 100 \Rightarrow x = \pm\sqrt{2}$
- $10^{5x^2} = 100\sqrt[3]{10} \Rightarrow x = \pm\sqrt{\frac{7}{15}}$

Section 3.5 Review: Exponential and logarithmic equations

"Solve" in the following means "Find all real solutions."

Example 3.5.1: Solve

- $2^x = 35 \Rightarrow x = \log_2 35$
- $2^x = 32 \Rightarrow x = 5$
- $2^x = -35 \Rightarrow$ No solution
- $4^x = 17 \Rightarrow x = \log_4 17$
- $5^x = 35 \Rightarrow x = \log_5 35$
- $7^x = 1 \Rightarrow x = 0$
- $5^x = \frac{1}{5} \Rightarrow x = -1$
- $7^x = -1 \Rightarrow$ No solution

Example 3.5.2: Solve

- $10^{2x+2} = 100\sqrt[3]{10} \Rightarrow x = \frac{1}{6}$
- $10^{x^2} = 100 \Rightarrow x = \pm\sqrt{2}$
- $10^{3x-1} = 10\sqrt[3]{1000} \Rightarrow x = 1$
- $10^{5x^2} = 100\sqrt[3]{10} \Rightarrow x = \pm\sqrt{\frac{7}{15}}$

Example 3.5.3: Solve

- $7e^{x+3} = 12 \Rightarrow$
- $7e^{4-x} = -12 \Rightarrow$
- $2e^{3x+3} = 9 \Rightarrow$
- $5e^{x+3} + e^3 = 0 \Rightarrow$

Section 3.5 Review: Exponential and logarithmic equations

"Solve" in the following means "Find all real solutions."

Example 3.5.1: Solve

- $2^x = 35 \Rightarrow x = \log_2 35$
- $2^x = -35 \Rightarrow$ No solution
- $5^x = 35 \Rightarrow x = \log_5 35$
- $5^x = \frac{1}{5} \Rightarrow x = -1$
- $2^x = 32 \Rightarrow x = 5$
- $4^x = 17 \Rightarrow x = \log_4 17$
- $7^x = 1 \Rightarrow x = 0$
- $7^x = -1 \Rightarrow$ No solution

Example 3.5.2: Solve

- $10^{2x+2} = 100\sqrt[3]{10} \Rightarrow x = \frac{1}{6}$
- $10^{3x-1} = 10\sqrt[3]{1000} \Rightarrow x = 1$
- $10^{x^2} = 100 \Rightarrow x = \pm\sqrt{2}$
- $10^{5x^2} = 100\sqrt[3]{10} \Rightarrow x = \pm\sqrt{\frac{7}{15}}$

Example 3.5.3: Solve

- $7e^{x+3} = 12 \Rightarrow x = \ln\frac{12}{7} - 3$
- $2e^{3x+3} = 9 \Rightarrow x = \frac{1}{3}\ln\frac{9}{2} - 1$
- $7e^{4-x} = -12 \Rightarrow$ No solution
- $5e^{x+3} + e^3 = 0 \Rightarrow$ No solution

Section 3.5 Review: Exponential and logarithmic equations

"Solve" in the following means "Find all real solutions."

Example 3.5.1: Solve

- $2^x = 35 \Rightarrow x = \log_2 35$
- $2^x = -35 \Rightarrow$ No solution
- $5^x = 35 \Rightarrow x = \log_5 35$
- $5^x = \frac{1}{5} \Rightarrow x = -1$
- $2^x = 32 \Rightarrow x = 5$
- $4^x = 17 \Rightarrow x = \log_4 17$
- $7^x = 1 \Rightarrow x = 0$
- $7^x = -1 \Rightarrow$ No solution

Example 3.5.2: Solve

- $10^{2x+2} = 100\sqrt[3]{10} \Rightarrow x = \frac{1}{6}$
- $10^{3x-1} = 10\sqrt[3]{1000} \Rightarrow x = 1$
- $10^{x^2} = 100 \Rightarrow x = \pm\sqrt{2}$
- $10^{5x^2} = 100\sqrt[3]{10} \Rightarrow x = \pm\sqrt{\frac{7}{15}}$

Example 3.5.3: Solve

- $7e^{x+3} = 12 \Rightarrow x = \ln\frac{12}{7} - 3$
- $2e^{3x+3} = 9 \Rightarrow x = \frac{1}{3}\ln\frac{9}{2} - 1$
- $7e^{4-x} = -12 \Rightarrow$ No solution
- $5e^{x+3} + e^3 = 0 \Rightarrow$ No solution

Example 3.5.4: Solve

- $2^{2x} - 5\cdot 2^x + 6 = 0. \Rightarrow$
- $3^{6x} + 5\cdot 3^{6x} + 6 = 0. \Rightarrow$
- $2^{4x} - 5\cdot 2^{2x} - 6 = 0. \Rightarrow$
- $2^{4x} - 7\cdot 2^{4x} + 12 = 0. \Rightarrow$

Section 3.5 Review: Exponential and logarithmic equations

"Solve" in the following means "Find all real solutions."

Example 3.5.1: Solve

- $2^x = 35 \Rightarrow x = \log_2 35$
- $2^x = -35 \Rightarrow$ No solution
- $5^x = 35 \Rightarrow x = \log_5 35$
- $5^x = \frac{1}{5} \Rightarrow x = -1$
- $2^x = 32 \Rightarrow x = 5$
- $4^x = 17 \Rightarrow x = \log_4 17$
- $7^x = 1 \Rightarrow x = 0$
- $7^x = -1 \Rightarrow$ No solution

Example 3.5.2: Solve

- $10^{2x+2} = 100\sqrt[3]{10} \Rightarrow x = \frac{1}{6}$
- $10^{3x-1} = 10\sqrt[3]{1000} \Rightarrow x = 1$
- $10^{x^2} = 100 \Rightarrow x = \pm\sqrt{2}$
- $10^{5x^2} = 100\sqrt[3]{10} \Rightarrow x = \pm\sqrt{\frac{7}{15}}$

Example 3.5.3: Solve

- $7e^{x+3} = 12 \Rightarrow x = \ln\frac{12}{7} - 3$
- $2e^{3x+3} = 9 \Rightarrow x = \frac{1}{3}\ln\frac{9}{2} - 1$
- $7e^{4-x} = -12 \Rightarrow$ No solution
- $5e^{x+3} + e^3 = 0 \Rightarrow$ No solution

Example 3.5.4: Solve

- $2^{2x} - 5 \cdot 2^x + 6 = 0. \Rightarrow x = \frac{\ln 3}{\ln 2}; 1$
- $3^{6x} + 5 \cdot 3^{6x} + 6 = 0. \Rightarrow$ No solution
- $2^{4x} - 5 \cdot 2^{2x} - 6 = 0. \Rightarrow x = \frac{\ln 6}{2\ln 2}$
- $2^{4x} - 7 \cdot 2^{4x} + 12 = 0. \Rightarrow x = \frac{1}{4}$

Example 3.5.5: Solve

- $xe^x + x^3e^x = 0. \Rightarrow$
- $6x^2e^x - xe^x + x^3e^x = 0. \Rightarrow$
- $4xe^x - x^2e^x = 0. \Rightarrow$
- $e^x - e^{-x} = 0. \Rightarrow$

Example 3.5.5: Solve

- $xe^x + x^3e^x = 0. \Rightarrow x = 0$
- $6x^2e^x - xe^x + x^3e^x = 0. \Rightarrow x = 0, -3 \pm \sqrt{10}$
- $4xe^x - x^2e^x = 0. \Rightarrow x = 0, 4$
- $e^x - e^{-x} = 0. \Rightarrow x = 0$

Example 3.5.5: Solve

- $xe^x + x^3 e^x = 0. \Rightarrow x = 0$
- $6x^2 e^x - xe^x + x^3 e^x = 0. \Rightarrow x = 0, -3 \pm \sqrt{10}$
- $4xe^x - x^2 e^x = 0. \Rightarrow x = 0, 4$
- $e^x - e^{-x} = 0. \Rightarrow x = 0$

Example 3.5.6: Solve

- $5\log(3x+2) = 7 \Rightarrow$
- $5\log(x^2) = 10 \Rightarrow$
- $5\log(3x+2) = -7 \Rightarrow$
- $5\log(3x+10) = 10 \Rightarrow$

Example 3.5.5: Solve

- $xe^x + x^3 e^x = 0. \Rightarrow x = 0$
- $6x^2 e^x - xe^x + x^3 e^x = 0. \Rightarrow x = 0, -3 \pm \sqrt{10}$
- $4xe^x - x^2 e^x = 0. \Rightarrow x = 0, 4$
- $e^x - e^{-x} = 0. \Rightarrow x = 0$

Example 3.5.6: Solve

- $5 \log(3x + 2) = 7 \Rightarrow x = \frac{10^{\frac{7}{5}} - 2}{3}$
- $5 \log(x^2) = 10 \Rightarrow x = \pm 10$
- $5 \log(3x + 2) = -7 \Rightarrow$ No solution
- $5 \log(3x + 10) = 10 \Rightarrow x = 30$

Example 3.5.5: Solve

- $xe^x + x^3e^x = 0. \Rightarrow x = 0$
- $6x^2e^x - xe^x + x^3e^x = 0. \Rightarrow x = 0, -3 \pm \sqrt{10}$
- $4xe^x - x^2e^x = 0. \Rightarrow x = 0, 4$
- $e^x - e^{-x} = 0. \Rightarrow x = 0$

Example 3.5.6: Solve

- $5\log(3x+2) = 7 \Rightarrow x = \frac{10^{\frac{7}{5}} - 2}{3}$
- $5\log(x^2) = 10 \Rightarrow x = \pm 10$
- $5\log(3x+2) = -7 \Rightarrow$ No solution
- $5\log(3x+10) = 10 \Rightarrow x = 30$

Example 3.5.7: Solve

- $\log_2 x = 3 - \log_2(x+2) \Rightarrow$
- $\log_2 x + \log_2(x-6) = 3 \Rightarrow$
- $\log_2 x + \log_2(x+3) = 1 \Rightarrow$
- $\log(x+2) + \log(x-2) = 1 \Rightarrow$

Example 3.5.5: Solve

- $xe^x + x^3e^x = 0. \Rightarrow x = 0$
- $6x^2e^x - xe^x + x^3e^x = 0. \Rightarrow x = 0, -3 \pm \sqrt{10}$
- $4xe^x - x^2e^x = 0. \Rightarrow x = 0, 4$
- $e^x - e^{-x} = 0. \Rightarrow x = 0$

Example 3.5.6: Solve

- $5\log(3x+2) = 7 \Rightarrow x = \frac{10^{\frac{7}{5}} - 2}{3}$
- $5\log(x^2) = 10 \Rightarrow x = \pm 10$
- $5\log(3x+2) = -7 \Rightarrow$ No solution
- $5\log(3x+10) = 10 \Rightarrow x = 30$

Example 3.5.7: Solve

- $\log_2 x = 3 - \log_2(x+2) \Rightarrow x = 2$
- $\log_2 x + \log_2(x-6) = 3 \Rightarrow x = 3 + \sqrt{17}$
- $\log_2 x + \log_2(x+3) = 1 \Rightarrow x = \frac{1}{2}(\sqrt{17} - 3)$
- $\log(x+2) + \log(x-2) = 1 \Rightarrow x = \sqrt{14}$

Example 3.5.5: Solve

- $xe^x + x^3 e^x = 0. \Rightarrow x = 0$
- $6x^2 e^x - xe^x + x^3 e^x = 0. \Rightarrow x = 0, -3 \pm \sqrt{10}$
- $4xe^x - x^2 e^x = 0. \Rightarrow x = 0, 4$
- $e^x - e^{-x} = 0. \Rightarrow x = 0$

Example 3.5.6: Solve

- $5\log(3x+2) = 7 \Rightarrow x = \frac{10^{\frac{7}{5}} - 2}{3}$
- $5\log(x^2) = 10 \Rightarrow x = \pm 10$
- $5\log(3x+2) = -7 \Rightarrow$ No solution
- $5\log(3x+10) = 10 \Rightarrow x = 30$

Example 3.5.7: Solve

- $\log_2 x = 3 - \log_2(x+2) \Rightarrow x = 2$
- $\log_2 x + \log_2(x-6) = 3 \Rightarrow x = 3 + \sqrt{17}$
- $\log_2 x + \log_2(x+3) = 1 \Rightarrow x = \frac{1}{2}(\sqrt{17} - 3)$
- $\log(x+2) + \log(x-2) = 1 \Rightarrow x = \sqrt{14}$

Example 3.5.8: Solve

- $\log_3(x^2) + 1 = \log_3(x^2 + 18) \Rightarrow$
- $\log_2(x^2) + 1 = \log_2(x^2 + 4) \Rightarrow$
- $\log_3 x + 2 = \log_3(x^2 + 18) \Rightarrow$
- $\log_2(x+3) - 1 = \log_2(x) \Rightarrow$

Example 3.5.5: Solve

- $xe^x + x^3 e^x = 0. \Rightarrow x = 0$
- $6x^2 e^x - xe^x + x^3 e^x = 0. \Rightarrow x = 0, -3 \pm \sqrt{10}$
- $4xe^x - x^2 e^x = 0. \Rightarrow x = 0, 4$
- $e^x - e^{-x} = 0. \Rightarrow x = 0$

Example 3.5.6: Solve

- $5\log(3x+2) = 7 \Rightarrow x = \frac{10^{\frac{7}{5}} - 2}{3}$
- $5\log(x^2) = 10 \Rightarrow x = \pm 10$
- $5\log(3x+2) = -7 \Rightarrow$ No solution
- $5\log(3x+10) = 10 \Rightarrow x = 30$

Example 3.5.7: Solve

- $\log_2 x = 3 - \log_2(x+2) \Rightarrow x = 2$
- $\log_2 x + \log_2(x-6) = 3 \Rightarrow x = 3 + \sqrt{17}$
- $\log_2 x + \log_2(x+3) = 1 \Rightarrow x = \frac{1}{2}(\sqrt{17} - 3)$
- $\log(x+2) + \log(x-2) = 1 \Rightarrow x = \sqrt{14}$

Example 3.5.8: Solve

- $\log_3(x^2) + 1 = \log_3(x^2 + 18) \Rightarrow x = \pm 3$
- $\log_2(x^2) + 1 = \log_2(x^2 + 4) \Rightarrow x = \pm 2$
- $\log_3 x + 2 = \log_3(x^2 + 18) \Rightarrow x = 3, 6$
- $\log_2(x+3) - 1 = \log_2(x) \Rightarrow x = 3$

Chapter 3 Section 6: Exponential growth and decay

- 3.6.1: Exponential growth and decay
- 3.6.2: The relative growth rate
- 3.6.3: The exponential growth factor
- 3.6.4: Radioactive decay
- 3.6.5: Section 3.6 Quiz

Section 3.6 Preview: Definitions

- Definition 3.6.1: The function $N(t) = N(0)K^t$ models exponential growth or decay.
- Definition 3.6.2: If the compound interest rate is r , the amount $N(t)$ in a bank account after t years is
- Definition 3.6.3: If $N(t) = N(0)K^t$ at times $t = 0$ and $t = T$, then $N(t) = N(0)\left(\frac{N(T)}{N(0)}\right)^{t/T}$
- Definition 3.6.4: The half-life of an exponentially decaying substance is the time it takes to decay to half its original amount.

3.6.1 Exponential growth and decay

The function $N(t) = N(0)K^t$ models

- exponential growth if $K > 1$;
- exponential decay if $K < 1$;
- The relative growth (or decay) factor is K.
- The relative growth (or decay) rate is $r = \ln K$.

When discussing bank interest, recall that $5\% = 5/100$, which we write as interest rate $r = .05$

Let $N(t)$ be the amount in a bank account t years after it is opened. If the annual interest rate is r, compounded

- continuously, $N(t) = N(0)e^{rt}$: $\quad K = e^r$
- annually, $N(t) = N(0)(1+r)^t$: $\quad K = 1 + r$
- k times a year, $N(t) = N(0)(1+\frac{r}{k})^{kt}$: $K = (1+\frac{r}{k})^k$

If interest is compounded a very large number k times a year, it is shown in calculus that $K = (1+\frac{r}{k})^k$, the growth factor for compounding k times a year, will be as close as you desire to $K = e^r$, the growth factor for continuous compounding.

Calculator activity: Let $r = .05$. Find e^r and compare it to $(1+\frac{r}{k})^k$ for $k = 10, 100, 1000$.

Example 1:
A bank account starts off with $\$100$ and receives 5 percent annual interest. How much is in the account
after 10 years? after 100 years; after 1000 years;
if interest is compounded a) annually b) continuously?

Solution:

Compounded	a) Annually	b) Continuously
$N(t)$	$100(1.05)^t$	$100e^{.05t}$
$t = 10$	$\$162.89$	$\$164.87$
$t = 100$	$\$13,150.12$	$\$14,841.31$
$t = 1000$	$\$1.546 \cdot 10^{23}$	$\$5.185 \cdot 10^{23}$

3.6.2 The relative growth factor for exponential growth

Example 2: A population of bacteria doubles after 10 hours. Assume it grows exponentially. Find its relative growth rate.

Solution: Let $N(t)$ be the population at time t. Since the population after 10 hours is twice the original, $N(10) = 2N(0)$.

The growth formula: $N(t) = N(0)e^{rt}$.

Given: $N(10) = 2N(0)$ $N(10) = N(0)e^{r\cdot 10} = 2N(0)$

Divide by $N(0)$: $e^{r\cdot 10} = 2$

Take ln: $r \cdot 10 = \ln 2 \Rightarrow r = \frac{\ln 2}{10}$

Answer: The relative growth rate is $\frac{\ln 2}{10} \approx .069 \approx 6.9\%$.

Example 3: My bank account starts off with 100 dollars at 2:00 P.M. and grows exponentially. At 4:00 P.M., its value is 110 dollars. Find its value at 10:00 P.M. Find its value at any time t.

Solution: Let $N(t)$ be the number of dollars in my bank account t hours after 2:00 P.M. The problem tells us:

- $N(0) = 100$, the starting amount at 2:00, when we set $t = 0$.
- $N(2) = 110$, the amount at 4:00 P.M., which is 2 hours later, when $t = 2$.

The growth formula: $N(t) = N(0)K^t$

Given $N(0) = 100$: $N(t) = 100K^t$

Plug in $t = 2$: $N(2) = 100K^2$

Given $N(2) = 110$: $110 = 100K^2$

Solve for K: $\frac{110}{100} = 1.1 = K^2$

$$\Rightarrow K = \sqrt{1.1} = 1.1^{\frac{1}{2}}$$

Therefore $N(t) = 100\left((1.1)^{\frac{1}{2}}\right)^t = 100(1.1)^{\frac{t}{2}}$

At 10:00, $t = 8$: $N(8) = 100(1.1)^{\frac{8}{2}} = 100(1.1)^4$

Answer:
- The account value t hours after 2:00 P.M. is $100(1.1)^{\frac{t}{2}}$ dollars.
- At 10:00P.M., $t = 8$ hours after 2:00 P.M., its value is $100(1.1)^4 \approx \$146.21$

3.6.3 Finding the growth factor K in $N(t) = N(0)K^t$

Here's another way to look at the last problem. What happens when you know the population $N(t)$ at initial time $t = 0$ as well as at a specific later time $t = T$?

At time $t = T$: $N(T) = N(0)K^T$

Divide by $N(0)$: $\dfrac{N(T)}{N(0)} = K^T$

Raise to $\dfrac{t}{T}$: $\left(\dfrac{N(T)}{N(0)}\right)^{t/T} = (K^T)^{t/T} = K^t$

Then $N(t) = N(0)K^t = N(0)\left(\dfrac{N(T)}{N(0)}\right)^{t/T}$

Given $N(t) = N(0)K^t$ at times $t = 0$ and $t = T$

$$N(t) = N(0)\left(\frac{N(T)}{N(0)}\right)^{t/T}$$

Example 4: A bacteria colony starts off with 100 million bacteria at 2:00 P.M. and grows exponentially. At 4:00 P.M., there are 110 million bacteria. How many bacteria are there at 10:00 P.M.? at any time t?

Solution: Since $N(0) = 100$ and $N(2) = 110$, $T = 2$ and $\frac{N(2)}{N(0)} = \frac{110}{100} = 1.1$. Thus

$N(t) = N(0)\left(\frac{N(T)}{N(0)}\right)^{t/T} = N(0)\left(\frac{N(2)}{N(0)}\right)^{t/2}$.

Answer: The bacteria population

- t hours after 2:00 P.M. is $N(t) = 100(1.1)^{t/2}$ million.
- at 10:00 P.M. is $N(8) = 100(1.1)^4$ million.

Example 5: Solve $N = CK^t$ for t.

Solution: Divide by C: $\frac{N}{C} = K^t$

Take log (to any base): $\log \frac{N}{C} = \log K^t$

Laws of logs $\log N - \log C = t \log K$

Divide by $\log K$: $\boxed{t = \frac{\log N - \log C}{\log K}}$

Example 6: A bank account starts off at 2:00 P.M. with 100 dollars and grows exponentially. At 4:00 P.M., its value is 110 dollars. When will its value be 140 dollars?

Solution: Let $N(t)$ be the number of dollars in my bank account t hours after 2:00 P.M. The problem tells us:

- $N(0) = 100$, the starting amount at 2:00 P.M., when we set $t = 0$.
- $N(2) = 110$, the amount at 4:00 P.M., which is 2 hours later, when $t = 2$.

We are asked to find at which time t (hours after 2:00 P.M.) does $N(t) = 140$.

We will apply the Exponential growth formula with $T = 2$.

Exponential growth formula: $N(t) = N(0)\left(\frac{N(T)}{N(0)}\right)^{\frac{t}{T}}$

Given: $N(T) = N(2) = 110$ $\quad N(t) = N(0)\left(\frac{N(2)}{N(0)}\right)^{\frac{t}{2}}$

Given: $\frac{N(2)}{N(0)} = \frac{110}{100} = 1.1$ $\quad N(t) = 100(1.1)^{t/2}$

Find out when $N(t) = 140$: $\quad 140 = 100(1.1)^{t/2}$

Divide by 100: $\quad \frac{140}{100} = 1.4 = (1.1)^{t/2}$

Take log of both sides: $\quad \log 1.4 = \log\,(1.1)^{t/2}$

Bring down the power: $\quad \log 1.4 = \frac{t}{2}\log 1.1$

Solve for t: $\quad t = \frac{2\log 1.4}{\log 1.1}$

Answer:

The account value is 140 dollars when the time is

$$t = \frac{2\log 1.4}{\log 1.1} \approx 7.06 \text{ hours after 2:00 P.M.}$$

The next slides studies a bank account problem. In this setting, $N(t) = N(0)e^{rt}$, where r is the continuously compounded interest rate. That's banking terminology for the relative growth rate.

Example 7: A bank account starts off at 2:00 P.M. with value 100 dollars and receives interest, compounded continuously. At 4:00 P.M.,its value is 110 dollars. What is the compound interest rate?

Solution: Let $N(t)$ be the account value t hours after 2:00 P.M. Let r be the interest rate.

Continuous compounding:	$N(t) = N(0)e^{rt} = 100e^{rt}$
Let $t = 2$:	$N(2) = 100e^{2r} = 110$
Divide by 100	$\frac{110}{100} = 1.1 = e^{2r}$
Take ln to find r	$\ln 1.1 = 2r$
Relative growth rate:	$r = \frac{1}{2} \ln 1.1$

Answer:

- The relative growth rate is $r = \frac{1}{2} \ln\ 1.1 \approx .0477 \approx 4.77\%$

.

- The interest rate on the account is 4.77% , compounded continuously.

The next page shows a very similar example involving radioactive decay. The only real difference is that the "growth" factor K is less than 1 since the mass of a radioactive element decreases as time passes.

3.6.4 Radioactive decay

The half-life of an exponentially decaying substance

is the time it takes to decay from the amount at any beginning time t to half that amount.

Example 8: A sample of element stuffium with mass 100 grams will take 11 days to decay exponentially to 70 grams.

a) Find the mass of stuffium t days from now.
b) What is the half-life of stuffium?
c) How long does it take for the original sample to lose 65% of its mass?

Solution to a): Let $N(t)$ = the number of grams of stuffium at time t days from now. Given: $N(0) = 100$ and $N(11) = 70$. Thus $N(t) = N(0)K^t = 100K^t$.

Exponential growth:	$N(t) = 100K^t$
$N(11) = 70$:	$70 = 100K^{11}$
	$K^{11} = 70/100 = .7$
Take 11^{th} roots:	$K = \sqrt[11]{.7} = (.7)^{1/11}$
Now we know	$N(t) = N(0)K^t = 100(.7^{1/11})^t$
Power to a power:	$N(t) = 100(.7)^{t/11}$

Answer to a):

The mass of Stuffium mass will be $100(.7)^{\frac{t}{11}}$ grams t days from now.

Solution to b): $N(t)$ is half of the starting amount 100 grams when $N(t) = 50$.

We need to solve for t:	$N(t) = 50 = 100(.7)^{\frac{t}{11}}$
Divide by 100	$\frac{1}{2} = (.7)^{\frac{t}{11}}$
Take logs:	$\log \frac{1}{2} = \log \left[(.7)^{\frac{t}{11}}\right]$
Rewrite	$-\log 2 = \frac{t}{11} \log .7 = \frac{t \log .7}{11}$
Solve for t	$t = -\frac{11 \log 2}{\log .7}$

Answer to b):

The half-life of stuffium is $-\frac{11 \log 2}{\log .7} \approx 21.4$ years.

Solution to c): If the 100-gram sample loses 65% of its mass, the amount remaining will be $100 - .65(100) = 100 - 65 = 35$ grams.

To find the time t when $N(t) = 100(.7)^{\frac{t}{11}} = 35$, redo problem b) with 50 replaced by 35.

Answer to c):

Stuffium loses 65% of its mass after $\frac{11 \log .35}{\log .7} \approx 32.4$ years.

3.6.5 Quiz

Example 3.6.1a:
My bank account starts off at $\$100$ and receives 5 percent interest, compounded annually. How much is in my account after 10 years? After 100 years?

Example 3.6.1b:
My bank account starts off with $\$100$ and receives 5 percent interest, compounded continuously. How much is in my account after 10 years? After 100 years?

Example 3.6.2: A population of bacteria doubles after 10 hours. Assume exponential growth. Find its relative growth rate.

Example 3.6.3: My bank account starts off with 100 dollars at 2:00 P.M. and grows exponentially. At 4:00 P.M., its value is 110 dollars. Find its value at 10:00 P.M. Find its value at any time t.

Example 3.6.4: My bacteria colony starts off with 100 million bacteria at 2:00 P.M. and grows exponentially. At 4:00 P.M., there are 110 million bacteria. How many bacteria are there at 10:00 P.M.? at any time t?

Example 3.6.5: Solve $N = CK^t$ for t.

Example 3.6.6: My bank account starts off at 2:00 P.M. with 100 dollars and grows exponentially. At 4:00 P.M., it is 110 dollars. When will its value be 140 dollars?

Example 3.6.7: My bank account starts off at 2:00 P.M. with value 100 dollars and grows exponentially. At 4:00 P.M., its value is 110 dollars. What is its relative growth rate? What is the compound interest rate?

Example 3.6.8: A sample of element stuffium with mass 100 grams will decay exponentially to 70 grams 11 days from now.

a) Find the formula for $N(t) =$ the mass of stuffium t days from now.

b) What is the half-life of stuffium?

c) How long does it take for the original sample to lose 65% of its mass?

Section 3.6 Review: Exponential growth and decay

In all problems: • $N(t) = N(0)K^t$ is the population or amount at time t. • If $T > 0$: $N(t) = N(0)\left(\frac{N(T)}{N(0)}\right)^{\frac{t}{T}}$

Example 1: A bank account starts off with \$ 100 and receives 5 percent annual interest. How much is in the account after t years if interest is compounded

- a) annually $\boxed{N(t) = 100(1.05)^t}$
- b) continuously? $\boxed{N(t) = 100e^{.05t}}$

Example 2: A population doubles after 10 hours. Find its relative growth rate r: $N(t) = N(0)e^{rt}; \Rightarrow$

$N(10) = N(0)e^{r\cdot 10} = 2N(0) \Rightarrow e^{r\cdot 10} = 2$ $\boxed{r = \frac{\ln 2}{10}}$

Example 3: My bank account starts off with \$100 at 2:00 P.M. At 4:00 P.M., its value is \$110.

Solution: $N(0) = 100$; $N(2) = 110$; $N(t) = N(0)K^t$

$N(t) = 100(\frac{110}{100})^{\frac{t}{2}} = 100(1.1)^{\frac{t}{2}}$

What is the account value

- t hours after 2:00 P.M.? $\Rightarrow$ $\boxed{100(1.1)^{\frac{t}{2}} \text{ dollars}}$
- At 10:00P.M.? $t = 8 \Rightarrow$ $\boxed{100(1.1)^4 \approx \$146.21}$

Example 4: There are 100 million bacteria at 2:00 P.M. and 110 million at 4:00 P.M.

$N(0) = 100$;$N(2) = 110$; $\frac{N(2)}{N(0)} = 1.1$

$N(t) = N(0)\left(\frac{N(2)}{N(0)}\right)^{t/T} = 100(1.1)^{t/2}$.

How many bacteria are there at

- any time t? $\boxed{N(t) = 100(1.1)^{t/2} \text{ million}}$.
- at 10:00 P.M.? $\boxed{N(8) = 100(1.1)^4 \text{ million.}}$

Example 5: Solve $N = CK^t$ for t. $\boxed{t = \frac{\log N - \log C}{\log K}}$

Example 6: A bank account starts off at 2:00 P.M. with 100 dollars. At 4:00 P.M., its value is 110 dollars. When will its value be 140 dollars?

$N(0) = 100, N(T) = N(2) = 110; N(t) = 100(1.1)^{t/2}$

Find out when

$N(t) = 140 = 100(1.1)^{t/2} \Rightarrow t = \frac{2\log 1.4}{\log 1.1}$

The account value is 140 when the time is

$t = \boxed{\frac{2\log 1.4}{\log 1.1} \approx 7.06 \text{ hours after 2:00 P.M.}}$

Example 7:A bank account starts off at 2:00 P.M. with value 100 dollars and receives interest, compounded continuously. At 4:00 P.M.,its value is 110 dollars.

Continuous compounding: $N(t) = N(0)e^{rt} = 100e^{rt}$
Given $N(2) = 110 = 100e^{2r} \Rightarrow e^{2r} = 1.1$.

What is the relative growth rate? $\Rightarrow \frac{1}{2} \ln 1.1 \approx .0477$

What is the interest rate compounded continuously?
$\Rightarrow$ $\boxed{4.77 \text{ per cent}}$

Example 8: A sample of element stuffium with mass 100 grams will take 11 days to decay exponentially to 70 grams.

a) Find the mass of stuffium t days from now.
$N(0) = 100$
$N(t) = N(0)K^t = 100K^t$.
$N(11) = 70 \Rightarrow 70 = 100K^t \Rightarrow K = \sqrt[11]{.7} = (.7)^{1/11} \Rightarrow$
$N(t) = 100(.7)^{t/11}$ Stuffium mass will be

$\boxed{100(.7)^{\frac{t}{11}} \text{ grams } t \text{ days from now.}}$

b) What is the half-life of stuffium?

$N(t) = 50 = 100(.7)^{\frac{t}{11}} \Rightarrow t = \boxed{-\frac{11 \log 2}{\log .7} \approx 21.4 \text{ days.}}$

c) How long does it take for the original sample to lose 65% of its mass?

At what time t does $N(t) = 100(.7)^{\frac{t}{11}} = 35$. Stuffium

loses 65% of its mass after $\boxed{\frac{11 \log .35}{\log .7} \approx 32.4 \text{ days.}}$

Chapter 3 Review

- Precalculus Section 3.1 Review: Exp and log functions
- Precalculus Section 3.2 Review: Natural exp functions
- Precalculus Section 3.3 Review: Logarithmic functions
- Precalculus Section 3.4 Review: Laws of logarithms
- Precalculus Section 3.5 Review: Logarithmic equations
- Precalculus Section 3.6 Review: Exponential growth and decay

To review a section listed above:

Click on its button to view the first Example in that section as well as three similar questions. Work out the answers, then click again to see if you are correct. If so, keep on clicking.

If you have trouble answering a question, click on the to its left to access its solution in the text.

Then click on the faint ⥀ Adobe control at the bottom right of the text screen to continue your review.

Precalculus Chapter 4: Trigonometric functions. graphs, identities

Section 4.1: Right triangle trigonometry

4.1.1: Right triangle trigonometric ratios
4.1.2: Special right triangles
4.1 Quiz 4.1 Review

Section 4.2: Angles and circles

4.2.1: Angle measure
4.2.2: Angles in standard position
4.2.3: Circle arcs and sectors
4.2.4: Circular motion
4.2 Quiz 4.2 Review

Section 4.3: Trig functions of general angles

4.3.1: Radian measure of general angles
4.3.2: Trig functions of general angles
4.3.3: Angles in Quadrants 2,3,4
4.3.4: Finding the reference angle
4.3.5: Trig functions generalize SohCahToa
4.3.6: Trigonometric identities
4.3.7: Evaluating trigonometric functions
4.3.8: How your calculator computes cosines
4.3 Quiz 4.3 Review

Section 4.4: Trig functions and graphs

4.4.1: The sine of circular angles
4.4.2: The cosine of circular angles
4.4.3: Four basic sine and cosine graphs
4.4.4: Symmetry and periodicity
4.4.5: Graphing $A \sin Bx$ and $A \cos Bx$
4.4.6 Graphing the standard wave
4.4.7: Graphing $y = A \sin(Bx + C)$
4.4.8: Graphing $y = \tan x$
4.4 Quiz 4.4 Review

Section 4.5: Inverse trigonometric functions

4.5.1: Solving simple trigonometric equations
4.5.2: Inverse trigonometric functions
4.5.3: Solving harder trigonometric equations
4.5 Quiz 4.5 Review

Precalculus Chapter 4: Trigonometric functions, graphs, identities

Section 4.6: Trigonometric identities

- 4.6.1: Trig functions of 30°, 45°, and 60°.
- 4.6.2: Basic trigonometric identities
- 4.6.3: Even-odd identities
- 4.6.4: Even and odd functions
- 4.6.5 Simplifying trigonometric expressions
- 4.6 Quiz 4.6 Review

Section 4.7: Sum and difference formulas

- 4.7.1: Trig functions of general angles
- 4.7.2: Sum and difference formulas
- 4.7.3: Multiples of 15°
- 4.7 Quiz 4.7 Review

Section 4.8: Trig functions of general angles

- 4.8.1: Expressing one trig function in terms of another
- 4.8.2: Double-angle formulas
- 4.8.3: Half-angle formulas
- 4.8.4: Composing trig functions and their inverses
- 4.8 Quiz 4.8 Review

Chapter 4 Review

Chapter 4 Section 1: Right triangle trigonometry

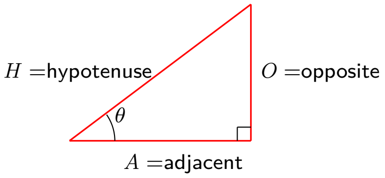
- 4.1.1: Right triangle trigonometric ratios
- 4.1.2: Special right triangles
- 4.1.3: Section 4.1 Quiz

Section 4.1 Preview: Definitions and Procedures

Definition 4.1.1: Trigonometric ratios for acute angle θ in a right triangle:

Procedure 4.1.1: To solve a triangle, find all angles and side lengths from the given information.

4.1.1 Right triangle trigonometric ratios (with Tamara Kucherenko)



In the following, each of the words *opposite, adjacent, hypotenuse* is the positive real number equal to the length of the correspondingly labeled side of the right triangle above.

Trigonometric ratios for acute angle θ in a right triangle:

$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$ $\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$

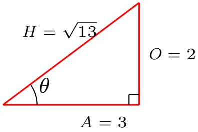
$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$ $\cot\theta = \frac{\text{Adjacent}}{\text{Opposite}}$

$\csc\theta = \frac{\text{Hypotenuse}}{\text{Opposite}}$ $\sec\theta = \frac{\text{Hypotenuse}}{\text{Adjacent}}$

The complete English names for these trig ratios are sine, cosine, tangent, cotangent, cosecant, and secant.

They depend *only* on the angle θ and not on the size of the triangle, since any two right triangles with angle θ are similar and the ratios of their corresponding sides are the same.

Example 1. Find the six trigonometric ratios of the angle θ.



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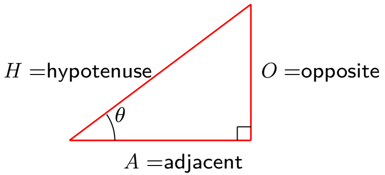

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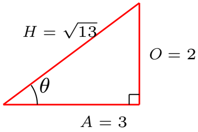
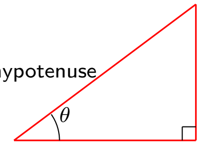


Solution.

$\sin\theta = \frac{2}{\sqrt{13}}$ $\cos\theta = \frac{3}{\sqrt{13}}$ $\tan\theta = \frac{2}{3}$

$\csc\theta = \frac{\sqrt{13}}{2}$ $\sec\theta = \frac{\sqrt{13}}{3}$ $\cot\theta = \frac{3}{2}$

4.1.1 Right triangle trigonometric ratios (with Tamara Kucherenko)



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$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$ $\cot\theta = \frac{\text{Adjacent}}{\text{Opposite}}$

$\csc\theta = \frac{\text{Hypotenuse}}{\text{Opposite}}$ $\sec\theta = \frac{\text{Hypotenuse}}{\text{Adjacent}}$

The complete English names for these trig ratios are sine, cosine, tangent, cotangent, cosecant, and secant.

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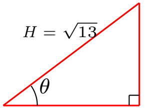


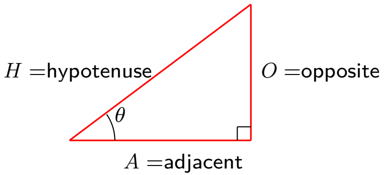
Solution.

$\sin\theta = \frac{2}{\sqrt{13}}$ $\cos\theta = \frac{3}{\sqrt{13}}$ $\tan\theta = \frac{2}{3}$

$\csc\theta = \frac{\sqrt{13}}{2}$ $\sec\theta = \frac{\sqrt{13}}{3}$ $\cot\theta = \frac{3}{2}$

Example 2. If $\sin\theta = \frac{4}{7}$, sketch a right triangle with acute angle θ.

4.1.1 Right triangle trigonometric ratios (with Tamara Kucherenko)



In the following, each of the words *opposite, adjacent, hypotenuse* is the positive real number equal to the length of the correspondingly labeled side of the right triangle above.

Trigonometric ratios for acute angle θ in a right triangle:

$\sin\theta = \dfrac{\text{Opposite}}{\text{Hypotenuse}}$ $\cos\theta = \dfrac{\text{Adjacent}}{\text{Hypotenuse}}$

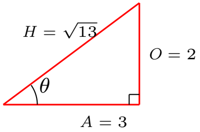
$\tan\theta = \dfrac{\text{Opposite}}{\text{Adjacent}}$ $\cot\theta = \dfrac{\text{Adjacent}}{\text{Opposite}}$

$\csc\theta = \dfrac{\text{Hypotenuse}}{\text{Opposite}}$ $\sec\theta = \dfrac{\text{Hypotenuse}}{\text{Adjacent}}$

The complete English names for these trig ratios are sine, cosine, tangent, cotangent, cosecant, and secant.

They depend *only* on the angle θ and not on the size of the triangle, since any two right triangles with angle θ are similar and the ratios of their corresponding sides are the same.

Example 1. Find the six trigonometric ratios of the angle θ.



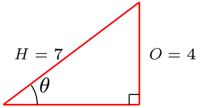
Solution.

$\sin\theta = \frac{2}{\sqrt{13}}$ $\cos\theta = \frac{3}{\sqrt{13}}$ $\tan\theta = \frac{2}{3}$

$\csc\theta = \frac{\sqrt{13}}{2}$ $\sec\theta = \frac{\sqrt{13}}{3}$ $\cot\theta = \frac{3}{2}$

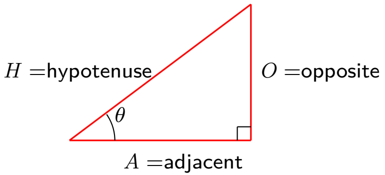
Example 2. If $\sin\theta = \frac{4}{7}$, sketch a right triangle with acute angle θ.

Solution. Since $\sin\theta$ is the ratio of the opposite side to the hypotenuse, sketch a triangle with a side of length 4 opposite to θ and hypotenuse length 7.



To find the adjacent side, use the Pythagorean Theorem: $O^2 + A^2 = H^2$. Then
$A = \sqrt{H^2 - O^2} = \sqrt{7^2 - 4^2} = \sqrt{33}$.

4.1.1 Right triangle trigonometric ratios (with Tamara Kucherenko)



In the following, each of the words *opposite, adjacent, hypotenuse* is the positive real number equal to the length of the correspondingly labeled side of the right triangle above.

Trigonometric ratios for acute angle θ in a right triangle:

$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$ $\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$

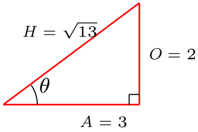
$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$ $\cot\theta = \frac{\text{Adjacent}}{\text{Opposite}}$

$\csc\theta = \frac{\text{Hypotenuse}}{\text{Opposite}}$ $\sec\theta = \frac{\text{Hypotenuse}}{\text{Adjacent}}$

The complete English names for these trig ratios are sine, cosine, tangent, cotangent, cosecant, and secant.

They depend *only* on the angle θ and not on the size of the triangle, since any two right triangles with angle θ are similar and the ratios of their corresponding sides are the same.

Example 1. Find the six trigonometric ratios of the angle θ.

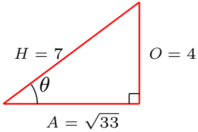


Solution.
$\sin\theta = \frac{2}{\sqrt{13}}$ $\cos\theta = \frac{3}{\sqrt{13}}$ $\tan\theta = \frac{2}{3}$

$\csc\theta = \frac{\sqrt{13}}{2}$ $\sec\theta = \frac{\sqrt{13}}{3}$ $\cot\theta = \frac{3}{2}$

Example 2. If $\sin\theta = \frac{4}{7}$, sketch a right triangle with acute angle θ.

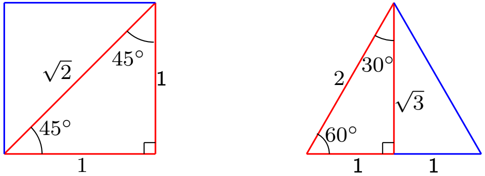
Solution. Since $\sin\theta$ is the ratio of the opposite side to the hypotenuse, sketch a triangle with a side of length 4 opposite to θ and hypotenuse length 7.



To find the adjacent side, use the Pythagorean Theorem: $O^2 + A^2 = H^2$. Then
$A = \sqrt{H^2 - O^2} = \sqrt{7^2 - 4^2} = \sqrt{33}$.

4.1.2 Special right triangles

Use the special triangles below to calculate the trigonometric ratios for angles with measures $30^\circ, 45^\circ$, and 60°.



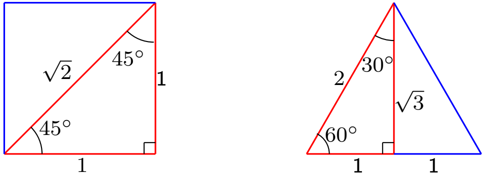
θ°	θ rad	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Fractions have been rationalized: $\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

Memorize the two red triangles and be able to quickly calculate all trig functions of $30, 45$, or 60 degrees.

4.1.2 Special right triangles

Use the special triangles below to calculate the trigonometric ratios for angles with measures $30^\circ, 45^\circ$, and 60°.



θ°	θ rad	sin θ	cos θ	tan θ	csc θ	sec θ	cot θ
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Fractions have been rationalized: $\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

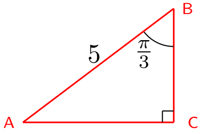
Memorize the two red triangles and be able to quickly calculate all trig functions of $30, 45$, or 60 degrees.

If you are given some sides and angles of a triangle, you may want to find the other sides and angles.

To *solve* a triangle:

Find all angles and side lengths from the given information.

Example 3: Solve the right triangle



Solution:

The unknown sides are AC and BC.
To find AC, look for an equation that relates AC to given lengths and angles.

$$\sin\frac{\pi}{3} = \frac{AC}{AB} = \frac{AC}{5} \quad \text{so} \quad AC = 5\sin\frac{\pi}{3} = \frac{5\sqrt{3}}{2}$$

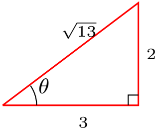
Similarly, $$BC = 5\cos\frac{\pi}{3} = \frac{5}{2}$$

Since the acute angles in a right triangle add to $90^\circ = \frac{\pi}{2}$ radians, $$\text{angle } A = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

The next section presents the more general discussion of angle measure that is needed in mathematics and science.

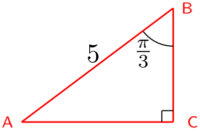
Section 4.1 Quiz

Example 4.1.1: Find the six trigonometric ratios of the angle θ.



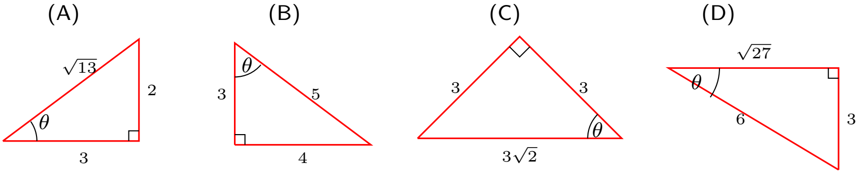
Example 4.1.2: If $\sin\theta = \frac{4}{7}$, sketch a right triangle with acute angle θ.

Example 3: Solve the right triangle



Section 4.1 Review: Right triangle trigonometry

Example 4.1.1: Find the six trigonometric ratios of the angle θ.



(A) • $\sin\theta =$ • $\cos\theta =$ • $\tan\theta =$ • $\csc\theta =$ • $\sec\theta =$ • $\cot\theta =$

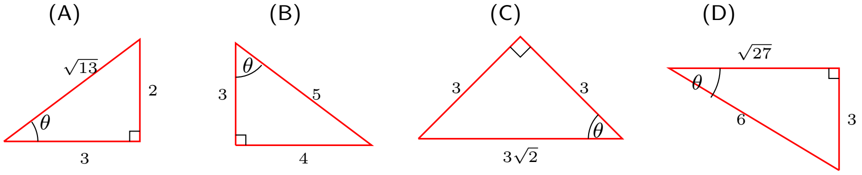
(B) • $\sin\theta =$ • $\cos\theta =$ • $\tan\theta =$ • $\csc\theta =$ • $\sec\theta =$ • $\cot\theta =$

(C) • $\sin\theta =$ • $\cos\theta =$ • $\tan\theta = 1$ • $\csc\theta =$ • $\sec\theta =$ • $\cot\theta =$

(D) • $\sin\theta =$ • $\cos\theta =$ • $\tan\theta =$ • $\csc\theta =$ • $\sec\theta =$ • $\cot\theta =$

Section 4.1 Review: Right triangle trigonometry

Example 4.1.1: Find the six trigonometric ratios of the angle θ.



(A) • $\sin\theta = \frac{2}{\sqrt{13}}$ • $\cos\theta = \frac{3}{\sqrt{13}}$ • $\tan\theta = \frac{2}{3}$ • $\csc\theta = \frac{\sqrt{13}}{2}$ • $\sec\theta = \frac{\sqrt{13}}{3}$ • $\cot\theta = \frac{3}{2}$

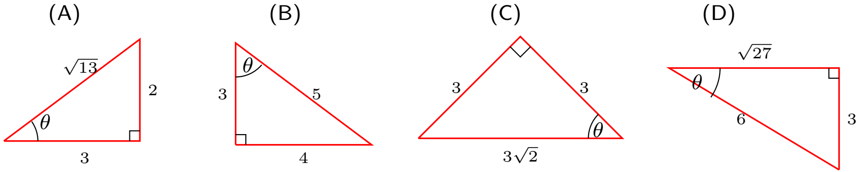
(B) • $\sin\theta = \frac{4}{5}$ • $\cos\theta = \frac{3}{5}$ • $\tan\theta = \frac{4}{3}$ • $\csc\theta = \frac{5}{4}$ • $\sec\theta = \frac{5}{3}$ • $\cot\theta = \frac{3}{4}$

(C) • $\sin\theta = \frac{1}{\sqrt{2}}$ • $\cos\theta = \frac{1}{\sqrt{2}}$ • $\tan\theta = 1$ • $\csc\theta = \sqrt{2}$ • $\sec\theta = \sqrt{2}$ • $\cot\theta = 1$

(D) • $\sin\theta = \frac{1}{2}$ • $\cos\theta = \frac{\sqrt{3}}{2}$ • $\tan\theta = \frac{1}{\sqrt{3}}$ • $\csc\theta = 2$ • $\sec\theta = \frac{2}{\sqrt{3}}$ • $\cot\theta = \sqrt{3}$

Section 4.1 Review: Right triangle trigonometry

Example 4.1.1: Find the six trigonometric ratios of the angle θ.



(A) • $\sin\theta = \frac{2}{\sqrt{13}}$ • $\cos\theta = \frac{3}{\sqrt{13}}$ • $\tan\theta = \frac{2}{3}$ • $\csc\theta = \frac{\sqrt{13}}{2}$ • $\sec\theta = \frac{\sqrt{13}}{3}$ • $\cot\theta = \frac{3}{2}$

(B) • $\sin\theta = \frac{4}{5}$ • $\cos\theta = \frac{3}{5}$ • $\tan\theta = \frac{4}{3}$ • $\csc\theta = \frac{5}{4}$ • $\sec\theta = \frac{5}{3}$ • $\cot\theta = \frac{3}{4}$

(C) • $\sin\theta = \frac{1}{\sqrt{2}}$ • $\cos\theta = \frac{1}{\sqrt{2}}$ • $\tan\theta = 1$ • $\csc\theta = \sqrt{2}$ • $\sec\theta = \sqrt{2}$ • $\cot\theta = 1$

(D) • $\sin\theta = \frac{1}{2}$ • $\cos\theta = \frac{\sqrt{3}}{2}$ • $\tan\theta = \frac{1}{\sqrt{3}}$ • $\csc\theta = 2$ • $\sec\theta = \frac{2}{\sqrt{3}}$ • $\cot\theta = \sqrt{3}$

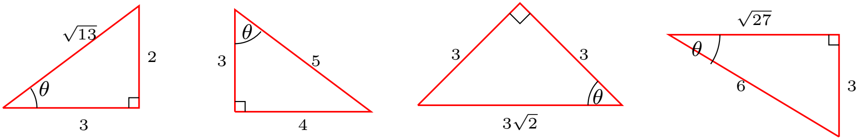
Example 4.1.2:sketch a right triangle with acute angle θ if

$\sin\theta = \frac{4}{7}$ $\cos\theta = \frac{2}{3}$ $\sin\theta = \frac{2}{5}$ $\sec\theta = \frac{7}{\sqrt{33}}$

Section 4.1 Review: Right triangle trigonometry

Example 4.1.1: Find the six trigonometric ratios of the angle θ.

(A) (B) (C) (D)



(A) • $\sin\theta = \frac{2}{\sqrt{13}}$ • $\cos\theta = \frac{3}{\sqrt{13}}$ • $\tan\theta = \frac{2}{3}$ • $\csc\theta = \frac{\sqrt{13}}{2}$ • $\sec\theta = \frac{\sqrt{13}}{3}$ • $\cot\theta = \frac{3}{2}$

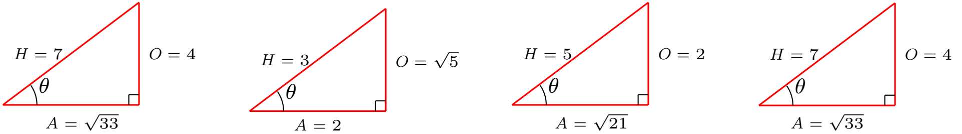
(B) • $\sin\theta = \frac{4}{5}$ • $\cos\theta = \frac{3}{5}$ • $\tan\theta = \frac{4}{3}$ • $\csc\theta = \frac{5}{4}$ • $\sec\theta = \frac{5}{3}$ • $\cot\theta = \frac{3}{4}$

(C) • $\sin\theta = \frac{1}{\sqrt{2}}$ • $\cos\theta = \frac{1}{\sqrt{2}}$ • $\tan\theta = 1$ • $\csc\theta = \sqrt{2}$ • $\sec\theta = \sqrt{2}$ • $\cot\theta = 1$

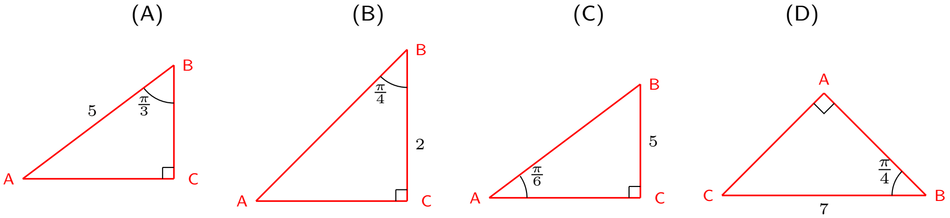
(D) • $\sin\theta = \frac{1}{2}$ • $\cos\theta = \frac{\sqrt{3}}{2}$ • $\tan\theta = \frac{1}{\sqrt{3}}$ • $\csc\theta = 2$ • $\sec\theta = \frac{2}{\sqrt{3}}$ • $\cot\theta = \sqrt{3}$

Example 4.1.2: sketch a right triangle with acute angle θ if

$\sin\theta = \frac{4}{7}$ $\cos\theta = \frac{2}{3}$ $\sin\theta = \frac{2}{5}$ $\sec\theta = \frac{7}{\sqrt{33}}$



Example 3: Solve the right triangle



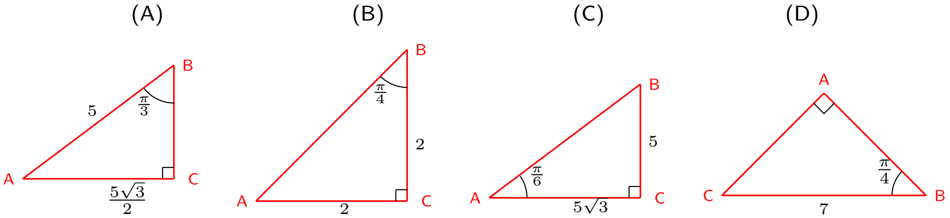
(A) • $AB =$ • $BC =$ • $AC =$ • $\angle A =$ • $\angle B =$ • $\angle C =$

(B) • $AB =$ • $BC =$ • $AC =$ • $\angle A =$ • $\angle B =$ • $\angle C =$

(C) • $AB =$ • $BC =$ • $AC =$ • $\angle A =$ • $\angle B =$ • $\angle C =$

(D) • $AB =$ • $BC =$ • $AC =$ • $\angle A =$ • $\angle B =$ • $\angle C =$

Example 3: Solve the right triangle



(A) • $AB = 5$ • $BC = \frac{5}{2}$ • $AC = \frac{5\sqrt{3}}{2}$ • $\angle A = \frac{\pi}{6}$ • $\angle B = \frac{\pi}{3}$ • $\angle C = \frac{\pi}{2}$

(B) • $AB = 2\sqrt{2}$ • $BC = 2$ • $AC = 2$ • $\angle A = \frac{\pi}{4}$ • $\angle B = \frac{\pi}{4}$ • $\angle C = \frac{\pi}{2}$

(C) • $AB = 10$ • $BC = 5$ • $AC = 5\sqrt{3}$ • $\angle A = \frac{\pi}{6}$ • $\angle B = \frac{\pi}{3}$ • $\angle C = \frac{\pi}{2}$

(D) • $AB = \frac{7}{\sqrt{2}}$ • $BC = 7$ • $AC = \frac{7}{\sqrt{2}}$ • $\angle A = \frac{\pi}{2}$ • $\angle B = \frac{\pi}{4}$ • $\angle C = \frac{\pi}{4}$

Section 4.2: Angles and circles

- 4.2.1: Angle measure
- 4.2.2: Angles in standard position
- 4.2.3: Circle arcs and sectors
- 4.2.4: Circular motion
- 4.2.5: Section 4.2 Quiz

Section 4.2 Preview: Definitions and Procedures

- Definition 4.2.1: An angle θ is the path of a point moving in one direction on a circle.
- Definition 4.2.2: If angle θ goes once around a radius r circle, its angle measure is 2π and its path length is $2\pi r$.
- Definition 4.2.3: On a circle with center O, if θ goes from point P to Q, its initial side is OP and its terminal side is OQ.
- Definition 4.2.4: Angle measure in radians is written as a pure number; in degrees as ^{o}.
- Definition 4.2.5: Angles are coterminal if their terminal sides are identical.
- Definition 4.2.6: Angles are coterminal if their measures differ by an integer multiple of 2π.
- Definition 4.2.7: A disc consists of all points on *and inside* a circle.
- Definition 4.2.8: A sector of a disc is the region between two radii and the arc joining their endpoints.
- Definition 4.2.9: In a radius r circle, the length s of an arc with central angle θ is $s = r\theta$.
- Definition 4.2.10: In a radius r disc, the area of a sector with central angle θ is $A = \frac{1}{2}r^2\theta$.
- Definition 4.2.11: The linear speed of a point moving θ radians in time t along a radius r circle is $\frac{r|\theta|}{t}$.

- Procedure 4.2.1: To convert angle measure from degrees to radians, multiply by $\frac{\pi}{180}$.
- Procedure 4.2.2: To find angles coterminal with θ, add an integer multiple of 2π.

4.2.1 Circular angles (with Tamara Kucherenko)

An angle θ is the path of a point moving in one direction on a radius r circle. If the path length $= s$,

the angle measure (size in radians) is

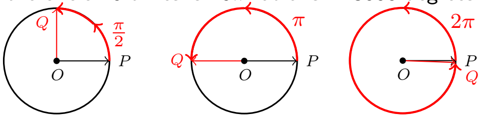
- $\theta = \frac{s}{r}$ if the point moves counterclockwise ;
- $\theta = -\frac{s}{r}$ if the point moves clockwise.
- The path length is always positive: $s = r|\theta|$.

If angle θ goes once counterclockwise around a radius r circle, its

- angle measure $= 2\pi$ radians $= 360$ degrees;
- path length $=$ circle circumference $= 2\pi r$.

Examples: For any circle, the angle measure of a path that goes counterclockwise around

- one quarter of the circle is $\frac{\pi}{2}$ radians $= 90$ degrees.
- half the circle is π radians $= 180$ degrees.
- the whole circle once is 2π radians $= 360$ degrees.
- the circle 10 times is 20π radians $= 3600$ degrees.



On a circle with center O, If θ goes from point P to Q,

its • initial side is OP; • terminal side is OQ;
its • initial point is P; • end (terminal) point is Q;

Angle measure in radians is written as a pure number

Angle measure in degrees is written followed by $^\circ$.

Memorize the following to avoid fraction calculations.

- $\pi = 180^\circ$ • $\frac{\pi}{2} = 90^\circ$ • $\frac{\pi}{3} = 60^\circ$ • $\frac{\pi}{4} = 45^\circ$ • $\frac{\pi}{6} = 30^\circ$
- $\left(\frac{180}{\pi}\right)^\circ = 1$ (radian) • $180^\circ = \pi$ • $1^\circ = \frac{\pi}{180}$

Angle measure: to convert

- degrees to radians, multiply by $\frac{\pi}{180^\circ}$.
- radians to degrees, multiply by $\frac{180^\circ}{\pi}$.

Example 1:

Express a) 75° in radians; b) $\frac{\pi}{15}$ radians in degrees.

a) $75^\circ = 75^\circ \cdot \frac{\pi}{180^\circ} = \boxed{\frac{5\pi}{12}}$ b) $\frac{\pi}{15} = \frac{\not\pi}{15} \cdot \frac{180^\circ}{\not\pi} = \boxed{12^\circ}$

Example 2:

Convert a) $\frac{11\pi}{4}$ to degrees; b) 330° to radians.

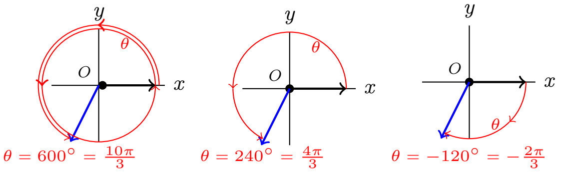
a) $\frac{11\pi}{4} = 11 \cdot \frac{\pi}{4} = 11 \cdot 45^\circ = \boxed{495^\circ}$

b) $330^\circ = 11 \cdot 30^\circ = 11 \cdot \frac{\pi}{6} = \boxed{\frac{11\pi}{6}}$

4.2.2 Angles in Standard Position

Definition of coterminal angles

- An angle is in standard position if it is drawn in the xy-plane with its vertex at the origin and its initial side on the positive x-axis.
- Two angles in standard position are coterminal if their terminal sides coincide.



Angles are in standard position.
Initial sides are black, terminal sides are blue.
All three angles are coterminal.

Angles are *coterminal* if their measures differ by

$2\pi n$ radians, or by $360n$ degrees, where n is an integer.

That's because rotation around one or more complete circles brings the terminal ray back to its initial position.

Example 3. a) Are angles $-\frac{5\pi}{3}$ and $\frac{7\pi}{3}$ coterminal?
b) Are angles 400° and 1300° coterminal?

Solution. a) The difference of the angles is $-\frac{5\pi}{3} - \frac{7\pi}{3} = -\frac{12\pi}{3} = -4\pi = -2\cdot 2\pi$.

$\boxed{-\frac{5\pi}{3} \text{ and } \frac{7\pi}{3} \text{ are coterminal.}}$

b) The difference of the angles is $1300^\circ - 400^\circ = 900^\circ$, but $\frac{900^\circ}{360^\circ} = \frac{15}{6} = 2.5$ is not an integer.

$\boxed{400^\circ \text{ and } 1300^\circ \text{ are not coterminal.}}$

To find angles coterminal with θ, add to θ

an integer times $360^\circ = 2\pi$ radians.

Example 4. Find an angle with measure between 0° and 360° that is coterminal with the angle of measure 1290° in standard position.

Solution. Keep subtracting 360° from 1290° until the resulting angle measure is in the requested range.

$1290^\circ - 360^\circ - 360^\circ - 360^\circ = \boxed{210^\circ}$.

Mathematical language: Technically speaking, an angle is a path and its measure is a number. That distinction is a bit clumsy in practice. We will write $\theta = 2\pi$ when the measure of θ is 2π.

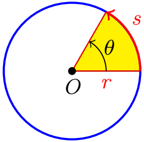
4.2.3 Measuring circle arcs and sectors

A disc consists of all points on *and inside* a circle.

- A circle is a curve in the plane: The equation $x^2 + y^2 = r^2$ defines the circle with radius r and center the origin. Its circumference is $2\pi r$.
- A disc is a region in the plane: The inequality $x^2 + y^2 \leq r^2$ defines the disc with radius r and center the origin. Its area is πr^2.
  It is technically incorrect but customary to say that a radius r circle's area is πr^2.
- The circle is the boundary (or edge) of the disc.

A *sector of a disc* is the region between and including

an angle θ in $(0, 2\pi)$ (arc on the circle) and the sides of θ.



Assume $\theta > 0$. Then θ is the central angle of both

- the red arc and
- the yellow circular sector.

As stated in the previous section:

In a radius r circle, the length s of an arc

with central angle θ (radians) is $s = r\theta$.

The fraction of the disk covered by the yellow sector equals the fraction of the circle covered by the red arc. Therefore

$$\frac{\text{sector area A}}{\text{total disc area } \pi r^2} = \frac{\text{arc length } r\theta}{\text{total circumference } 2\pi r}.$$

Multiply both sides of the equation by πr^2:

The area A of a sector of a radius r disc

with central angle θ (radians) is $A = \frac{1}{2} r^2 \theta$.

Example 5: Find the length of an arc with central angle 30° on a circle with radius 10 meters.

Solution.
Convert 30° into radians: $30 \cdot \frac{\pi}{180} = \frac{\pi}{6}$.

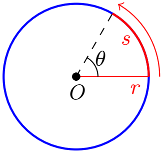
Then $s = r\theta = (10)\frac{\pi}{6}$. The arc length is $\frac{5\pi}{3}$ meters.

Example 6: Find the area of a sector of a circle with central angle 60° and radius 3 meters.

Solution. Convert 60° into radians: $60 \cdot \frac{\pi}{180} = \frac{\pi}{3}$.
Then $A = \frac{1}{2} r^2 \theta = \frac{1}{2}(3)^2 \left(\frac{\pi}{3}\right)$.

The sector area is $\frac{3\pi}{2}$ square meters.

4.2.4 Circular Motion



Two ways to describe motion of a point moving at constant speed along a circle:

Linear speed $= \frac{\text{distance traveled along the circle}}{\text{time}}$

Angular speed $= \frac{|\text{change of angle (in radians)}|}{\text{time}}$

If a point moves θ radians along a radius r circle in every elapsed time t

- the distance it travels in time t is $s = r|\theta|$.
- The point's angular speed is $\omega = \frac{|\theta|}{t}$.
- The point's linear speed is $v = \frac{s}{t} = \frac{r|\theta|}{t} = r\omega$.
- Speed is always positive!

Example 7: A comet moves 15 revolutions every 100 years along a circular orbit with radius 10^9 miles. Find the comet's angular and linear speeds.

Solution. In $t = 100$ years, the comet moves $\theta = 15 \cdot 2\pi = 30\pi$ radians. Thus $\omega = \frac{\theta}{t} = \frac{30\pi \text{ radians}}{100 \text{ years}}$.

Angular speed $= \frac{3\pi}{10}$ radians per year.

$v = r\omega = 10^9$ miles $\cdot \frac{3\pi}{10}$ (radians) per year.

Linear speed $= 3 \cdot 10^8\pi$ miles per year.

Example 8: A bicycle's wheels are 24 inches in diameter and rotate 125 revolutions per minute. Find the speed of the bicycle in miles per hour.

Solution. Each revolution is 2π radians, and so the wheels' angular speed is $\omega = 2\pi \cdot 125 = 250\pi$ radians per minute. Since the wheels have radius 12 inches, the linear speed is $r\omega = 12 \cdot 250\pi$ inches per minute $=$ 250π feet per minute.

To convert feet per minute to miles per hour use
1 mile $= 5280$ feet $= 88 \cdot 60$ feet; 1 hour $= 60$ minutes.

$$250\pi\ \frac{\text{feet}}{\text{minute}} \cdot 1 \cdot 1 = 250\pi\ \frac{\text{feet}}{\text{minute}} \cdot \frac{1 \text{ mile}}{5280\text{feet}} \cdot \frac{60 \text{ minutes}}{\text{hour}}$$

$$= \frac{250\pi \cdot 60}{88 \cdot 60}\ \frac{\text{miles}}{\text{hour}} = \frac{2 \cdot 125\pi}{2 \cdot 44}\ \frac{\text{miles}}{\text{hour}}$$

The bicycle's speed is $\frac{125\pi}{44}$ miles per hour.

Section 4.2 Quiz

Example 4.2.1: Express a) $75°$ in radians; b) $\pi/15$ radians in degrees.

Example 4.2.2: Convert a) $\frac{11\pi}{4}$ rad to degrees; b) $330°$ to radians.

Example 4.2.3: a) Are angles $-\frac{5\pi}{3}$ and $\frac{7\pi}{3}$ coterminal?
b) Are angles $400°$ and $1300°$ coterminal?

Example 4.2.4: Find an angle with measure between $0°$ and $360°$ that is coterminal with the angle of measure $1290°$ in standard position.

Example 4.2.5: Find the length of an arc of a circle with radius 10 meters and central angle $30°$.

Example 4.2.6: Find the area of a sector of a circle with central angle $60°$ and radius 3 meters.

Example 4.2.7: A comet moves 15 revolutions every 100 years along a circular orbit with radius 10^9 miles. Find the comet's angular and linear speeds.

Example 4.2.8: A bicycle's wheels are 22 inches in diameter. If the wheels rotate 125 revolutions per minute, find the speed of the bicycle in miles per hour.

Section 4.2 Review: Circular angles

Convert radians to degrees or degrees to radians:

Example 4.2.1:

- $75^\circ =$
- $\pi/15 =$
- $30^\circ =$
- $\pi/10 =$

Section 4.2 Review: Circular angles

Convert radians to degrees or degrees to radians:

Example 4.2.1:

- $75^\circ = \frac{5\pi}{12}$
- $\pi/15 = 12^\circ$
- $30^\circ = \frac{\pi}{6}$
- $\pi/10 = 18^\circ$

Section 4.2 Review: Circular angles

Convert radians to degrees or degrees to radians:

Example 4.2.1:

- $75^\circ = \frac{5\pi}{12}$
- $135^\circ =$
- $\pi/15 = 12^\circ$
- $3\pi =$
- $30^\circ = \frac{\pi}{6}$
- $90^\circ =$
- $\pi/10 = 18^\circ$
- $\pi/5 =$

Section 4.2 Review: Circular angles

Convert radians to degrees or degrees to radians:

Example 4.2.1:

- $75° = \frac{5\pi}{12}$
- $135° = \frac{3\pi}{4}$
- $\pi/15 = 12°$
- $3\pi = 540°$
- $30° = \frac{\pi}{6}$
- $90° = \frac{\pi}{2}$
- $\pi/10 = 18°$
- $\pi/5 = 36°$

Section 4.2 Review: Circular angles

Convert radians to degrees or degrees to radians:

Example 4.2.1:

- $75° = \frac{5\pi}{12}$
- $\pi/15 = 12°$
- $30° = \frac{\pi}{6}$
- $\pi/10 = 18°$
- $135° = \frac{3\pi}{4}$
- $3\pi = 540°$
- $90° = \frac{\pi}{2}$
- $\pi/5 = 36°$

Example 4.2.2:

- $\frac{11\pi}{4} =$
- $330° =$
- $\frac{3\pi}{8} =$
- $75° =$

Convert radians to degrees or degrees to radians:

Example 4.2.1:

- $75^\circ = \frac{5\pi}{12}$
- $135^\circ = \frac{3\pi}{4}$
- $\pi/15 = 12^\circ$
- $3\pi = 540^\circ$
- $30^\circ = \frac{\pi}{6}$
- $90^\circ = \frac{\pi}{2}$
- $\pi/10 = 18^\circ$
- $\pi/5 = 36^\circ$

Example 4.2.2:

- $\frac{11\pi}{4} = 495^\circ$
- $330^\circ = \frac{11\pi}{6}$
- $\frac{3\pi}{8} = 67.5^\circ$
- $75^\circ = \frac{5\pi}{12}$

Section 4.2 Review: Circular angles

Convert radians to degrees or degrees to radians:

Example 4.2.1:

- $75^\circ = \frac{5\pi}{12}$
- $135^\circ = \frac{3\pi}{4}$
- $\pi/15 = 12^\circ$
- $3\pi = 540^\circ$
- $30^\circ = \frac{\pi}{6}$
- $90^\circ = \frac{\pi}{2}$
- $\pi/10 = 18^\circ$
- $\pi/5 = 36^\circ$

Example 4.2.2:

- $\frac{11\pi}{4} = 495^\circ$
- $\frac{\pi}{12} =$
- $330^\circ = \frac{11\pi}{6}$
- $3645^\circ =$
- $\frac{3\pi}{8} = 67.5^\circ$
- $7 =$
- $75^\circ = \frac{5\pi}{12}$
- $7^\circ =$

Section 4.2 Review: Circular angles

Convert radians to degrees or degrees to radians:

Example 4.2.1:

- $75° = \frac{5\pi}{12}$
- $135° = \frac{3\pi}{4}$
- $\pi/15 = 12°$
- $3\pi = 540°$
- $30° = \frac{\pi}{6}$
- $90° = \frac{\pi}{2}$
- $\pi/10 = 18°$
- $\pi/5 = 36°$

Example 4.2.2:

- $\frac{11\pi}{4} = 495°$
- $\frac{\pi}{12} = 15°$
- $330° = \frac{11\pi}{6}$
- $3645° = \frac{41\pi}{4}$
- $\frac{3\pi}{8} = 67.5°$
- $7 = \frac{1260}{\pi}°$
- $75° = \frac{5\pi}{12}$
- $7° = \frac{7\pi}{180}$

Section 4.2 Review: Circular angles

Convert radians to degrees or degrees to radians:

Example 4.2.1:

- $75° = \frac{5\pi}{12}$
- $135° = \frac{3\pi}{4}$
- $\pi/15 = 12°$
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- $30° = \frac{\pi}{6}$
- $90° = \frac{\pi}{2}$
- $\pi/10 = 18°$
- $\pi/5 = 36°$

Example 4.2.2:

- $\frac{11\pi}{4} = 495°$
- $\frac{\pi}{12} = 15°$
- $330° = \frac{11\pi}{6}$
- $3645° = \frac{41\pi}{4}$
- $\frac{3\pi}{8} = 67.5°$
- $7 = \frac{1260}{\pi}°$
- $75° = \frac{5\pi}{12}$
- $7° = \frac{7\pi}{180}$

Example 4.2.3: Are the angles coterminal?

- $-\frac{5\pi}{3}$ and $\frac{7\pi}{3} \Rightarrow$
- $400°$ and $1300° \Rightarrow$
- $-\frac{\pi}{3}$ and $\frac{17\pi}{3} \Rightarrow$
- $400°$ and $40° \Rightarrow$

Section 4.2 Review: Circular angles

Convert radians to degrees or degrees to radians:

Example 4.2.1:

- $75° = \frac{5\pi}{12}$
- $135° = \frac{3\pi}{4}$
- $\pi/15 = 12°$
- $3\pi = 540°$
- $30° = \frac{\pi}{6}$
- $90° = \frac{\pi}{2}$
- $\pi/10 = 18°$
- $\pi/5 = 36°$

Example 4.2.2:

- $\frac{11\pi}{4} = 495°$
- $\frac{\pi}{12} = 15°$
- $330° = \frac{11\pi}{6}$
- $3645° = \frac{41\pi}{4}$
- $\frac{3\pi}{8} = 67.5°$
- $7 = \frac{1260}{\pi}°$
- $75° = \frac{5\pi}{12}$
- $7° = \frac{7\pi}{180}$

Example 4.2.3: Are the angles coterminal?

- $-\frac{5\pi}{3}$ and $\frac{7\pi}{3}$ $\Rightarrow$ Yes
- $400°$ and $1300°$ $\Rightarrow$ No
- $-\frac{\pi}{3}$ and $\frac{17\pi}{3}$ $\Rightarrow$ Yes
- $400°$ and $40°$ $\Rightarrow$ Yes

Section 4.2 Review: Circular angles

Convert radians to degrees or degrees to radians:

Example 4.2.1:

- $75^\circ = \frac{5\pi}{12}$
- $135^\circ = \frac{3\pi}{4}$
- $\pi/15 = 12^\circ$
- $3\pi = 540^\circ$
- $30^\circ = \frac{\pi}{6}$
- $90^\circ = \frac{\pi}{2}$
- $\pi/10 = 18^\circ$
- $\pi/5 = 36^\circ$

Example 4.2.2:

- $\frac{11\pi}{4} = 495^\circ$
- $\frac{\pi}{12} = 15^\circ$
- $330^\circ = \frac{11\pi}{6}$
- $3645^\circ = \frac{41\pi}{4}$
- $\frac{3\pi}{8} = 67.5^\circ$
- $7 = \frac{1260}{\pi}^\circ$
- $75^\circ = \frac{5\pi}{12}$
- $7^\circ = \frac{7\pi}{180}$

Example 4.2.3: Are the angles coterminal?

- $-\frac{5\pi}{3}$ and $\frac{7\pi}{3} \Rightarrow$ Yes
- $-\frac{5\pi}{6}$ and $\frac{19\pi}{6} \Rightarrow$
- 400° and $1300^\circ \Rightarrow$ No
- 100° and $-320^\circ \Rightarrow$
- $-\frac{\pi}{3}$ and $\frac{17\pi}{3} \Rightarrow$ Yes
- -11π and $15\pi \Rightarrow$
- 400° and $40^\circ \Rightarrow$ Yes
- 1800° and $-1800^\circ \Rightarrow$

Section 4.2 Review: Circular angles

Convert radians to degrees or degrees to radians:

Example 4.2.1:

- $75° = \frac{5\pi}{12}$
- $135° = \frac{3\pi}{4}$
- $\pi/15 = 12°$
- $3\pi = 540°$
- $30° = \frac{\pi}{6}$
- $90° = \frac{\pi}{2}$
- $\pi/10 = 18°$
- $\pi/5 = 36°$

Example 4.2.2:

- $\frac{11\pi}{4} = 495°$
- $\frac{\pi}{12} = 15°$
- $330° = \frac{11\pi}{6}$
- $3645° = \frac{41\pi}{4}$
- $\frac{3\pi}{8} = 67.5°$
- $7 = \frac{1260}{\pi}°$
- $75° = \frac{5\pi}{12}$
- $7° = \frac{7\pi}{180}$

Example 4.2.3: Are the angles coterminal?

- $-\frac{5\pi}{3}$ and $\frac{7\pi}{3}$ $\Rightarrow$ Yes
- $-\frac{5\pi}{6}$ and $\frac{19\pi}{6}$ $\Rightarrow$ Yes
- $400°$ and $1300°$ $\Rightarrow$ No
- $100°$ and $-320°$ $\Rightarrow$ No
- $-\frac{\pi}{3}$ and $\frac{17\pi}{3}$ $\Rightarrow$ Yes
- -11π and 15π $\Rightarrow$ Yes
- $400°$ and $40°$ $\Rightarrow$ Yes
- $1800°$ and $-1800°$ $\Rightarrow$ Yes

Section 4.2 Review: Circular angles

Convert radians to degrees or degrees to radians:

Example 4.2.1:

- $75° = \frac{5\pi}{12}$
- $135° = \frac{3\pi}{4}$
- $\pi/15 = 12°$
- $3\pi = 540°$
- $30° = \frac{\pi}{6}$
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- $\pi/10 = 18°$
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Example 4.2.2:

- $\frac{11\pi}{4} = 495°$
- $\frac{\pi}{12} = 15°$
- $330° = \frac{11\pi}{6}$
- $3645° = \frac{41\pi}{4}$
- $\frac{3\pi}{8} = 67.5°$
- $7 = \frac{1260}{\pi}°$
- $75° = \frac{5\pi}{12}$
- $7° = \frac{7\pi}{180}$

Example 4.2.3: Are the angles coterminal?

- $-\frac{5\pi}{3}$ and $\frac{7\pi}{3}$ $\Rightarrow$ Yes
- $-\frac{5\pi}{6}$ and $\frac{19\pi}{6}$ $\Rightarrow$ Yes
- $400°$ and $1300°$ $\Rightarrow$ No
- $100°$ and $-320°$ $\Rightarrow$ No
- $-\frac{\pi}{3}$ and $\frac{17\pi}{3}$ $\Rightarrow$ Yes
- -11π and 15π $\Rightarrow$ Yes
- $400°$ and $40°$ $\Rightarrow$ Yes
- $1800°$ and $-1800°$ $\Rightarrow$ Yes

Example 4.2.4: Find an angle θ with

- $0° \le \theta \le 360°$ coterminal with $1290°$ $\Rightarrow$
- $-\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ coterminal with $\frac{27\pi}{4}$ $\Rightarrow$
- $400° \le \theta \le 760°$ coterminal with $80°$ $\Rightarrow$
- $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$ coterminal with $-650°$ $\Rightarrow$

Section 4.2 Review: Circular angles

Convert radians to degrees or degrees to radians:

Example 4.2.1:

- $75° = \frac{5\pi}{12}$
- $135° = \frac{3\pi}{4}$
- $\pi/15 = 12°$
- $3\pi = 540°$
- $30° = \frac{\pi}{6}$
- $90° = \frac{\pi}{2}$
- $\pi/10 = 18°$
- $\pi/5 = 36°$

Example 4.2.2:

- $\frac{11\pi}{4} = 495°$
- $\frac{\pi}{12} = 15°$
- $330° = \frac{11\pi}{6}$
- $3645° = \frac{41\pi}{4}$
- $\frac{3\pi}{8} = 67.5°$
- $7 = \frac{1260}{\pi}°$
- $75° = \frac{5\pi}{12}$
- $7° = \frac{7\pi}{180}$

Example 4.2.3: Are the angles coterminal?

- $-\frac{5\pi}{3}$ and $\frac{7\pi}{3}$ $\Rightarrow$ Yes
- $-\frac{5\pi}{6}$ and $\frac{19\pi}{6}$ $\Rightarrow$ Yes
- $400°$ and $1300°$ $\Rightarrow$ No
- $100°$ and $-320°$ $\Rightarrow$ No
- $-\frac{\pi}{3}$ and $\frac{17\pi}{3}$ $\Rightarrow$ Yes
- -11π and 15π $\Rightarrow$ Yes
- $400°$ and $40°$ $\Rightarrow$ Yes
- $1800°$ and $-1800°$ $\Rightarrow$ Yes

Example 4.2.4: Find an angle θ with

- $0° \leq \theta \leq 360°$ coterminal with $1290°$ $\Rightarrow \theta = 210°$
- $-\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$ coterminal with $\frac{27\pi}{4} \Rightarrow \theta = \frac{3\pi}{4}$
- $400° \leq \theta \leq 760°$ coterminal with $80°$ $\Rightarrow \theta = 440°$
- $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$ coterminal with $-650° \Rightarrow \theta = 70°$

Section 4.2 Review: Circular angles

Convert radians to degrees or degrees to radians:

Example 4.2.1:

- $75° = \frac{5\pi}{12}$
- $135° = \frac{3\pi}{4}$
- $\pi/15 = 12°$
- $3\pi = 540°$
- $30° = \frac{\pi}{6}$
- $90° = \frac{\pi}{2}$
- $\pi/10 = 18°$
- $\pi/5 = 36°$

Example 4.2.2:

- $\frac{11\pi}{4} = 495°$
- $\frac{\pi}{12} = 15°$
- $330° = \frac{11\pi}{6}$
- $3645° = \frac{41\pi}{4}$
- $\frac{3\pi}{8} = 67.5°$
- $7 = \frac{1260}{\pi}°$
- $75° = \frac{5\pi}{12}$
- $7° = \frac{7\pi}{180}$

Example 4.2.3: Are the angles coterminal?

- $-\frac{5\pi}{3}$ and $\frac{7\pi}{3} \Rightarrow$ Yes
- $-\frac{5\pi}{6}$ and $\frac{19\pi}{6} \Rightarrow$ Yes
- $400°$ and $1300° \Rightarrow$ No
- $100°$ and $-320° \Rightarrow$ No
- $-\frac{\pi}{3}$ and $\frac{17\pi}{3} \Rightarrow$ Yes
- -11π and $15\pi \Rightarrow$ Yes
- $400°$ and $40° \Rightarrow$ Yes
- $1800°$ and $-1800° \Rightarrow$ Yes

Example 4.2.4: Find an angle θ with

- $0° \leq \theta \leq 360°$ coterminal with $1290° \Rightarrow \theta = 210°$
- $-\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$ coterminal with $\frac{27\pi}{4} \Rightarrow \theta = \frac{3\pi}{4}$
- $400° \leq \theta \leq 760°$ coterminal with $80° \Rightarrow \theta = 440°$
- $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$ coterminal with $-650° \Rightarrow \theta = 70°$

Example 4.2.5:

- Find the length of an arc with central angle $30°$ on a circle with radius 10 meters. $\Rightarrow$
- Find the length of an arc with central angle $\frac{\pi}{4}$ on a circle with radius 10 meters. $\Rightarrow$
- Find the central angle of an arc with length 5π meters on a circle with radius 10 meters. $\Rightarrow$
- Find the central angle of an arc with length 5 inches on a circle with radius 10 feet. $\Rightarrow$

Section 4.2 Review: Circular angles

Convert radians to degrees or degrees to radians:

Example 4.2.1:

- $75° = \frac{5\pi}{12}$
- $135° = \frac{3\pi}{4}$
- $\pi/15 = 12°$
- $3\pi = 540°$
- $30° = \frac{\pi}{6}$
- $90° = \frac{\pi}{2}$
- $\pi/10 = 18°$
- $\pi/5 = 36°$

Example 4.2.2:

- $\frac{11\pi}{4} = 495°$
- $\frac{\pi}{12} = 15°$
- $330° = \frac{11\pi}{6}$
- $3645° = \frac{41\pi}{4}$
- $\frac{3\pi}{8} = 67.5°$
- $7 = \frac{1260}{\pi}°$
- $75° = \frac{5\pi}{12}$
- $7° = \frac{7\pi}{180}$

Example 4.2.3: Are the angles coterminal?

- $-\frac{5\pi}{3}$ and $\frac{7\pi}{3} \Rightarrow$ Yes
- $-\frac{5\pi}{6}$ and $\frac{19\pi}{6} \Rightarrow$ Yes
- $400°$ and $1300° \Rightarrow$ No
- $100°$ and $-320° \Rightarrow$ No
- $-\frac{\pi}{3}$ and $\frac{17\pi}{3} \Rightarrow$ Yes
- -11π and $15\pi \Rightarrow$ Yes
- $400°$ and $40° \Rightarrow$ Yes
- $1800°$ and $-1800° \Rightarrow$ Yes

Example 4.2.4: Find an angle θ with

- $0° \le \theta \le 360°$ coterminal with $1290° \Rightarrow \theta = 210°$
- $-\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ coterminal with $\frac{27\pi}{4} \Rightarrow \theta = \frac{3\pi}{4}$
- $400° \le \theta \le 760°$ coterminal with $80° \Rightarrow \theta = 440°$
- $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$ coterminal with $-650° \Rightarrow \theta = 70°$

Example 4.2.5:

- Find the length of an arc with central angle $30°$ on a circle with radius 10 meters. $\Rightarrow \frac{5\pi}{3}$
- Find the length of an arc with central angle $\frac{\pi}{4}$ on a circle with radius 10 meters. $\Rightarrow \frac{5\pi}{2}$
- Find the central angle of an arc with length 5π meters on a circle with radius 10 meters. $\Rightarrow \frac{\pi}{2}$
- Find the central angle of an arc with length 5 inches on a circle with radius 10 feet. $\Rightarrow \frac{1}{24}$

Example 4.2.6:

- Find the area of a sector of a circle with central angle 60° and radius 3 meters. $\Rightarrow$
- Find the area of a sector of a circle with central angle $\frac{\pi}{4}$ and radius 3 meters. $\Rightarrow$
- Find the central angle of a sector with area 3π square meters on a circle with radius 2 meters. $\Rightarrow$
- Find the central angle of a sector with area π square inches on a circle with radius 2 feet. $\Rightarrow$

Example 4.2.6:

- Find the area of a sector of a circle with central angle 60° and radius 3 meters. $\Rightarrow \frac{3\pi}{2} m^2$
- Find the area of a sector of a circle with central angle $\frac{\pi}{4}$ and radius 3 meters. $\Rightarrow \frac{9\pi}{8} m^2$
- Find the central angle of a sector with area 3π square meters on a circle with radius 2 meters. $\Rightarrow \frac{3\pi}{2}$
- Find the central angle of a sector with area π square inches on a circle with radius 2 feet. $\Rightarrow \frac{\pi}{288}$

Example 4.2.6:

- Find the area of a sector of a circle with central angle 60° and radius 3 meters. $\Rightarrow \frac{3\pi}{2}m^2$
- Find the area of a sector of a circle with central angle $\frac{\pi}{4}$ and radius 3 meters. $\Rightarrow \frac{9\pi}{8}m^2$
- Find the central angle of a sector with area 3π square meters on a circle with radius 2 meters. $\Rightarrow \frac{3\pi}{2}$
- Find the central angle of a sector with area π square inches on a circle with radius 2 feet. $\Rightarrow \frac{\pi}{288}$

Example 4.2.7:

- Find the angular and linear speeds of a comet that moves 15 revolutions every 100 years along a circular orbit with radius 10^9 miles. $\Rightarrow$
- Find the angular and linear speeds of a comet that moves 1 revolutions every 140 years along a circular orbit with radius 10^7 miles. $\Rightarrow$
- If a comet moves with angular speed π radians per year along a circular orbit with radius 10^9 meters, how many revolutions (complete circles) does the comet move in 5 years? $\Rightarrow$
- If a comet moves with angular speed 8 radians per year moves a total of 10^9 meters in 5 years along a circular orbit, what is the radius of the circle? $\Rightarrow$

Example 4.2.6:

- Find the area of a sector of a circle with central angle 60° and radius 3 meters. $\Rightarrow \frac{3\pi}{2} m^2$
- Find the area of a sector of a circle with central angle $\frac{\pi}{4}$ and radius 3 meters. $\Rightarrow \frac{9\pi}{8} m^2$
- Find the central angle of a sector with area 3π square meters on a circle with radius 2 meters. $\Rightarrow \frac{3\pi}{2}$
- Find the central angle of a sector with area π square inches on a circle with radius 2 feet. $\Rightarrow \frac{\pi}{288}$

Example 4.2.7:

- Find the angular and linear speeds of a comet that moves 15 revolutions every 100 years along a circular orbit with radius 10^9 miles. $\Rightarrow \omega = \frac{3\pi}{10}$ radians/year ; $v = 3 \cdot 10^8 \pi$ miles/year
- Find the angular and linear speeds of a comet that moves 1 revolutions every 140 years along a circular orbit with radius 10^7 miles. $\Rightarrow \omega = \frac{\pi}{70}$ radians/year ; $v = \frac{10^6}{7}\pi$miles/year
- If a comet moves with angular speed π radians per year along a circular orbit with radius 10^9 meters, how many revolutions (complete circles) does the comet move in 5 years? $\Rightarrow \frac{5}{2}$ revolutions .
- If a comet moves with angular speed 8 radians per year moves a total of 10^9 meters in 5 years along a circular orbit, what is the radius of the circle? $\Rightarrow 2.5 \cdot 10^7$ meters

Example 4.2.6:

- Find the area of a sector of a circle with central angle 60° and radius 3 meters. $\Rightarrow \frac{3\pi}{2} m^2$
- Find the area of a sector of a circle with central angle $\frac{\pi}{4}$ and radius 3 meters. $\Rightarrow \frac{9\pi}{8} m^2$
- Find the central angle of a sector with area 3π square meters on a circle with radius 2 meters. $\Rightarrow \frac{3\pi}{2}$
- Find the central angle of a sector with area π square inches on a circle with radius 2 feet. $\Rightarrow \frac{\pi}{288}$

Example 4.2.7:

- Find the angular and linear speeds of a comet that moves 15 revolutions every 100 years along a circular orbit with radius 10^9 miles. $\Rightarrow \omega = \frac{3\pi}{10}$ radians/year ; $v = 3 \cdot 10^8 \pi$ miles/year
- Find the angular and linear speeds of a comet that moves 1 revolutions every 140 years along a circular orbit with radius 10^7 miles. $\Rightarrow \omega = \frac{\pi}{70}$ radians/year ; $v = \frac{10^6}{7} \pi$miles/year
- If a comet moves with angular speed π radians per year along a circular orbit with radius 10^9 meters, how many revolutions (complete circles) does the comet move in 5 years? $\Rightarrow \frac{5}{2}$ revolutions .
- If a comet moves with angular speed 8 radians per year moves a total of 10^9 meters in 5 years along a circular orbit, what is the radius of the circle? $\Rightarrow 2.5 \cdot 10^7$ meters

Example 4.2.8:

- A bicycle's wheels are 24 inches in diameter and rotate 125 RPM (revolutions per minute) . Find the speed of the bicycle in miles per hour. $\Rightarrow$
- A bicycle's wheels are 3 feet in diameter and rotate 60 RPM. Find the speed of the bicycle in feet per minute. $\Rightarrow$
- A bicycle with wheels with radius 2 feet has speed 30 miles per hour. How many RPM do the wheels rotate? $\Rightarrow$ $\Rightarrow$
- A bicycle with wheels rotating 30 RPM has speed 30 miles per hour. What is the radius of the wheels? $\Rightarrow$

Example 4.2.6:

- Find the area of a sector of a circle with central angle 60° and radius 3 meters. $\Rightarrow \frac{3\pi}{2}m^2$
- Find the area of a sector of a circle with central angle $\frac{\pi}{4}$ and radius 3 meters. $\Rightarrow \frac{9\pi}{8}m^2$
- Find the central angle of a sector with area 3π square meters on a circle with radius 2 meters. $\Rightarrow \frac{3\pi}{2}$
- Find the central angle of a sector with area π square inches on a circle with radius 2 feet. $\Rightarrow \frac{\pi}{288}$

Example 4.2.7:

- Find the angular and linear speeds of a comet that moves 15 revolutions every 100 years along a circular orbit with radius 10^9 miles. $\Rightarrow \omega = \frac{3\pi}{10}$ radians/year ; $v = 3 \cdot 10^8 \pi$ miles/year
- Find the angular and linear speeds of a comet that moves 1 revolutions every 140 years along a circular orbit with radius 10^7 miles. $\Rightarrow \omega = \frac{\pi}{70}$ radians/year ; $v = \frac{10^6}{7}\pi$miles/year
- If a comet moves with angular speed π radians per year along a circular orbit with radius 10^9 meters, how many revolutions (complete circles) does the comet move in 5 years? $\Rightarrow \frac{5}{2}$ revolutions .
- If a comet moves with angular speed 8 radians per year moves a total of 10^9 meters in 5 years along a circular orbit, what is the radius of the circle? $\Rightarrow 2.5 \cdot 10^7$ meters

Example 4.2.8:

- A bicycle's wheels are 24 inches in diameter and rotate 125 RPM (revolutions per minute) . Find the speed of the bicycle in miles per hour. $\Rightarrow \frac{125\pi}{44}$ miles/hour.
- A bicycle's wheels are 3 feet in diameter and rotate 60 RPM. Find the speed of the bicycle in feet per minute. $\Rightarrow 1584$ feet/minute
- A bicycle with wheels with radius 2 feet has speed 30 miles per hour. How many RPM do the wheels rotate? $\Rightarrow \frac{660}{\pi}$ RPM $\Rightarrow$
- A bicycle with wheels rotating 30 RPM has speed 30 miles per hour. What is the radius of the wheels? $\Rightarrow \frac{44}{\pi}$ feet

Chapter 4 Section 3: Trigonometric functions of circular angles

- 4.3.1: A dictionary of angles
- 4.3.2: Trig functions of general angles
- 4.3.3: Angles in Quadrants 2,3,4
- 4.3.4: Finding the reference angle
- 4.3.5: Trig functions generalize SohCahToa
- 4.3.6: Trigonometric identities
- 4.3.7: Evaluating trigonometric functions
- 4.3.8: How your calculator computes cosines
- 4.3.9: Section 4.3 Quiz

Section 4.3 Preview: Definitions and Procedures

- Definition 4.3.1: An angle on a circle is in standard position if its initial side is the ray from $(0,0)$ to $(r,0)$.
- Definition 4.3.2: The trigonometric functions of angle θ are
- Definition 4.3.3: if θ is in standard position with endpoint (x,y) on a radius 1 circle
  - $\cos\theta = x$;
  - $\sin\theta = y$.
- Definition 4.3.4: The quadrant of an angle in standard position is:
- Definition 4.3.5: The reference angle Ref θ of angle θ is:
- Definition 4.3.6: If θ is in Q2, Ref $\theta = \pi - \theta$.
- Definition 4.3.7: If θ is in Q3, Ref $\theta = \theta - \pi$.
- Definition 4.3.8: If θ is in Q4, Ref $\theta = 2\pi - \theta$.
- Definition 4.3.9: If Q4 angle θ is in $(-\frac{\pi}{2}, 0)$ Ref $\theta = -\theta$.
- Definition 4.3.10: If Q4 angle θ is in $(-\frac{\pi}{2}, 0)$ then $\theta = -$ Ref θ.
- Definition 4.3.11: Review of Reference angles for angles in $(0, 2\pi)$:
- Definition 4.3.12: Reference angles for other angles θ:

Section 4.3 Preview: Definitions and Procedures, continued

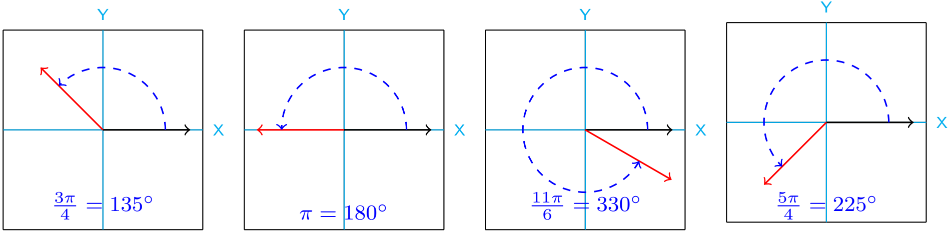
- Definition 4.3.13: If θ in $(0, \frac{\pi}{2})$ and Ref $\theta = 37^\circ$ then $\theta =$Ref $\theta = 37^\circ$.
- Definition 4.3.14: If θ in $(\frac{\pi}{2}, \pi)$ and Ref $\theta = 37^\circ$ then $\theta == 180^\circ - 37^\circ = 143^\circ$.
- Definition 4.3.15: If θ in$(\pi, \frac{3\pi}{2})$ and Ref $\theta = 37^\circ$ then $\theta = 180^\circ + 37^\circ = 217^\circ$.
- Definition 4.3.16: If θ in $(\frac{3\pi}{2}, 2\pi)$ and Ref $\theta = 37^\circ$ then $\theta = 360^\circ - 37^\circ = 323^\circ$.
- Definition 4.3.17: If θ in $(-\frac{\pi}{2}, 0)$ and Ref $\theta = 37^\circ$ then $\theta = -$Ref $\theta = -37^\circ$.
- Definition 4.3.18: Summary:1 Trig functions of angles θ in terms of Ref θ.

- Procedure 4.3.1: To find signs of trig functions in each Quadrant:

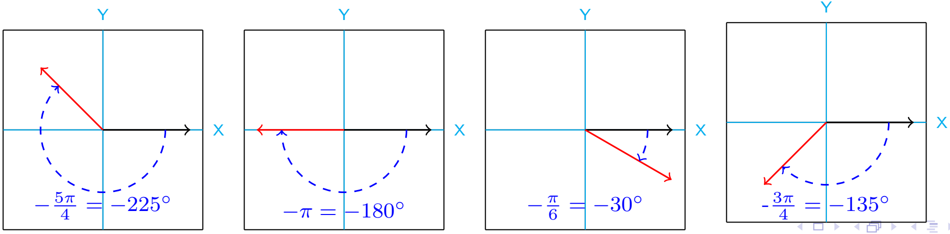
4.3.1. Examples of positive and negative angles

Be comfortable recognizing and drawing the angles below. Remember that π radians $= 180°$.
An angle's initial and terminal sides are often referred to as rays.
The following angles are in standard position: the initial black ray points along the positive x-axis.

If $\theta > 0$,rotate the initial black ray counterclockwise.



If $\theta < 0$, rotate the initial black ray clockwise.



4.3.2 Trigonometric functions of general (= circular) angles

If angle $\theta > 0$ on circle $x^2 + y^2 = r^2$ is in standard position,

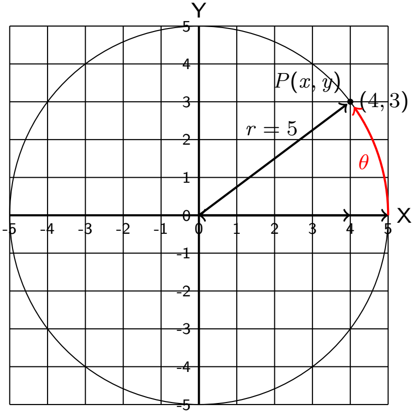
- its initial side is the ray from $(0,0)$ to $(r,0)$.
- its terminal side is the ray from $(0,0)$ to point $P(x,y)$ on the circle.
- Angle θ is the counterclockwise path from the endpoint of the initial ray to the endpoint of the terminal ray.

The trigonometric functions of angle θ are

$$\cos(\theta) = \frac{x}{r} \qquad \sec(\theta) = \frac{1}{\cos\theta} = \frac{r}{x}$$
$$\sin(\theta) = \frac{y}{r} \qquad \csc(\theta) = \frac{1}{\sin\theta} = \frac{r}{y}$$
$$\tan(\theta) = \frac{y}{x} \qquad \cot(\theta) = \frac{1}{\tan\theta} = \frac{x}{y}$$

In the picture at the right, the endpoint of angle θ (the arrow tip of the terminal side) is $P(4,3)$. Thus $x = 4, y = 3, r = \sqrt{x^2 + y^2} = 5$ and so

$$\cos(\theta) = \frac{x}{r} = \frac{4}{5} \qquad \sec(\theta) = \frac{r}{x} = \frac{5}{4}$$
$$\sin(\theta) = \frac{y}{r} = \frac{3}{5} \qquad \csc(\theta) = \frac{r}{y} = \frac{5}{3}$$
$$\tan(\theta) = \frac{y}{x} = \frac{3}{4} \qquad \cot(\theta) = \frac{r}{y} = \frac{4}{3}$$

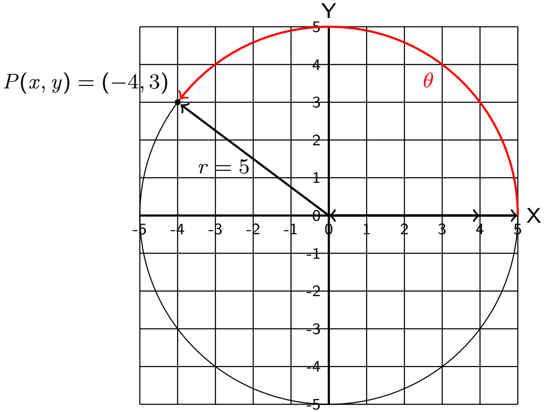


The previous example used angle $\theta \approx 37°$ with terminal side in Quadrant 1. At the right is a Quadrant 2 angle with the same reference angle. The arrow tip of the terminal side is at $P(-4, 3)$. Thus $x = -4, y = 3, r = \sqrt{x^2 + y^2} = 5$ and so

$\cos(\theta) = \frac{x}{r} = -\frac{4}{5}$ $\quad$ $\sec(\theta) = \frac{r}{x} = -\frac{5}{4}$

$\sin(\theta) = \frac{y}{r} = \frac{3}{5}$ $\quad$ $\csc(\theta) = \frac{r}{y} = \frac{5}{3}$

$\tan(\theta) = \frac{y}{x} = -\frac{3}{4}$ $\quad$ $\cot(\theta) = \frac{x}{y} = -\frac{4}{3}$



4.3.3: Signs of trig functions of angles in Quadrants 2,3,4

Basic meaning of sin and cos: On a radius 1 circle

if θ is in standard position with endpoint (x, y)

- $\cos\theta = x$;
- $\sin\theta = y$.

- If $r = 1$, $\cos(\theta) = x/r = x$,
  positive in Q1 and Q4, negative in Q2 and Q3
- If $r = 1$, $\sin(\theta) = y/r = y$,
  positive in Q1 and Q2, negative in Q3 and Q4.
- $\tan(\theta) = y/x$
  positive in Q1 and Q3, negative in Q2 and Q4.

4.3.3: Signs of trig functions of angles in Quadrants 2,3,4

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  positive in Q1 and Q2, negative in Q3 and Q4.
- $\tan(\theta) = y/x$
  positive in Q1 and Q3, negative in Q2 and Q4.

To find signs of trig functions in each Quadrant:

If the terminal side of the angle is in

- Q1: x is $+$, y is $+$, so All cos, sin, tan are $+$.

4.3.3: Signs of trig functions of angles in Quadrants 2,3,4

Basic meaning of sin and cos: On a radius 1 circle

if θ is in standard position with endpoint (x, y)

- $\cos\theta = x$;
- $\sin\theta = y$.

- If $r = 1$, $\cos(\theta) = x/r = x$,
  positive in Q1 and Q4, negative in Q2 and Q3
- If $r = 1$, $\sin(\theta) = y/r = y$,
  positive in Q1 and Q2, negative in Q3 and Q4.
- $\tan(\theta) = y/x$
  positive in Q1 and Q3, negative in Q2 and Q4.

To find signs of trig functions in each Quadrant:

If the terminal side of the angle is in

- Q1: x is $+$, y is $+$, so All cos, sin, tan are $+$.
- Q2: x is $-$, y is $+$, so Sin is $+$ but cos and tan are $-$.

4.3.3: Signs of trig functions of angles in Quadrants 2,3,4

Basic meaning of sin and cos: On a radius 1 circle

if θ is in standard position with endpoint (x, y)

- $\cos\theta = x$;
- $\sin\theta = y$.

- If $r = 1$, $\cos(\theta) = x/r = x$, positive in Q1 and Q4, negative in Q2 and Q3
- If $r = 1$, $\sin(\theta) = y/r = y$, positive in Q1 and Q2, negative in Q3 and Q4.
- $\tan(\theta) = y/x$ positive in Q1 and Q3, negative in Q2 and Q4.

To find signs of trig functions in each Quadrant:

If the terminal side of the angle is in

- Q1: x is $+$, y is $+$, so All cos, sin, tan are $+$.
- Q2: x is $-$, y is $+$, so Sin is $+$ but cos and tan are $-$.
- Q3: x is $-$, y is $-$, so Tan is $+$, but cos and sin are $-$.

4.3.3: Signs of trig functions of angles in Quadrants 2,3,4

Basic meaning of sin and cos: On a radius 1 circle

if θ is in standard position with endpoint (x, y)

- $\cos\theta = x$;
- $\sin\theta = y$.

- If $r = 1$, $\cos(\theta) = x/r = x$,
  positive in Q1 and Q4, negative in Q2 and Q3
- If $r = 1$, $\sin(\theta) = y/r = y$,
  positive in Q1 and Q2, negative in Q3 and Q4.
- $\tan(\theta) = y/x$
  positive in Q1 and Q3, negative in Q2 and Q4.

To find signs of trig functions in each Quadrant:

If the terminal side of the angle is in

- Q1: x is $+$, y is $+$, so All cos, sin, tan are $+$.
- Q2: x is $-$, y is $+$, so Sin is $+$ but cos and tan are $-$.
- Q3: x is $-$, y is $-$, so Tan is $+$, but cos and sin are $-$.
- Q4: x is $+$, y is $-$, so Cos is $+$, but sin and tan are $-$.

4.3.3: Signs of trig functions of angles in Quadrants 2,3,4

Basic meaning of sin and cos: On a radius 1 circle

if θ is in standard position with endpoint (x, y)

- $\cos\theta = x$;
- $\sin\theta = y$.

- If $r = 1$, $\cos(\theta) = x/r = x$,
  positive in Q1 and Q4, negative in Q2 and Q3
- If $r = 1$, $\sin(\theta) = y/r = y$,
  positive in Q1 and Q2, negative in Q3 and Q4.
- $\tan(\theta) = y/x$
  positive in Q1 and Q3, negative in Q2 and Q4.

To find signs of trig functions in each Quadrant:

If the terminal side of the angle is in

- Q1: x is $+$, y is $+$, so All cos, sin, tan are $+$.
- Q2: x is $-$, y is $+$, so Sin is $+$ but cos and tan are $-$.
- Q3: x is $-$, y is $-$, so Tan is $+$, but cos and sin are $-$.
- Q4: x is $+$, y is $-$, so Cos is $+$, but sin and tan are $-$.
- The first letters of the trig functions that are positive in Q1, Q2, Q3, Q4 are A, S, T, C, sometimes remembered as All Students Take Calculus.

4.3.3: Signs of trig functions of angles in Quadrants 2,3,4

Basic meaning of sin and cos: On a radius 1 circle

if θ is in standard position with endpoint (x, y)

- $\cos\theta = x$;
- $\sin\theta = y$.

- If $r = 1$, $\cos(\theta) = x/r = x$,
  positive in Q1 and Q4, negative in Q2 and Q3
- If $r = 1$, $\sin(\theta) = y/r = y$,
  positive in Q1 and Q2, negative in Q3 and Q4.
- $\tan(\theta) = y/x$
  positive in Q1 and Q3, negative in Q2 and Q4.

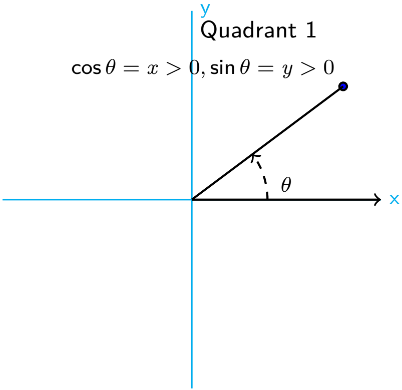
To find signs of trig functions in each Quadrant:

If the terminal side of the angle is in

- Q1: x is $+$, y is $+$, so All cos, sin, tan are $+$.
- Q2: x is $-$, y is $+$, so Sin is $+$ but cos and tan are $-$.
- Q3: x is $-$, y is $-$, so Tan is $+$, but cos and sin are $-$.
- Q4: x is $+$, y is $-$, so Cos is $+$, but sin and tan are $-$.
- The first letters of the trig functions that are positive in Q1, Q2, Q3, Q4 are A, S, T, C, sometimes remembered as All Students Take Calculus.

An angle in standard position is called

a Q1 angle if its terminal side is in Quadrant 1



4.3.3: Signs of trig functions of angles in Quadrants 2,3,4

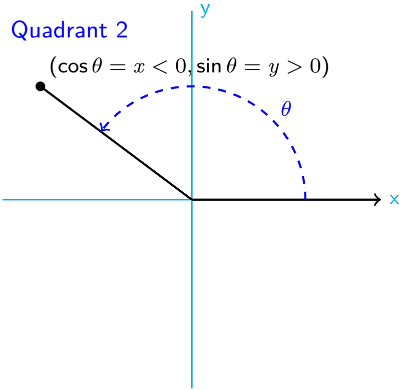
Basic meaning of sin and cos: On a radius 1 circle

if θ is in standard position with endpoint (x, y)

- $\cos\theta = x$;
- $\sin\theta = y$.

An angle in standard position is called

a Q2 angle if its terminal side is in Quadrant 2



- If $r = 1$, $\cos(\theta) = x/r = x$,
  positive in Q1 and Q4, negative in Q2 and Q3
- If $r = 1$, $\sin(\theta) = y/r = y$,
  positive in Q1 and Q2, negative in Q3 and Q4.
- $\tan(\theta) = y/x$
  positive in Q1 and Q3, negative in Q2 and Q4.

To find signs of trig functions in each Quadrant:

If the terminal side of the angle is in

- Q1: x is $+$, y is $+$, so All cos, sin, tan are $+$.
- Q2: x is $-$, y is $+$, so Sin is $+$ but cos and tan are $-$.
- Q3: x is $-$, y is $-$, so Tan is $+$, but cos and sin are $-$.
- Q4: x is $+$, y is $-$, so Cos is $+$, but sin and tan are $-$.
- The first letters of the trig functions that are positive in Q1, Q2, Q3, Q4 are A, S, T, C, sometimes remembered as All Students Take Calculus.

4.3.3: Signs of trig functions of angles in Quadrants 2,3,4

Basic meaning of sin and cos: On a radius 1 circle

if θ is in standard position with endpoint (x, y)

- $\cos\theta = x$;
- $\sin\theta = y$.

- If $r = 1$, $\cos(\theta) = x/r = x$, positive in Q1 and Q4, negative in Q2 and Q3
- If $r = 1$, $\sin(\theta) = y/r = y$, positive in Q1 and Q2, negative in Q3 and Q4.
- $\tan(\theta) = y/x$ positive in Q1 and Q3, negative in Q2 and Q4.

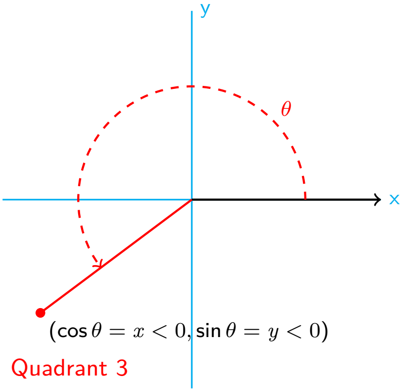
To find signs of trig functions in each Quadrant:

If the terminal side of the angle is in

- Q1: x is $+$, y is $+$, so All cos, sin, tan are $+$.
- Q2: x is $-$, y is $+$, so Sin is $+$ but cos and tan are $-$.
- Q3: x is $-$, y is $-$, so Tan is $+$, but cos and sin are $-$.
- Q4: x is $+$, y is $-$, so Cos is $+$, but sin and tan are $-$.
- The first letters of the trig functions that are positive in Q1, Q2, Q3, Q4 are A, S, T, C, sometimes remembered as All Students Take Calculus.

An angle in standard position is called

a Q3 angle if its terminal side is in Quadrant 3



4.3.3: Signs of trig functions of angles in Quadrants 2,3,4

Basic meaning of sin and cos: On a radius 1 circle

if θ is in standard position with endpoint (x, y)

- $\cos\theta = x$;
- $\sin\theta = y$.

- If $r = 1$, $\cos(\theta) = x/r = x$, positive in Q1 and Q4, negative in Q2 and Q3
- If $r = 1$, $\sin(\theta) = y/r = y$, positive in Q1 and Q2, negative in Q3 and Q4.
- $\tan(\theta) = y/x$ positive in Q1 and Q3, negative in Q2 and Q4.

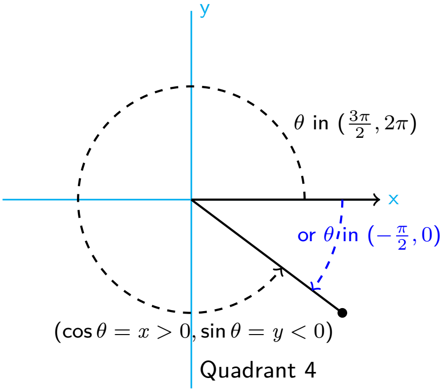
To find signs of trig functions in each Quadrant:

If the terminal side of the angle is in

- Q1: x is $+$, y is $+$, so All cos, sin, tan are $+$.
- Q2: x is $-$, y is $+$, so Sin is $+$ but cos and tan are $-$.
- Q3: x is $-$, y is $-$, so Tan is $+$, but cos and sin are $-$.
- Q4: x is $+$, y is $-$, so Cos is $+$, but sin and tan are $-$.
- The first letters of the trig functions that are positive in Q1, Q2, Q3, Q4 are A, S, T, C, sometimes remembered as All Students Take Calculus.

An angle in standard position is called

a Q4 angle if its terminal side is in Quadrant 4



4.3.4: Finding the reference angle of a given angle

Trigonometric functions of any angle are easily computed from the corresponding functions of the related angle Ref θ, the *reference angle* of θ.

Definition of reference angle Ref θ

- If the endpoint of θ is on either the x-axis or the y-axis (θ is an integer multiple of $\frac{\pi}{2} = 90°$), then Ref θ is *undefined.*
- For any other angle θ, Ref θ in $(0, 90°) = (0, \frac{\pi}{2})$ is the smaller (acute) angle between the x-axis and the terminal line of θ.
- All trigonometric functions of Ref θ are positive.

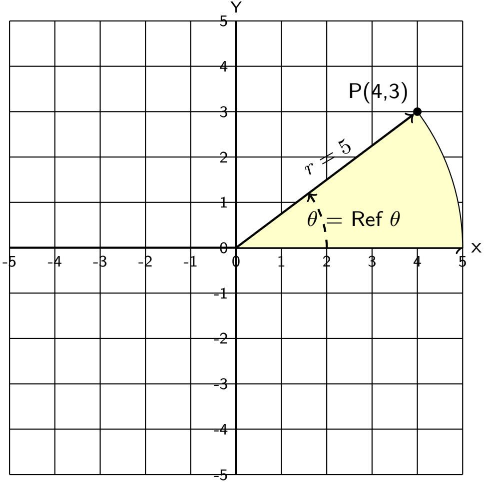
The angle θ at the right has endline in Q1.
Therefore Ref $\theta = \theta$.

This reference angle will reappear in the next 4 pictures.

On this slide, in Q1, all trig functions are positive.

$\cos(\theta) = \frac{x}{r} = \frac{4}{5}$; $\sin(\theta) = \frac{y}{r} = \frac{3}{5}$; $\tan(\theta) = \frac{y}{x} = \frac{3}{4}$

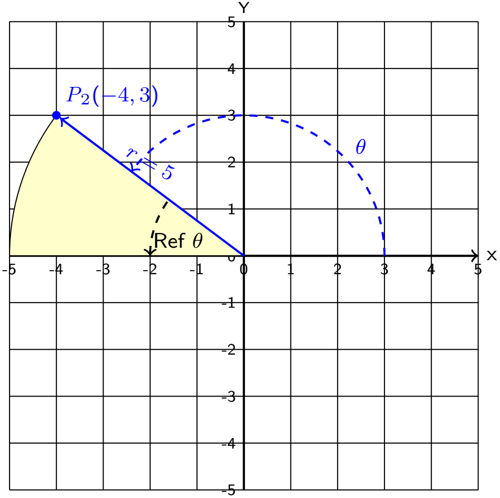
Ref θ, colored yellow on these slides, is > 0 and so is a counterclockwise arc.



Quadrant 2 angles: suppose θ in $(\frac{\pi}{2}, \pi)$

- Ref $\theta + \theta = 180^\circ = \pi$.
- $\theta = 180^\circ -$ Ref $\theta = \pi -$ Ref θ.
- Ref $\theta = 180^\circ - \theta = \pi - \theta$.

The endpoint of the terminal side of θ is $P_2(-4, 3)$
Thus $x = -4, y = 3, r = \sqrt{(-4)^2 + 3^2} = 5$.
$\cos(\theta) = \frac{x}{r} = \frac{-4}{5} = -\frac{4}{5}$ is Negative
$\sin(\theta) = \frac{y}{r} = \frac{3}{5}$ is Positive.
$\tan(\theta) = \frac{y}{x} = \frac{3}{-4} = -\frac{3}{4}$ is Negative.
These statements also follow from ASTC.



Quadrant 3 angles: suppose θ in $(\pi, \frac{3\pi}{2})$

- $\theta-$Ref $\theta = 180^\circ = \pi$.
- $\theta =$Ref $\theta + 180^\circ =$ Ref $\theta + \pi$.
- Ref $\theta = \theta - 180^\circ = \theta - \pi$.

The endpoint of the terminal side of θ is $P_3(-4,-3)$

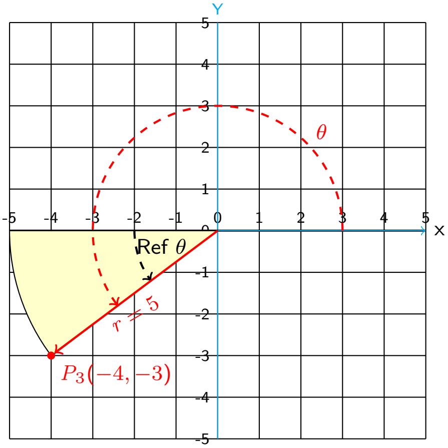
Thus $x = -4, y = -3, r = \sqrt{(-4)^2 + (-3)^2} = 5$.

$\cos(\theta) = \frac{x}{r} = \frac{-4}{5} = -\frac{4}{5}$ is Negative.

$\sin(\theta) = \frac{y}{r} = \frac{-3}{5} = -\frac{3}{5}$ is Negative.

$\tan(\theta) = \frac{y}{x} = \frac{-3}{-4} = \frac{3}{4}$ is Positive.

These statements also follow from ASTC.



Quadrant 4 angles: suppose θ in $(\frac{3\pi}{2}, \pi)$

- $\theta +$ Ref $\theta = 2\pi$.
- Ref $\theta = 2\pi - \theta = 360° - \theta$.
- $\theta = 2\pi -$ Ref $\theta = 360° -$ Ref θ.

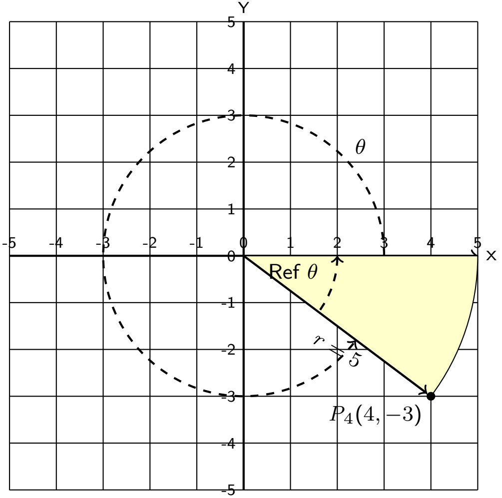
The endpoint of the terminal side of θ is $P_4(4, -3)$
Thus $x = 4, y = -3, r = \sqrt{4^2 + (-3)^2} = 5$.

$\cos(\theta) = \frac{x}{r} = \frac{4}{5} = \frac{4}{5}$ is Positive.

$\sin(\theta) = \frac{y}{r} = \frac{-3}{5} = -\frac{3}{5}$ is Negative.

$\tan(\theta) = \frac{y}{x} = \frac{-3}{4} = -\frac{3}{4}$ is Negative.

These statements also follow from ASTC.



If θ is a Quadrant 4 (Q4) angle in $(-\frac{\pi}{2}, 0)$ then

Ref $\theta = -\theta$ and $\theta = -$ Ref θ.

The endpoint of the terminal side of θ is $P_4(4, -3)$

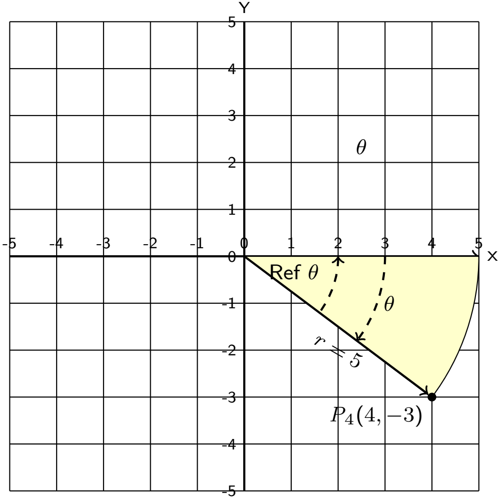
Thus $x = 4, y = -3, r = \sqrt{4^2 + (-3)^2} = 5$.

$\cos(\theta) = \frac{x}{r} = \frac{4}{5} = \frac{4}{5}$ is Positive.

$\sin(\theta) = \frac{y}{r} = \frac{-3}{5} = -\frac{3}{5}$ is Negative.

$\tan(\theta) = \frac{y}{x} = \frac{3}{-4} = -\frac{3}{4}$ is Positive.

These statements also follow from ASTC.

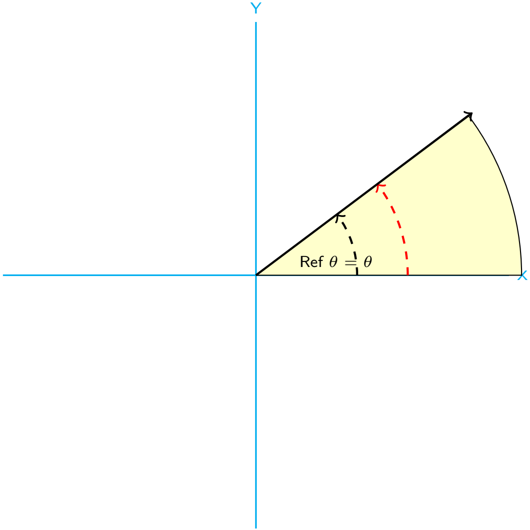


Summary: Finding the reference angle of a given angle.

Review of Reference angles

In the diagram at the right, angle θ is the red arc.

- Q1: If θ in $(0, 90^\circ) = (0, \frac{\pi}{2})$: Ref $\theta = \theta$.

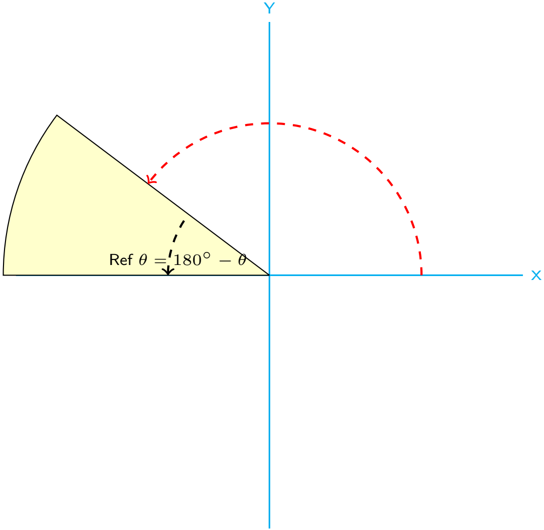


Summary: Finding the reference angle of a given angle.

Review of Reference angles

In the diagram at the right, angle θ is the red arc.

- Q1: If θ in $(0, 90^\circ) = (0, \frac{\pi}{2})$: Ref $\theta = \theta$.
- Q2: If θ in $(90^\circ, 180^\circ) = (\frac{\pi}{2}, \pi)$:
  Ref $\theta = 180^\circ - \theta = \pi - \theta$.

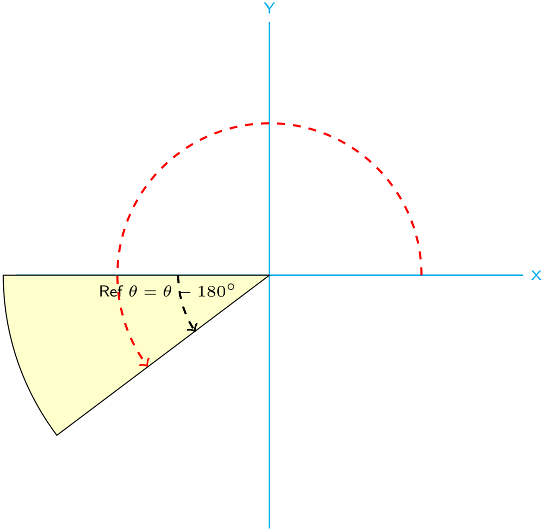


Summary: Finding the reference angle of a given angle.

Review of Reference angles

In the diagram at the right, angle θ is the red arc.

- Q1: If θ in $(0, 90^\circ) = (0, \frac{\pi}{2})$: Ref $\theta = \theta$.
- Q2: If θ in $(90^\circ, 180^\circ) = (\frac{\pi}{2}, \pi)$: Ref $\theta = 180^\circ - \theta = \pi - \theta$.
- Q3: If θ in $(180^\circ, 270^\circ) = (\pi, \frac{3\pi}{2})$: Ref $\theta = \theta - 180^\circ = \theta - \pi$.

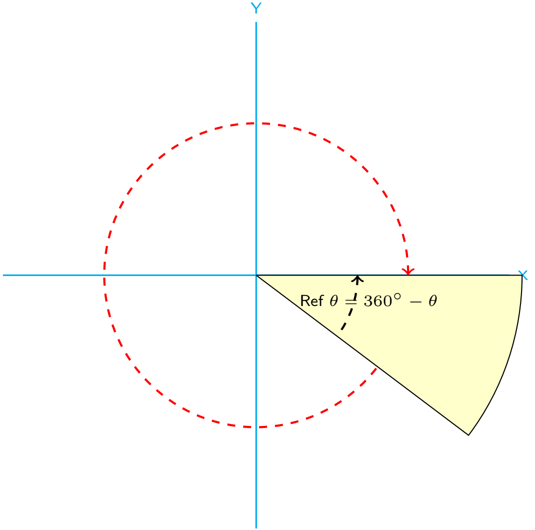


Summary: Finding the reference angle of a given angle.

Review of Reference angles

In the diagram at the right, angle θ is the red arc.

- Q1: If θ in $(0, 90^\circ) = (0, \frac{\pi}{2})$: Ref $\theta = \theta$.
- Q2: If θ in $(90^\circ, 180^\circ) = (\frac{\pi}{2}, \pi)$:
  Ref $\theta = 180^\circ - \theta = \pi - \theta$.
- Q3: If θ in $(180^\circ, 270^\circ) = (\pi, \frac{3\pi}{2})$:
  Ref $\theta = \theta - 180^\circ = \theta - \pi$.
- Q4: If θ in $(270^\circ, 360^\circ) = (\frac{3\pi}{2}, 2\pi)$:
  Ref $\theta = 360^\circ - \theta = 2\pi - \theta$.



Summary: Finding the reference angle of a given angle.

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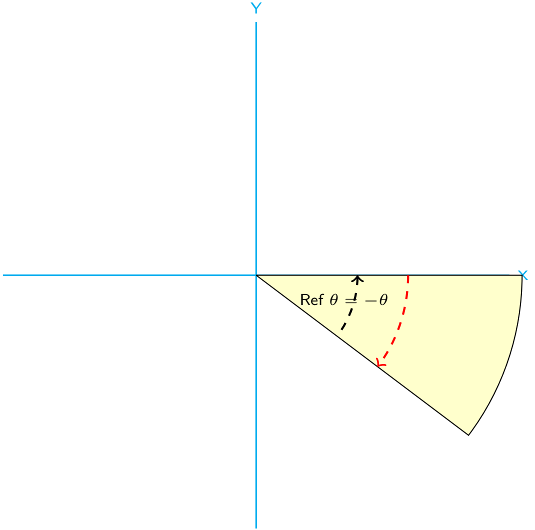
  Ref $\theta = \theta - 180^\circ = \theta - \pi$.
- Q4: If θ in $(270^\circ, 360^\circ) = (\frac{3\pi}{2}, 2\pi)$:
  Ref $\theta = 360^\circ - \theta = 2\pi - \theta$.
- Q4: If θ in $(-90^\circ, 0^\circ) = (-\frac{\pi}{2} 0)$:
  Ref $\theta = -\theta$.



Summary: Finding the reference angle of a given angle.

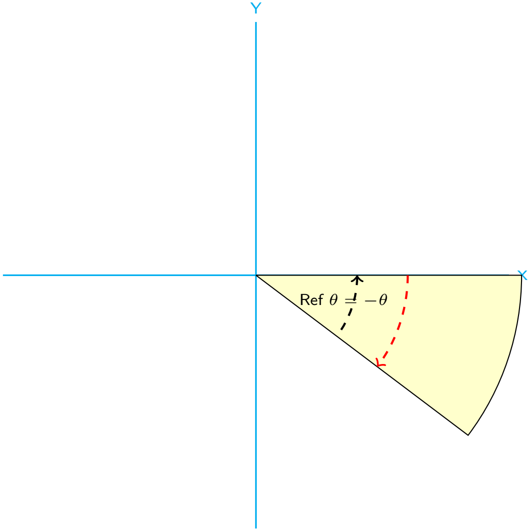
Review of Reference angles

In the diagram at the right, angle θ is the red arc.

- Q1: If θ in $(0, 90^\circ) = (0, \frac{\pi}{2})$: Ref $\theta = \theta$.
- Q2: If θ in $(90^\circ, 180^\circ) = (\frac{\pi}{2}, \pi)$:
  Ref $\theta = 180^\circ - \theta = \pi - \theta$.
- Q3: If θ in $(180^\circ, 270^\circ) = (\pi, \frac{3\pi}{2})$:
  Ref $\theta = \theta - 180^\circ = \theta - \pi$.
- Q4: If θ in $(270^\circ, 360^\circ) = (\frac{3\pi}{2}, 2\pi)$:
  Ref $\theta = 360^\circ - \theta = 2\pi - \theta$.
- Q4: If θ in $(-90^\circ, 0^\circ) = (-\frac{\pi}{2} 0)$:
  Ref $\theta = -\theta$.

Reference angles for other angles θ

- If $\theta < 0$ or $\theta > 2\pi = 360^\circ$, add or subtract a multiple of $2\pi = 360^\circ$ to/from θ to get a coterminal angle θ' with $0^\circ \leq \theta' < 2\pi = 360^\circ$. Then Ref θ = Ref θ'.
- If $\theta < 0$, Ref θ =Ref $(-\theta)$.

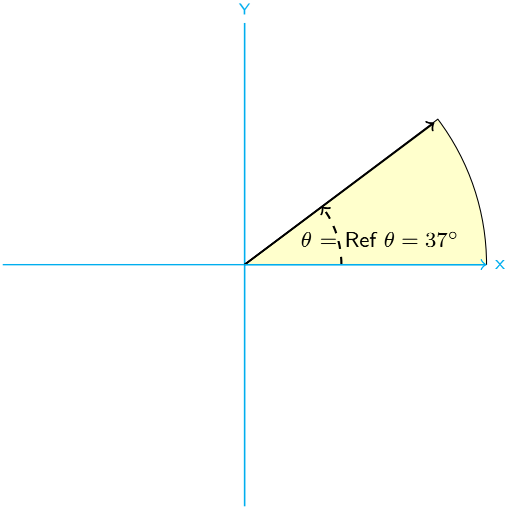


Finding θ if you know its reference angle

Now we work in reverse. If θ is an angle with endline in a given Quadrant, and we are given Ref $\theta = 37°$, what is the measure of angle θ?

If θ in $(0, \frac{\pi}{2})$ is a Q1 angle and Ref $\theta = 37°$

then $\theta =$Ref $\theta = 37°$.



Finding θ if you know its reference angle

Now we work in reverse. If θ is an angle with endline in a given Quadrant, and we are given Ref $\theta = 37°$, what is the measure of angle θ?

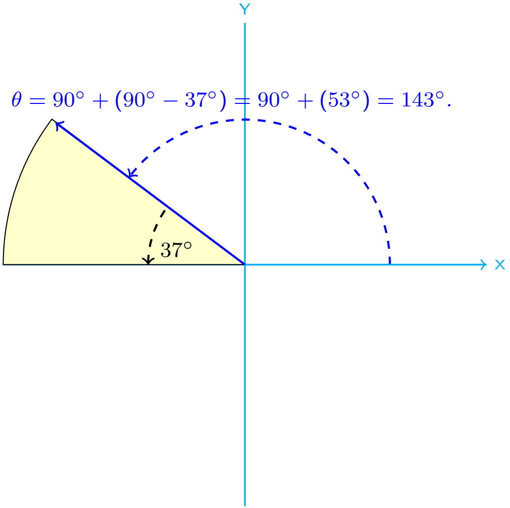
If θ in $(0, \frac{\pi}{2})$ is a Q1 angle and Ref $\theta = 37°$

then $\theta =$Ref $\theta = 37°$.

In the next 4 frames, study the graph to understand how to compute θ as a sum.

If θ in $(\frac{\pi}{2}, \pi)$ is a Q2 angle and Ref $\theta = 37°$

then $\theta = 180°-$Ref $\theta = 180° - 37° = 143°$.



Finding θ if you know its reference angle

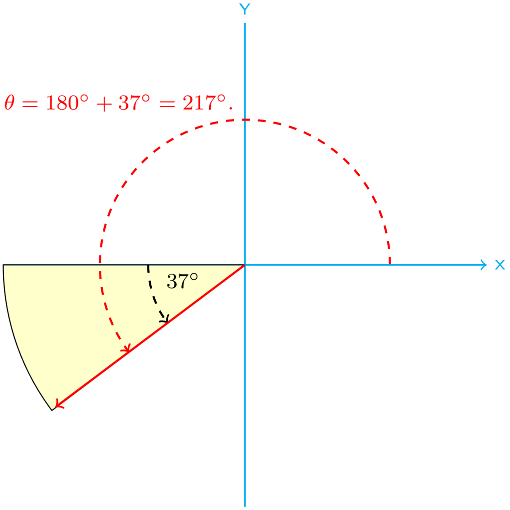
Now we work in reverse. If θ is an angle with endline in a given Quadrant, and we are given Ref $\theta = 37°$, what is the measure of angle θ?

If θ in $(0, \frac{\pi}{2})$ is a Q1 angle and Ref $\theta = 37°$

then $\theta =$Ref $\theta = 37°$.

If θ in$(\pi, \frac{3\pi}{2})$ a Q3 angle and Ref $\theta = 37°$

then $\theta = 180°+$ Ref $\theta = 180° + 37° = 217°$.



Finding θ if you know its reference angle

Now we work in reverse. If θ is an angle with endline in a given Quadrant, and we are given Ref $\theta = 37°$, what is the measure of angle θ?

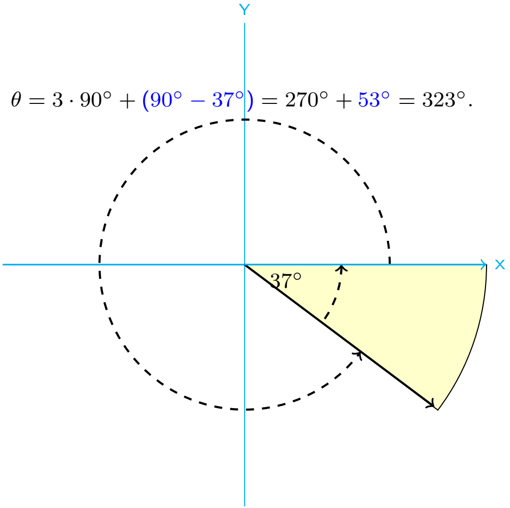
If θ in $(0, \frac{\pi}{2})$ is a Q1 angle and Ref $\theta = 37°$

then $\theta =$Ref $\theta = 37°$.

If θ in $(\frac{3\pi}{2}, 2\pi)$ is a Q4 angle and Ref $\theta = 37°$

then $\theta = 360° -$Ref $\theta = 360° - 37° = 323°$.

$\theta = 3 \cdot 90° + (90° - 37°) = 270° + 53° = 323°$.

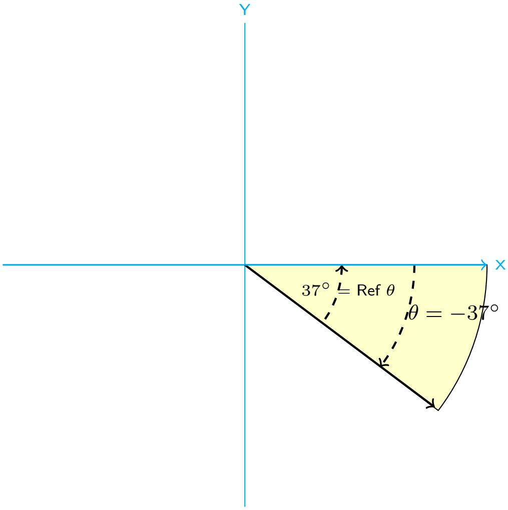


Finding θ if you know its reference angle

Now we work in reverse. If θ is an angle with endline in a given Quadrant, and we are given Ref $\theta = 37°$, what is the measure of angle θ?

If θ in $(0, \frac{\pi}{2})$ is a Q1 angle and Ref $\theta = 37°$

then $\theta =$Ref $\theta = 37°$.



If θ in $(-\frac{\pi}{2}, 0)$ is a Q4 angle and Ref $\theta = 37°$

then $\theta = -$Ref $\theta = -37°$.

Example 1: Find the cosines of :

- $150°$: Terminal line in Q2:
Ref $150° = 180° - 150° = 30°$. Since cosine is negative in Q2, $\cos(150°) = -\cos(30°) = -\frac{\sqrt{3}}{2}$
- $225°$:Terminal line in Q3:
Ref $225° = 225° - 180° = 45°$. Since cosine is negative in Q3, $\cos(225°) = -\cos(45°) = -\frac{1}{\sqrt{2}}$
- $\frac{5\pi}{3}$: Terminal line in Q4:
Ref $\frac{5\pi}{3} = 2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$.
Since cosine is positive in Q4,$\cos(\frac{5\pi}{3}) = \cos(\frac{\pi}{3}) = \frac{1}{2}$
- 7π : Ref $7\pi = 7\pi - 2\pi - 2\pi - 2\pi = \pi$. On the unit circle, the endpoint of angle π is $(x, y) = (-1, 0)$:
$\cos 7\pi = \cos \pi = x = -1$.

Example 2: Find the reference angle of $920°$.

Solution: $920° - 360° - 360° = 200°$. Since $180° < 200° < 270°$, angle $200°$ is in Q3.

Its reference angle is $200° - 180° = \boxed{20°}$.

Example 3: Find the reference angle of $\frac{19\pi}{3}$.

Solution Method 1: Convert to degrees:
$19(\frac{\pi}{3}) = 19(60°) = 1140°$.

Then $1140° - 360° - 360° - 360° = 60°$

is in Q1, and so its reference angle is $\boxed{60°}$.

Solution Method 2: Keep radians.

$\frac{19\pi}{3} - 2\pi - 2\pi - 2\pi = \frac{19\pi}{3} - 6\pi$

$= \frac{19\pi}{3} - \frac{18\pi}{3} = \frac{\pi}{3}$ is in Q1

and so its reference angle is $\boxed{\frac{\pi}{3}}$.

The following summary need not be memorized if you understand the previous slides. As usual, θ is in standard position, and the angle's terminal line is in the specified quadrant.

Trig functions of angles θ in terms of Ref θ.

- $\cos\theta = \cos(\text{Ref }\theta)$ for Q1 and Q4 angles θ.
- $\cos\theta = -\cos(\text{Ref }\theta)$ for Q2 and Q3 angles θ.
- $\sin\theta = \sin(\text{Ref }\theta)$ for Q1 and Q2.angles θ.
- $\sin\theta = -\sin(\text{Ref }\theta)$ for Q3 and Q4 angles θ.
- $\tan\theta = \tan(\text{Ref }\theta)$ for Q1 and Q3 angles θ.
- $\tan\theta = -\tan(\text{Ref }\theta)$ for Q2 and Q4 angles θ.

4.3.5 Trig functions generalize SohCahToa

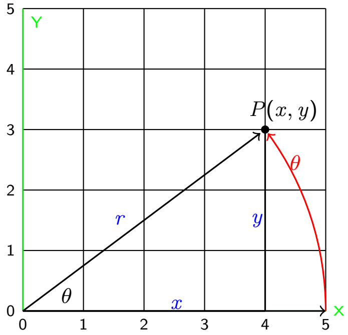
The diagram at the right shows an acute triangle in Quadrant 1.
The terminal side of angle θ goes from the origin to $P(x, y)$.
The distance from P to the origin is $r = \sqrt{x^2 + y^2}$.

The red arc shows the angle θ in standard position with terminal line OP.
For any angle θ, with endpoint (x, y) in any Quadrant, we defined trig functions as follows:

$$\cos(\theta) = \frac{x}{r} \qquad \sin(\theta) = \frac{y}{r} \qquad \tan(\theta) = \frac{y}{x}$$
$$\sec(\theta) = \frac{r}{x} \qquad \sec(\theta) = \frac{r}{y} \qquad \cot(\theta) = \frac{x}{y}$$

The above formulas agree with the SohCahToa definitions of trig functions of *acute angles* θ $(0 < \theta < \frac{\pi}{2})$ since, in the diagram:

- x is the length of the side *Adjacent* to angle θ.
- y is the length of the side *Opposite* angle θ.
- r is the length of the *Hypotenuse.*



Therefore, in Quadrant 1,

$$\cos(\theta) = \frac{x}{r} = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \sin(\theta) = \frac{y}{r} = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \tan(\theta) = \frac{y}{x} = \frac{\text{Opposite}}{\text{Adjacent}}$$

$$\sec(\theta) = \frac{r}{x} = \frac{\text{Hypotenuse}}{\text{Adjacent}} \qquad \sec(\theta) = \frac{r}{y} = \frac{\text{Hypotenuse}}{\text{Opposite}} \qquad \cot(\theta) = \frac{x}{y} = \frac{\text{Adjacent}}{\text{Opposite}}$$

4.3.6 Pythagorean trigonometric identities

On a radius 1 circle, $x^2 + y^2 = 1$,
$x = \cos\theta$, $y = \sin\theta$, $\frac{y}{x} = \tan\theta$
$\frac{1}{x} = \sec\theta$, $\frac{1}{y} = \csc\theta$, $\frac{x}{y} = \cot\theta$.

Therefore $\boxed{\cos^2\theta + \sin^2\theta = 1}$: Solve to obtain

$$\boxed{\cos(\theta) = \pm\sqrt{1 - \sin^2\theta} \text{ and } \sin(\theta) = \pm\sqrt{1 - \cos^2\theta}}$$

Divide $x^2 + y^2 = 1$ by x^2 : $1 + \left(\frac{y}{x}\right)^2 = \left(\frac{1}{x}\right)^2$:

Therefore $\boxed{1 + \tan^2\theta = \sec^2\theta}$: Solve to obtain

$$\boxed{\tan\theta = \pm\sqrt{\sec^2\theta - 1} \text{ and } \sec\theta = \pm\sqrt{1 + \tan^2\theta}}$$

Divide $x^2 + y^2 = 1$ by y^2 : $\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{1}{y}\right)^2$:

Therefore $\boxed{\cot^2\theta + 1 = \csc^2\theta}$: Solve to obtain

$$\boxed{\cot(\theta) = \pm\sqrt{\csc^2\theta - 1} \text{ and } \csc(\theta) = \pm\sqrt{1 + \cot^2\theta}}$$

In all cases, the choice of plus or minus depends on the quadrant of angle θ, most easily remembered using ASTC.

Example 4: If θ is a Quadrant 3 angle, express $\tan(\theta)$ in terms of $\sin(\theta)$.

Solution:

- On the unit circle $x^2 + y^2 = 1$, let the endpoint of angle θ be $(x, y) = (\cos\theta, \sin\theta)$. To find $\tan\theta = \frac{y}{x}$ in terms of $\sin\theta = y$, we must express x in terms of y.
- Solve $x^2 + y^2 = 1$ for y to get $x = \pm\sqrt{1 - y^2}$ Point (x, y) is in Q3, so x is negative. Since $\sqrt{()}$is positive, $x = -\sqrt{1 - y^2}$
- $\tan\theta = \dfrac{y}{x} = \dfrac{y}{-\sqrt{1 - y^2}} = -\dfrac{y}{\sqrt{1 - y^2}}$

Answer: Since $y = \sin\theta$, $\boxed{\tan\theta = -\dfrac{\sin\theta}{\sqrt{1 - \sin^2\theta}}}$

4.3.7 Evaluating trigonometric functions

Example 5: If $\sec(\theta) = -\frac{5}{3}$ and θ is in Quadrant 2, find the other 5 trig functions of θ.

Solution: If the terminal line of θ goes from the origin to $P(x, y)$, the distance from P to the origin is $r = \sqrt{x^2 + y^2}$

Then $\sec(\theta) = \frac{r}{x} = -\frac{5}{3}$.

Choose the simplest r and x that satisfy this equation, but remember that r is positive. The easiest solution is $r = 5$ and $x = -3$.
Substitute these values in

$x^2 + y^2 = r^2$. to get

$(-3)^2 + y^2 = 5^2$

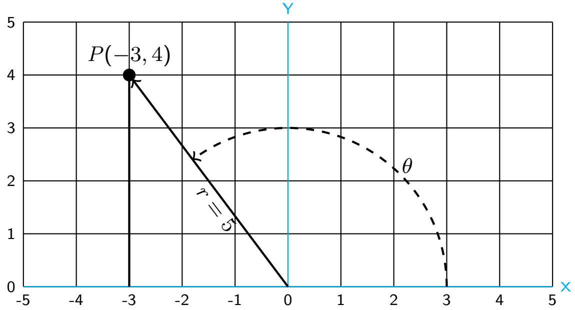
$9 + y^2 = 25$ and so $y = \pm 4$.

Since $P(x, y)$ is in quadrant 2, where y is positive, $y = 4$.
Therefore $x = -3, y = 4, r = 5$.

Note that $\sec(\theta) = \frac{r}{x} = \frac{5}{-3} = -\frac{5}{3}$ checks with the given information.

The other five trig functions are

$$\cos(\theta) = \frac{x}{r} = \frac{-3}{5} = -\frac{3}{5}$$
$$\sin(\theta) = \frac{y}{r} = \frac{4}{5}$$
$$\tan(\theta) = \frac{y}{x} = \frac{4}{-3} = -\frac{4}{3}$$
$$\cot(\theta) = \frac{x}{y} = \frac{-3}{4} = -\frac{3}{4}$$
$$\csc(\theta) = \frac{r}{y} = \frac{5}{4}$$



4.3.8 How your calculator computes cosines

If θ is the radian measure of an angle, it is shown in Calculus III that an approximate formula for cosine is

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!}$$

You know that $\cos(\pi/3) = \frac{1}{2}$.

Example 6: Test the accuracy of the above formula with $\theta = \frac{\pi}{3} \approx 1.047$.

Solution: Compute by hand with 3 decimal place accuracy, or (if you must) use a calculator to check that

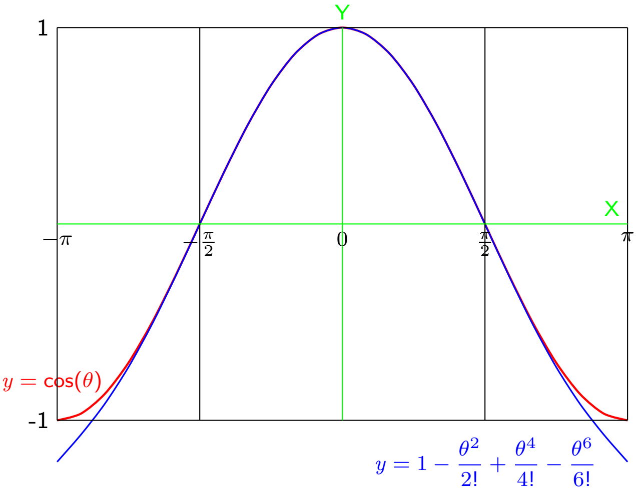
$$1 - \frac{1.047^2}{2} + \frac{1.047^4}{24} - \frac{1.047^6}{720} \approx 0.500$$

The functions

$y = \cos(\theta)$ and $y = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!}$

are graphed at the right. Note that the graphs look identical for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

How close to identical, really?



The difference of the functions $y = \cos(\theta)$ and

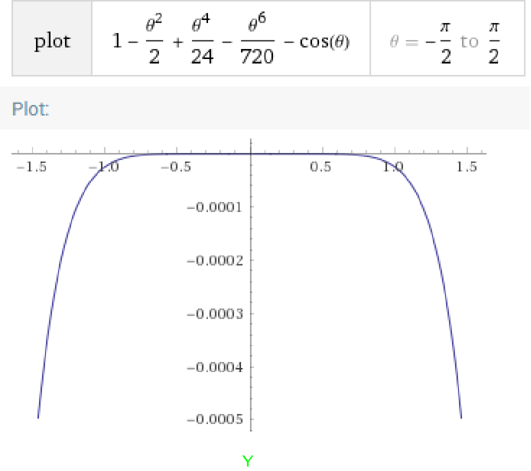
$$y = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \quad \text{for } -\pi/2 < \theta < \pi/2$$

is graphed at the right, using

▸ Wolfram Calculator .

Look at the scale on the y-axis!

When $\theta = \pi/3$, the difference is ≈ 0.000035.
The above simple formula gives the value of
$\cos\frac{\pi}{3} = 0.5$ correct to better than 99.99 per cent accuracy!



Section 4.3 Quiz

Example 4.3.1: Find the cosines of the following angles: $150°, 225°, 300°$

Example 4.3.2: Find the reference angle of $920°$.

Example 4.3.3: Find the reference angle of $\frac{19\pi}{3}$.

Example 4.3.4: If θ is a Quadrant 3 angle, express $\tan(\theta)$ in terms of $\sin(\theta)$.

Example 4.3.5: If $\sec(\theta) = -\frac{5}{3}$ and θ is in Quadrant 2, find the other 5 trig functions of θ.

Section 4.3 Review: Trig functions of general angles

Example 4.3.1: Find

- $\cos 150^\circ =$
- $\cos 225^\circ =$
- $\cos 300^\circ =$
- $\sin 150^\circ =$
- $\sin 225^\circ =$
- $\sin 300^\circ =$
- $\sec 150^\circ =$
- $\tan 225^\circ =$
- $\tan 300^\circ =$
- $\sec 0^\circ =$
- $\tan 225^\circ =$
- $\tan 330^\circ =$

Section 4.3 Review: Trig functions of general angles

Example 4.3.1: Find

- $\cos 150° = -\frac{\sqrt{3}}{2}$
- $\cos 225° = -\frac{1}{\sqrt{2}}$
- $\cos 300° = \frac{1}{2}-$
- $\sin 150° = \frac{1}{2}$
- $\sin 225° = \frac{1}{\sqrt{2}}$
- $\sin 300° = -\frac{\sqrt{3}}{2}$
- $\sec 150° = -\frac{2}{\sqrt{3}}$
- $\tan 225° = 1$
- $\tan 300° = -\sqrt{3}$
- $\sec 0° = 1$
- $\tan 225° = 1$
- $\tan 330° = -\frac{1}{\sqrt{3}}$

Section 4.3 Review: Trig functions of general angles

Example 4.3.1: Find

- $\cos 150^\circ = -\frac{\sqrt{3}}{2}$
- $\cos 225^\circ = -\frac{1}{\sqrt{2}}$
- $\cos 300^\circ = \frac{1}{2} -$
- $\sin 150^\circ = \frac{1}{2}$
- $\sin 225^\circ = \frac{1}{\sqrt{2}}$
- $\sin 300^\circ = -\frac{\sqrt{3}}{2}$
- $\sec 150^\circ = -\frac{2}{\sqrt{3}}$
- $\tan 225^\circ = 1$
- $\tan 300^\circ = -\sqrt{3}$
- $\sec 0^\circ = 1$
- $\tan 225^\circ = 1$
- $\tan 330^\circ = -\frac{1}{\sqrt{3}}$

Example 4.3.2: Find the reference angle of

- $920^\circ \Rightarrow$
- $-200^\circ \Rightarrow$
- $-140^\circ \Rightarrow$
- $-45^\circ \Rightarrow$

Section 4.3 Review: Trig functions of general angles

Example 4.3.1: Find

- $\cos 150^\circ = -\frac{\sqrt{3}}{2}$
- $\cos 225^\circ = -\frac{1}{\sqrt{2}}$
- $\cos 300^\circ = \frac{1}{2}-$
- $\sin 150^\circ = \frac{1}{2}$
- $\sin 225^\circ = \frac{1}{\sqrt{2}}$
- $\sin 300^\circ = -\frac{\sqrt{3}}{2}$
- $\sec 150^\circ = -\frac{2}{\sqrt{3}}$
- $\tan 225^\circ = 1$
- $\tan 300^\circ = -\sqrt{3}$
- $\sec 0^\circ = 1$
- $\tan 225^\circ = 1$
- $\tan 330^\circ = -\frac{1}{\sqrt{3}}$

Example 4.3.2: Find the reference angle of

- $920^\circ \Rightarrow 20^\circ$
- $-200^\circ \Rightarrow 20^\circ$
- $-140^\circ \Rightarrow 40^\circ$
- $-45^\circ \Rightarrow 45^\circ$

Section 4.3 Review: Trig functions of general angles

Example 4.3.1: Find

- $\cos 150° = -\frac{\sqrt{3}}{2}$
- $\cos 225° = -\frac{1}{\sqrt{2}}$
- $\cos 300° = \frac{1}{2} -$
- $\sin 150° = \frac{1}{2}$
- $\sin 225° = \frac{1}{\sqrt{2}}$
- $\sin 300° = -\frac{\sqrt{3}}{2}$
- $\sec 150° = -\frac{2}{\sqrt{3}}$
- $\tan 225° = 1$
- $\tan 300° = -\sqrt{3}$
- $\sec 0° = 1$
- $\tan 225° = 1$
- $\tan 330° = -\frac{1}{\sqrt{3}}$

Example 4.3.2: Find the reference angle of

- $920° \Rightarrow 20°$
- $-200° \Rightarrow 20°$
- $-140° \Rightarrow 40°$
- $-45° \Rightarrow 45°$

Example 4.3.3: Find the reference angle of

- $\frac{19\pi}{3} \Rightarrow$
- $\frac{3\pi}{2} \Rightarrow$
- $\frac{-5\pi}{6} \Rightarrow$
- $\frac{19\pi}{12} \Rightarrow$

Section 4.3 Review: Trig functions of general angles

Example 4.3.1: Find

- $\cos 150^\circ = -\frac{\sqrt{3}}{2}$
- $\cos 225^\circ = -\frac{1}{\sqrt{2}}$
- $\cos 300^\circ = \frac{1}{2} -$
- $\sin 150^\circ = \frac{1}{2}$
- $\sin 225^\circ = \frac{1}{\sqrt{2}}$
- $\sin 300^\circ = -\frac{\sqrt{3}}{2}$
- $\sec 150^\circ = -\frac{2}{\sqrt{3}}$
- $\tan 225^\circ = 1$
- $\tan 300^\circ = -\sqrt{3}$
- $\sec 0^\circ = 1$
- $\tan 225^\circ = 1$
- $\tan 330^\circ = -\frac{1}{\sqrt{3}}$

Example 4.3.2: Find the reference angle of

- $920^\circ \Rightarrow 20^\circ$
- $-200^\circ \Rightarrow 20^\circ$
- $-140^\circ \Rightarrow 40^\circ$
- $-45^\circ \Rightarrow 45^\circ$

Example 4.3.3: Find the reference angle of

- $\frac{19\pi}{3} \Rightarrow \frac{\pi}{3}$
- $\frac{3\pi}{2} \Rightarrow$ not defined
- $\frac{-5\pi}{6} \Rightarrow \frac{\pi}{6}$
- $\frac{19\pi}{12} \Rightarrow \frac{5\pi}{12}$

Section 4.3 Review: Trig functions of general angles

Example 4.3.1: Find

- $\cos 150^\circ = -\frac{\sqrt{3}}{2}$
- $\cos 225^\circ = -\frac{1}{\sqrt{2}}$
- $\cos 300^\circ = \frac{1}{2} -$
- $\sin 150^\circ = \frac{1}{2}$
- $\sin 225^\circ = \frac{1}{\sqrt{2}}$
- $\sin 300^\circ = -\frac{\sqrt{3}}{2}$
- $\sec 150^\circ = -\frac{2}{\sqrt{3}}$
- $\tan 225^\circ = 1$
- $\tan 300^\circ = -\sqrt{3}$
- $\sec 0^\circ = 1$
- $\tan 225^\circ = 1$
- $\tan 330^\circ = -\frac{1}{\sqrt{3}}$

Example 4.3.2: Find the reference angle of

- $920^\circ \Rightarrow 20^\circ$
- $-200^\circ \Rightarrow 20^\circ$
- $-140^\circ \Rightarrow 40^\circ$
- $-45^\circ \Rightarrow 45^\circ$

Example 4.3.3: Find the reference angle of

- $\frac{19\pi}{3} \Rightarrow \frac{\pi}{3}$
- $\frac{3\pi}{2} \Rightarrow$ not defined
- $\frac{-5\pi}{6} \Rightarrow \frac{\pi}{6}$
- $\frac{19\pi}{12} \Rightarrow \frac{5\pi}{12}$

Example 4.3.4: If θ is a

- Q3 angle, express $\tan\theta$ in terms of $\sin(\theta) \Rightarrow$
- Q2 angle, express $\sec\theta$ in terms of $\sin(\theta) \Rightarrow$
- Q4 angle, express $\tan\theta$ in terms of $\cos(\theta) \Rightarrow$
- Q2 angle, express $\cos\theta$ in terms of $\tan(\theta) \Rightarrow$

Section 4.3 Review: Trig functions of general angles

Example 4.3.1: Find

- $\cos 150° = -\frac{\sqrt{3}}{2}$
- $\cos 225° = -\frac{1}{\sqrt{2}}$
- $\cos 300° = \frac{1}{2}-$
- $\sin 150° = \frac{1}{2}$
- $\sin 225° = \frac{1}{\sqrt{2}}$
- $\sin 300° = -\frac{\sqrt{3}}{2}$
- $\sec 150° = -\frac{2}{\sqrt{3}}$
- $\tan 225° = 1$
- $\tan 300° = -\sqrt{3}$
- $\sec 0° = 1$
- $\tan 225° = 1$
- $\tan 330° = -\frac{1}{\sqrt{3}}$

Example 4.3.2: Find the reference angle of

- $920° \Rightarrow 20°$
- $-200° \Rightarrow 20°$
- $-140° \Rightarrow 40°$
- $-45° \Rightarrow 45°$

Example 4.3.3: Find the reference angle of

- $\frac{19\pi}{3} \Rightarrow \frac{\pi}{3}$
- $\frac{3\pi}{2} \Rightarrow$ not defined
- $\frac{-5\pi}{6} \Rightarrow \frac{\pi}{6}$
- $\frac{19\pi}{12} \Rightarrow \frac{5\pi}{12}$

Example 4.3.4: If θ is a

- Q3 angle, express $\tan\theta$ in terms of $\sin(\theta) \Rightarrow \tan\theta = -\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$
- Q2 angle, express $\sec\theta$ in terms of $\sin(\theta) \Rightarrow \sec\theta = -\frac{1}{\sqrt{1-\sin^2\theta}}$
- Q4 angle, express $\tan\theta$ in terms of $\cos(\theta) \Rightarrow \tan\theta = -\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$
- Q2 angle, express $\cos\theta$ in terms of $\tan(\theta) \Rightarrow \cos\theta = -\frac{1}{\sqrt{1+\tan^2\theta}}$

Example 4.3.5: find the other 5 trig functions of θ if

- $\sec(\theta) = -\frac{5}{3}$ and θ is in Q2; $\Rightarrow$
- $\cos(\theta) = -\frac{1}{3}$ and θ is in Q2; $\Rightarrow$
- $\tan(\theta) = \frac{5}{7}$ and θ is in Q3; $\Rightarrow$
- $\sin(\theta) = -\frac{5}{9}$ and θ is in Q3; $\Rightarrow$

Section 4.3 Review

Example 4.3.5: find the other 5 trig functions of θ if

- $\sec(\theta) = -\frac{5}{3}$ and θ is in Q2; $\Rightarrow \cos\theta = -\frac{3}{5}$; $\sin\theta = \frac{4}{5}$; $\tan\theta = -\frac{4}{3}$; $\csc\theta = \frac{5}{4}$; $\cot\theta = -\frac{3}{4}$
- $\cos(\theta) = -\frac{1}{3}$ and θ is in Q2; $\Rightarrow$ $\sin\theta = \frac{2\sqrt{2}}{3}$; $\tan\theta = -2\sqrt{2}$; $\sec\theta = -3$; $\csc\theta = \frac{3}{2\sqrt{2}}$; $\cot\theta = -\frac{1}{2\sqrt{2}}$
- $\tan(\theta) = \frac{5}{7}$ and θ is in Q3; $\Rightarrow \cos\theta = -\frac{7}{\sqrt{74}}$; $\sin\theta = -\frac{5}{\sqrt{74}}$; $\sec\theta = -\frac{\sqrt{74}}{7}$; $\csc\theta = -\frac{\sqrt{74}}{5}$; $\cot\theta = \frac{7}{5}$
- $\sin(\theta) = -\frac{5}{9}$ and θ is in Q3; $\Rightarrow \cos\theta = -\frac{2\sqrt{14}}{9}$; $\tan\theta = \frac{5}{2\sqrt{14}}$; $\sec\theta = -\frac{9}{2\sqrt{14}}$; $\csc\theta = -\frac{9}{5}$; $\cot\theta = \frac{2\sqrt{14}}{5}$

Chapter 4 Section 4: Trigonometric functions and graphs

- 4.4.1: The sine of circular angles
- 4.4.2: The cosine of circular angles
- 4.4.3: Four basic sine and cosine graphs
- 4.4.4: Symmetry and periodicity
- 4.4.5: Graphing $A\sin Bx$ and $A\cos Bx$
- 4.4.6: Graphing the standard wave
- 4.4.7: Graphing $y = A\sin(Bx + C)$
- 4.4.8: Graphing $y = \tan x$
- 4.4.9: Section 4.4 Review and Quiz

Section 4.4 Preview: Definitions and Procedures

- Definition 4.4.1: The cosine graph is y-axis symmetric.
- Definition 4.4.2: The sine graph is origin symmetric.
- Definition 4.4.3: Sine and Cosine graphs are periodic with period 2π.
- Definition 4.4.4: The effect on the graph when you modify equation $y = f(x)$ is
- Definition 4.4.5: The standard wave of $y = \sin x$ or $y = \cos x$ is
- Definition 4.4.6: To graph one wave of $y = f(x) = A\cos Bx$ or $y = g(x) = A\sin Bx$ by relabeling axes
- Definition 4.4.7: If $0 < C < 2\pi$, the standard wave of $y = A\sin(Bx + C)$ or $y = A\cos(Bx + C)$ is
- Definition 4.4.8: Assume C arbitrary. The standard wave of $y = A\sin(Bx + C)$ or $y = A\cos(Bx + C)$ is
- Definition 4.4.9: $\arctan x = \theta$ provided $\tan\theta = x$ and $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

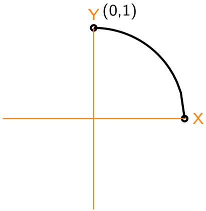
- Procedure 4.4.1: To graph the standard wave of $y = A\cos(Bx + C)$ or $y = A\sin(Bx + C)$ on a 4 x 2 grid:

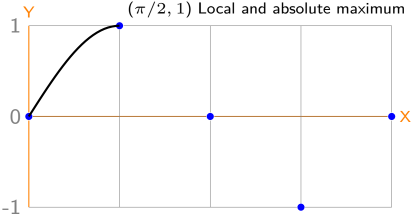
4.4.1 Graphing $\sin\theta$ as a function of circular angle θ

If the endline of angle θ joins the origin to point (x, y) on the circle $x^2 + y^2 = 1$, then $\cos\theta = x$ and $\sin\theta = y$. To start drawing the graph of $y = \sin\theta$ for $0 \le \theta \le 2\pi$, move around the circle $x^2 + y^2 = 1$ as follows.

Start at point $(1, 0)$ (corresponding to $\theta = 0$), move counterclockwise around the circle until you return to your starting point. The following landmarks occur each quarter-circle turn:

As θ goes from:	On the circle $x^2 + y^2 = 1$ point $(x, y) = (\cos\theta, \sin\theta)$ goes from	On the graph of $y = \sin\theta$ $(\theta, \sin\theta)$ goes from
0 to $\pi/2$	$(1, 0)$ to $(0, 1)$	$(0, 0)$ to $(\pi/2, 1)$





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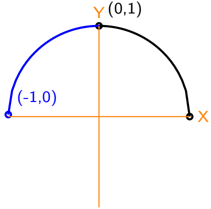
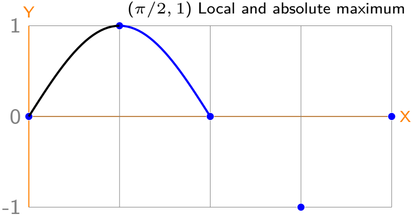
As θ goes from:	On the circle $x^2 + y^2 = 1$ point $(x, y) = (\cos\theta, \sin\theta)$ goes from	On the graph of $y = \sin\theta$ $(\theta, \sin\theta)$ goes from
0 to $\pi/2$	$(1, 0)$ to $(0, 1)$	$(0, 0)$ to $(\pi/2, 1)$
$\pi/2$ to π	$(0, 1)$ to $(-1, 0)$	$(\pi/2, 1)$ to $(\pi, 0)$





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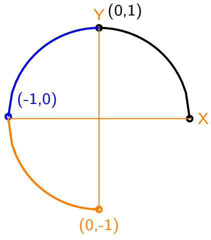
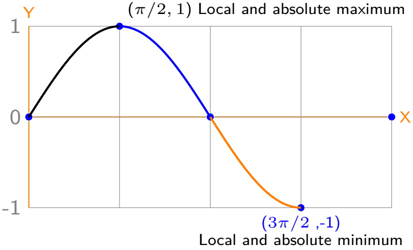
As θ goes from:	On the circle $x^2 + y^2 = 1$ point $(x, y) = (\cos\theta, \sin\theta)$ goes from	On the graph of $y = \sin\theta$ $(\theta, \sin\theta)$ goes from
0 to $\pi/2$	$(1, 0)$ to $(0, 1)$	$(0, 0)$ to $(\pi/2, 1)$
$\pi/2$ to π	$(0, 1)$ to $(-1, 0)$	$(\pi/2, 1)$ to $(\pi, 0)$
π to $3\pi/2$	$(-1, 0)$ to $(0, -1)$	$(\pi, 0)$ to $(3\pi/2, -1)$





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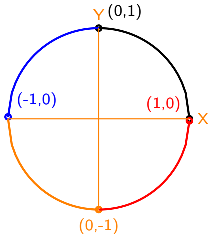
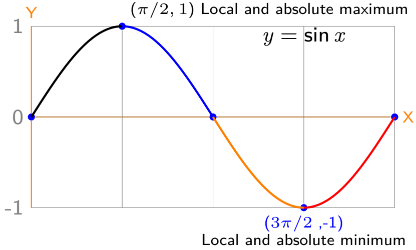
$\pi/2$ to π	$(0, 1)$ to $(-1, 0)$	$(\pi/2, 1)$ to $(\pi, 0)$
π to $3\pi/2$	$(-1, 0)$ to $(0, -1)$	$(\pi, 0)$ to $(3\pi/2, -1)$
$3\pi/2$ to 2π	$(0, -1)$ to $(1, 0)$	$(3\pi/2, -1)$ to $(2\pi, 0)$





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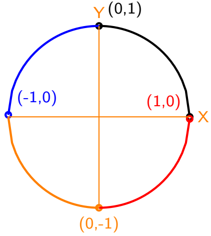

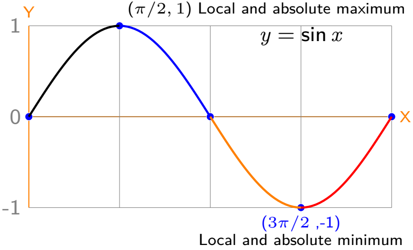

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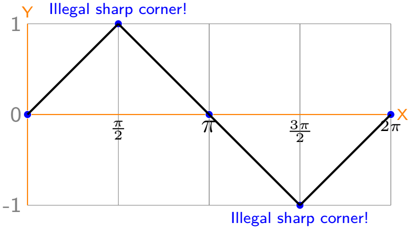

4.4.2 Graphing $\cos\theta$ as a function of circular angle θ

If the endline of angle θ joins the origin to point (x, y) on the circle $x^2 + y^2 = 1$, then $\cos\theta = x$ and $\sin\theta = y$.
To start drawing the graph of $y = \sin\theta$ for $0 \le \theta \le 2\pi$,travel around the circle $x^2 + y^2 = 1$ as follows.

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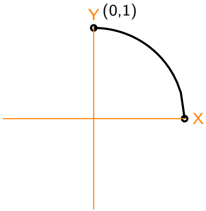

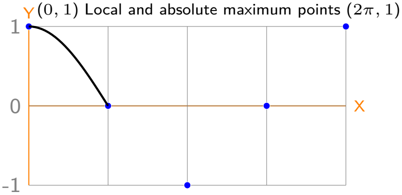

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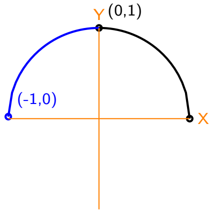

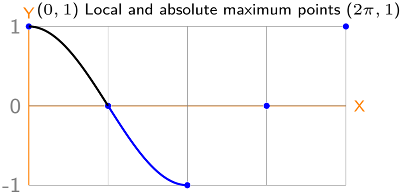

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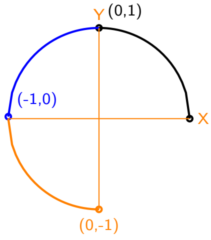

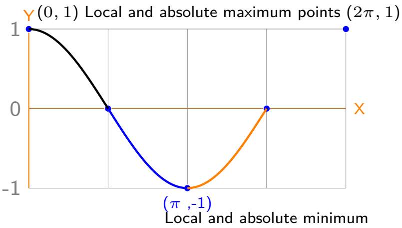

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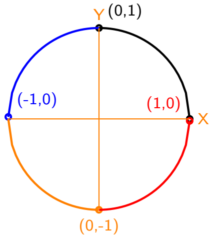

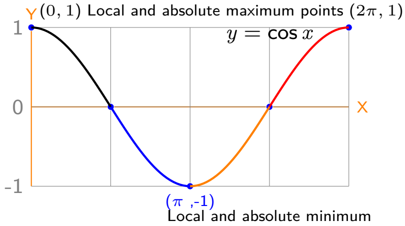

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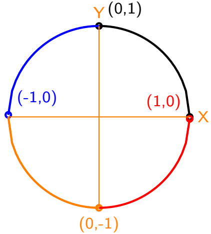

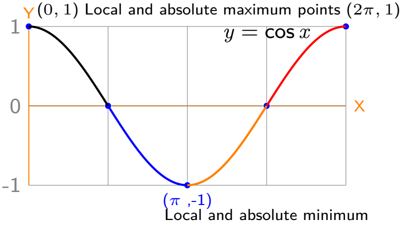

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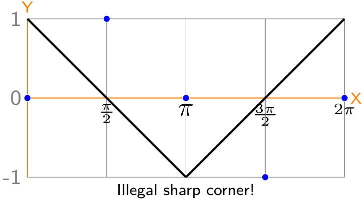
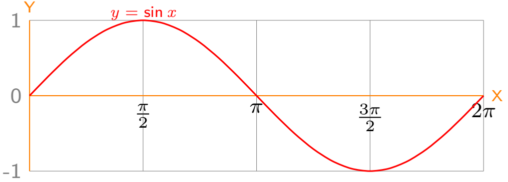
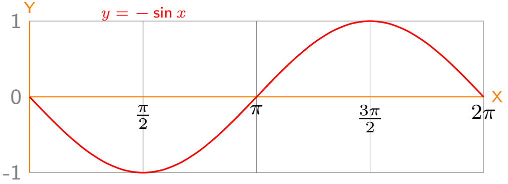


4.4.3 Four basic sine and cosine graphs

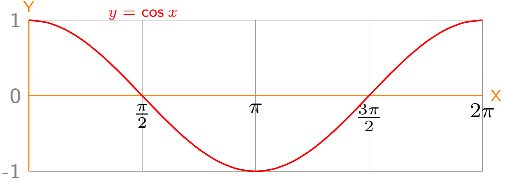
Example 1: Graph $y = \sin x$ and $y = -\sin x$.
Solution: One wave of $y = \sin x$:



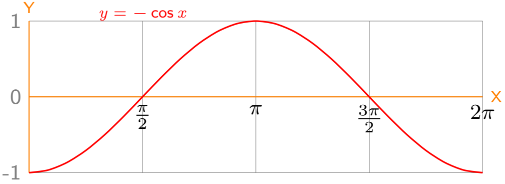
Reflect it through the x-axis to get the following graph of $y = -\sin x$.



Example 2: Graph $y = \cos x$ and $y = -\cos x$.
Solution: One wave of $y = \cos x$:



Reflect it through the x-axis to get the following graph of $y = -\cos x$.



4.4.4 Cosine and sine graphs are symmetric and periodic

Cosine graph is y-axis symmetric: $\cos(-\theta) = \cos(\theta)$

Reason: If the endpoint of angle θ is $(x, y) = (\cos\theta, \sin\theta)$, then the endpoint of angle $-\theta$ is $(x, -y)$ Thus $\cos(\theta) = \cos(-\theta) = x$

Sine graph is origin symmetric: $\sin(-\theta) = -\sin(\theta)$

Reason: If the endpoint of angle θ is $(x, y) = (\cos\theta, \sin\theta)$, then the endpoint of angle $-\theta$ is $(x, -y)$. Thus $\sin(-\theta) = -\sin(\theta) = -y$

Sine and Cosine graphs are periodic with period 2π: $\sin(\theta + 2\pi) = \sin\theta$ and $\cos(\theta + 2\pi) = \sin(\theta)$

Reason: θ and $\theta + 2\pi$ are coterminal angles: if angle θ has terminal point $(x, y) = (\cos\theta, \sin\theta)$, adding 2π yields angle $\theta + 2\pi$ with the same terminal ray and endpoint $(x, y) = (\cos\theta + 2\pi, \sin\theta + 2\pi)$. Thus

- $\cos(\theta) = \cos(\theta + 2\pi) = x$
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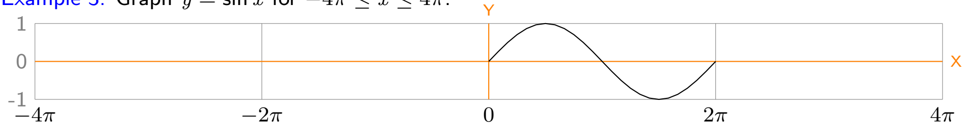
Example 3: Graph $y = \sin x$ for $-4\pi \leq x \leq 4\pi$.



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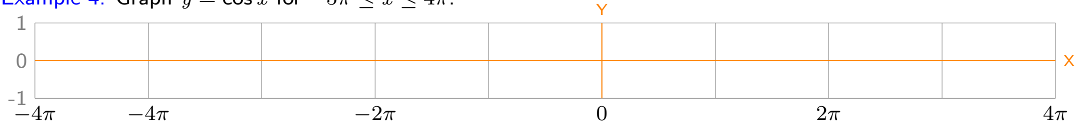

4.4.4 Cosine and sine graphs are symmetric and periodic

Cosine graph is y-axis symmetric: $\cos(-\theta) = \cos(\theta)$

Reason: If the endpoint of angle θ is $(x, y) = (\cos\theta, \sin\theta)$, then the endpoint of angle $-\theta$ is $(x, -y)$ Thus $\cos(\theta) = \cos(-\theta) = x$

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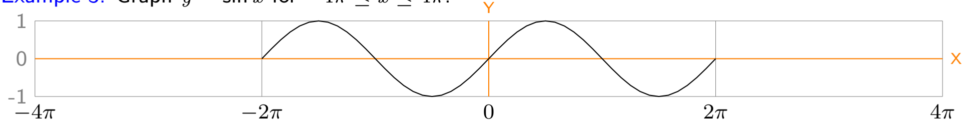

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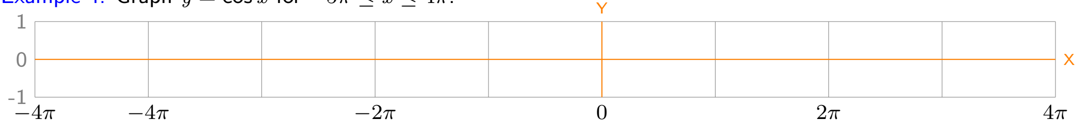

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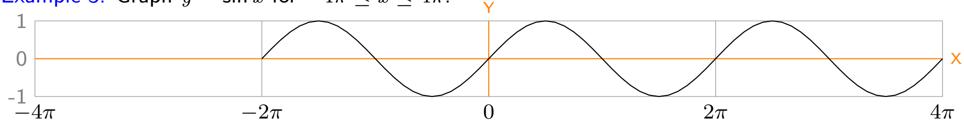

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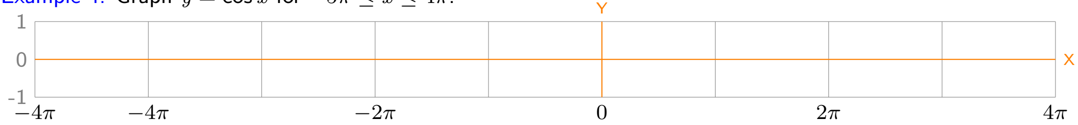

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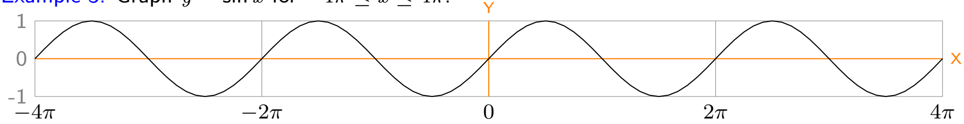

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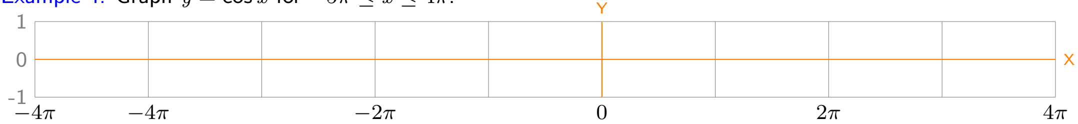

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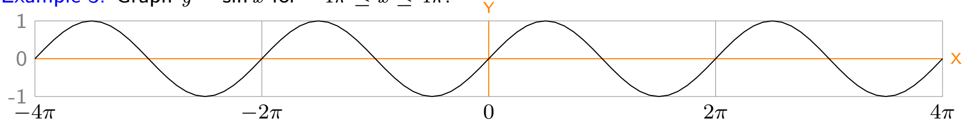

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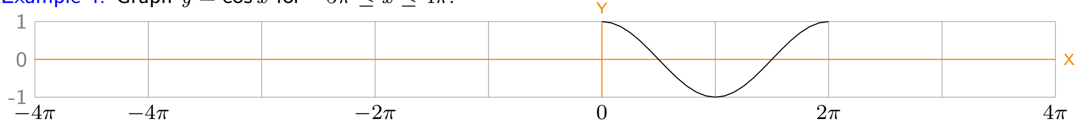

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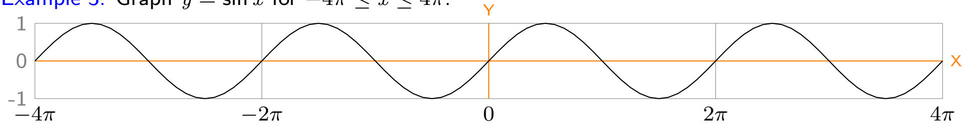

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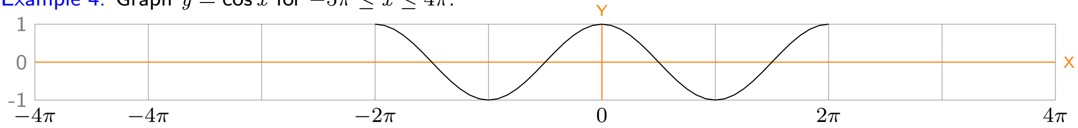

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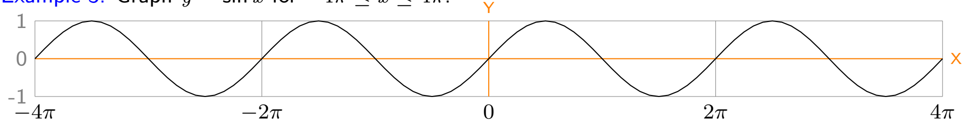

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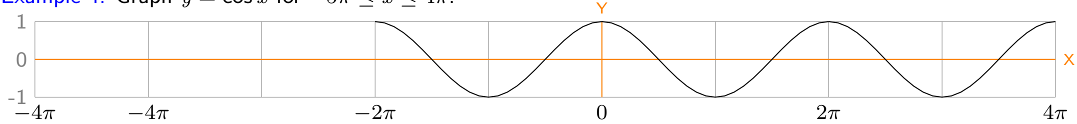

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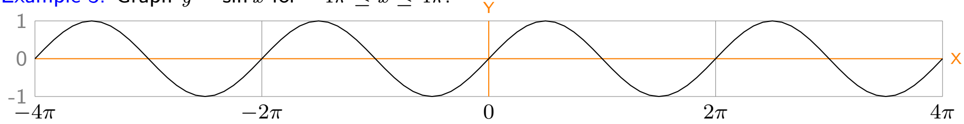

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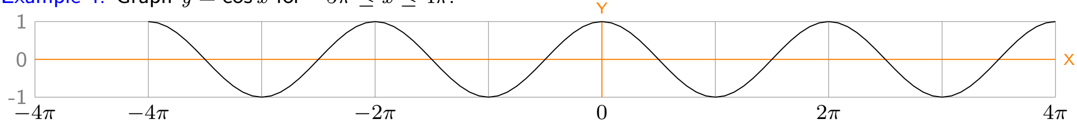

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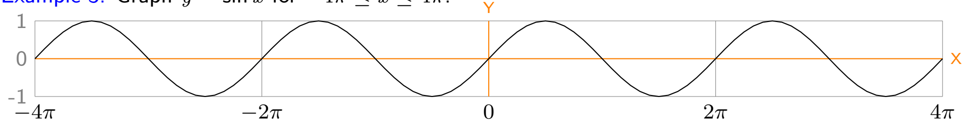

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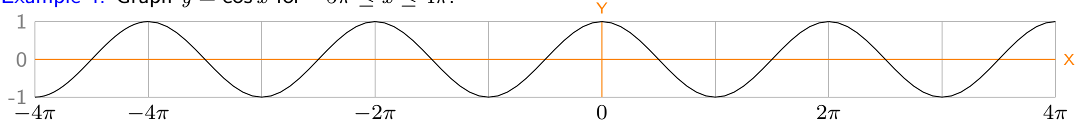

4.4.5: Graphing $y = A\sin Bx$ and $y = A\cos Bx$

Chapter 2 demonstrated the

Effect on the graph when you modify equation $y = f(x)$

Action	Change in $y = f(x)$	Point (x, y)
x-shift right 2	$\Rightarrow y = f(x-2)$	$\Rightarrow (x+2, y)$
x-shift left 2	$\Rightarrow y = f(x+2)$	$\Rightarrow (x-2, y)$
y-shift up 2	$\Rightarrow y = f(x)+2$	$\Rightarrow (x, y+2)$
y-shift down 2	$\Rightarrow y = f(x)-2$	$\Rightarrow (x, y-2)$
x-stretch 2	$\Rightarrow y = f(x/2)$	$\Rightarrow (2x, y)$
x-shrink 2	$\Rightarrow y = f(2x)$	$\Rightarrow (x/2, y)$
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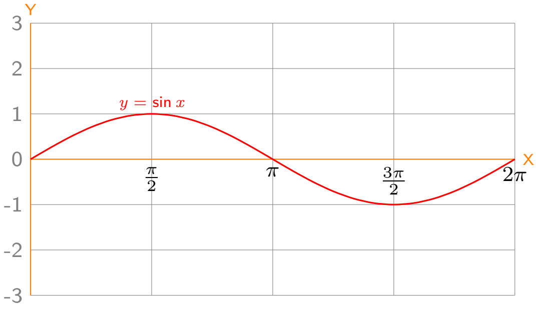
The standard wave of $y = \sin x$ or $y = \cos x$ is

the graph of the function's restriction to domain $0 \le x \le 2\pi$.

Example 5: Transform the standard wave of $y = \sin x$ to the graph of $y = 3\sin x$.

Solution: Start with $y = f(x) = \sin x$. According to the table, y-stretch 3 to get the graph of $y = 3\sin x$.



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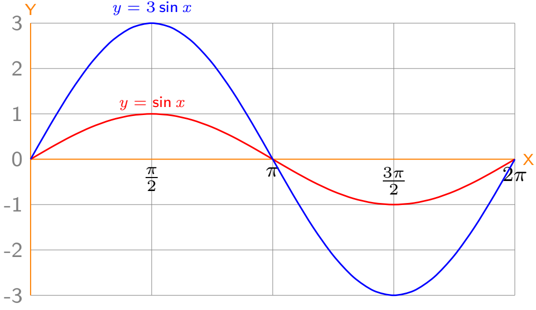

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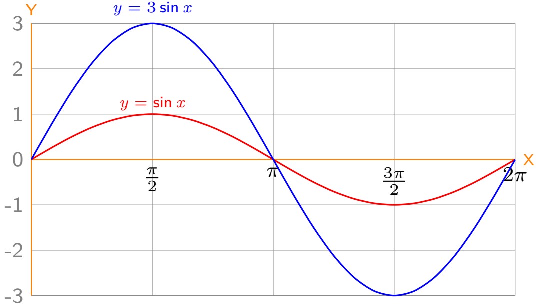

Example 6: Transform the graph of $y = \sin x; 0 \le x \le 2\pi$ to the graph of $y = \sin 2x$.

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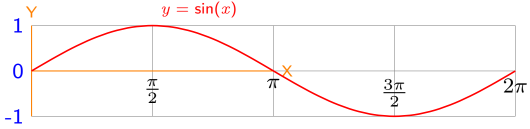

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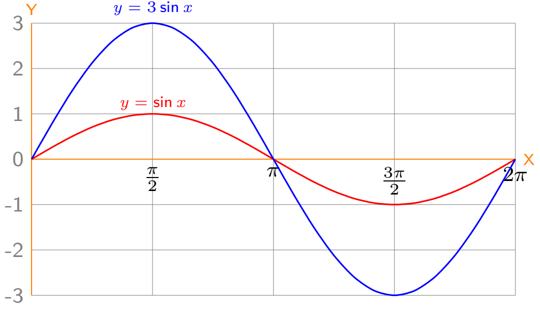

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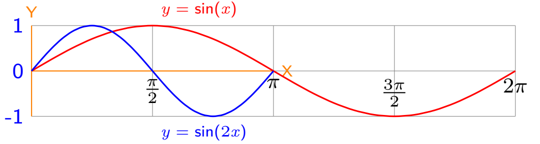
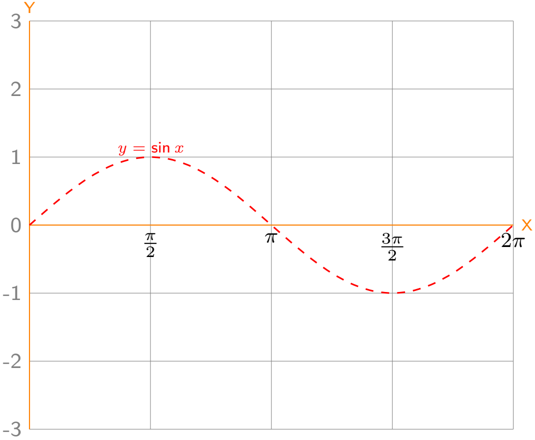


4.4.6 Graphing the standard wave



Example 7: Graph $y = -3\sin 2x$ by transforming the standard wave of $y = \sin x$.

Solution:

- Begin with the graph of $y = \sin x$

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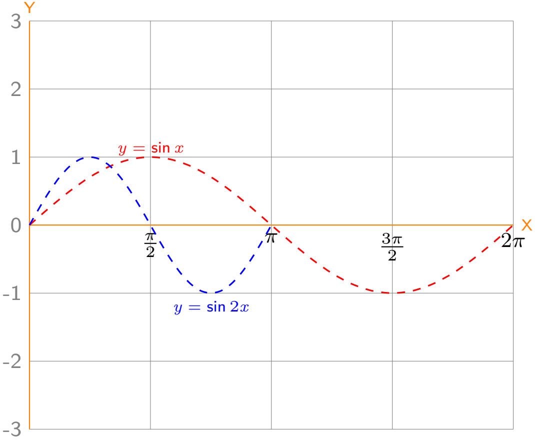

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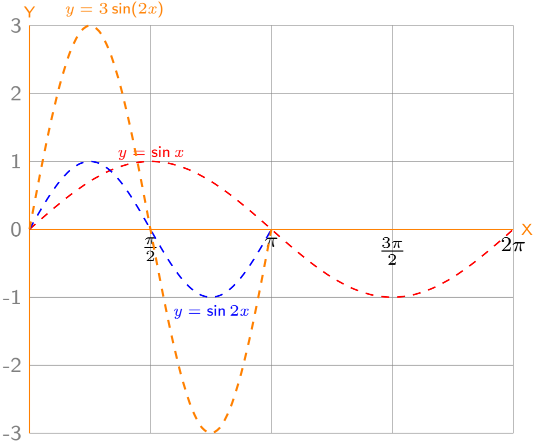
Solution:

- Begin with the graph of $y = \sin x$
- x-shrink 2: Substitute $2x$ for x in $y = \sin x$ to get the blue graph $y = \sin 2x$.

4.4.6 Graphing the standard wave

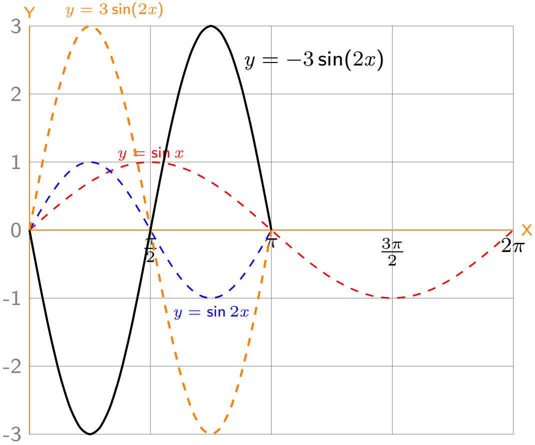


Example 7: Graph $y = -3\sin 2x$ by transforming the standard wave of $y = \sin x$.

Solution:

- Begin with the graph of $y = \sin x$
- x-shrink 2: Substitute $2x$ for x in $y = \sin x$ to get the blue graph $y = \sin 2x$.
- y-stretch 3: Multiply the RHS of $y = \sin 2x$ by 3 to get the graph of $y = 3\sin 2x$, drawn in orange.

4.4.6 Graphing the standard wave



Example 7: Graph $y = -3\sin 2x$ by transforming the standard wave of $y = \sin x$.

Solution:

- Begin with the graph of $y = \sin x$
- x-shrink 2: Substitute $2x$ for x in $y = \sin x$ to get the blue graph $y = \sin 2x$.
- y-stretch 3: Multiply the RHS of $y = \sin 2x$ by 3 to get the graph of $y = 3\sin 2x$, drawn in orange.
- y-axis reflect to transform $y = 3\sin 2x$ to the requested black graph $y = -3\sin 2x$, which shows a complete wave of the sine curve $y = -3\sin(2x)$. The domain has shrunk to $0 \leq x \leq \pi$.

Periodic functions such as sine and cosine are much easier to graph than polynomial, exp or log functions. Instead of transforming a starting graph such as $y = \cos x$ to $y = A\cos(Bx)$, take advantage of the fact that all cosine waves look pretty much alike and simply relabel the axes.

To graph one wave of $y = f(x) = A\cos Bx$ or $y = g(x) = A\sin Bx$ by *relabeling axes*

- Substitute the sign of A for A and the sign of B for B
- Draw the graph of the function so obtained with domain $0 \leq x \leq 2\pi$ and range $-1 \leq y \leq 1$.
- Relabel the x-axis so that $0 \leq x \leq 2\pi/|B|$.
- Relabel the y-axis so that $-|A| \leq y \leq |A|$.

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Example 8:
Sketch one wave of the graph of $y = -3\sin(2x)$.
Solution:

- The requested graph $y = A\sin Bx = -3\sin 2x$ will be obtained by reflecting the graph of $y = 3\sin 2x$ through the x-axis.
  The domain is $0 \le x \le 2\pi/B$, i.e., $0 \le x \le \pi$.
  The range is $-|A| \le y \le |A|$, i.e., $-3 \le y \le 3$.

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- Start by drawing the 4 by 2 box grid , which will accommodate every cosine or sine wave.

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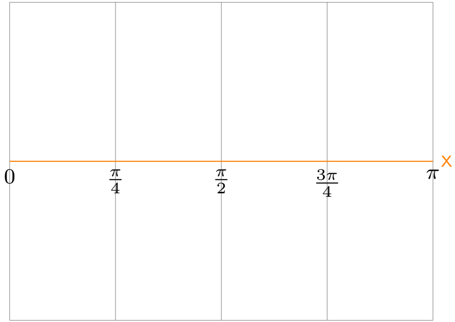

- Start by drawing the 4 by 2 box grid , which will accommodate every cosine or sine wave.
- Angle x goes from 0 to π. The width of the wave is π: each section of the graph has width $\pi/4$. Write x-labels $0, \pi/4, \pi/2, 3\pi/4$.

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Sketch one wave of the graph of $y = -3\sin(2x)$.
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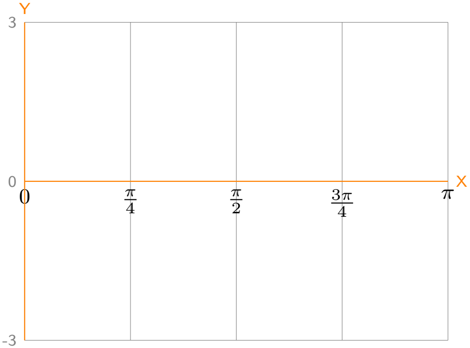

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- Write y-labels $-3, 0, 3$ and draw the axes.

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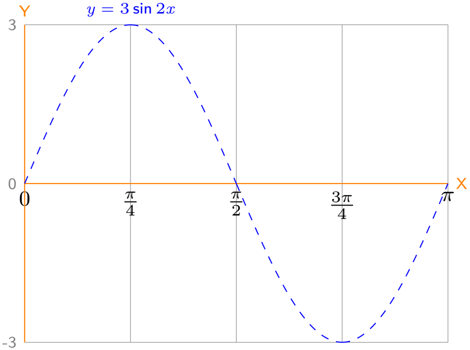

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- Now sketch $y = 3\sin 2x$ by filling up the box with the shape of the standard wave of $y = \sin x$.

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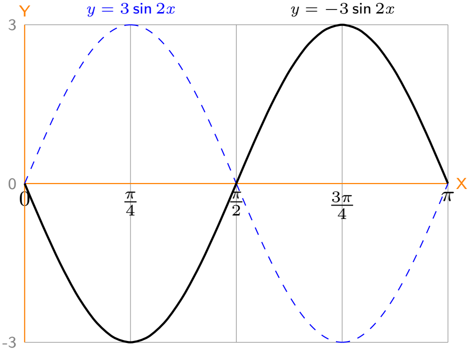

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- Write y-labels $-3, 0, 3$ and draw the axes.
- Now sketch $y = 3\sin 2x$ by filling up the box with the shape of the standard wave of $y = \sin x$.
- Reflect through the x-axis to get the desired graph $y = -3\sin 2x$.

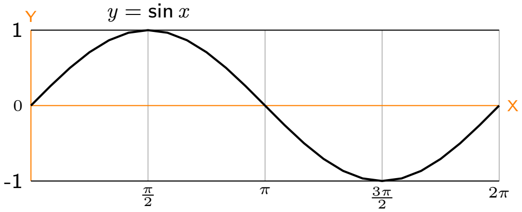
4.4.7 Graphing $y = A\sin(Bx + C)$

Example 9: Graph $y = 3\sin(2x + \pi/3)$ by transforming the graph of $y = \sin x$ with domain $0 \le x \le 2\pi$.
This will be difficult. The next two examples explain a much easier way to draw the graph.

Each new graph is shown as a thick solid line. The previous graph is shown as a thin dotted line.

Solution:
First rewrite the equation as $y = 3\sin(2(x + \pi/6))$.

- Start with the graph $y = \sin x; 0 \le x \le 2\pi$.



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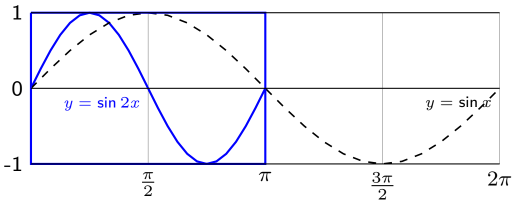
Solution:
First rewrite the equation as $y = 3\sin(2(x + \pi/6))$.

- Start with the graph $y = \sin x; 0 \le x \le 2\pi$.
- Replace x in $y = \sin x$ by $2x$. This shrinks its graph horizontally toward the y-axis by a factor of 2 to yield one wave $y = \sin(2x); 0 \le x \le \pi$.



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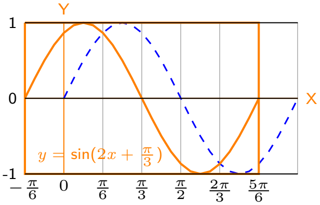
- Replace x in $y = \sin x$ by $2x$. This shrinks its graph horizontally toward the y-axis by a factor of 2 to yield one wave $y = \sin(2x); 0 \le x \le \pi$.
- Replace x in $y = \sin(2x)$ by $x + \pi/6$. This shifts its graph left $\pi/6$ to yield one wave of $y = \sin(2(x + \pi/6))$; $-\frac{\pi}{6} \le x \le \frac{5\pi}{6}$ Vertical lines are $\pi/6$ radians apart.



4.4.7 Graphing $y = A\sin(Bx + C)$

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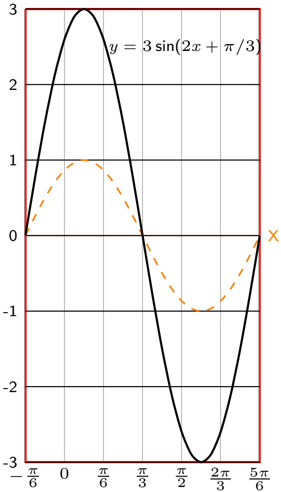
- Replace x in $y = \sin(2x)$ by $x + \pi/6$. This shifts its graph left $\pi/6$ to yield one wave of $y = \sin(2(x + \pi/6))$; $-\frac{\pi}{6} \le x \le \frac{5\pi}{6}$ Vertical lines are $\pi/6$ radians apart.
- Multiply RHS of $y = \sin(2x + \pi/3)$ by 3. This stretches the graph vertically away from the x-axis by a factor of 3 to yield one wave of $y = 3\sin(2x + \pi/3)$; with domain $-\frac{\pi}{6} \le x \le \frac{5\pi}{6}$.



4.4.8 Graphing the standard wave of function $y = A\sin(Bx + C)$ or $y = A\cos(Bx + C)$

Example 9 transformed the equation $y = \sin x$ with domain $0 \leq x \leq 2\pi$ by using shifts and stretches to obtain one wave of the graph $y = 3\sin(2x + \pi/3)$. That was tricky.
Earlier we found an easier way to graph $y = A\sin Bx$ by relabeling the axes of the graph of $y = \pm\sin x$ with domain $0 \leq x \leq 2\pi$.
We can do the same for $y = A\sin(Bx + C)$ by carefully choosing its domain and then relabeling axes.

Assume $0 < C < 2\pi$. To graph the *standard wave* of $y = A\sin(Bx + C)$ or $y = A\cos(Bx + C)$ restrict the domain to all x satisfying $0 \leq Bx + C \leq 2\pi$.

- If $B > 0$ The standard wave's
  - domain is $[-\frac{C}{B}, \frac{-C+2\pi}{B}]$ if $B > 0$;
  - amplitude is $|A|$, half the wave's height.
  - phase shift is the left endpoint of its domain.
  - period (= width) is $\frac{2\pi}{B}$.
- If $B < 0$, graph $y = -A\sin(-Bx - C) = A\sin(Bx + C)$. or $y = \cos(-Bx - C) = A\cos(Bx + C)$ and use the fact that $-B > 0$.

Assume $0 < C < 2\pi$. To graph the standard wave of $y = A\cos(Bx + C)$ or $y = A\sin(Bx + C)$ on a 4 by 2 grid:

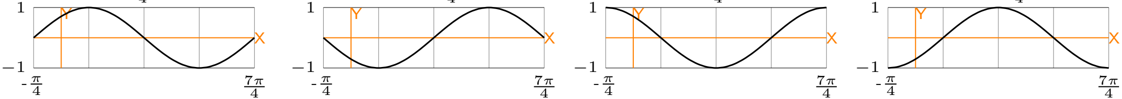
- Draw the standard wave of the function (with domain $0 \leq x \leq 2\pi$ and range $-1 \leq y \leq 1$), obtained by substituting 0 for C, the sign of A for A, and the sign of B for B.
- The left and right x-labels are phase shift and phase shift + $\frac{2\pi}{|B|}$.
- The top and bottom y-labels are $|A|$ and $-|A|$.

Assume C arbitrary. The *standard wave* of $y = A\sin(Bx + C)$ or $y = A\cos(Bx + C)$

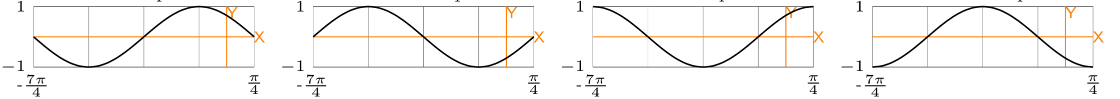
is the standard wave of the graph obtained by substituting for C a coterminal angle between 0 and 2π.

This page shows 16 graphs, $y = A\sin(Bx + C)$ and $y = A\cos(Bx + C)$ for $A = \pm 1, B = \pm 1$ and $C = \pm\frac{\pi}{4}$. Make sure you understand why each graph is obtained by shifting the standard wave of $y = \pm\cos x$ or $y = \pm\sin x$ left or right $\frac{\pi}{4}$.

$y = \sin(x + \frac{\pi}{4})$ $y = -\sin(x + \frac{\pi}{4})$ $y = \cos(x + \frac{\pi}{4})$ $y = -\cos(x + \frac{\pi}{4})$

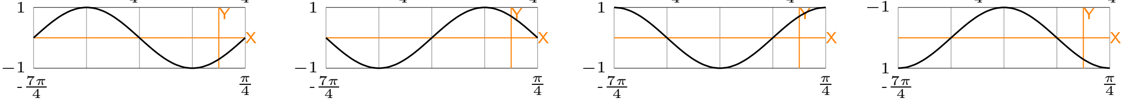


$y = \sin(-x + \frac{\pi}{4})$ $y = -\sin(-x + \frac{\pi}{4})$ $y = \cos(-x + \frac{\pi}{4})$ $y = -\cos(-x + \frac{\pi}{4})$

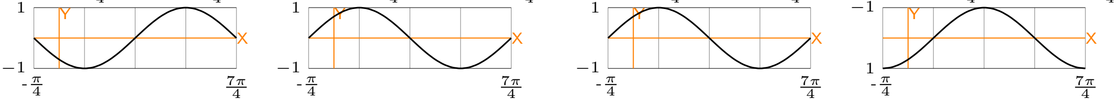


In the following 8 graphs, $C = -\frac{\pi}{4}$ is negative and so it is replaced by the coterminal angle $-\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}$

$y = \sin(x - \frac{\pi}{4}) = \sin(x + \frac{7\pi}{4})$ $y = -\sin(x - \frac{\pi}{4}) = -\sin(x + \frac{7\pi}{4})$ $y = \cos(x - \frac{\pi}{4}) = \cos(x + \frac{7\pi}{4})$ $y = -\cos(x - \frac{\pi}{4}) = -\cos(x + \frac{7\pi}{4})$



$y = \sin(\text{-}x - \frac{\pi}{4}) = \sin(\text{-}x + \frac{7\pi}{4})$ $y = -\sin(\text{-}x - \frac{\pi}{4}) = -\sin(\text{-}x + \frac{7\pi}{4})$ $y = \cos(\text{-}x + \frac{\pi}{4}) = \cos(\text{-}x + \frac{7\pi}{4})$ $y = -\cos(\text{-}x - \frac{\pi}{4}) = -\cos(\text{-}x + \frac{7\pi}{4})$



To prepare for graphing $y = A\sin(Bx + C)$ or $y = A\cos(Bx + C)$ by relabeling axes, it's important to practice finding the domain that will be used for those graphs.

Example 10: Find the domain of the standard waves of

- $y = 3\sin(2x + \frac{\pi}{3}) \Rightarrow 0 \le 2x + \frac{\pi}{3} \le 2\pi \Rightarrow$

  Domain: $-\frac{\pi}{6} \le x \le \frac{5\pi}{6}$

  Details are at the right, in the solution of Example 11.

- $y = 3\sin(\frac{\pi}{4} - 2x) \Rightarrow 0 \le -2x + \frac{\pi}{4} \le 2\pi$

  $\Rightarrow -\frac{\pi}{4} \le -2x \le \frac{7\pi}{4}$ Domain: $-\frac{7\pi}{8} \le x \le \frac{\pi}{8}$

- $y = 3\cos(2x + \frac{\pi}{3}) \Rightarrow 0 \le 2x + \frac{\pi}{3} \le 2\pi$

  Domain: $-\frac{\pi}{6} \le x \le \frac{5\pi}{6}$

- $y = 3\cos(-\frac{\pi}{3} - 2x)$

  Since $C = -\frac{\pi}{3} < 0$ replace C by $C + 2\pi = \frac{5\pi}{3}$.

  $0 \le -2x + \frac{5\pi}{3} \le 2\pi$ Domain: $-\frac{\pi}{6} \le x \le \frac{5\pi}{6}$

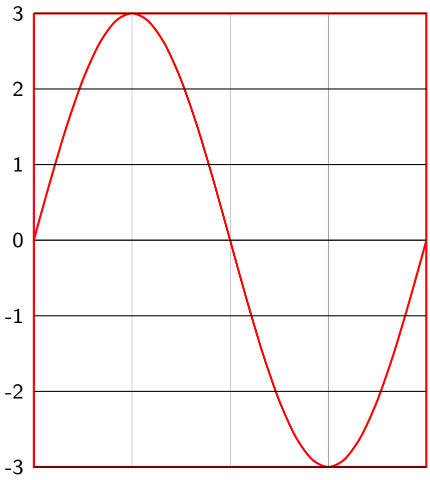
Example 11: Graph the standard wave of $y = 3\sin(2x + \pi/3)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = 3$; $B = 2$; $C = \pi/3$. Draw the 4 by 2 grid.

- The amplitude is $|A| = 3$, so y goes from -3 to 3.
- The period is $\frac{2\pi}{B} = \frac{2\pi}{2|} = \pi$.
- The phase shift is the x-value that solves $2x + \frac{\pi}{3} = 0$, namely $-\frac{\pi}{6}$.
- The standard wave has domain the interval [phase shift, phase shift + period] $= [-\frac{\pi}{6}, \frac{5\pi}{6}]$.
- Angle $Bx + C = 2x + \frac{\pi}{3}$ should go from 0 to 2π.

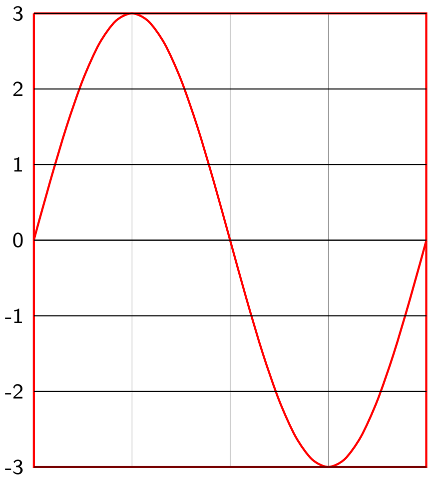
To solve	$0 \le 2x + \frac{\pi}{3} \le 2\pi$.
Subtract $\frac{\pi}{3}$	$0 - \frac{\pi}{3} \le 2x \le 2\pi - \frac{\pi}{3}$
Simplify	$-\frac{\pi}{3} \le 2x \le \frac{5\pi}{3}$
Divide by 2	$-\frac{\pi}{6} \le x \le \frac{5\pi}{6}$

- From the previous slide, the grid's left and right $x-$ labels are $-\frac{\pi}{6}$ and $\frac{5\pi}{6}$.



- From the previous slide, the grid's left and right $x-$ labels are $-\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

  The grid's width is $\frac{5\pi}{6} - (-\frac{\pi}{6}) = \pi$.



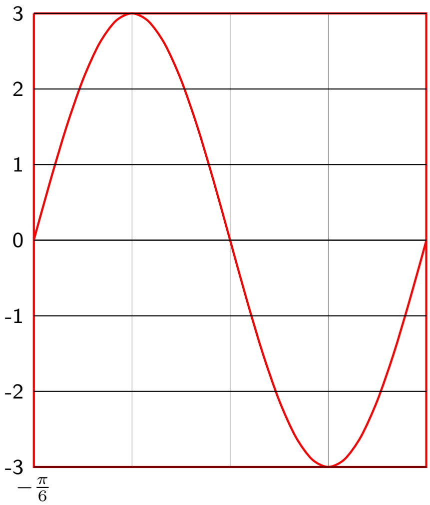
- From the previous slide, the grid's left and right $x-$ labels are $-\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

  The grid's width is $\frac{5\pi}{6} - (-\frac{\pi}{6}) = \pi$.

  Split that width π into four equal parts, each $\frac{\pi}{4}$ wide.

  The first x-axis label is $-\frac{\pi}{6}$.

  Keep adding $\frac{\pi}{4}$ to find the others. They are:



- From the previous slide, the grid's left and right $x-$ labels are $-\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

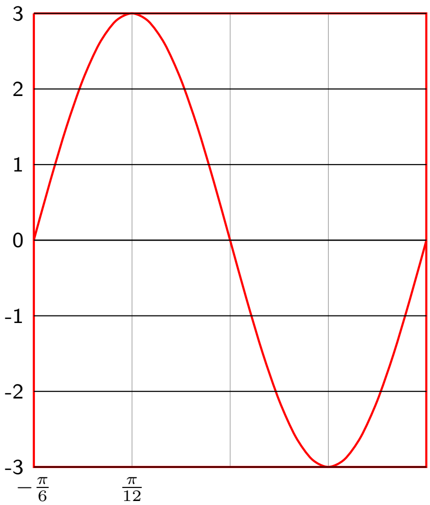
  The grid's width is $\frac{5\pi}{6} - (-\frac{\pi}{6}) = \pi$.

  Split that width π into four equal parts, each $\frac{\pi}{4}$ wide.

  The first x-axis label is $-\frac{\pi}{6}$.

  Keep adding $\frac{\pi}{4}$ to find the others. They are:

- $-\frac{\pi}{6} + \frac{\pi}{4} = -\frac{2\pi}{12} + \frac{3\pi}{12} = \frac{\pi}{12}$;



- From the previous slide, the grid's left and right $x-$ labels are $-\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

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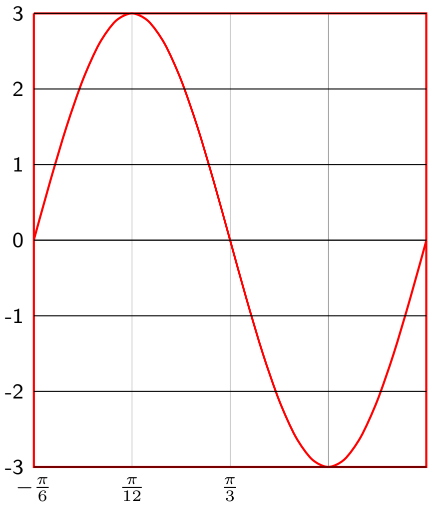
- $\frac{\pi}{12} + \frac{\pi}{4} = \frac{\pi}{12} + \frac{3\pi}{12} = \frac{4\pi}{12} = \frac{\pi}{3}$;



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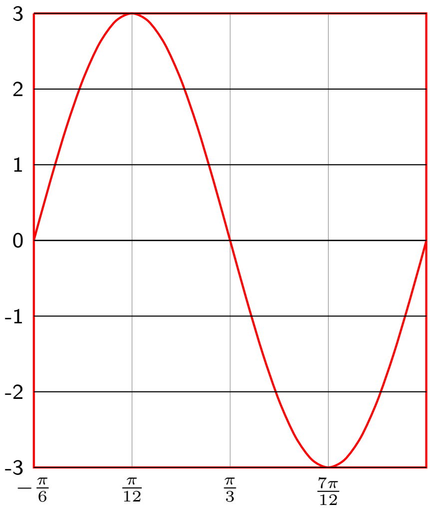

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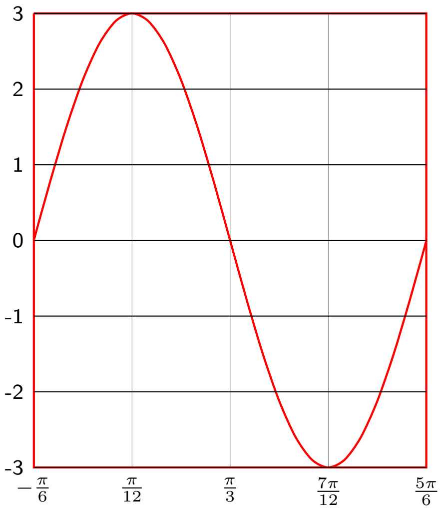

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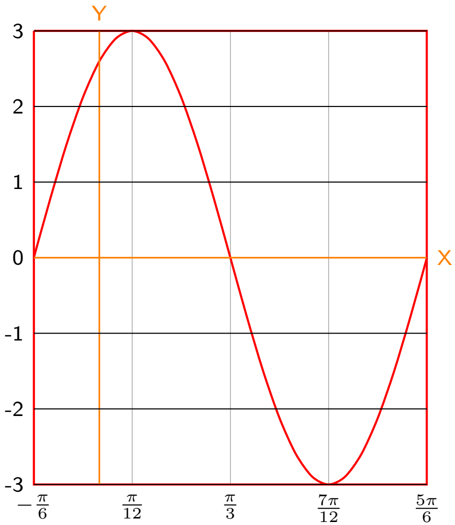
- Draw the axes: $x = 0$ and $y = 0$.



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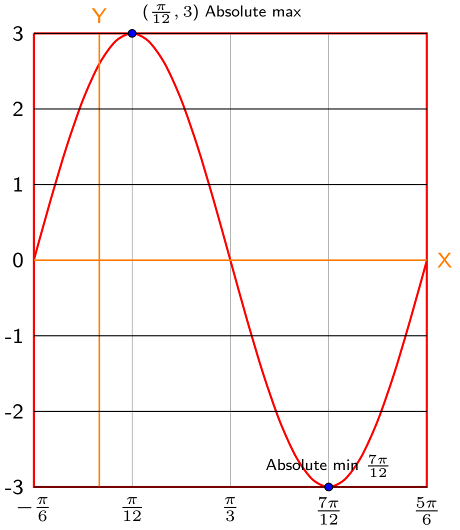
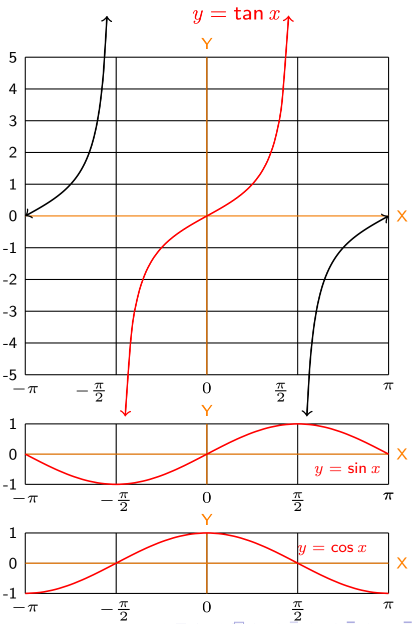
- Draw the axes: $x = 0$ and $y = 0$.
- Label the maximum and minimum points. Done!



4.4.8 Graphing $y = \tan x$

The function $f(x) = \tan x = \frac{\sin x}{\cos x}$ is defined when $\cos x \neq 0$. Therefore $\tan x$ is defined at all x-values other than $\frac{\pi}{2} \pm k\pi$ where $k = 0, \pm 1, \pm 2, \pm 3, \ldots$.

Example 12: Sketch the graph of $y = \tan x$ for $-\pi \leq x \leq \pi$. Look at the 3 graphs starting at $x = -\frac{\pi}{2}$ and click to move right to $x = \frac{\pi}{2}$.



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If $x = -\frac{\pi}{2}$, $\tan x = \frac{\sin(-\pi/2)}{\cos(-\pi/2)} = \frac{-1}{0}$ is undefined:
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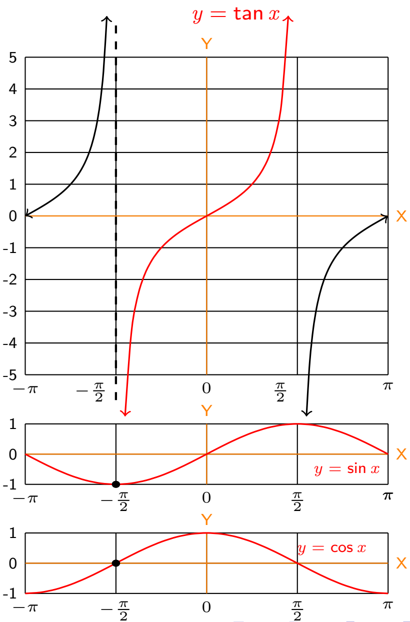

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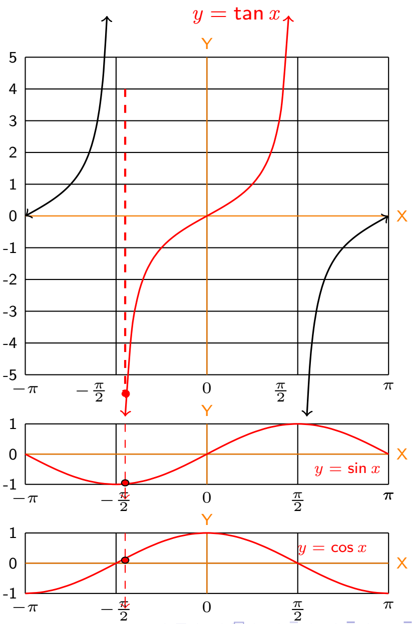
If $x = -\frac{\pi}{2} + .01$, on the vertical red dotted line, $\tan x$ is large negative because $\sin x \approx -1$ and $\cos x$ is very close to zero and positive.



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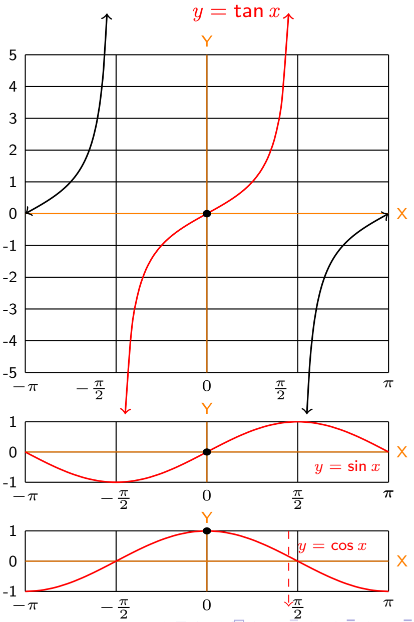

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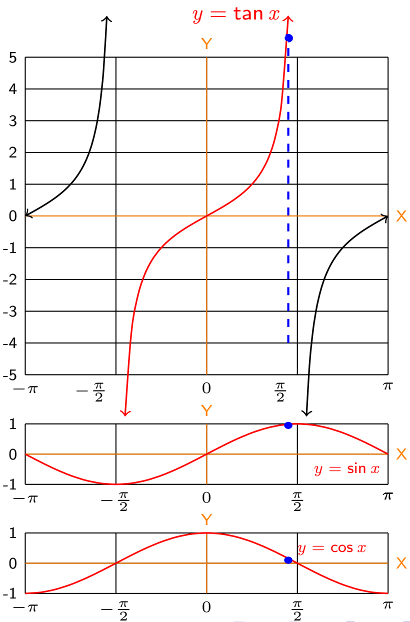
If $x = \frac{\pi}{2} - .01$, on the vertical blue dotted line, $\tan x$ is large positive because $\sin x \approx 1$ and $\cos x$ is very close to zero and positive.



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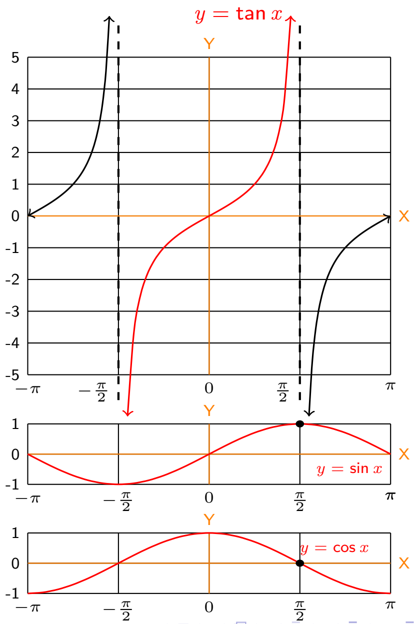

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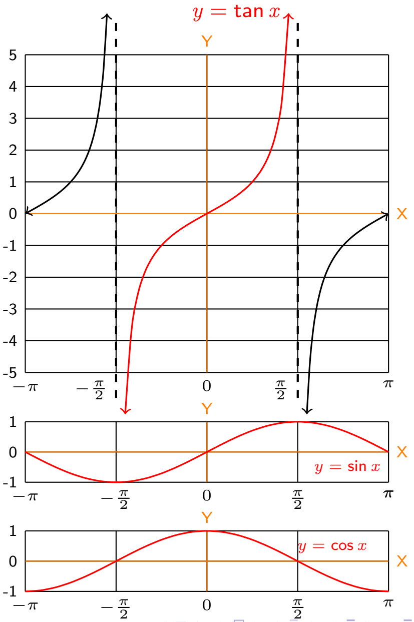
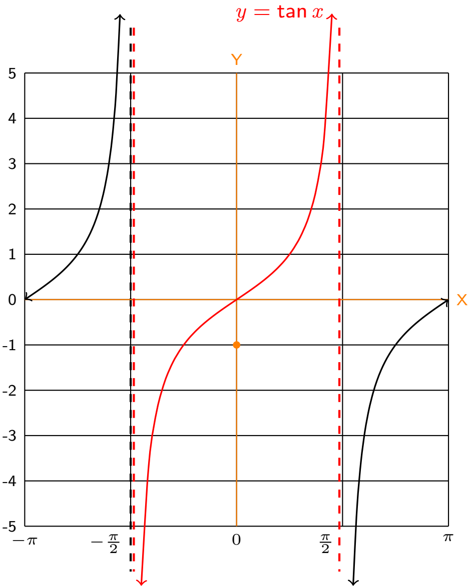
If $x = \frac{\pi}{2}$, $\tan x = \frac{\sin(\frac{\pi}{2})}{\cos(\frac{\pi}{2})} = \frac{1}{0}$ is undefined:
$x = \pi/2$ is a vertical asymptote.
$y = \tan x$ has period π. This means: $\tan(x + \pi) = \tan x$ provided both are defined.
$y = \sin x$ and $y = \cos x$ have period 2π.
One period of each of the 3 graphs has been drawn in red.



4.4.8 Graphing $y = \tan x$

Reminder:
$\arctan x = \theta$ **provided** $\tan\theta = x$ **and** $-\frac{\pi}{2} < x < \frac{\pi}{2}$

- To find $\arctan(-1)$, start on the $y-$axis at $y = -1$.



4.4.8 Graphing $y = \tan x$

Reminder:
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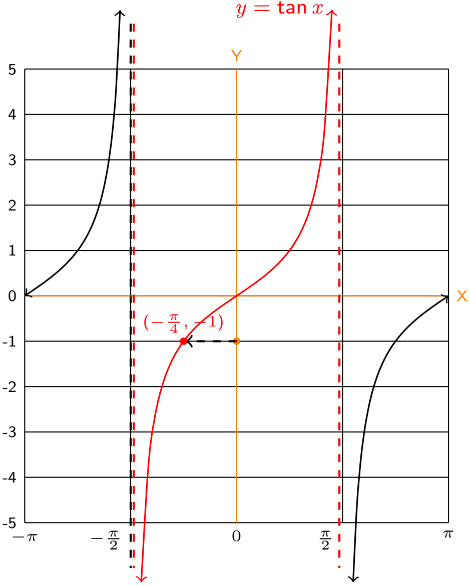
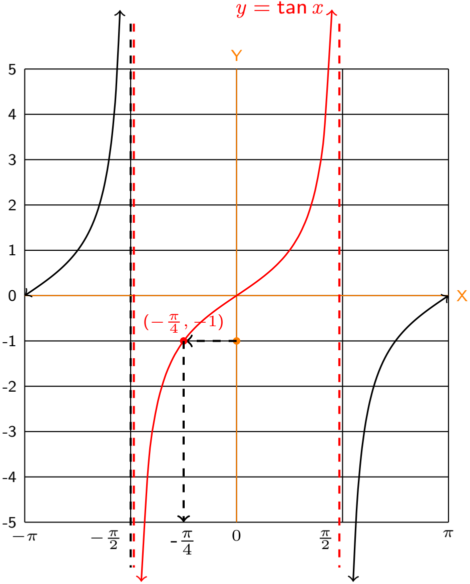
- To find $\arctan(-1)$, start on the $y-$axis at $y = -1$.
- Move horizontally (in this case, left) to the red point on the graph $y = \tan x$.



4.4.8 Graphing $y = \tan x$

Reminder:
arctan $x = \theta$ **provided** $\tan\theta = x$ **and** $-\frac{\pi}{2} < x < \frac{\pi}{2}$

- To find $\arctan(-1)$, start on the $y-$axis at $y = -1$.
- Move horizontally (in this case, left) to the red point on the graph $y = \tan x$.
- To find the $x-$ coordinate of that point, go down to the $x-$scale label $-\frac{\pi}{4}$.
  This shows $\tan(-\frac{\pi}{4}) = -1$ and therefore $\arctan(-1) = -\frac{\pi}{4}$.



Section 4.4 Quiz

Example 4.4.1: Graph $y = \sin x$ and $y = -\sin x$ for $0 \le x \le 2\pi$.

Example 4.4.2: Graph $y = \cos x$ and $y = -\cos x$ for $0 \le x \le 2\pi$.

Example 4.4.3: Graph $y = \sin x$ for $-4\pi \le x \le 4\pi$.

Example 4.4.4: Graph $y = \sin x$ for $-5\pi \le x \le 4\pi$.

Example 4.4.5: Graph one wave of $y = 3\sin x$.

Example 4.4.6: Graph one wave of $y = \sin 2x$.

Example 4.4.7: Starting with one wave of $y = \sin x$, draw one wave of the graph of $y = -3\sin(2x)$.

Example 4.4.8: Graph the standard wave of $y = -3\sin(2x)$ without using transformations.

Example 4.4.9: Graph $y = 3\sin(2x + \pi/3)$ by transforming the graph of $y = \sin x$ with domain $0 \le x \le 2\pi$.

Example 4.4.10: Find the domain of the standard waves of

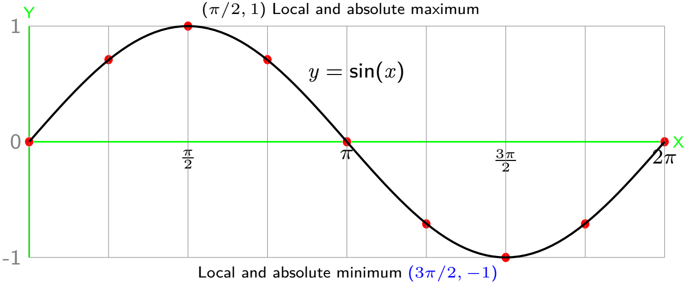
• $y = 3\sin(2x + \frac{\pi}{3})$ • $y = 3\sin(\frac{\pi}{4} - 2x)$ • $y = 3\cos(2x + \frac{\pi}{3})$ • $y = 3\cos(\frac{\pi}{4} - 2x)$

Example 4.4.11: Graph the standard wave of $y = 3\sin(2x + \pi/3)$ without using transformations.

Example 4.4.12: Graph $y = \tan x$ at all x-values in $[-\pi, \pi]$ where $\tan x$ is defined.

Section 4.4 Review: Graphing $y = \sin x$ and $y = \cos x$

Example 1: Sketch the graphs of $y = \sin x$ and $y = -\sin x$ for $0 \le x \le 2\pi$.



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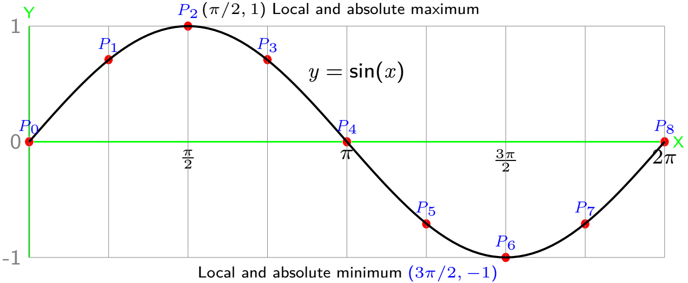

Find the exact coordinates of $P_0, P_1, ..., P_8 \Rightarrow$

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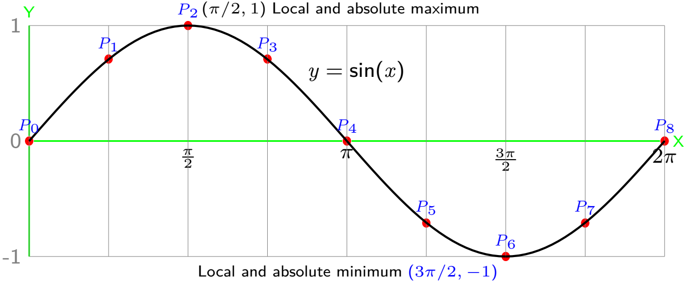

Find the exact coordinates of $P_0, P_1, \ldots, P_8 \Rightarrow$

$P_0(0,0)$; $P_1(\frac{\pi}{4}, \frac{1}{\sqrt{2}})$; $P_2(\frac{\pi}{2}, 1)$;
$P_3(\frac{3\pi}{4}, \frac{1}{\sqrt{2}})$; $P_4(\pi, 0)$; $P_5(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}})$,
$P_6(\frac{3\pi}{2}, -1)$; $P_7(\frac{7\pi}{4}, -\frac{1}{\sqrt{2}})$; $P_8(2\pi, 0)$

Section 4.4 Review: Graphing $y = \sin x$ and $y = \cos x$

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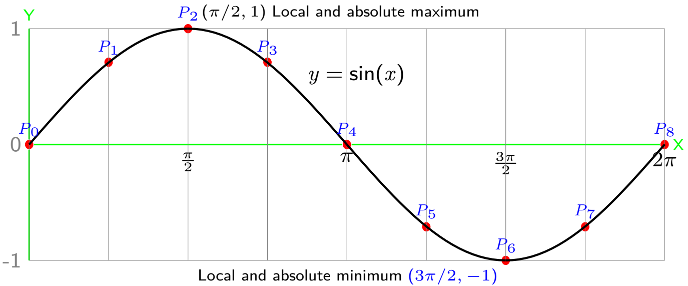

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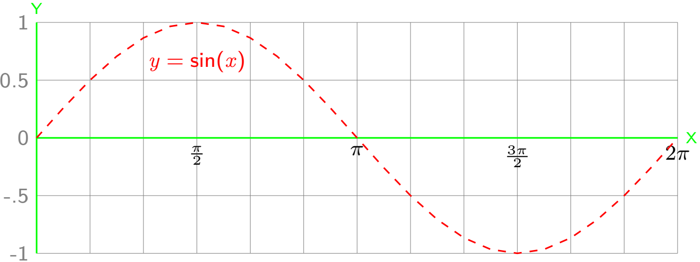

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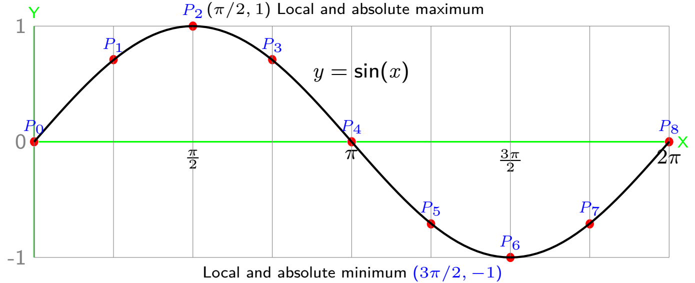

Find the exact coordinates of $P_0, P_1, \ldots, P_8 \Rightarrow$

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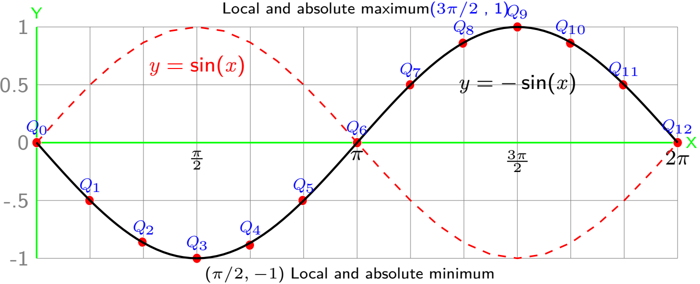
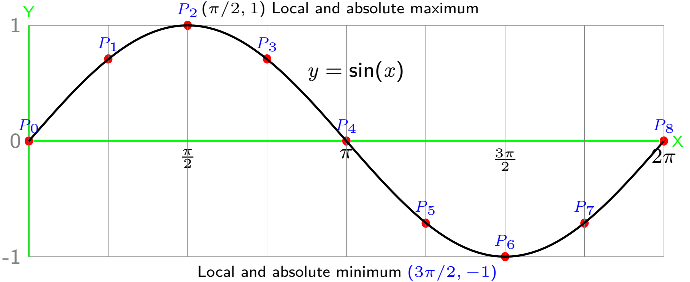


Find the exact coordinates of $Q_0, Q_1, \ldots, Q_{11} \Rightarrow$

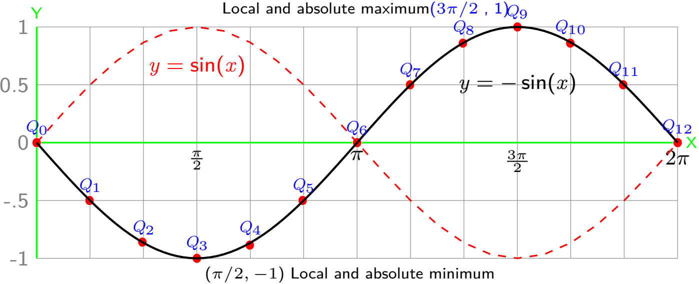
Section 4.4 Review: Graphing $y = \sin x$ and $y = \cos x$

Example 1: Sketch the graphs of $y = \sin x$ and $y = -\sin x$ for $0 \leq x \leq 2\pi$.



Find the exact coordinates of $P_0, P_1, ..., P_8 \Rightarrow$

$P_0(0,0)$; $P_1(\frac{\pi}{4}, \frac{1}{\sqrt{2}})$; $P_2(\frac{\pi}{2}, 1)$;
$P_3(\frac{3\pi}{4}, \frac{1}{\sqrt{2}})$; $P_4(\pi, 0)$; $P_5(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}})$,
$P_6(\frac{3\pi}{2}, -1)$; $P_7(\frac{7\pi}{4}, -\frac{1}{\sqrt{2}})$; $P_8(2\pi, 0)$

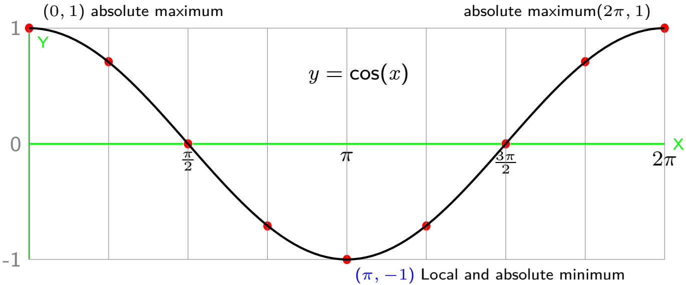


Find the exact coordinates of $Q_0, Q_1, ..., Q_{11} \Rightarrow$

$Q_0(0,0)$; $Q_1(\frac{\pi}{6}, -\frac{1}{2})$; $Q_2(\frac{\pi}{3}, -\frac{\sqrt{3}}{2})$;
$Q_3(\frac{\pi}{2}, -1)$; $Q_4(\frac{2\pi}{3}, -\frac{\sqrt{3}}{2})$; $Q_5(\frac{5\pi}{6}, -\frac{1}{2})$;
$Q_6(\pi, 0)$; $Q_7(\frac{7\pi}{6}, \frac{1}{2})$; $Q_8(\frac{4\pi}{3}, \frac{\sqrt{3}}{2})$;
$Q_9(\frac{3\pi}{2}, 1)$, $Q_{10}(\frac{5\pi}{3}, \frac{\sqrt{3}}{2})$;
$Q_{11}(\frac{11\pi}{6}, \frac{1}{2})$; $Q_{12}(2\pi, 0)$

Graphing $y = \sin x$ and $y = \cos x$ and recalling their values at familiar angles x.

Example 2: Sketch the graphs of $y = \cos x$ and $y = -\cos x$ for $0 \le x \le 2\pi$.



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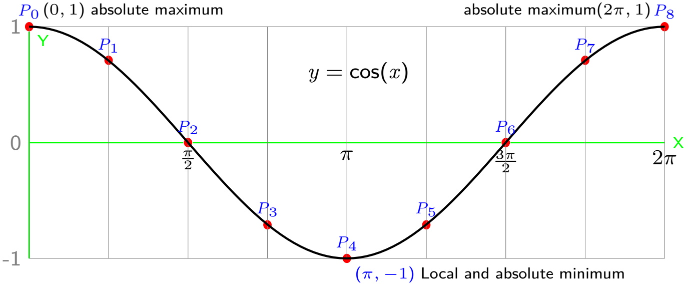

Find the exact coordinates of $P_0, P_1, ..., P_8 \Rightarrow$

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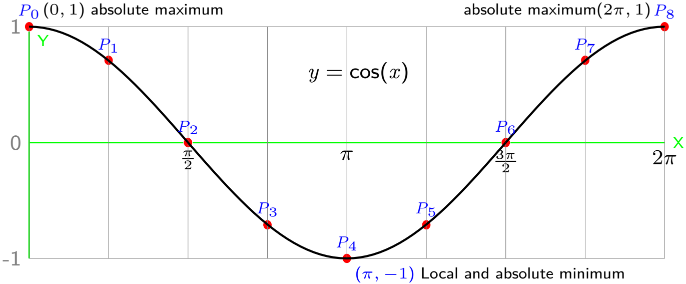

Find the exact coordinates of $P_0, P_1, ..., P_8 \Rightarrow$
$P_0(0, 1)$; $P_1(\frac{\pi}{4}, \frac{1}{\sqrt{2}})$; $P_2(\frac{\pi}{2}, 0)$;
$P_3(\frac{3\pi}{4}, -\frac{1}{\sqrt{2}})$; $P_4(\pi, 0)$; $P_5(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}})$,
$P_6(\frac{3\pi}{2}, 0)$; $P_7(\frac{7\pi}{4}, \frac{1}{\sqrt{2}})$; $P_8(2\pi, 1)$

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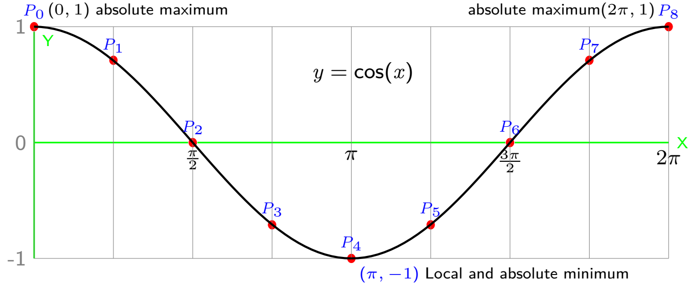

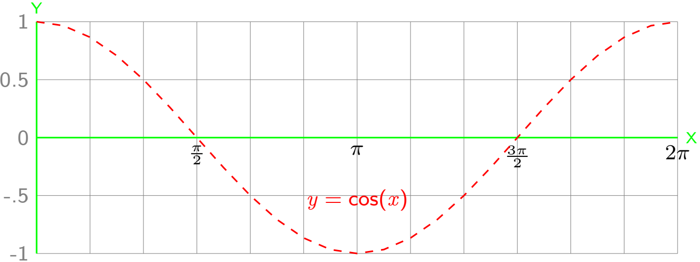

Find the exact coordinates of $P_0, P_1, ..., P_8 \Rightarrow$
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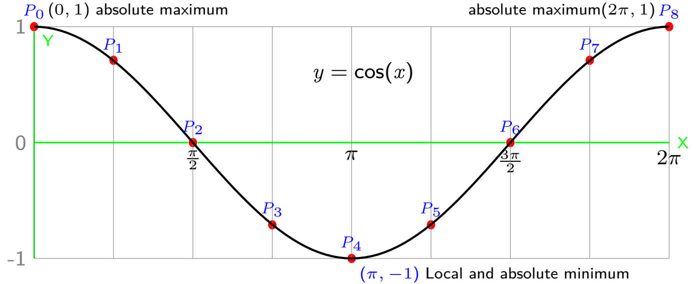

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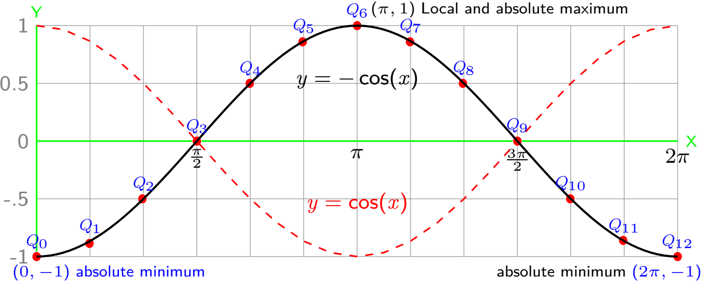

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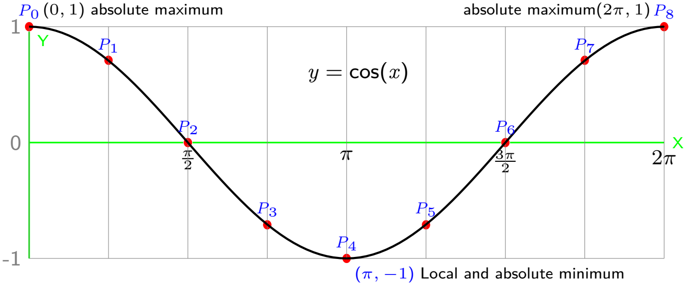

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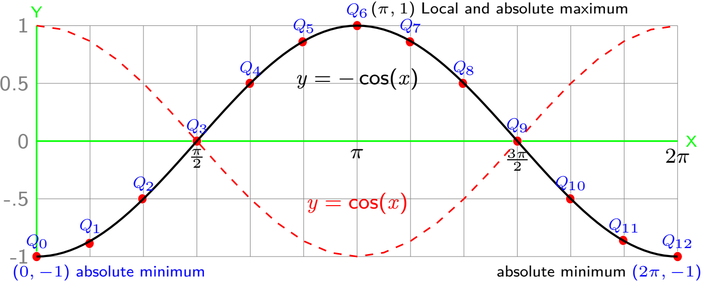

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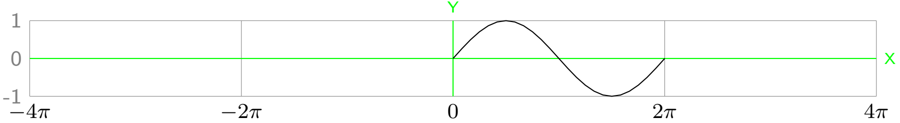
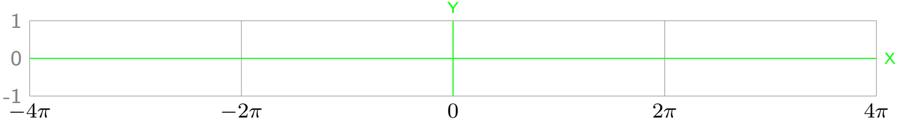
$Q_9(\frac{3\pi}{2}, 0)$, $Q_{10}(\frac{5\pi}{3}, -\frac{1}{2})$
$Q_{11}(\frac{11\pi}{6}, -\frac{\sqrt{3}}{2})$; $Q_{12}(2\pi, -1)$

Graphing sine and cosine with a variety of domains

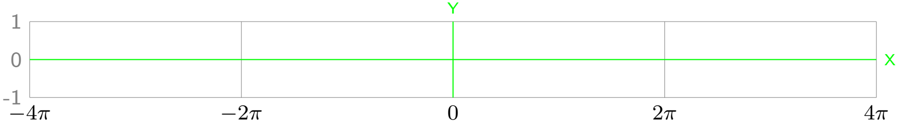
Example 3: Graph $y = \sin x$ for $-4\pi \leq x \leq 4\pi$.



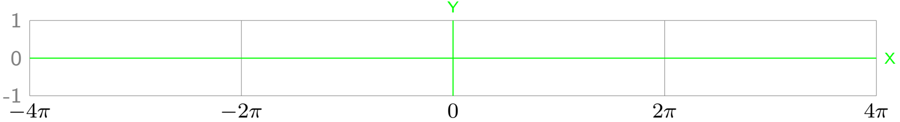
Graph $y = \sin x$ for $-4\pi \leq x \leq -\pi$.



Graph $y = \sin x$ for $-\pi \leq x \leq \pi$.



Graph $y = \sin x$ for $-3\pi \leq x \leq 3\pi$.



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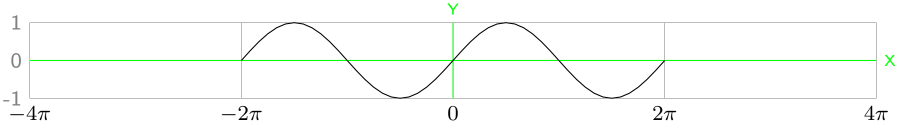

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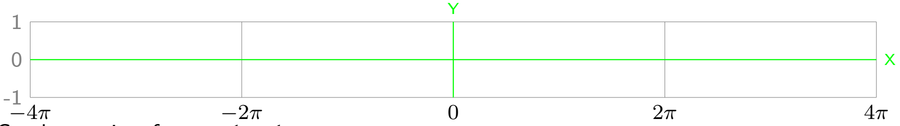

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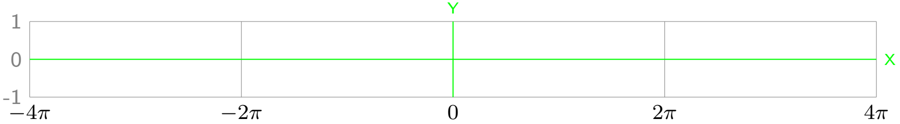

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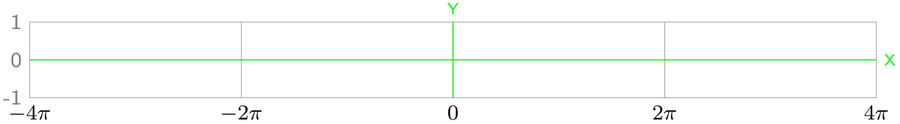

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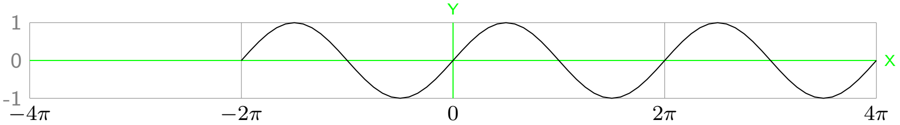

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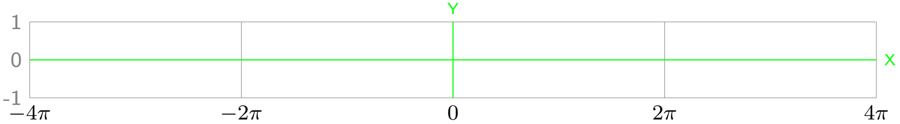

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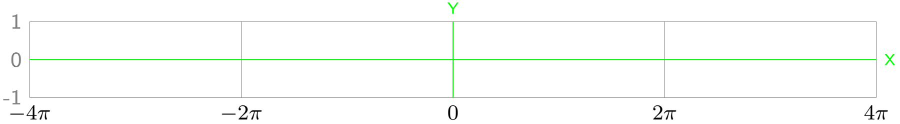

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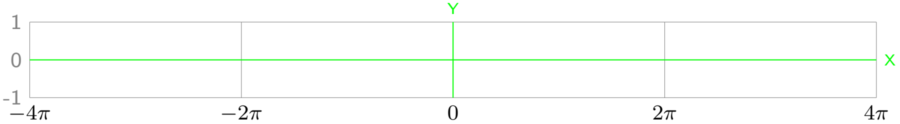

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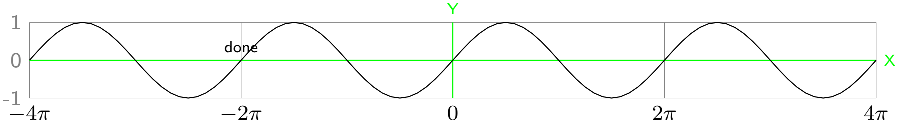

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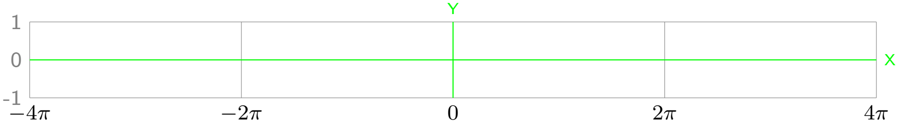

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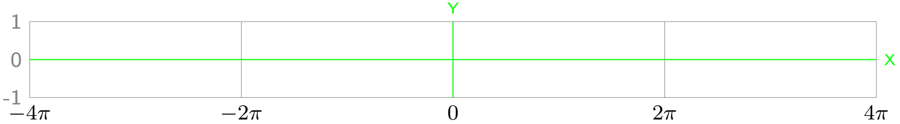

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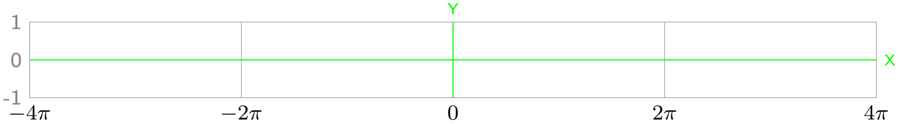

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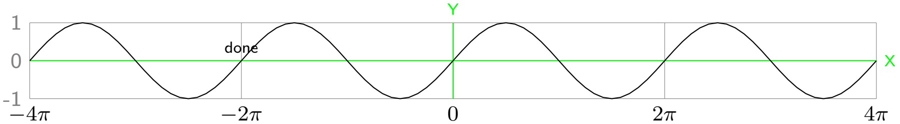

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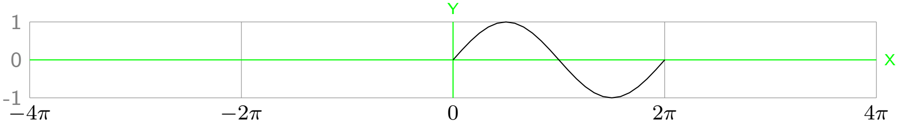

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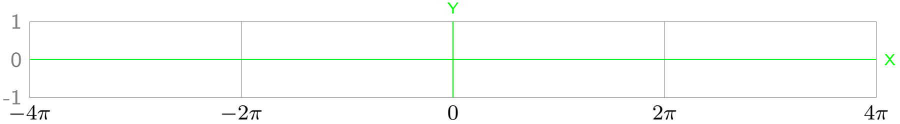

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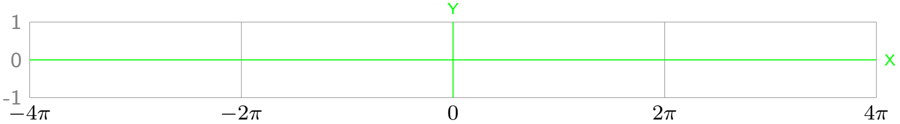

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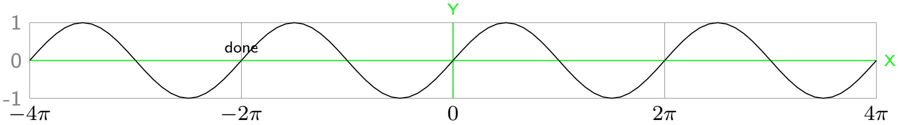

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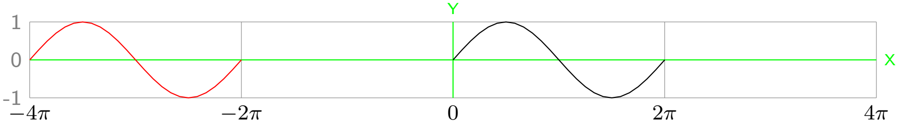

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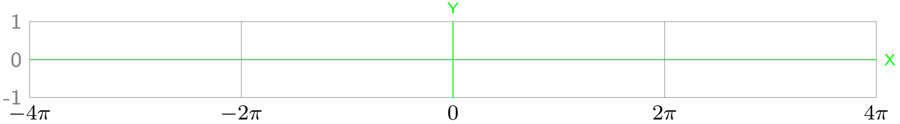

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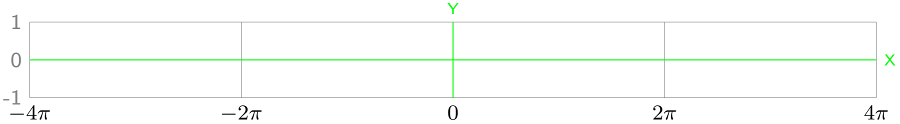

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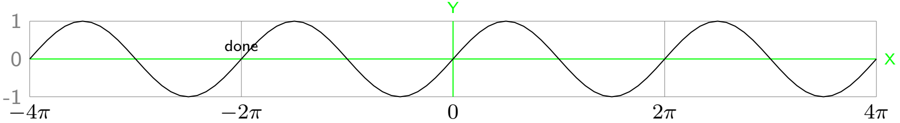

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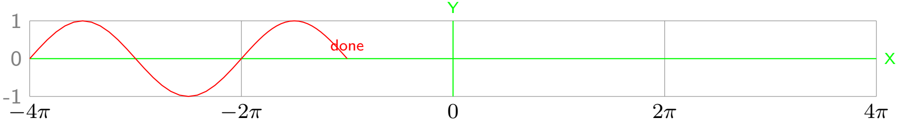

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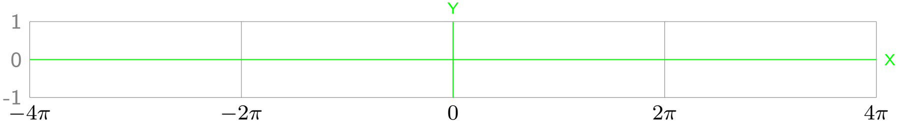

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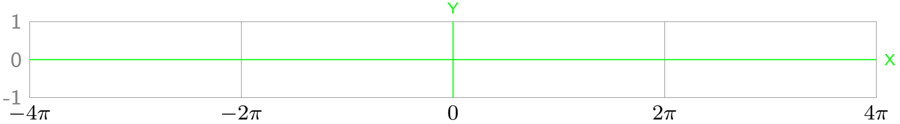

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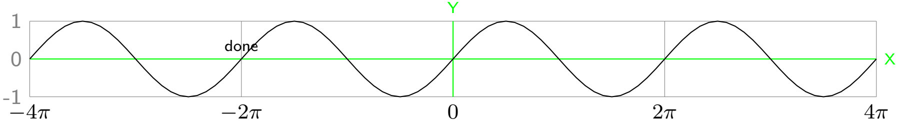

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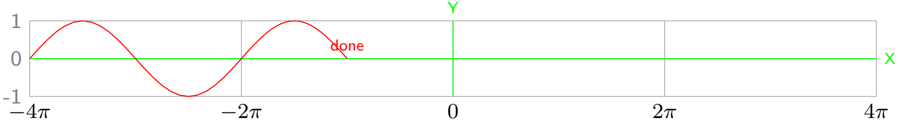

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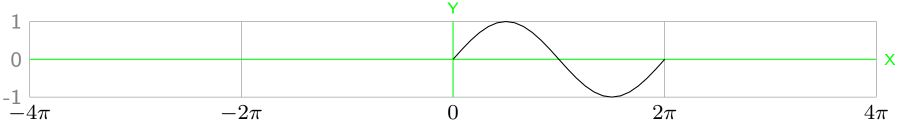

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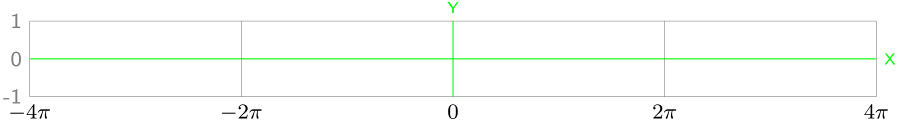

Graphing sine and cosine with a variety of domains

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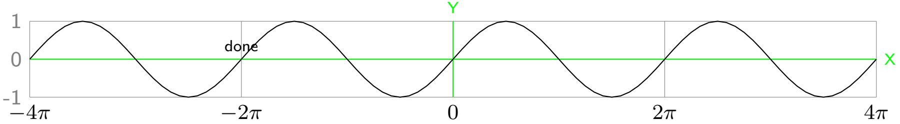

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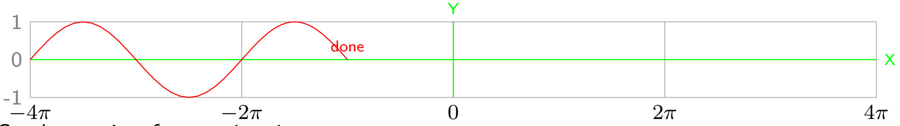

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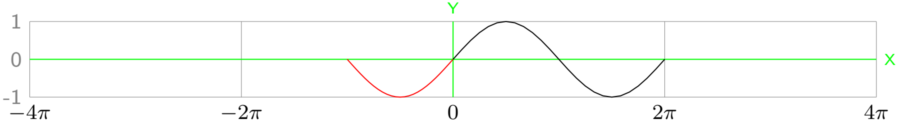

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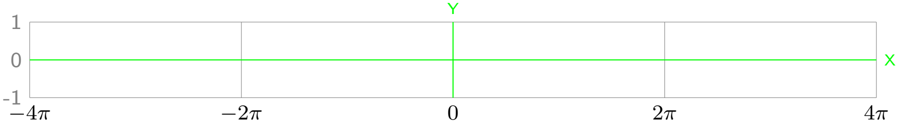

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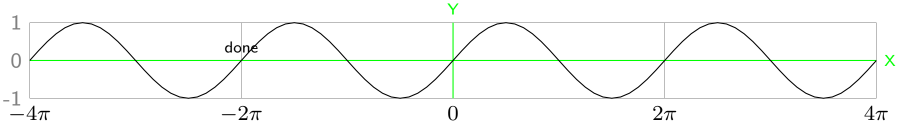

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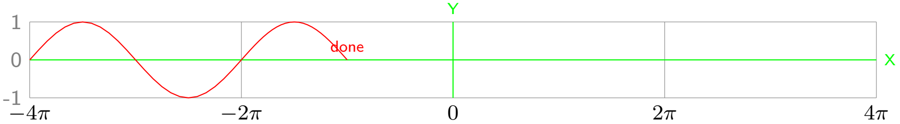

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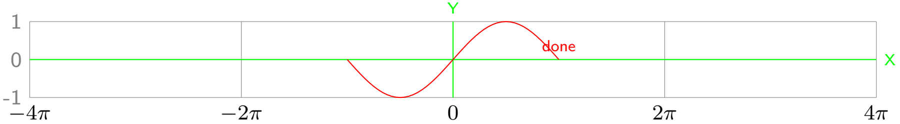
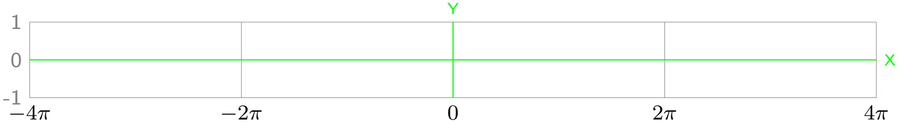
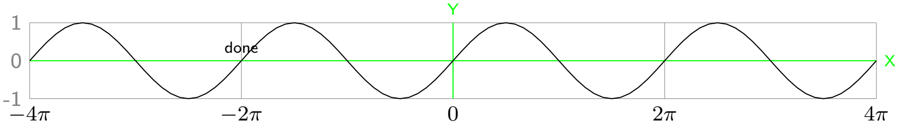


Graph $y = \sin x$ for $-3\pi \le x \le 3\pi$.

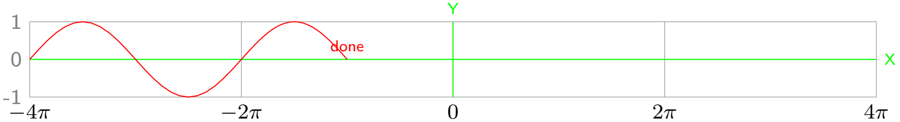


Graphing sine and cosine with a variety of domains

Example 3: Graph $y = \sin x$ for $-4\pi \le x \le 4\pi$.



Graph $y = \sin x$ for $-4\pi \le x \le -\pi$.



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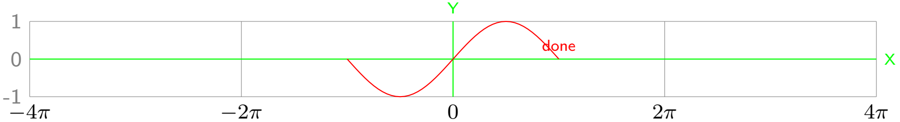
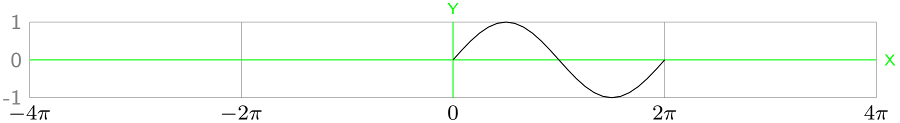


Graph $y = \sin x$ for $-3\pi \le x \le 3\pi$.



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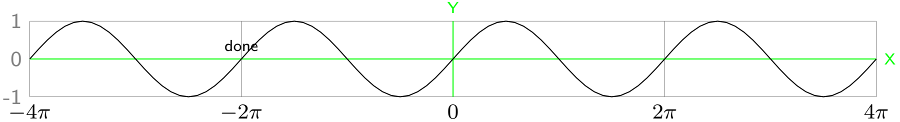

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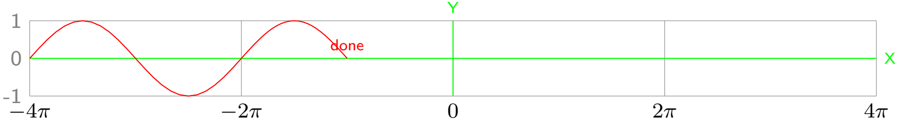

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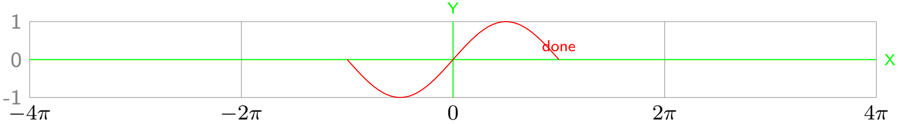

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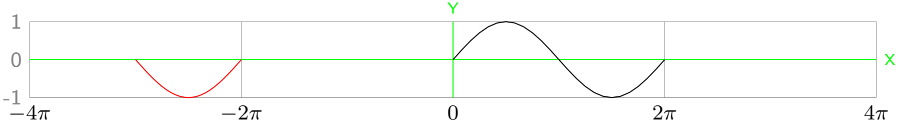

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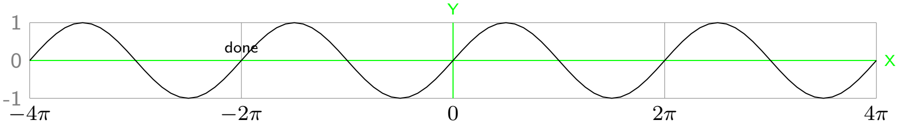

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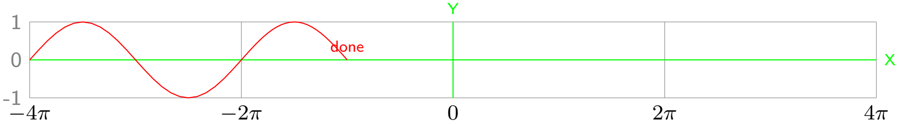

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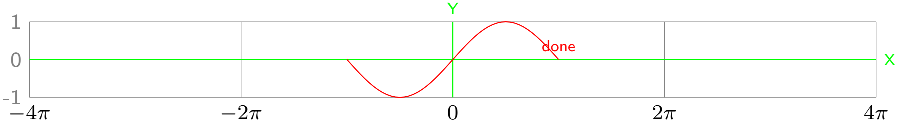

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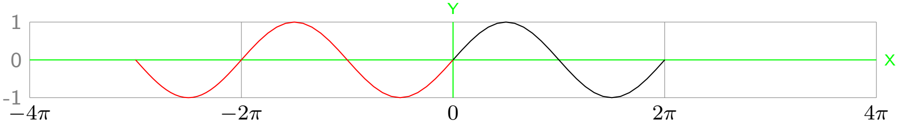

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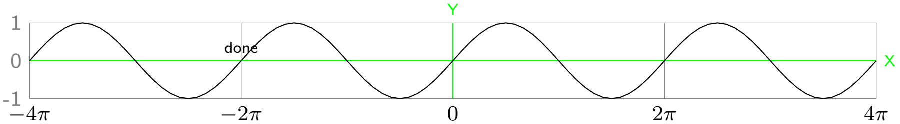

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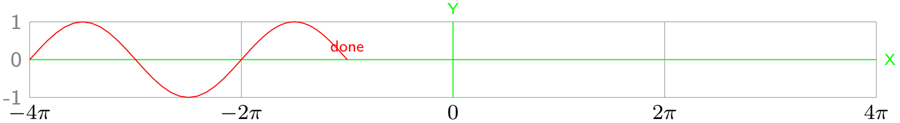

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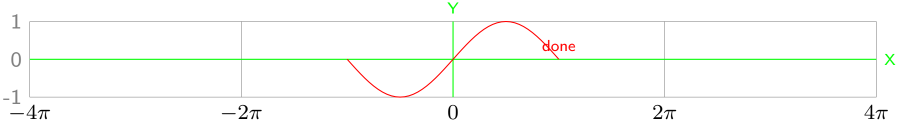

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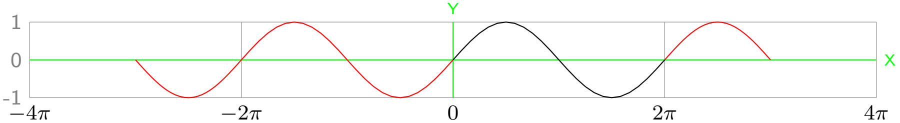

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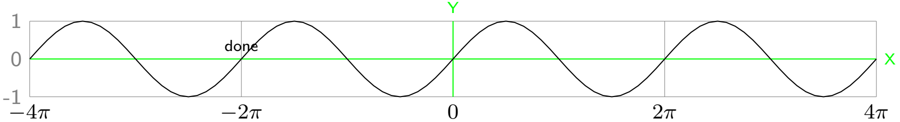

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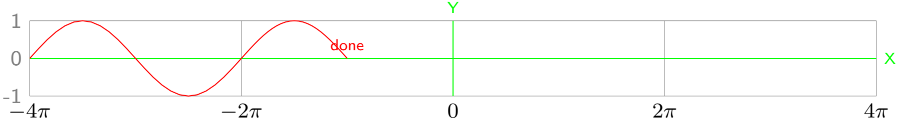

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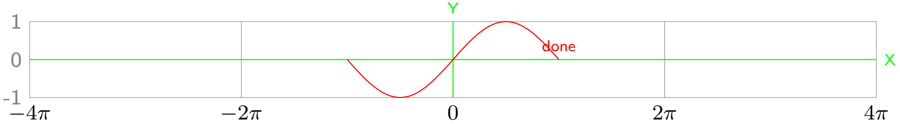

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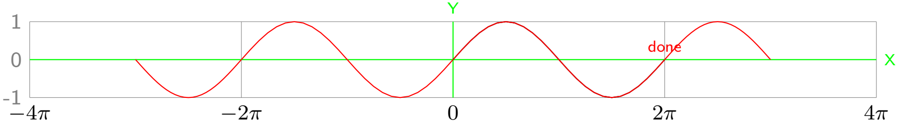
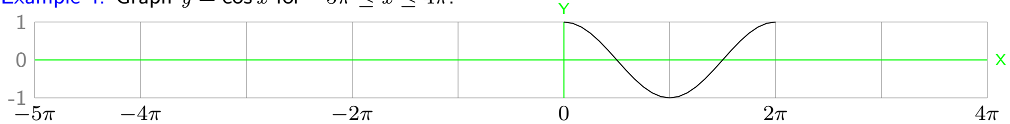


Graphing sine and cosine with a variety of domains

Example 4: Graph $y = \cos x$ for $-5\pi \le x \le 4\pi$.



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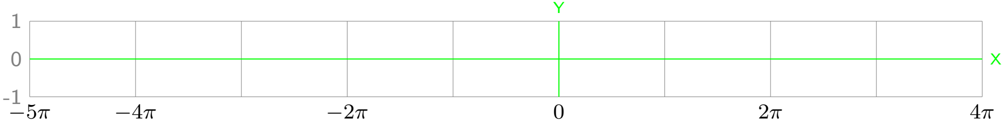

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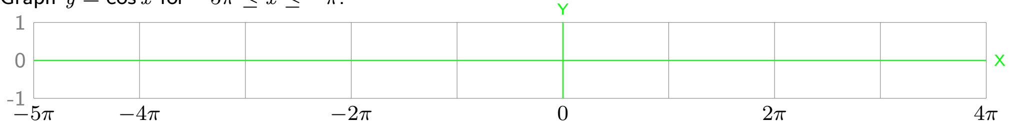

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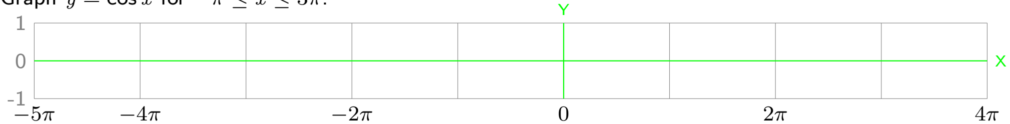

Graphing sine and cosine with a variety of domains

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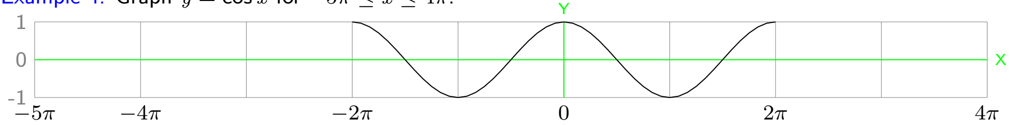

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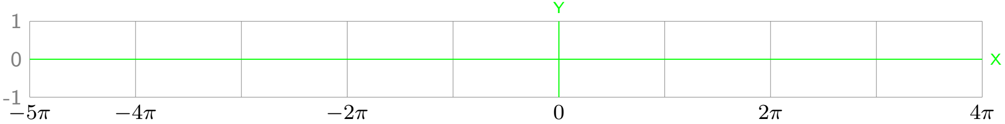

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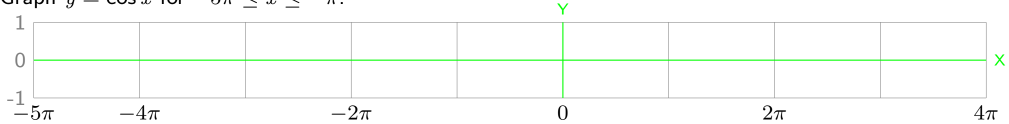

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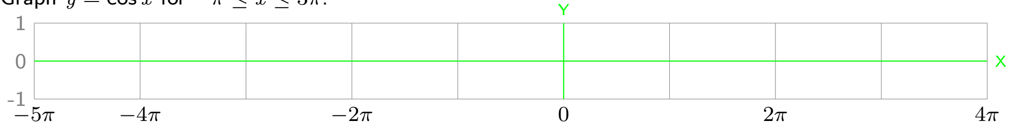

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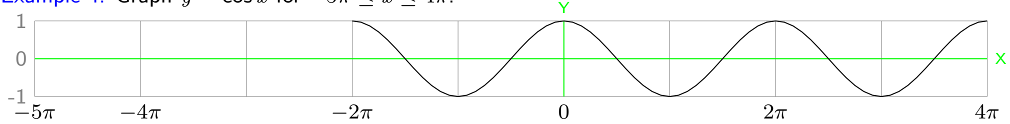

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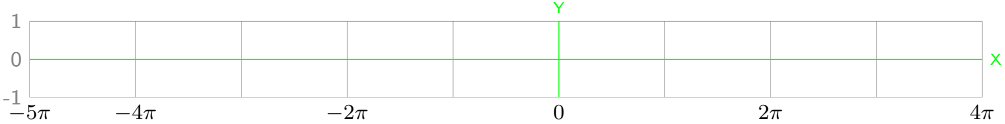

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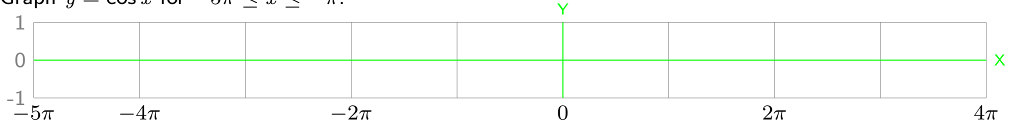

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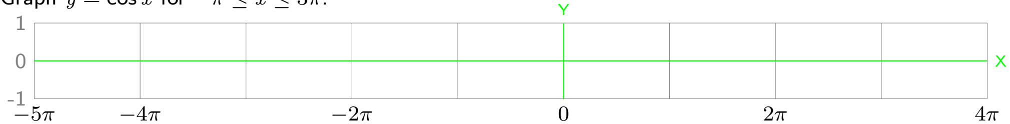

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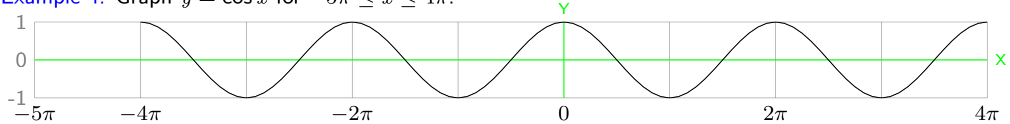

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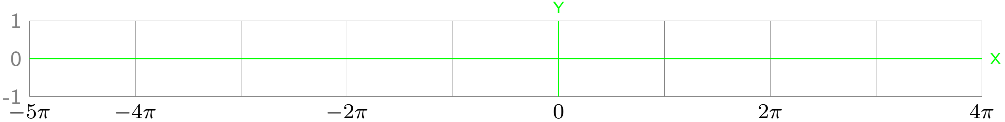

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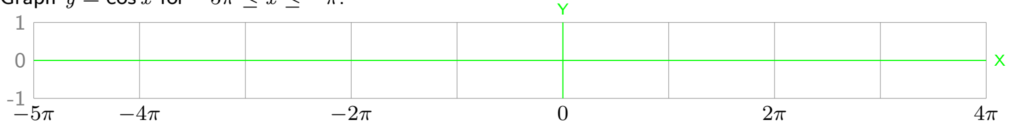

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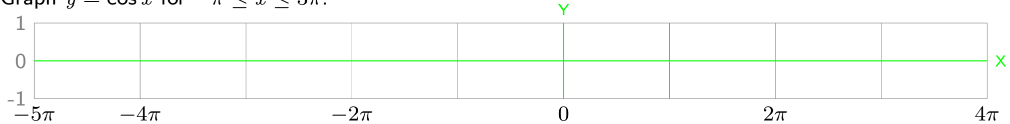

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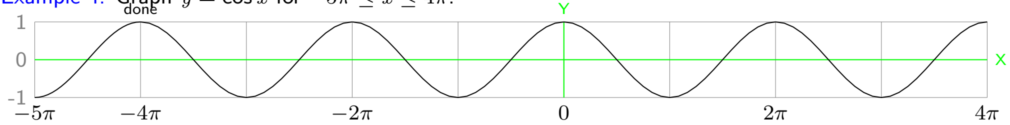

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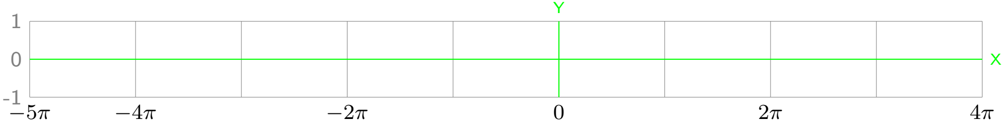

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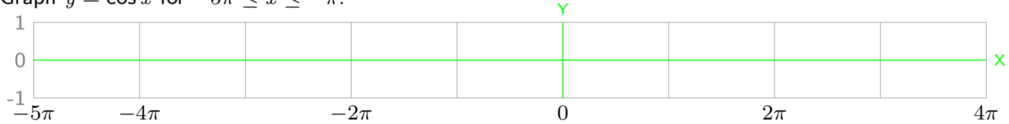

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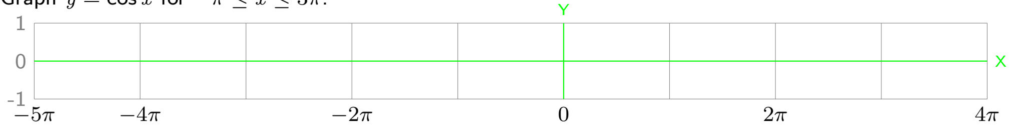

Graphing sine and cosine with a variety of domains

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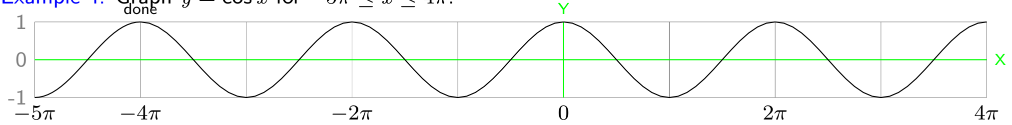

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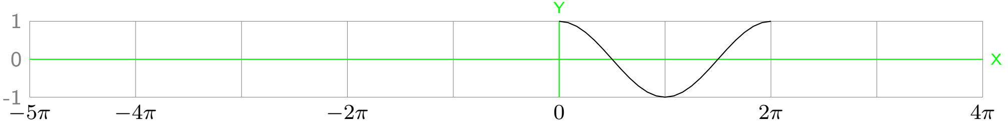

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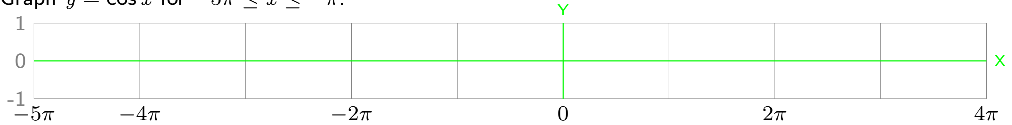

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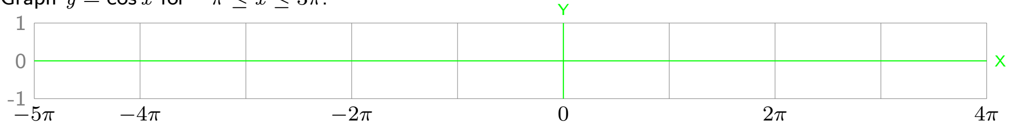

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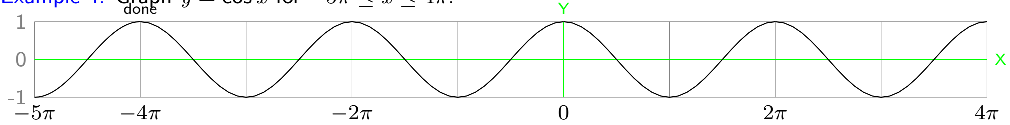

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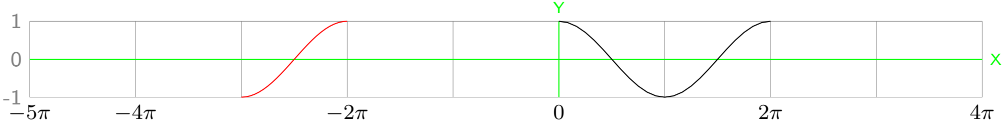

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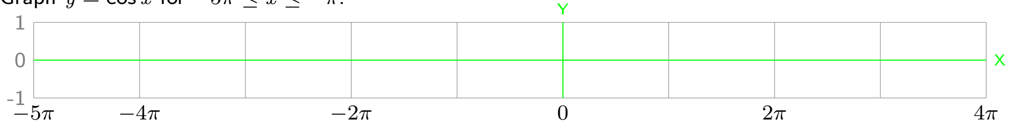

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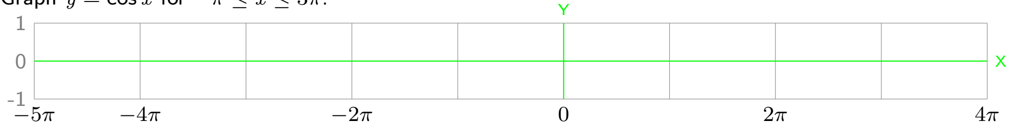

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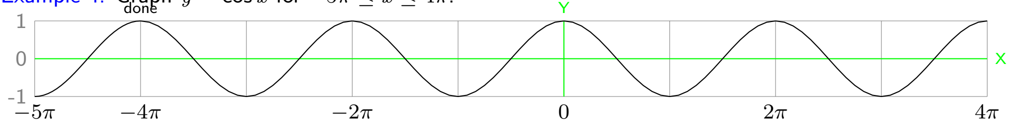

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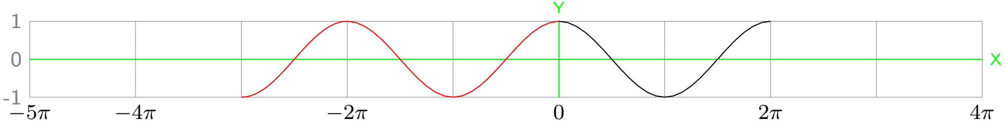

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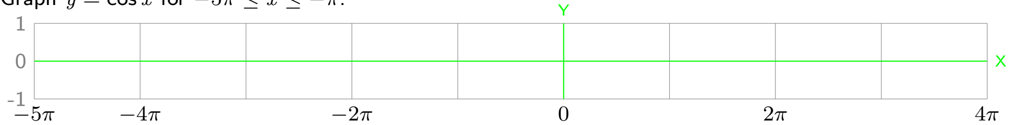

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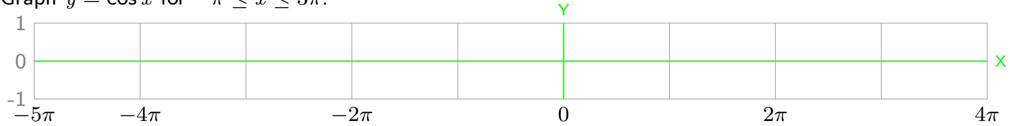

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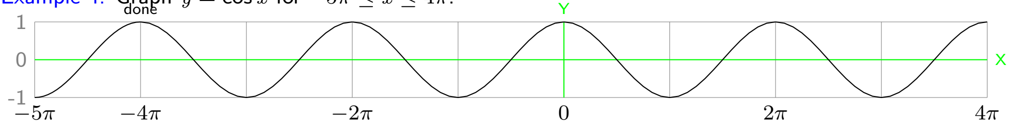

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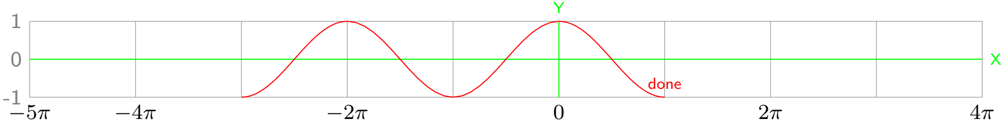

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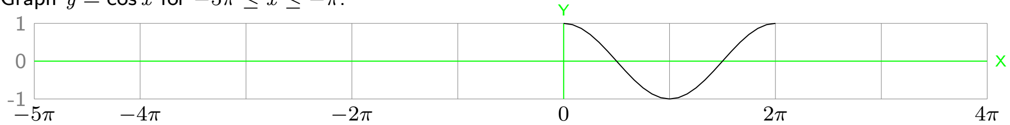

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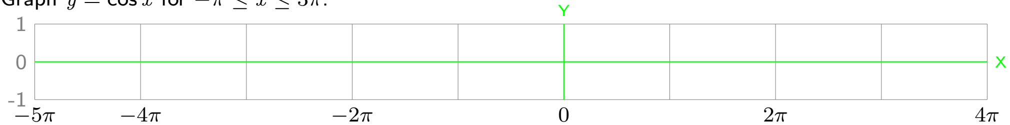

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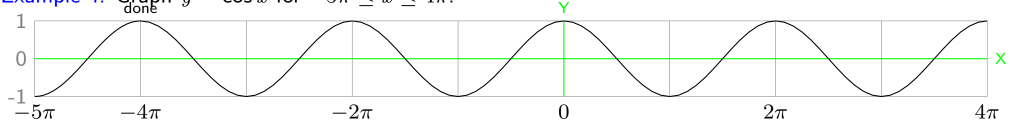

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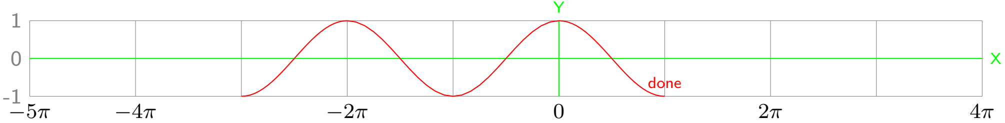

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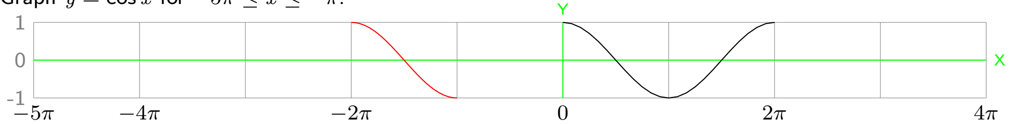
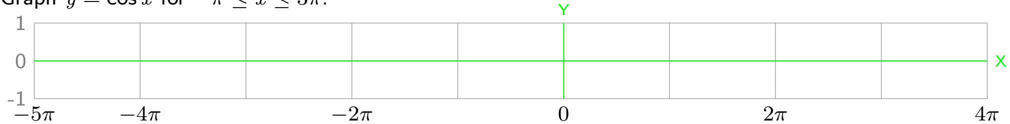
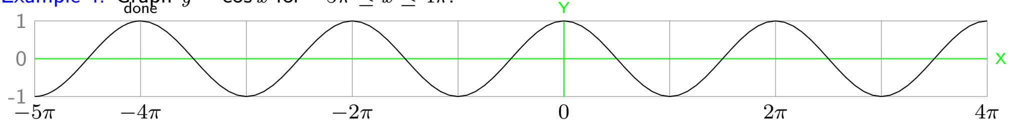


Graph $y = \cos x$ for $-\pi \leq x \leq 3\pi$.

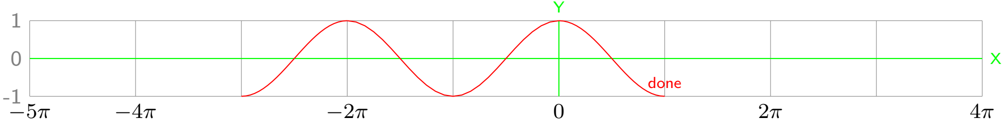


Graphing sine and cosine with a variety of domains

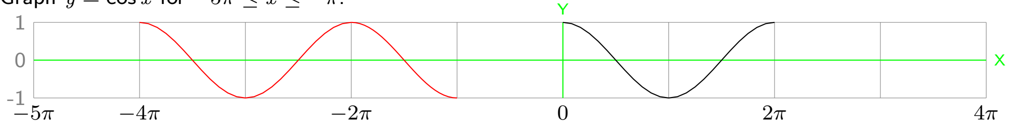
Example 4: Graph $y = \cos x$ for $-5\pi \leq x \leq 4\pi$.



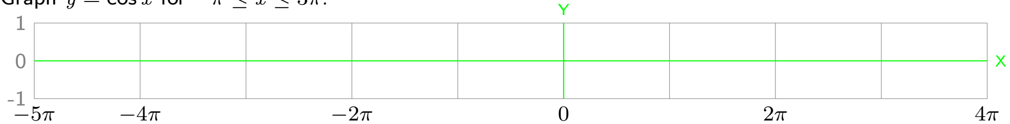
Graph $y = \cos x$ for $-3\pi \leq x \leq \pi$.



Graph $y = \cos x$ for $-5\pi \leq x \leq -\pi$.

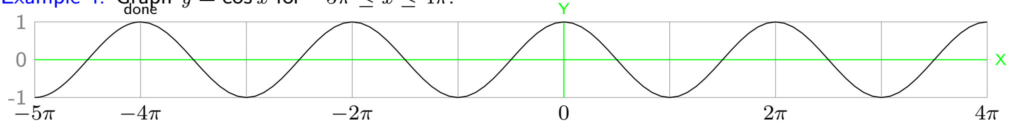


Graph $y = \cos x$ for $-\pi \leq x \leq 3\pi$.

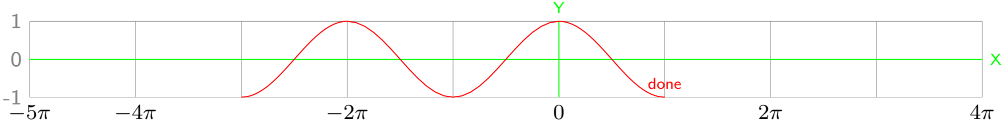


Graphing sine and cosine with a variety of domains

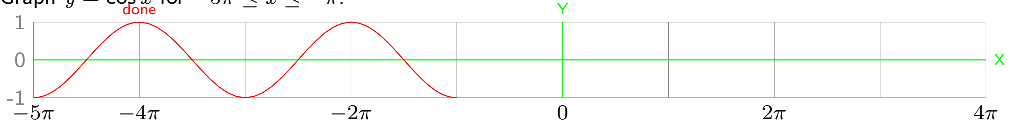
Example 4: Graph $y = \cos x$ for $-5\pi \le x \le 4\pi$.



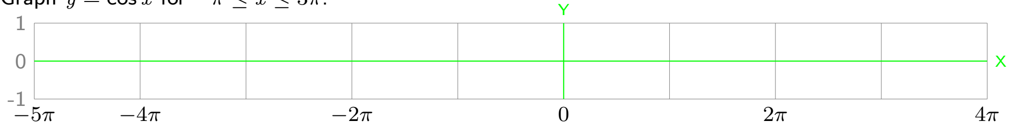
Graph $y = \cos x$ for $-3\pi \le x \le \pi$.



Graph $y = \cos x$ for $-5\pi \le x \le -\pi$.



Graph $y = \cos x$ for $-\pi \le x \le 3\pi$.



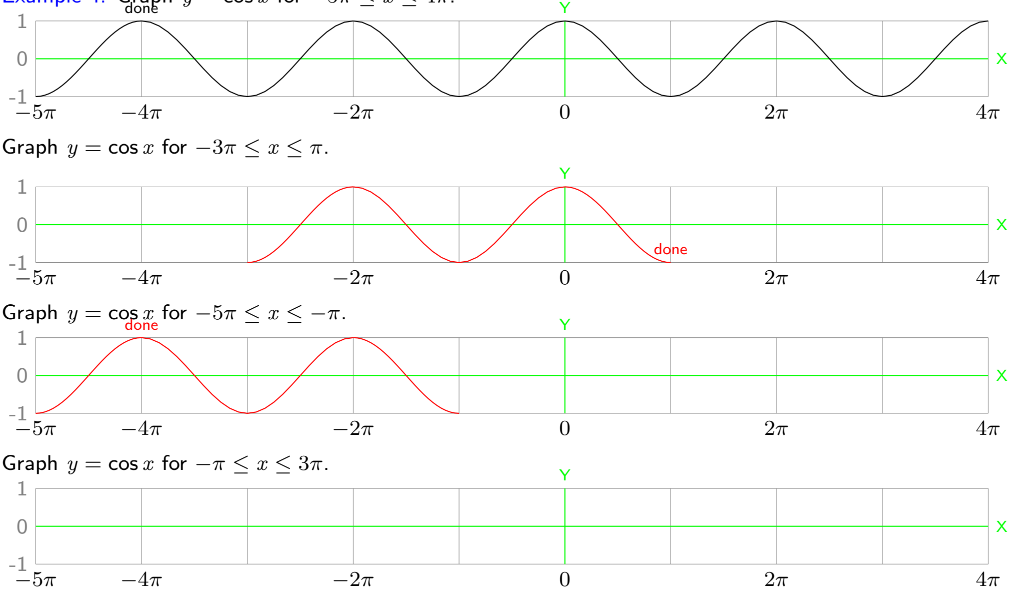
Graphing sine and cosine with a variety of domains

Example 4: Graph $y = \cos x$ for $-5\pi \le x \le 4\pi$.

Graph $y = \cos x$ for $-3\pi \le x \le \pi$.

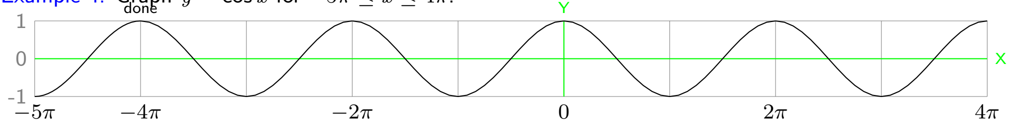
Graph $y = \cos x$ for $-5\pi \le x \le -\pi$.

Graph $y = \cos x$ for $-\pi \le x \le 3\pi$.

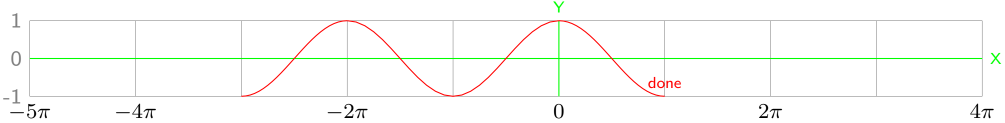


Graphing sine and cosine with a variety of domains

Example 4: Graph $y = \cos x$ for $-5\pi \le x \le 4\pi$.



Graph $y = \cos x$ for $-3\pi \le x \le \pi$.



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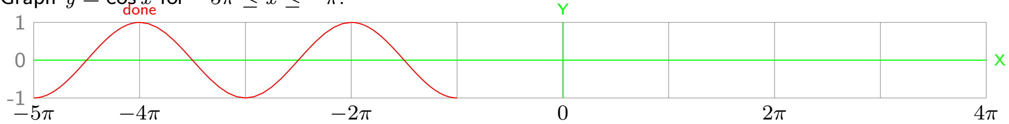

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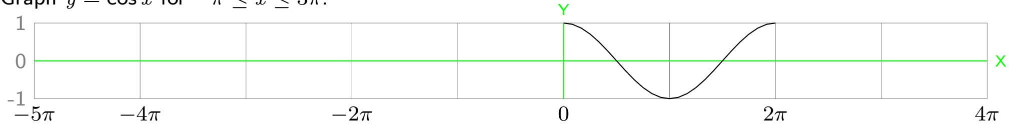

Graphing sine and cosine with a variety of domains

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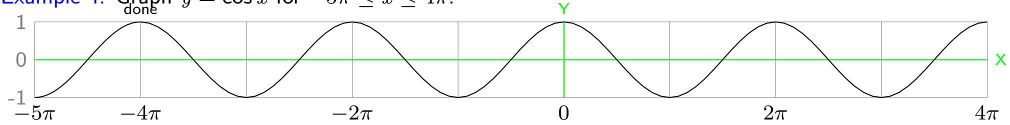

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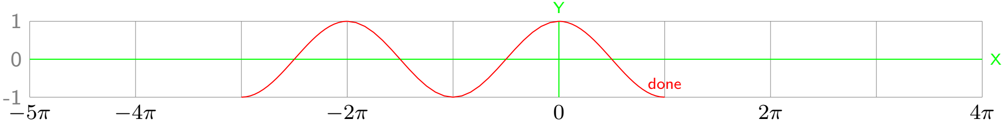

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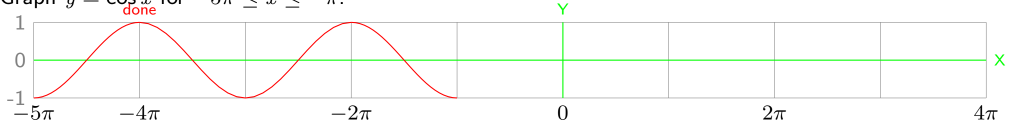

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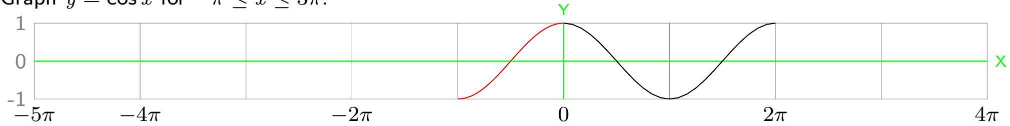
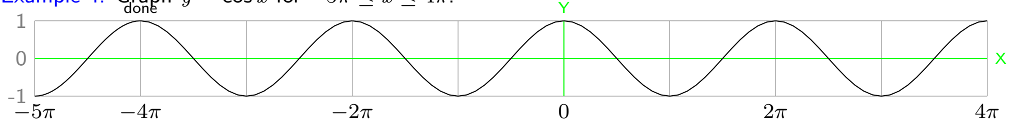


Graphing sine and cosine with a variety of domains

Example 4: Graph $y = \cos x$ for $-5\pi \le x \le 4\pi$.



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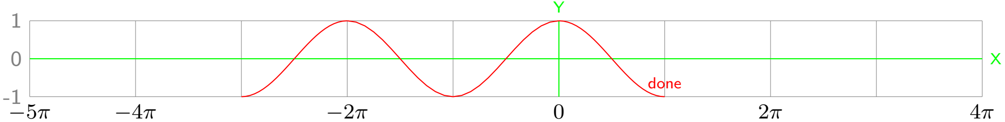
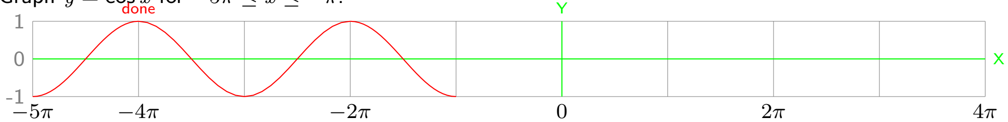
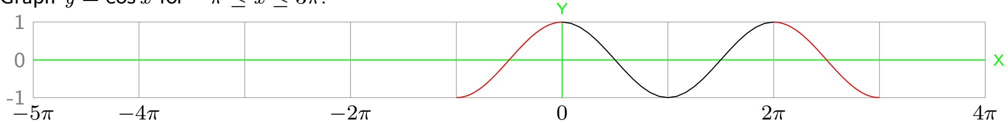


Graph $y = \cos x$ for $-5\pi \le x \le -\pi$.

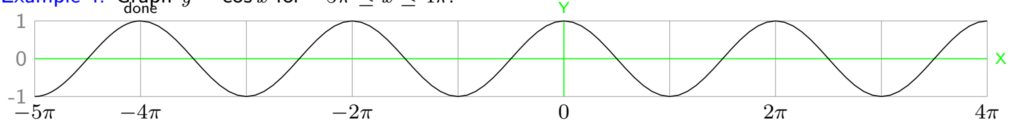


Graph $y = \cos x$ for $-\pi \le x \le 3\pi$.



Graphing sine and cosine with a variety of domains

Example 4: Graph $y = \cos x$ for $-5\pi \leq x \leq 4\pi$.



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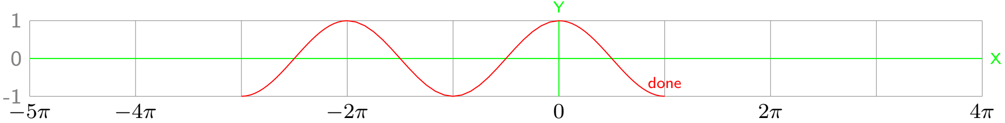

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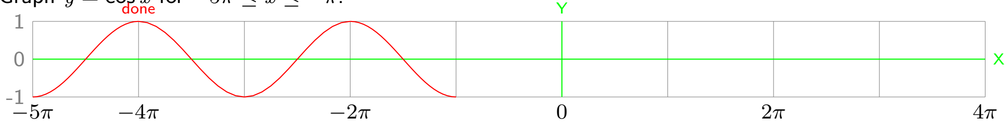

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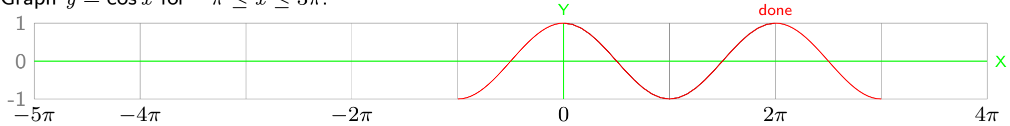
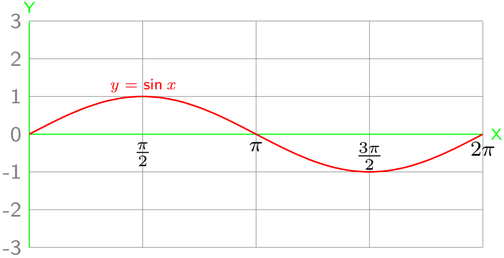


Graphing $y = A\sin x$ and $y = A\cos x$

Example 5: Graph one wave of $y = 3\sin x$



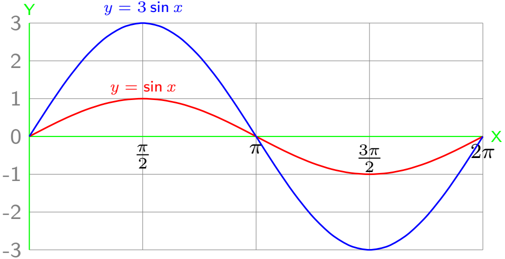
Graph one wave of $y = 2\cos x$

Graph one wave of $y = -2\sin x$

Graph one wave of $y = -3\cos x$

Graphing $y = A\sin x$ and $y = A\cos x$

Example 5: Graph one wave of $y = 3\sin x$



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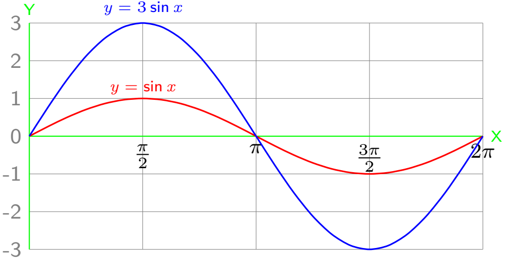

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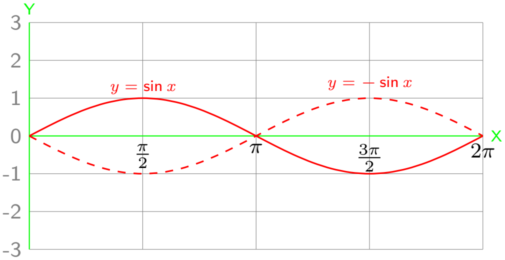

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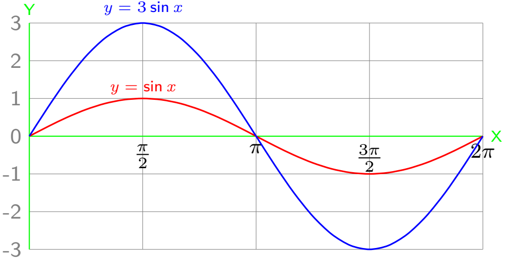

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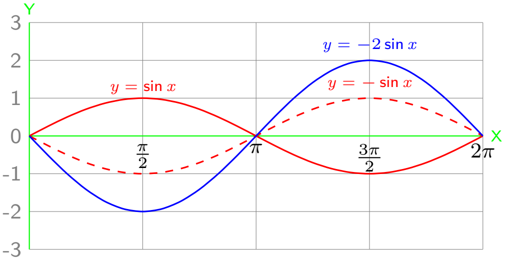

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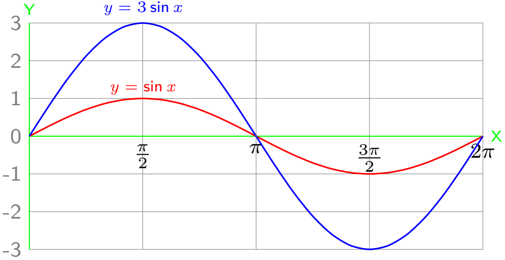

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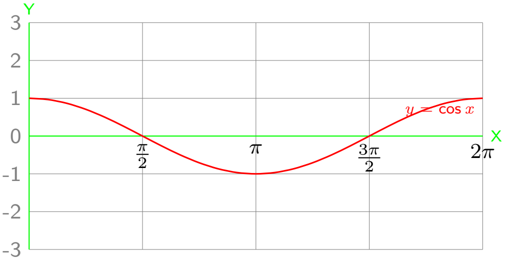
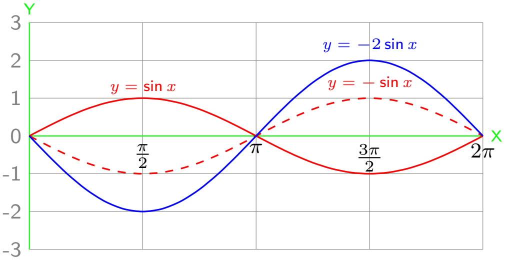


Graph one wave of $y = -2\sin x$



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Graphing $y = A\sin x$ and $y = A\cos x$

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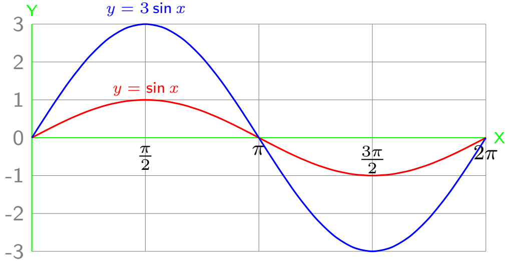
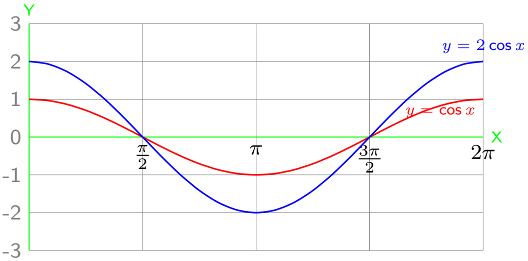
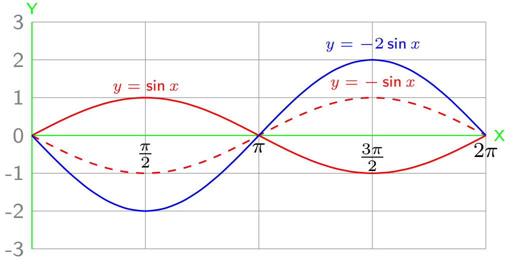


Graph one wave of $y = 2\cos x$



Graph one wave of $y = -2\sin x$



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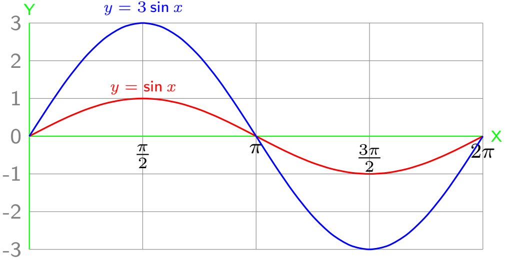

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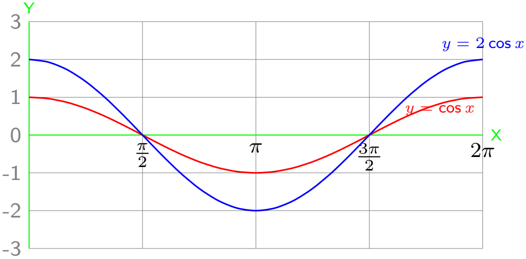
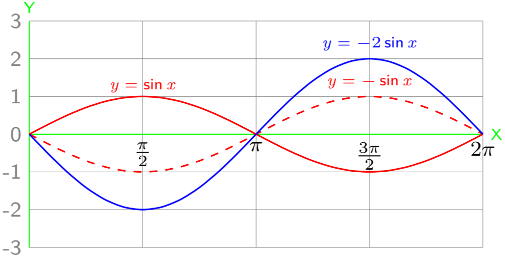
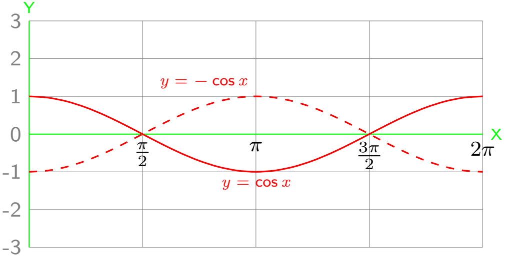


Graph one wave of $y = -2\sin x$



Graph one wave of $y = -3\cos x$



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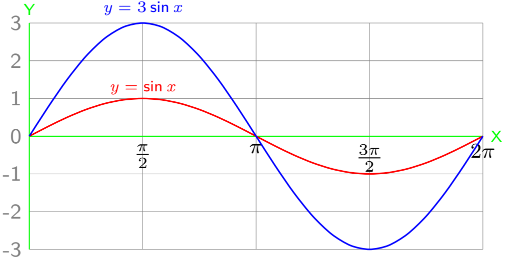
Example 5: Graph one wave of $y = 3 \sin x$



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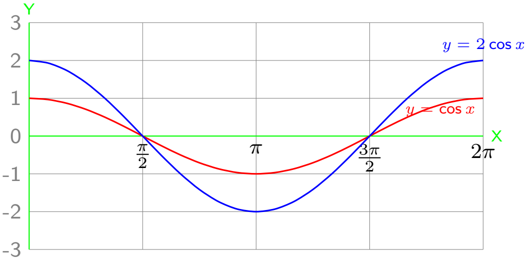
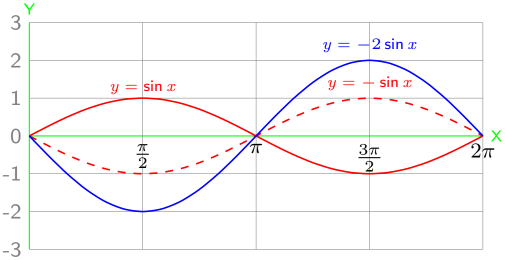
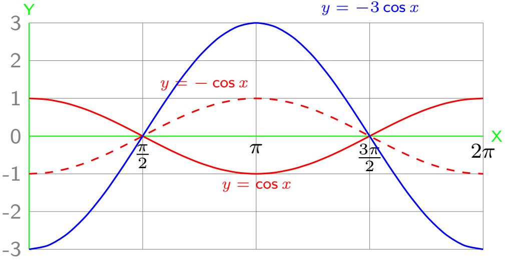


Graph one wave of $y = -2 \sin x$

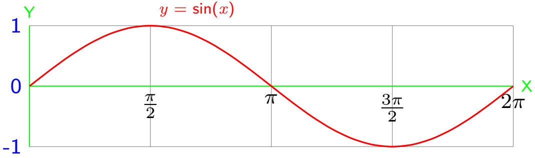


Graph one wave of $y = -3 \cos x$



Graphing $y = \sin Bx$ and $y = \cos Bx$

Example 6 : Graph one wave of $y = \sin 2x$



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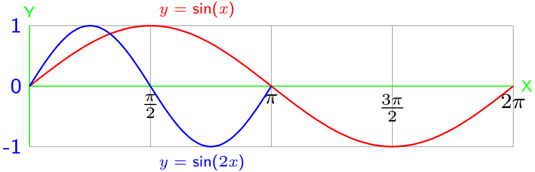

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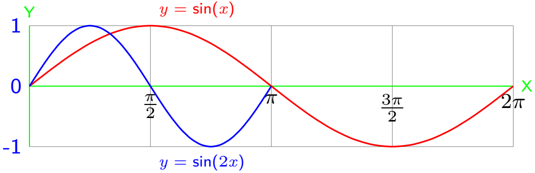
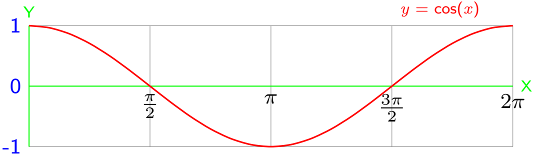


Graph one wave of $y = \cos 2x$



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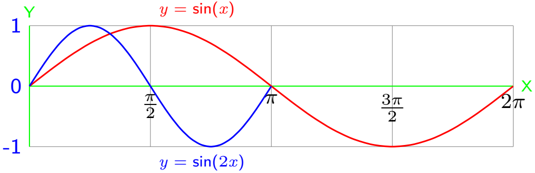

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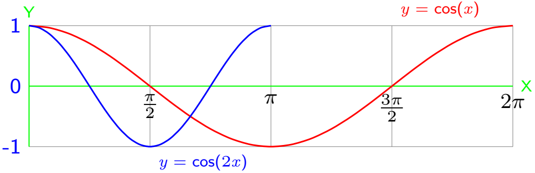

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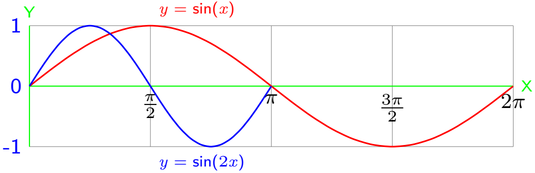

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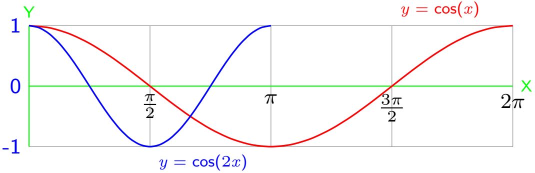
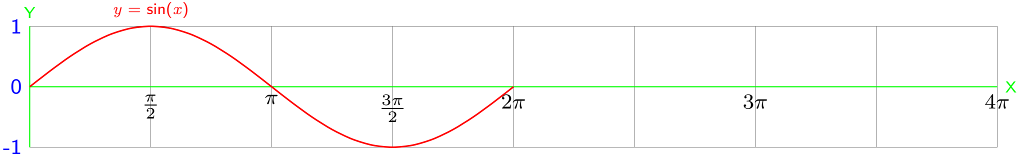
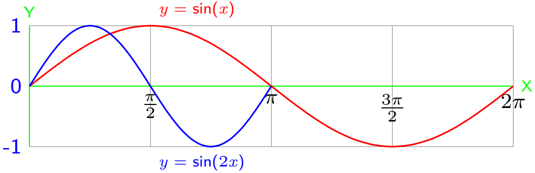


Graph one wave of $y = \sin \frac{x}{2}$

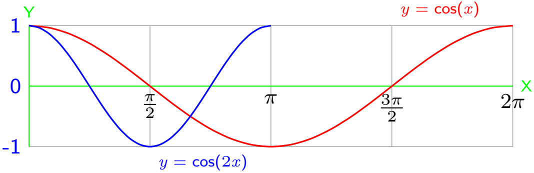


Graphing $y = \sin Bx$ and $y = \cos Bx$

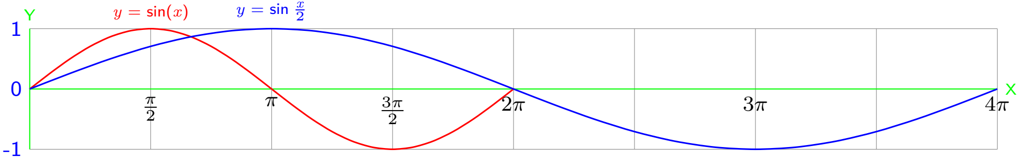
Example 6 : Graph one wave of $y = \sin 2x$



Graph one wave of $y = \cos 2x$

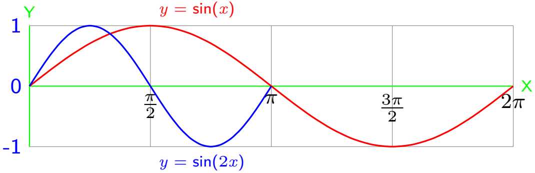


Graph one wave of $y = \sin \frac{x}{2}$

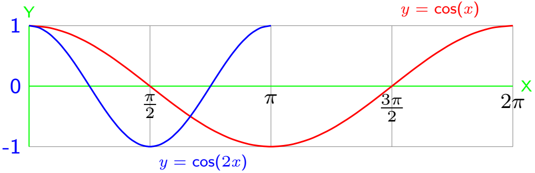


Graphing $y = \sin Bx$ and $y = \cos Bx$

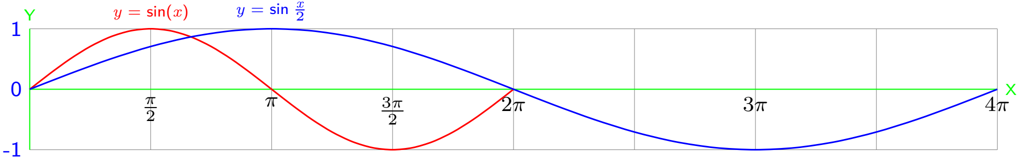
Example 6 : Graph one wave of $y = \sin 2x$



Graph one wave of $y = \cos 2x$



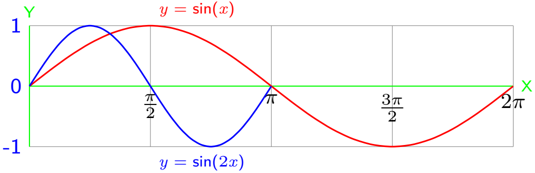
Graph one wave of $y = \sin \frac{x}{2}$



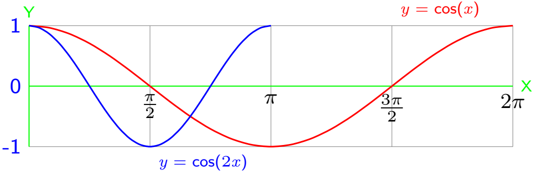
Graph one wave of $y = -\cos \frac{x}{2}$

Graphing $y = \sin Bx$ and $y = \cos Bx$

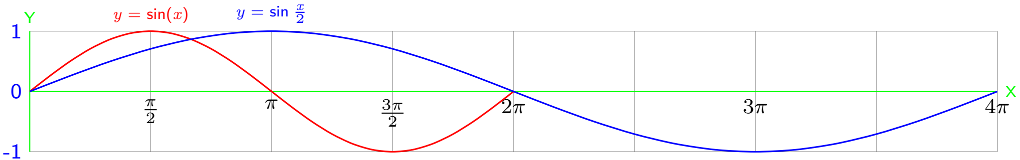
Example 6 : Graph one wave of $y = \sin 2x$



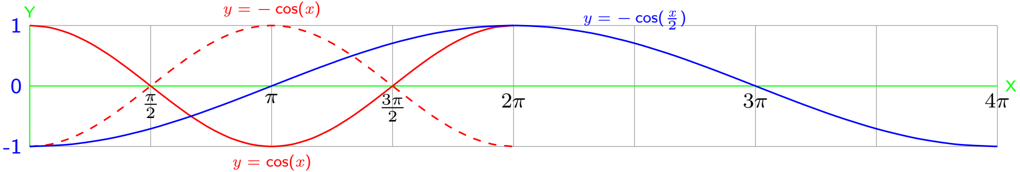
Graph one wave of $y = \cos 2x$



Graph one wave of $y = \sin \frac{x}{2}$



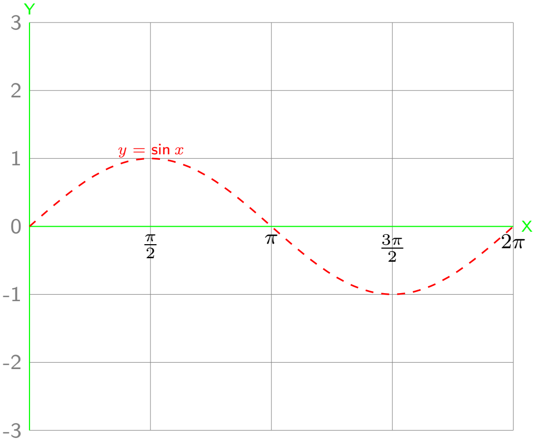
Graph one wave of $y = -\cos \frac{x}{2}$



4.4.6a Graphing the standard wave

The *standard wave* of graph $y = A\sin(Bx)$ or $y = A\cos(Bx)$ is obtained by restricting x

to the domain $0 \le x \le 2\pi/|B|$ so that the angle $|Bx|$ goes from 0 to 2π.

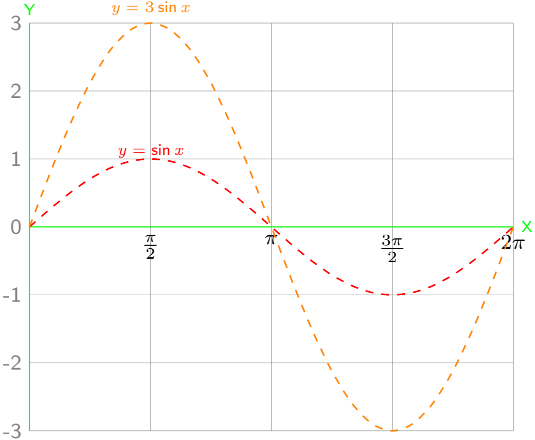


Example 7: Starting with the standard wave of $y = \sin x$, graph the standard wave of $y = -3\sin(2x)$.

4.4.6a Graphing the standard wave

The *standard wave* of graph $y = A\sin(Bx)$ or $y = A\cos(Bx)$ is obtained by restricting x

to the domain $0 \leq x \leq 2\pi/|B|$ so that the angle $|Bx|$ goes from 0 to 2π.



Example 7: Starting with the standard wave of $y = \sin x$, graph the standard wave of $y = -3\sin(2x)$.

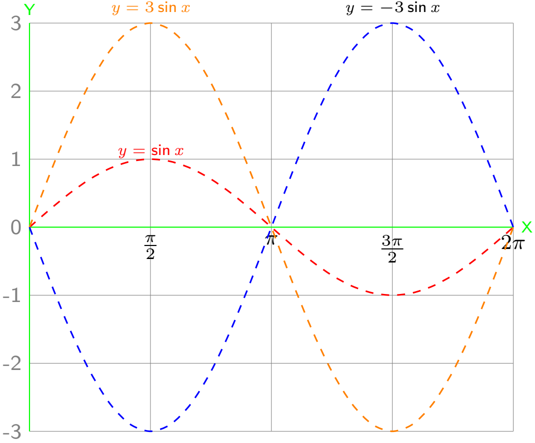
Solution:

- Multiply the RHS of $y = \sin x$ by 3 to stretch it vertically, away fromthe $x-$axis, by a factor of 3 to the orange graph of $y = 3\sin x$.

4.4.6a Graphing the standard wave

The *standard wave* of graph $y = A\sin(Bx)$ or $y = A\cos(Bx)$ is obtained by restricting x

to the domain $0 \le x \le 2\pi/|B|$ so that the angle $|Bx|$ goes from 0 to 2π.



Example 7: Starting with the standard wave of $y = \sin x$, graph the standard wave of $y = -3\sin(2x)$.

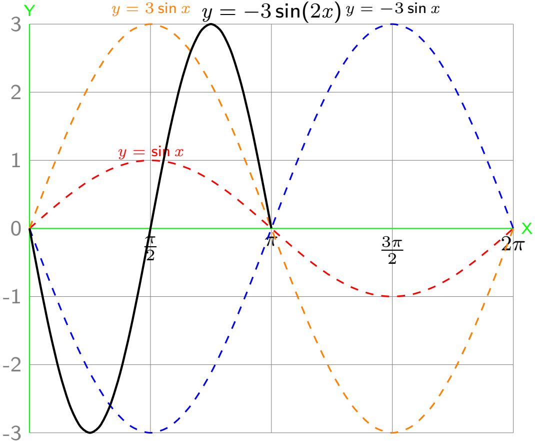
Solution:

- Multiply the RHS of $y = \sin x$ by 3 to stretch it vertically, away fromthe $x-$axis, by a factor of 3 to the orange graph of $y = 3\sin x$.
- Multiply the RHS of the equation $y = 3\sin x$ by -1 to reflect its graph through the$x-$axis to the blue graph of $y = -3\sin x$.

4.4.6a Graphing the standard wave

The *standard wave* of graph $y = A\sin(Bx)$ or $y = A\cos(Bx)$ is obtained by restricting x

to the domain $0 \le x \le 2\pi/|B|$ so that the angle $|Bx|$ goes from 0 to 2π.



Example 7: Starting with the standard wave of $y = \sin x$, graph the standard wave of $y = -3\sin(2x)$.

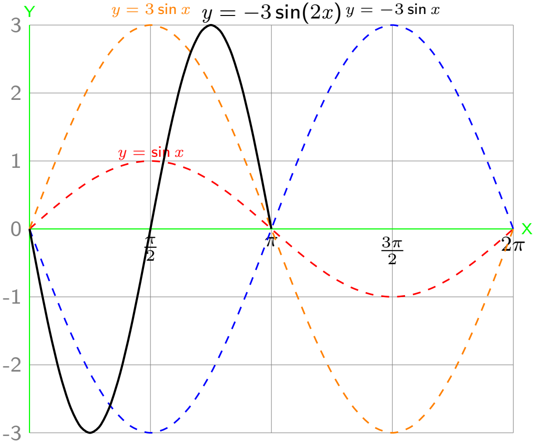
Solution:

- Multiply the RHS of $y = \sin x$ by 3 to stretch it vertically, away fromthe $x-$axis, by a factor of 3 to the orange graph of $y = 3\sin x$.
- Multiply the RHS of the equation $y = 3\sin x$ by -1 to reflect its graph through the$x-$axis to the blue graph of $y = -3\sin x$.
- Replace x in the equation $y = -3\sin x$ by $2x$. The graph of $y = -3\sin x$ shrinks by a factor of 2 in the x-direction to obtain the black graph at the left.

4.4.6a Graphing the standard wave

The *standard wave* of graph $y = A\sin(Bx)$ or $y = A\cos(Bx)$ is obtained by restricting x

to the domain $0 \le x \le 2\pi/|B|$ so that the angle $|Bx|$ goes from 0 to 2π.



Example 7: Starting with the standard wave of $y = \sin x$, graph the standard wave of $y = -3\sin(2x)$.

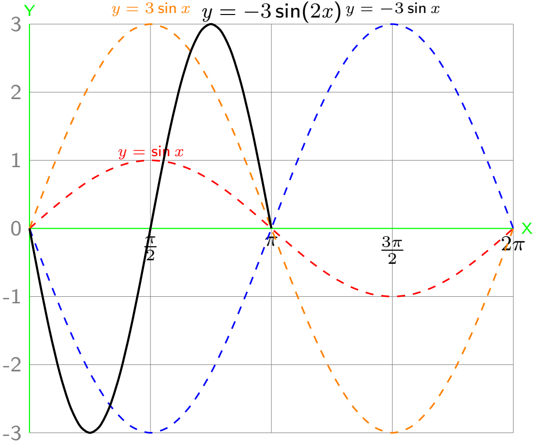
Solution:

- Multiply the RHS of $y = \sin x$ by 3 to stretch it vertically, away fromthe $x-$axis, by a factor of 3 to the orange graph of $y = 3\sin x$.
- Multiply the RHS of the equation $y = 3\sin x$ by -1 to reflect its graph through the$x-$axis to the blue graph of $y = -3\sin x$.
- Replace x in the equation $y = -3\sin x$ by $2x$. The graph of $y = -3\sin x$ shrinks by a factor of 2 in the x-direction to obtain the black graph at the left.
- The original graph of $y = \sin x$ has been stretched by a factor of 3 in the y-direction, reflected through the x-axis, and shrunk by a factor of 2 in the x-direction.

4.4.6a Graphing the standard wave

The *standard wave* of graph $y = A\sin(Bx)$ or $y = A\cos(Bx)$ is obtained by restricting x

to the domain $0 \le x \le 2\pi/|B|$ so that the angle $|Bx|$ goes from 0 to 2π.



Example 7: Starting with the standard wave of $y = \sin x$, graph the standard wave of $y = -3\sin(2x)$.

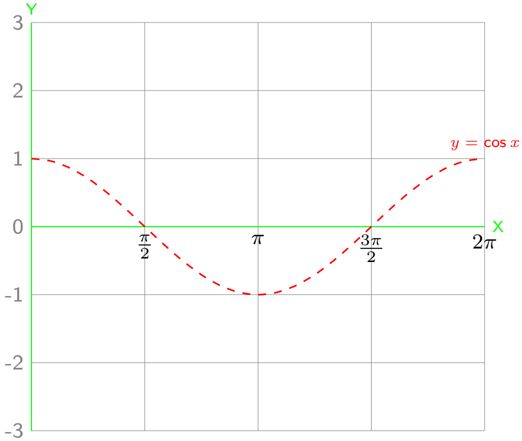
Solution:

- Multiply the RHS of $y = \sin x$ by 3 to stretch it vertically, away fromthe $x-$axis, by a factor of 3 to the orange graph of $y = 3\sin x$.
- Multiply the RHS of the equation $y = 3\sin x$ by -1 to reflect its graph through the$x-$axis to the blue graph of $y = -3\sin x$.
- Replace x in the equation $y = -3\sin x$ by $2x$. The graph of $y = -3\sin x$ shrinks by a factor of 2 in the x-direction to obtain the black graph at the left.
- The original graph of $y = \sin x$ has been stretched by a factor of 3 in the y-direction, reflected through the x-axis, and shrunk by a factor of 2 in the x-direction.
- The resulting black graph shows a complete wave of the sine curve $y = -3\sin(2x)$.

  However, the domain has shrunk to $0 \le x \le \pi$.

4.4.6b Graphing the standard wave

Example 7b: Starting with the standard wave of $y = \cos x$, graph the standard wave of $y = 3\cos(3x)$.

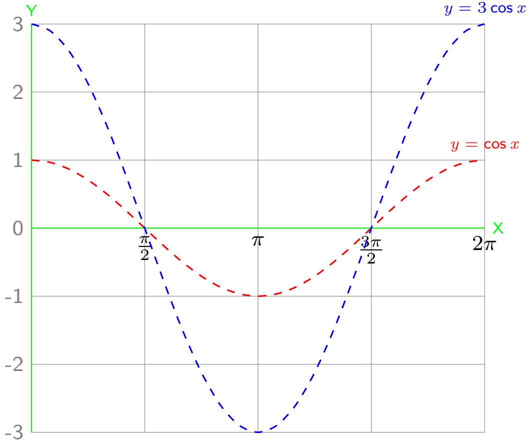


4.4.6b Graphing the standard wave

Example 7b: Starting with the standard wave of $y = \cos x$, graph the standard wave of $y = 3\cos(3x)$.

Solution:

- Multiply the RHS of the equation $y = \cos x$ by 3 to get the new equation $y = 3\cos x$. Multiply each y-coordinate on the red graph by 3 to obtain the blue graph.

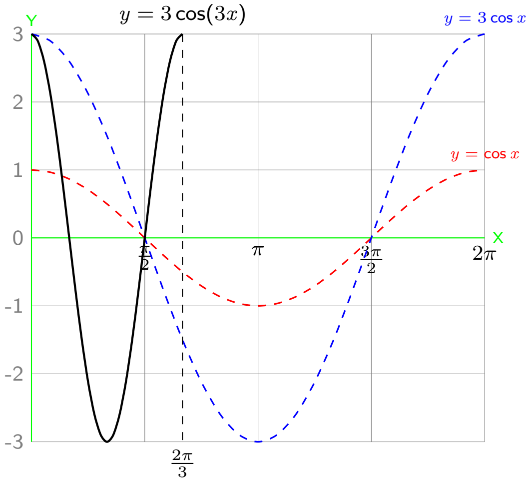


4.4.6b Graphing the standard wave

Example 7b: Starting with the standard wave of $y = \cos x$, graph the standard wave of $y = 3\cos(3x)$.

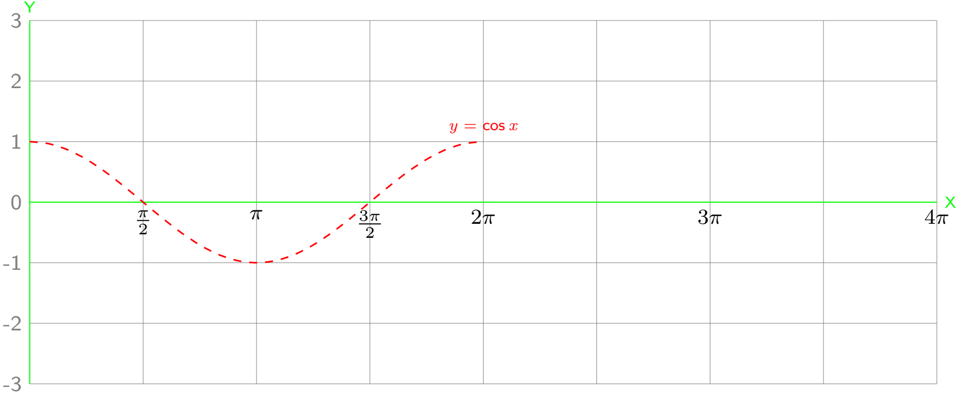
Solution:

- Multiply the RHS of the equation $y = \cos x$ by 3 to get the new equation $y = 3\cos x$. Multiply each y-coordinate on the red graph by 3 to obtain the blue graph.
- Replace x in the equation $y = 3\cos x$ by $3x$ to shrink its graph by a factor of 3 toward the x-axis. The result is the standard wave of $y = 3\cos 3x$, with domain $0 \le x \le \frac{2\pi}{3}$



4.4.6c Graphing the standard wave

Example 7c: Starting with the standard wave of $y = \cos x$, graph the standard wave of $y = 3\cos(\frac{x}{2})$.

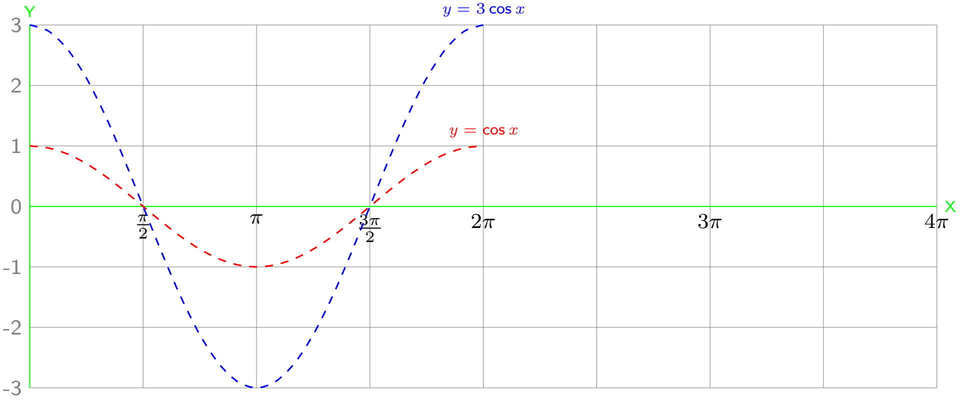


4.4.6c Graphing the standard wave

Example 7c: Starting with the standard wave of $y = \cos x$, graph the standard wave of $y = 3\cos(\frac{x}{2})$.

Solution:

- Multiply the RHS of the equation $y = \cos x$ by 3 to get the new equation $y = 3\cos x$. Multiply each y-coordinate on the red graph by 2 to obtain the blue graph.

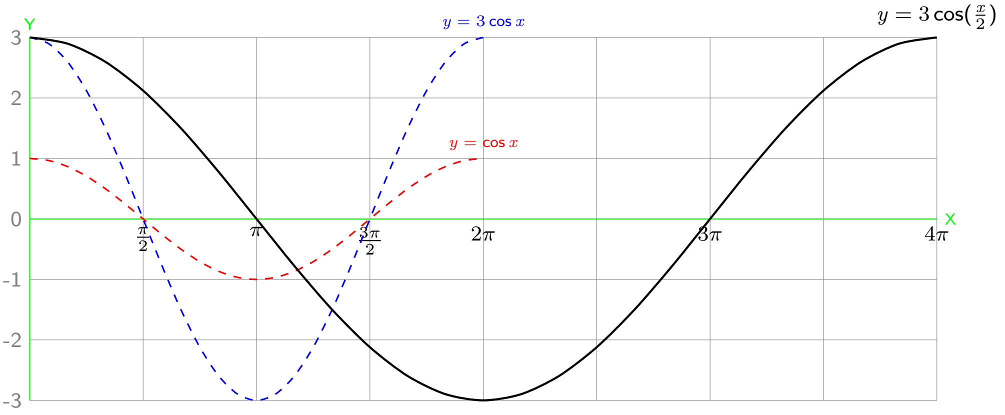


4.4.6c Graphing the standard wave

Example 7c: Starting with the standard wave of $y = \cos x$, graph the standard wave of $y = 3\cos(\frac{x}{2})$.

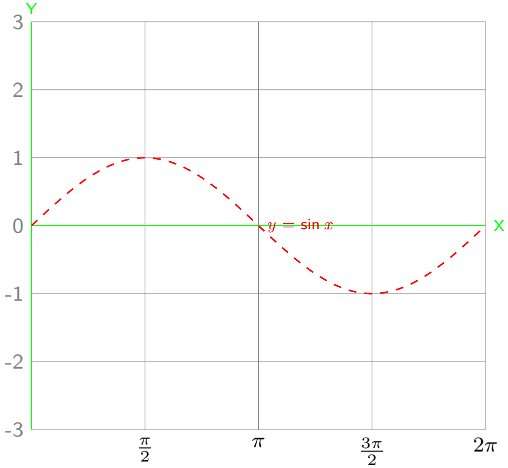
Solution:

- Multiply the RHS of the equation $y = \cos x$ by 3 to get the new equation $y = 3\cos x$. Multiply each y-coordinate on the red graph by 2 to obtain the blue graph.
- Replace x in the equation $y = 3\cos x$ by $\frac{x}{2}$ to stretch its graph by a factor of 2 in the x-direction. Done!



4.4.6d Graphing the standard wave

Example 7d: Starting with the standard wave of $y = \sin x$, graph the standard wave of $y = 3\sin(-3x)$.

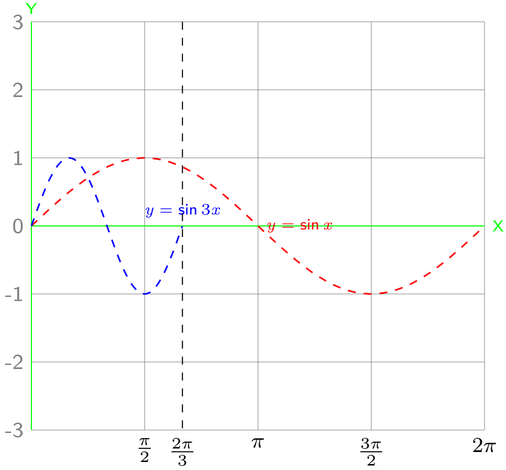


4.4.6d Graphing the standard wave

Example 7d: Starting with the standard wave of $y = \sin x$, graph the standard wave of $y = 3\sin(-3x)$.

Solution: When graphing the standard wave of $\sin -3x = \sin Bx$ with $B < 0$, graph instead the standard wave of $y = -\sin|B|x = -\sin 3x$.

- Start with $y = \sin x$ and shrink it by a factor of 3 toward the y-axis to get $y = \sin 3x$

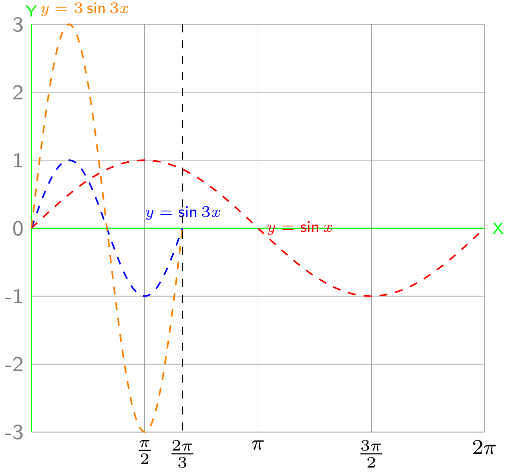


4.4.6d Graphing the standard wave

Example 7d: Starting with the standard wave of $y = \sin x$, graph the standard wave of $y = 3\sin(-3x)$.

Solution: When graphing the standard wave of $\sin -3x = \sin Bx$ with $B < 0$, graph instead the standard wave of $y = -\sin |B|x = -\sin 3x$.

- Start with $y = \sin x$ and shrink it by a factor of 3 toward the y-axis to get $y = \sin 3x$
- Stretch away from the $x-$axis by a factor of 3 to get $y = 3\sin 3x$.

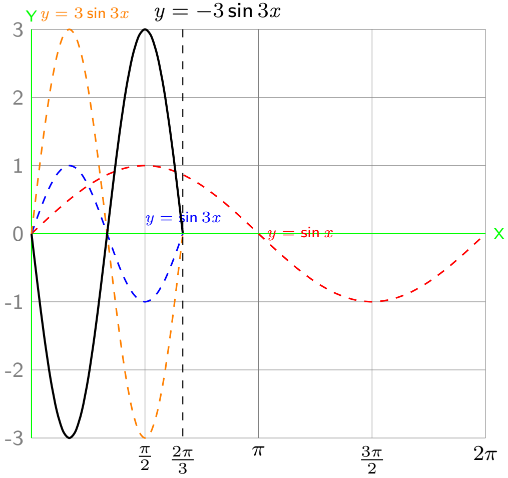


4.4.6d Graphing the standard wave

Example 7d: Starting with the standard wave of $y = \sin x$, graph the standard wave of $y = 3\sin(-3x)$.

Solution: When graphing the standard wave of $\sin -3x = \sin Bx$ with $B < 0$, graph instead the standard wave of $y = -\sin|B|x = -\sin 3x$.

- Start with $y = \sin x$ and shrink it by a factor of 3 toward the y-axis to get $y = \sin 3x$
- Stretch away from the $x-$axis by a factor of 3 to get $y = 3\sin 3x$.
- Reflect across the x-axis to get $y = -3\sin 3x = 3\sin(-3x)$, the requested standard wave, with domain $0 \le x \le \frac{2\pi}{3}$.

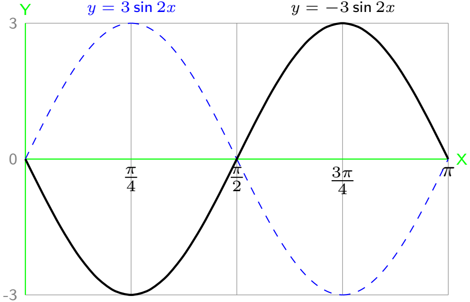


Example 8a: Sketch one wave of the graph of $y = -3\sin(2x)$ in one step, by first finding the domain and range of the standard wave.

Example 8a: Sketch one wave of the graph of $y = -3\sin(2x)$ in one step, by first finding the domain and range of the standard wave.

Solution: The domain is $0 \le x \le 2\pi/B$, i.e., $0 \le x \le \pi$.
The range is $-|A| \le y \le |A|$, i.e., $-3 \le y \le 3$.

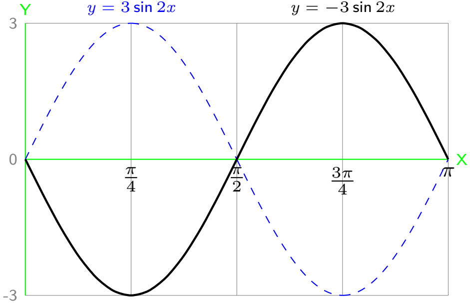
- Angle x goes from 0 to π. The width of the wave is π.
- Now sketch $y = 3\sin 2x$ by filling up the box with a sine wave.
- Reflect through the x-axis to get the desired graph $y = -3\sin 2x$.



Example 8a: Sketch one wave of the graph of $y = -3\sin(2x)$ in one step, by first finding the domain and range of the standard wave.

Solution: The domain is $0 \le x \le 2\pi/B$, i.e., $0 \le x \le \pi$. The range is $-|A| \le y \le |A|$, i.e., $-3 \le y \le 3$.

- Angle x goes from 0 to π. The width of the wave is π.
- Now sketch $y = 3\sin 2x$ by filling up the box with a sine wave.
- Reflect through the x-axis to get the desired graph $y = -3\sin 2x$.

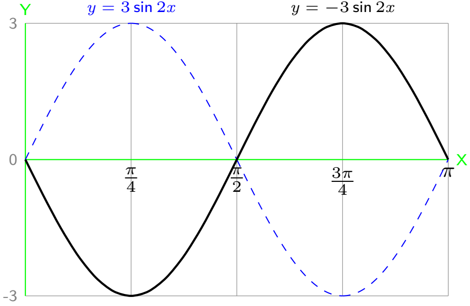


Example 8b: Sketch one wave of the graph of $y = A\sin Bx = \frac{\sin(3x)}{2}$.

Example 8a: Sketch one wave of the graph of $y = -3\sin(2x)$ in one step, by first finding the domain and range of the standard wave.

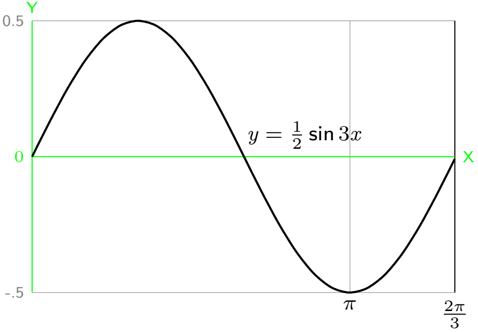
Solution: The domain is $0 \le x \le 2\pi/B$, i.e., $0 \le x \le \pi$.
The range is $-|A| \le y \le |A|$, i.e., $-3 \le y \le 3$.

- Angle x goes from 0 to π. The width of the wave is π.
- Now sketch $y = 3\sin 2x$ by filling up the box with a sine wave.
- Reflect through the x-axis to get the desired graph $y = -3\sin 2x$.



Example 8b: Sketch one wave of the graph of $y = A\sin Bx = \frac{\sin(3x)}{2}$. Solution: Here $A = \frac{1}{2}$ and $B = 3$.
The domain is $0 \le x \le 2\pi/B$, i.e., $0 \le x \le \frac{2\pi}{3}$.
The range is $-|A| \le y \le |A|$, i.e., $-\frac{1}{2} \le y \le \frac{1}{2}$.

- Angle x goes from 0 to $\frac{2\pi}{3}$
  The width of the wave is $\frac{2\pi}{3}$.
- Now sketch $y = \frac{\sin 3x}{2}$ by filling up the box with a sine wave.



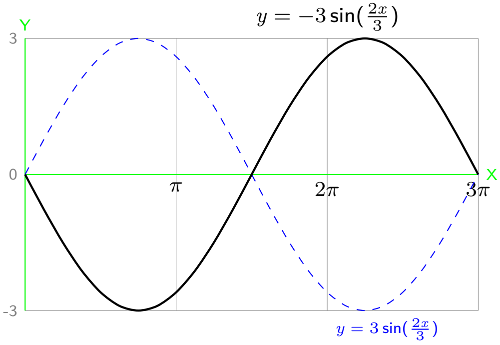
Example 8c: Sketch one wave of the graph of
$y = A \sin Bx = -3 \sin(\frac{2x}{3})$.

Example 8c: Sketch one wave of the graph of
$y = A \sin Bx = -3 \sin(\frac{2x}{3})$. Here $A = -3$ and $B = \frac{2}{3}$.
Solution: The domain is $0 \leq x \leq 2\pi/B$, i.e., $0 \leq x \leq 3\pi$.
The range is $-|A| \leq y \leq |A|$, i.e., $-3 \leq y \leq 3$.

- Angle x goes from 0 to 3π. The width of the wave is 3π.
- Now sketch $y = 3 \sin(\frac{2x}{3})$ by filling up the box with a sine wave.
- Reflect through the x-axis to get the desired graph $y = -3 \sin(\frac{2x}{3})$.

Example 8c: Sketch one wave of the graph of $y = A\sin Bx = -3\sin(\frac{2x}{3})$. Here $A = -3$ and $B = \frac{2}{3}$.
Solution: The domain is $0 \le x \le 2\pi/B$, i.e., $0 \le x \le 3\pi$.
The range is $-|A| \le y \le |A|$, i.e., $-3 \le y \le 3$.

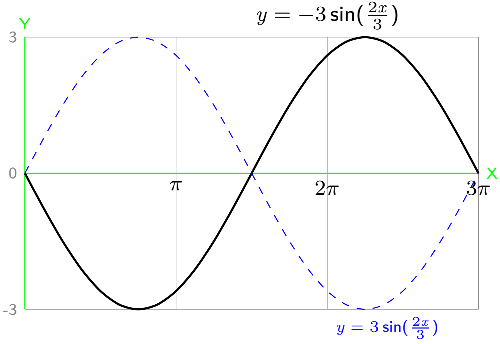
- Angle x goes from 0 to 3π. The width of the wave is 3π.
- Now sketch $y = 3\sin(\frac{2x}{3})$ by filling up the box with a sine wave.
- Reflect through the x-axis to get the desired graph $y = -3\sin(\frac{2x}{3})$.



Example 8c: Sketch one wave of the graph of $y = A\sin Bx = -3\sin(\frac{2x}{3})$. Here $A = -3$ and $B = \frac{2}{3}$.

Solution: The domain is $0 \le x \le 2\pi/B$, i.e., $0 \le x \le 3\pi$. The range is $-|A| \le y \le |A|$, i.e., $-3 \le y \le 3$.

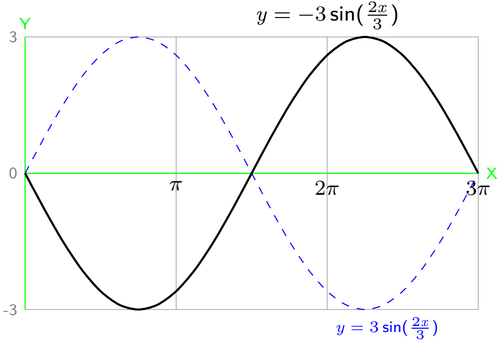
- Angle x goes from 0 to 3π. The width of the wave is 3π.
- Now sketch $y = 3\sin(\frac{2x}{3})$ by filling up the box with a sine wave.
- Reflect through the x-axis to get the desired graph $y = -3\sin(\frac{2x}{3})$.



Example 8d: Sketch one wave of the graph of $y = A\cos Bx = -\cos(\frac{x}{4})$.

Example 8c: Sketch one wave of the graph of $y = A\sin Bx = -3\sin(\frac{2x}{3})$. Here $A = -3$ and $B = \frac{2}{3}$.
Solution: The domain is $0 \le x \le 2\pi/B$, i.e., $0 \le x \le 3\pi$.
The range is $-|A| \le y \le |A|$, i.e., $-3 \le y \le 3$.

- Angle x goes from 0 to 3π. The width of the wave is 3π.
- Now sketch $y = 3\sin(\frac{2x}{3})$ by filling up the box with a sine wave.
- Reflect through the x-axis to get the desired graph $y = -3\sin(\frac{2x}{3})$.

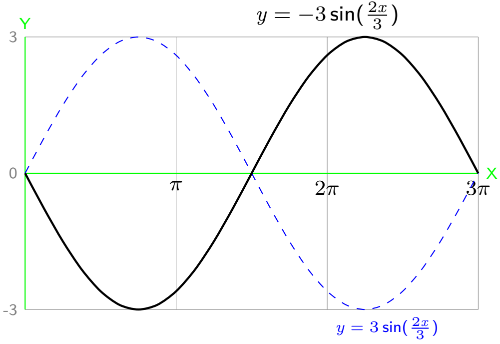


Example 8d: Sketch one wave of the graph of $y = A\cos Bx = -\cos(\frac{x}{4})$. Here $A = -1$ and $B = \frac{1}{4}$.
Solution: The domain is $0 \le x \le 2\pi/B$, i.e., $0 \le x \le 8\pi$.
The range is $-|A| \le y \le |A|$, i.e., $-3 \le y \le 3$.

- Angle x goes from 0 to 8π. The width of the wave is 8π.
- Now sketch $y = \cos(\frac{x}{4})$ by filling up the box with a cosine wave.
- Reflect through the x-axis to get the desired graph $y = -\cos(\frac{x}{4})$.

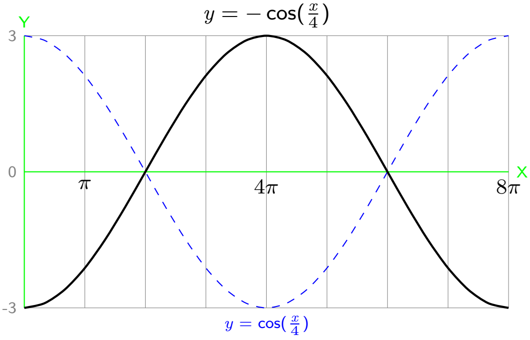
Example 8c: Sketch one wave of the graph of $y = A\sin Bx = -3\sin(\frac{2x}{3})$. Here $A = -3$ and $B = \frac{2}{3}$.
Solution: The domain is $0 \le x \le 2\pi/B$, i.e., $0 \le x \le 3\pi$. The range is $-|A| \le y \le |A|$, i.e., $-3 \le y \le 3$.

- Angle x goes from 0 to 3π. The width of the wave is 3π.
- Now sketch $y = 3\sin(\frac{2x}{3})$ by filling up the box with a sine wave.
- Reflect through the x-axis to get the desired graph $y = -3\sin(\frac{2x}{3})$.



Example 8d: Sketch one wave of the graph of $y = A\cos Bx = -\cos(\frac{x}{4})$. Here $A = -1$ and $B = \frac{1}{4}$.
Solution: The domain is $0 \le x \le 2\pi/B$, i.e., $0 \le x \le 8\pi$. The range is $-|A| \le y \le |A|$, i.e., $-3 \le y \le 3$.

- Angle x goes from 0 to 8π. The width of the wave is 8π.
- Now sketch $y = \cos(\frac{x}{4})$ by filling up the box with a cosine wave.
- Reflect through the x-axis to get the desired graph $y = -\cos(\frac{x}{4})$.



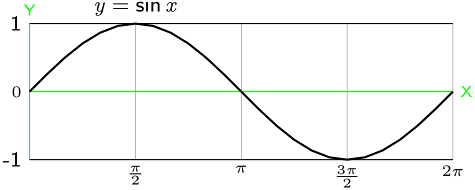
Graphing $y = A\sin(Bx + C)$ by modifying the standard wave.

Example 9a: Graph $y = 3\sin(2x + \pi/3)$ by transforming the graph of $y = \sin x$ with domain $0 \leq x \leq 2\pi$. This will be difficult. Problem 11 shows an easier way to draw the graph.

Solution: One method is to rewrite the equation as $y = 3\sin(2(x + \pi/6))$. For another method, see Example 9c in this review section.

- Start with the graph $y = \sin x; 0 \leq x \leq 2\pi$.

Each new graph is shown as a thick solid line. The previous graph is shown as a thin dotted line.



Graphing $y = A\sin(Bx + C)$ by modifying the standard wave.

Example 9a: Graph $y = 3\sin(2x + \pi/3)$ by transforming the graph of $y = \sin x$ with domain $0 \le x \le 2\pi$. This will be difficult. Problem 11 shows an easier way to draw the graph.

Solution: One method is to rewrite the equation as $y = 3\sin(2(x + \pi/6))$. For another method, see Example 9c in this review section.

- Start with the graph $y = \sin x; 0 \le x \le 2\pi$.
- Replace x in $y = \sin x$ by $2x$. This shrinks its graph horizontally toward the y-axis by a factor of 2 to yield one wave $y = \sin(2x); 0 \le x \le \pi$.

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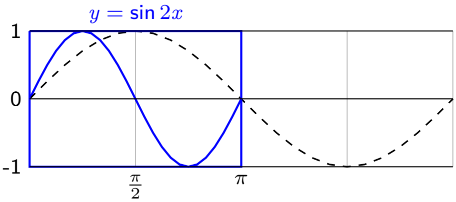


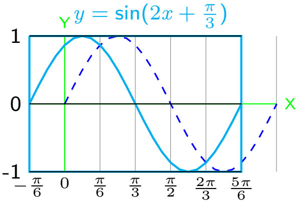
Graphing $y = A\sin(Bx + C)$ by modifying the standard wave.

Example 9a: Graph $y = 3\sin(2x + \pi/3)$ by transforming the graph of $y = \sin x$ with domain $0 \le x \le 2\pi$. This will be difficult. Problem 11 shows an easier way to draw the graph.

Solution: One method is to rewrite the equation as $y = 3\sin(2(x + \pi/6))$. For another method, see Example 9c in this review section.

- Start with the graph $y = \sin x; 0 \le x \le 2\pi$.
- Replace x in $y = \sin x$ by $2x$. This shrinks its graph horizontally toward the y-axis by a factor of 2 to yield one wave $y = \sin(2x); 0 \le x \le \pi$.
- Replace x in $y = \sin(2x)$ by $x + \pi/6$. This shifts its graph left $\pi/6$ to yield one wave of $y = \sin(2(x + \pi/6))$; $-\frac{\pi}{6} \le x \le \frac{5\pi}{6}$ Vertical lines are $\pi/6$ radians apart.

Each new graph is shown as a thick solid line. The previous graph is shown as a thin dotted line.



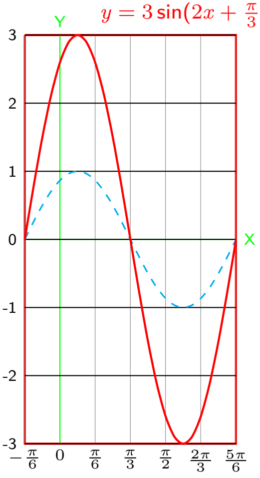
Graphing $y = A\sin(Bx + C)$ by modifying the standard wave.

Example 9a: Graph $y = 3\sin(2x + \pi/3)$ by transforming the graph of $y = \sin x$ with domain $0 \le x \le 2\pi$.
This will be difficult. Problem 11 shows an easier way to draw the graph.

Solution: One method is to rewrite the equation as $y = 3\sin(2(x + \pi/6))$. For another method, see Example 9c in this review section.

- Start with the graph $y = \sin x; 0 \le x \le 2\pi$.
- Replace x in $y = \sin x$ by $2x$. This shrinks its graph horizontally toward the y-axis by a factor of 2 to yield one wave $y = \sin(2x); 0 \le x \le \pi$.
- Replace x in $y = \sin(2x)$ by $x + \pi/6$. This shifts its graph left $\pi/6$ to yield one wave of $y = \sin(2(x + \pi/6))$; $-\frac{\pi}{6} \le x \le \frac{5\pi}{6}$ Vertical lines are $\pi/6$ radians apart.
- Multiply RHS of $y = \sin(2x + \pi/3)$ by 3. This stretches the graph vertically away from the x-axis by a factor of 3 to yield one wave of $y = 3\sin(2x + \pi/3)$; with domain $-\frac{\pi}{6} \le x \le \frac{5\pi}{6}$.

Each new graph is shown as a thick solid line. The previous graph is shown as a thin dotted line.



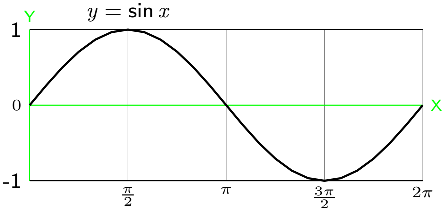
Graphing $y = A\sin(Bx + C)$ by modifying the standard wave.

Example 9b: Graph $y = -2\sin(x + \pi/2)$ by transforming the graph of $y = \sin x$ with domain $0 \leq x \leq 2\pi$.

Solution:

- Start with the graph $y = \sin x; 0 \leq x \leq 2\pi$.

Each new graph is shown as a thick solid line. The previous graph is shown as a thin dotted line.



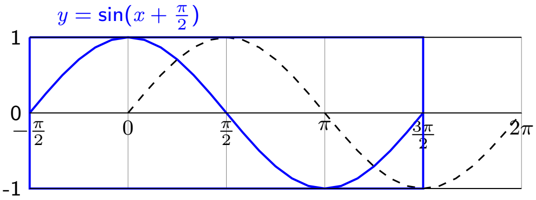
Graphing $y = A\sin(Bx + C)$ by modifying the standard wave.

Example 9b: Graph $y = -2\sin(x + \pi/2)$ by transforming the graph of $y = \sin x$ with domain $0 \le x \le 2\pi$.

Solution:

- Start with the graph $y = \sin x; 0 \le x \le 2\pi$.
- Replace x by $x + \frac{\pi}{2}$. The graph graph shifts left $\frac{\pi}{2}$ to yield one wave of $y = \sin(x + \frac{\pi}{2})$ with domain $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$.

Each new graph is shown as a thick solid line. The previous graph is shown as a thin dotted line.



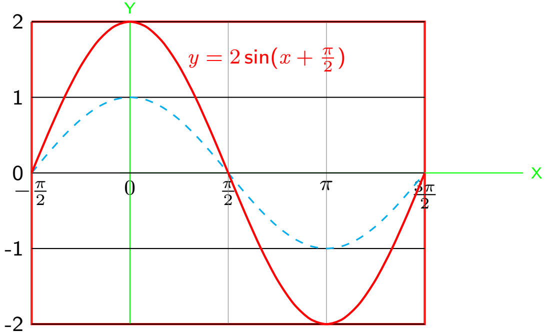
Graphing $y = A\sin(Bx + C)$ by modifying the standard wave.

Example 9b: Graph $y = -2\sin(x + \pi/2)$ by transforming the graph of $y = \sin x$ with domain $0 \le x \le 2\pi$.

Solution:

- Start with the graph $y = \sin x; 0 \le x \le 2\pi$.
- Replace x by $x + \frac{\pi}{2}$. The graph graph shifts left $\frac{\pi}{2}$ to yield one wave of $y = \sin(x + \frac{\pi}{2})$ with domain $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$.
- Multiply RHS of $y = \sin(x + \pi/2)$ by 2. This stretches the graph vertically away from the x-axis by a factor of 2 to yield one wave of $y = 2\sin(x + \pi/2)$ with domain $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

Each new graph is shown as a thick solid line. The previous graph is shown as a thin dotted line.



Graphing $y = A\sin(Bx + C)$ by modifying the standard wave.

Example 9b: Graph $y = -2\sin(x + \pi/2)$ by transforming the graph of $y = \sin x$ with domain $0 \le x \le 2\pi$.

Solution:

- Start with the graph $y = \sin x; 0 \le x \le 2\pi$.
- Replace x by $x + \frac{\pi}{2}$. The graph graph shifts left $\frac{\pi}{2}$ to yield one wave of $y = \sin(x + \frac{\pi}{2})$ with domain $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$.
- Multiply RHS of $y = \sin(x + \pi/2)$ by 2. This stretches the graph vertically away from the x-axis by a factor of 2 to yield one wave of $y = 2\sin(x + \pi/2)$ with domain $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$
- Multiply RHS of $y = 2\sin(x + \pi/2)$ by -1. This reflects the graph cross the x-axis to yield one wave of the desired graph $y = -2\sin(x + \pi/2)$ with domain $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

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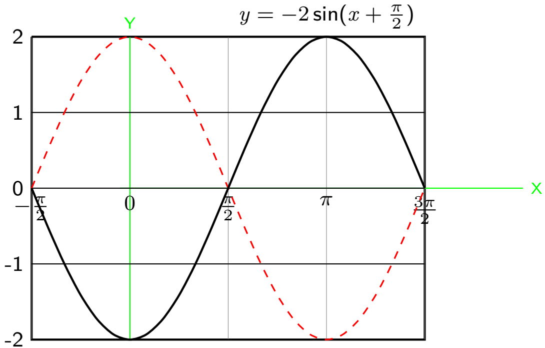

Graphing $y = A\sin(Bx + C)$ by modifying the standard wave.

Example 9c: Graph $y = \frac{\cos(2x - \pi/2)}{2}$ by transforming the graph of $y = \cos x$ with domain $0 \le x \le 2\pi$.
Solution:

- Start with the graph $y = \cos x; 0 \le x \le 2\pi$.

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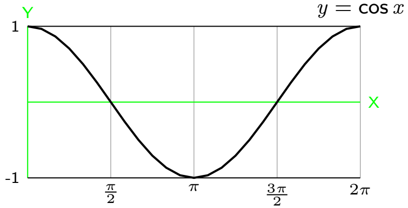
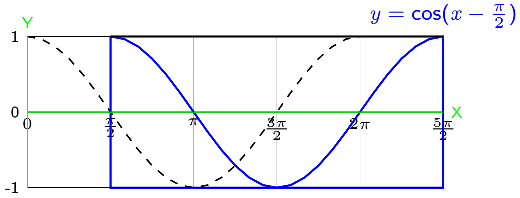


Graphing $y = A\sin(Bx + C)$ by modifying the standard wave.

Example 9c: Graph $y = \frac{\cos(2x-\pi/2)}{2}$ by transforming the graph of $y = \cos x$ with domain $0 \le x \le 2\pi$.
Solution:

- Start with the graph $y = \cos x; 0 \le x \le 2\pi$.
- Replace x by $x - \frac{\pi}{2}$. The graph shifts right $\frac{\pi}{2}$ to yield one wave of $y = \cos(x - \frac{\pi}{2})$ with domain $\frac{\pi}{2} \le x \le \frac{5\pi}{2}$.

Each new graph is shown as a thick solid line. The previous graph is shown as a thin dotted line.

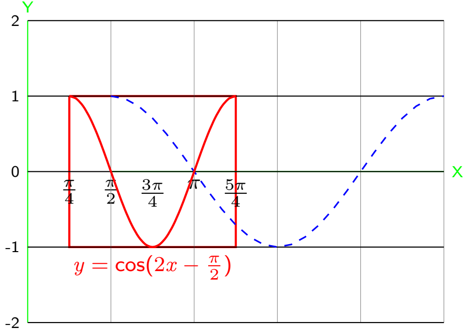


Graphing $y = A\sin(Bx + C)$ by modifying the standard wave.

Example 9c: Graph $y = \frac{\cos(2x-\pi/2)}{2}$ by transforming the graph of $y = \cos x$ with domain $0 \le x \le 2\pi$.
Solution:

- Start with the graph $y = \cos x; 0 \le x \le 2\pi$.
- Replace x by $x - \frac{\pi}{2}$. The graph shifts right $\frac{\pi}{2}$ to yield one wave of $y = \cos(x - \frac{\pi}{2})$ with domain $\frac{\pi}{2} \le x \le \frac{5\pi}{2}$.
- Replace x by $2x$ in $y = \cos(x - \pi/2)$ by 2. This shrinks the graph horizontally toward the y-axis by a factor of 2 to yield one wave of $y = 2\cos(2x - \pi/2)$ with domain $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$

Each new graph is shown as a thick solid line. The previous graph is shown as a thin dotted line.



Graphing $y = A\sin(Bx + C)$ by modifying the standard wave.

Example 9c: Graph $y = \frac{\cos(2x-\pi/2)}{2}$ by transforming the graph of $y = \cos x$ with domain $0 \leq x \leq 2\pi$.
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- Replace x by $x - \frac{\pi}{2}$. The graph shifts right $\frac{\pi}{2}$ to yield one wave of $y = \cos(x - \frac{\pi}{2})$ with domain $\frac{\pi}{2} \leq x \leq \frac{5\pi}{2}$.
- Replace x by $2x$ in $y = \cos(x - \pi/2)$ by 2. This shrinks the graph horizontally toward the y-axis by a factor of 2 to yield one wave of $y = 2\cos(2x - \pi/2)$ with domain $\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$
- Divide RHS of $y = \cos(2x - \pi/2)$ by 2 This shrinks the graph vertically by a factor of 2 to yield one wave of the desired graph $y = \frac{\cos(2x-\pi/2)}{2}$ with domain $-\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$

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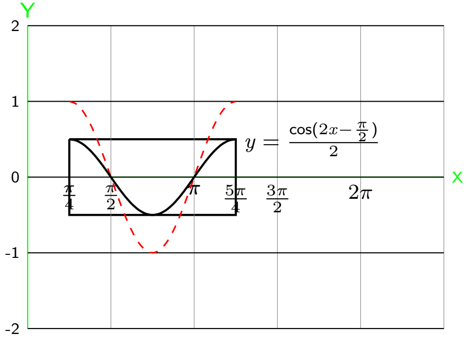


Example 10: Find the domain of the standard waves of the following cosine or sine functions.
If $B > 0$ solve $0 \le Bx + C \le 2\pi$.
If $B < 0$, solve $0 \le -Bx - C \le 2\pi$ since $\cos(Bx + C) = \cos(-Bx - C)$; $\sin(Bx + C) = -\sin(-Bx - C)$

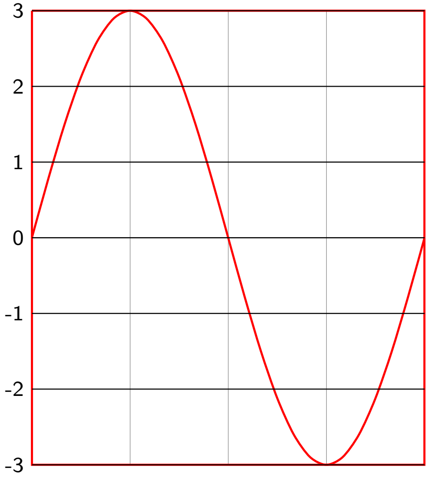
- $y = 3\sin(2x + \frac{\pi}{3})$
  $0 \le 2x + \frac{\pi}{3} \le 2\pi \Rightarrow$ $\boxed{-\frac{\pi}{6} \le x \le \frac{5\pi}{6}}$
- $y = 3\sin(\frac{\pi}{4} - 2x) = -3\sin(2x - \frac{\pi}{4})$
  $0 \le 2x - \frac{\pi}{4} \le 2\pi \Rightarrow$ $\boxed{\frac{\pi}{8} \le x \le \pi + \frac{\pi}{8}}$
- $y = 3\cos(2x + \frac{\pi}{3})$
  $0 \le 2x + \frac{\pi}{3} \le 2\pi \Rightarrow$ $\boxed{-\frac{\pi}{6} \le x \le \frac{5\pi}{6}}$
- $y = 3\cos(\frac{\pi}{4} - 2x) = 3\cos(2x - \frac{\pi}{4})$
  $0 \le 2x - \frac{\pi}{4} \le 2\pi \Rightarrow$ $\boxed{\frac{\pi}{8} \le x \le \pi + \frac{\pi}{8}}$
- $y = -3\cos(\pi - 2x) = -3\cos(2x - \pi)$
  $0 \le 2x - \pi \le 2\pi \Rightarrow$ $\boxed{\frac{\pi}{2} \le x \le \frac{3\pi}{2}}$
- $y = -3\sin(-2x) = 3\sin(2x)$
  $0 \le 2x \le 2\pi \Rightarrow$ $\boxed{0 \le x \le \pi}$
- $y = 3\sin(2x + \frac{\pi}{4})$
  $0 \le 2x + \frac{\pi}{4} \le 2\pi \Rightarrow$ $\boxed{-\frac{\pi}{8} \le x \le \frac{7\pi}{8}}$
- $y = 3\sin(5x - \pi)$
  $0 \le 5x - \pi \le 2\pi \Rightarrow$ $\boxed{\frac{\pi}{5} \le x \le \frac{3\pi}{5}}$
- $y = 3\sin(\pi - 2x) = -3\sin(2x - \pi)$
  $0 \le 2x - \pi \le 2\pi \Rightarrow$ $\boxed{\frac{\pi}{2} \le x \le \frac{3\pi}{2}}$
- $y = 3\cos(\pi - 2x) = 3\cos(2x - \pi)$
  $0 \le 2x - \pi \le 2\pi \Rightarrow$ $\boxed{\frac{\pi}{2} \le x \le \frac{3\pi}{2}}$
- $y = \cos(\frac{\pi}{2} + \frac{x}{2})$
  $0 \le \frac{x}{2} + \frac{\pi}{2} \le 2\pi \Rightarrow$ $\boxed{-\pi \le x \le 3\pi}$
- $y = \sin(\frac{\pi}{6} - \frac{x}{2}) = -\sin(\frac{x}{2} - \frac{\pi}{6})$
  $0 \le \frac{x}{2} - \frac{\pi}{6} \le 2\pi \Rightarrow$ $\boxed{\frac{\pi}{3} \le x \le \pi + \frac{13\pi}{3}}$

Graphing the standard wave of $y = A\sin(Bx + C)$ by finding its domain and range.

Example 11a: Graph the standard wave of $y = 3\sin(2x + \pi/3)$ by first finding its domain and range.

Solution: The function is $y = A\sin(Bx + C)$ with $A = 3; B = 2; C = \pi/3$. Draw the 4 by 2 grid.

- The amplitude is $|A| = 3$, so y goes from -3 to 3.



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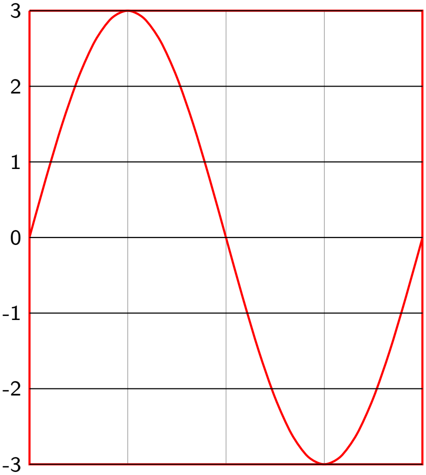
- The period is $\frac{2\pi}{B} = \frac{2\pi}{2|} = \pi$.



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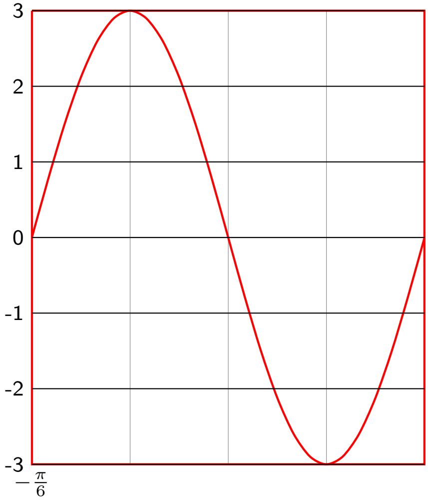
- The amplitude is $|A| = 3$, so y goes from -3 to 3.
- The period is $\frac{2\pi}{B} = \frac{2\pi}{2|} = \pi$.
- The phase shift is the x-value that solves $2x + \frac{\pi}{3} = 0$, namely $-\frac{\pi}{6}$.



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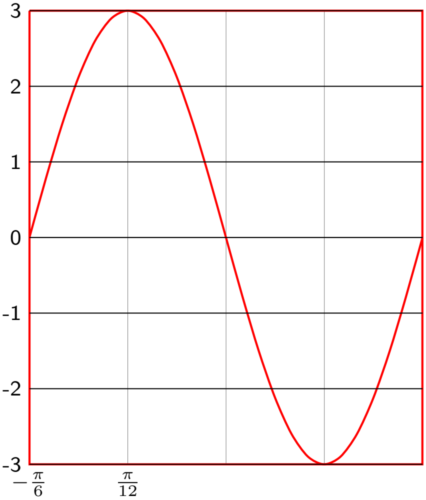
- The period is $\frac{2\pi}{B} = \frac{2\pi}{2|} = \pi$.
- The phase shift is the x-value that solves $2x + \frac{\pi}{3} = 0$, namely $-\frac{\pi}{6}$.
- The standard wave has domain the interval [phase shift, phase shift + period] $= [-\frac{\pi}{6}, \frac{5\pi}{6}]$.



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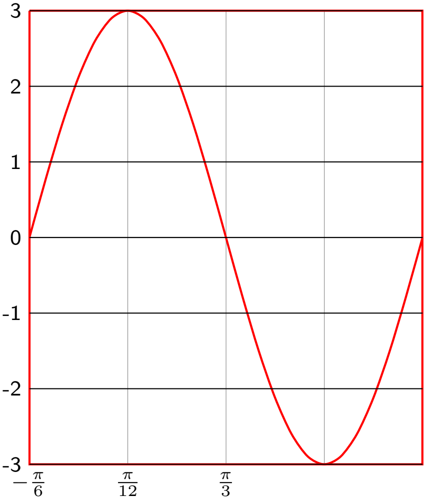
- The amplitude is $|A| = 3$, so y goes from -3 to 3.
- The period is $\frac{2\pi}{B} = \frac{2\pi}{2|} = \pi$.
- The phase shift is the x-value that solves $2x + \frac{\pi}{3} = 0$, namely $-\frac{\pi}{6}$.
- The standard wave has domain the interval [phase shift, phase shift + period] $= [-\frac{\pi}{6}, \frac{5\pi}{6}]$.
- Angle $Bx + C = 2x + \frac{\pi}{3}$ should go from 0 to 2π.

To solve	$0 \le 2x + \frac{\pi}{3} \le 2\pi$.
Subtract $\frac{\pi}{3}$	$0 - \frac{\pi}{3} \le 2x \le 2\pi - \frac{\pi}{3}$
Simplify	$-\frac{\pi}{3} \le 2x \le \frac{5\pi}{3}$
Divide by 2	$-\frac{\pi}{6} \le x \le \frac{5\pi}{6}$



Graphing the standard wave of $y = A\sin(Bx + C)$ by finding its domain and range.

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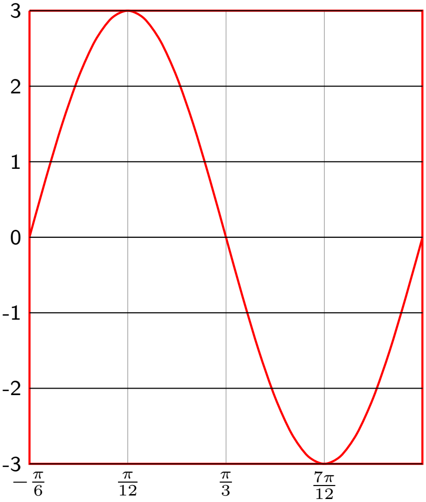
- The standard wave has domain the interval [phase shift, phase shift + period] $= [-\frac{\pi}{6}, \frac{5\pi}{6}]$.
- Angle $Bx + C = 2x + \frac{\pi}{3}$ should go from 0 to 2π.

To solve	$0 \le 2x + \frac{\pi}{3} \le 2\pi$.
Subtract $\frac{\pi}{3}$	$0 - \frac{\pi}{3} \le 2x \le 2\pi - \frac{\pi}{3}$
Simplify	$-\frac{\pi}{3} \le 2x \le \frac{5\pi}{3}$
Divide by 2	$-\frac{\pi}{6} \le x \le \frac{5\pi}{6}$



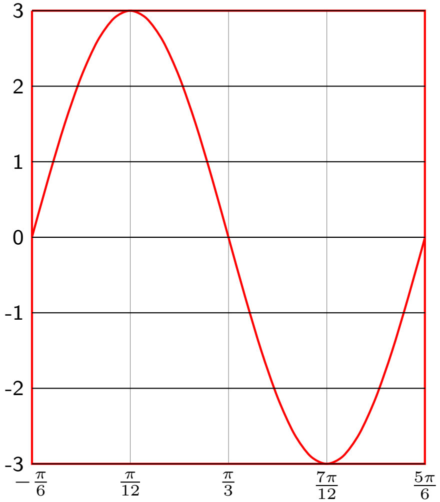
Graphing the standard wave of $y = A\sin(Bx + C)$ by finding its domain and range.

Example 11a: Graph the standard wave of $y = 3\sin(2x + \pi/3)$ by first finding its domain and range.

Solution: The function is $y = A\sin(Bx + C)$ with $A = 3$; $B = 2$; $C = \pi/3$. Draw the 4 by 2 grid.

- The amplitude is $|A| = 3$, so y goes from -3 to 3.
- The period is $\frac{2\pi}{B} = \frac{2\pi}{2|} = \pi$.
- The phase shift is the x-value that solves $2x + \frac{\pi}{3} = 0$, namely $-\frac{\pi}{6}$.
- The standard wave has domain the interval [phase shift, phase shift + period] $= [-\frac{\pi}{6}, \frac{5\pi}{6}]$.
- Angle $Bx + C = 2x + \frac{\pi}{3}$ should go from 0 to 2π.

To solve	$0 \le 2x + \frac{\pi}{3} \le 2\pi$.
Subtract $\frac{\pi}{3}$	$0 - \frac{\pi}{3} \le 2x \le 2\pi - \frac{\pi}{3}$
Simplify	$-\frac{\pi}{3} \le 2x \le \frac{5\pi}{3}$
Divide by 2	$-\frac{\pi}{6} \le x \le \frac{5\pi}{6}$



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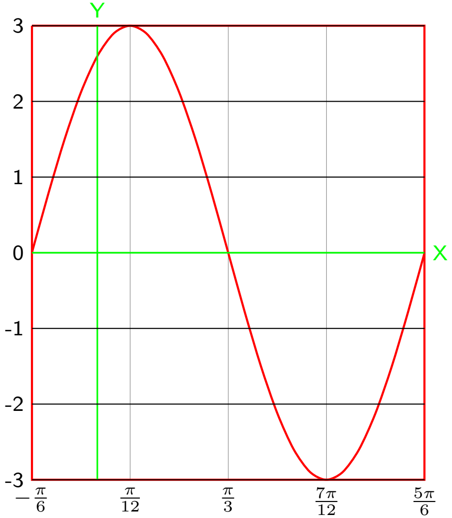
- The amplitude is $|A| = 3$, so y goes from -3 to 3.
- The period is $\frac{2\pi}{B} = \frac{2\pi}{2|} = \pi$.
- The phase shift is the x-value that solves $2x + \frac{\pi}{3} = 0$, namely $-\frac{\pi}{6}$.
- The standard wave has domain the interval [phase shift, phase shift + period] $= [-\frac{\pi}{6}, \frac{5\pi}{6}]$.
- Angle $Bx + C = 2x + \frac{\pi}{3}$ should go from 0 to 2π.

To solve	$0 \leq 2x + \frac{\pi}{3} \leq 2\pi.$
Subtract $\frac{\pi}{3}$	$0 - \frac{\pi}{3} \leq 2x \leq 2\pi - \frac{\pi}{3}$
Simplify	$-\frac{\pi}{3} \leq 2x \leq \frac{5\pi}{3}$
Divide by 2	$-\frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$



Graphing the standard wave of $y = A\sin(Bx + C)$ by finding its domain and range.

Example 11a: Graph the standard wave of $y = 3\sin(2x + \pi/3)$ by first finding its domain and range.

Solution: The function is $y = A\sin(Bx + C)$ with $A = 3$; $B = 2$; $C = \pi/3$. Draw the 4 by 2 grid.

- The amplitude is $|A| = 3$, so y goes from -3 to 3.
- The period is $\frac{2\pi}{B} = \frac{2\pi}{2|} = \pi$.
- The phase shift is the x-value that solves $2x + \frac{\pi}{3} = 0$, namely $-\frac{\pi}{6}$.
- The standard wave has domain the interval [phase shift, phase shift + period] $= [-\frac{\pi}{6}, \frac{5\pi}{6}]$.
- Angle $Bx + C = 2x + \frac{\pi}{3}$ should go from 0 to 2π.

To solve	$0 \leq 2x + \frac{\pi}{3} \leq 2\pi$.
Subtract $\frac{\pi}{3}$	$0 - \frac{\pi}{3} \leq 2x \leq 2\pi - \frac{\pi}{3}$
Simplify	$-\frac{\pi}{3} \leq 2x \leq \frac{5\pi}{3}$
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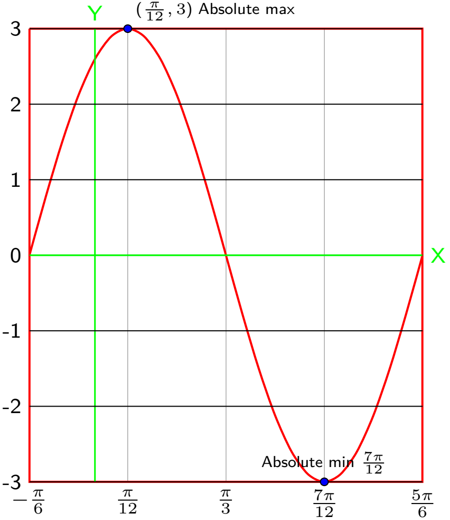
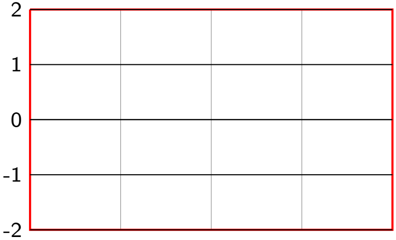


Graphing the standard wave of $y = A\sin(Bx + C)$ by first finding its domain and range.

11b: Graph the standard wave of $y = -2\sin(x + \pi/2)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = -2$; $B = 1$; $C = \pi/2$. Draw the 4 by 2 grid.

- The amplitude is $|A| = 2$: y goes from -2 to 2.



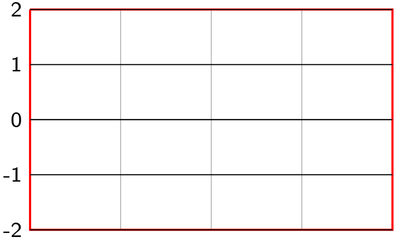
Example 11c: Graph the standard wave of $y = \frac{\cos(2x-\pi/2)}{2} = \frac{1}{2}\cos(2x - \pi/2)$.

Graphing the standard wave of $y = A\sin(Bx + C)$ by first finding its domain and range.

11b: Graph the standard wave of $y = -2\sin(x + \pi/2)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = -2$; $B = 1$; $C = \pi/2$. Draw the 4 by 2 grid.

- The amplitude is $|A| = 2$: y goes from -2 to 2.
- The period is $\frac{2\pi}{B} = 2\pi$.



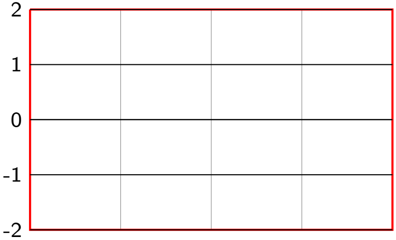
Example 11c: Graph the standard wave of $y = \frac{\cos(2x-\pi/2)}{2} = \frac{1}{2}\cos(2x - \pi/2)$.

Graphing the standard wave of $y = A\sin(Bx + C)$ by first finding its domain and range.

11b: Graph the standard wave of $y = -2\sin(x + \pi/2)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = -2$; $B = 1$; $C = \pi/2$. Draw the 4 by 2 grid.

- The amplitude is $|A| = 2$: y goes from -2 to 2.
- The period is $\frac{2\pi}{B} = 2\pi$.
- The phase shift is the x-value that solves $2 + \frac{\pi}{2} = 0$, namely $-\frac{\pi}{2}$.



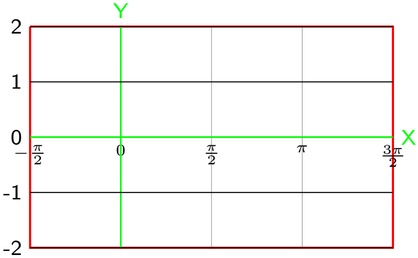
Example 11c: Graph the standard wave of $y = \frac{\cos(2x - \pi/2)}{2} = \frac{1}{2}\cos(2x - \pi/2)$.

Graphing the standard wave of $y = A\sin(Bx + C)$ by first finding its domain and range.

11b: Graph the standard wave of $y = -2\sin(x + \pi/2)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = -2$; $B = 1$; $C = \pi/2$. Draw the 4 by 2 grid.

- The amplitude is $|A| = 2$: y goes from -2 to 2.
- The period is $\frac{2\pi}{B} = 2\pi$.
- The phase shift is the x-value that solves $2 + \frac{\pi}{2} = 0$, namely $-\frac{\pi}{2}$.
- The standard wave has domain the interval [phase shift, phase shift + period] = $[-\frac{\pi}{2}, \frac{\pi}{2} + 2\pi] = [-\frac{\pi}{2}, \frac{9\pi}{2}]$.



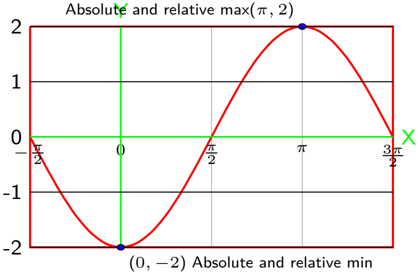
Example 11c: Graph the standard wave of $y = \frac{\cos(2x-\pi/2)}{2} = \frac{1}{2}\cos(2x - \pi/2)$.

Graphing the standard wave of $y = A\sin(Bx + C)$ by first finding its domain and range.

11b: Graph the standard wave of $y = -2\sin(x + \pi/2)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = -2$; $B = 1$; $C = \pi/2$. Draw the 4 by 2 grid.

- The amplitude is $|A| = 2$: y goes from -2 to 2.
- The period is $\frac{2\pi}{B} = 2\pi$.
- The phase shift is the x-value that solves $2 + \frac{\pi}{2} = 0$, namely $-\frac{\pi}{2}$.
- The standard wave has domain the interval [phase shift, phase shift + period] = $[-\frac{\pi}{2}, \frac{\pi}{2} + 2\pi] = [-\frac{\pi}{2}, \frac{9\pi}{2}]$.
- Angle $Bx + C = x + \frac{\pi}{2}$ goes from 0 to 2π.



Example 11c: Graph the standard wave of $y = \frac{\cos(2x-\pi/2)}{2} = \frac{1}{2}\cos(2x - \pi/2)$.

Graphing the standard wave of $y = A\sin(Bx + C)$ by first finding its domain and range.

11b: Graph the standard wave of $y = -2\sin(x + \pi/2)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = -2$; $B = 1$; $C = \pi/2$. Draw the 4 by 2 grid.

- The amplitude is $|A| = 2$: y goes from -2 to 2.
- The period is $\frac{2\pi}{B} = 2\pi$.
- The phase shift is the x-value that solves $2 + \frac{\pi}{2} = 0$, namely $-\frac{\pi}{2}$.
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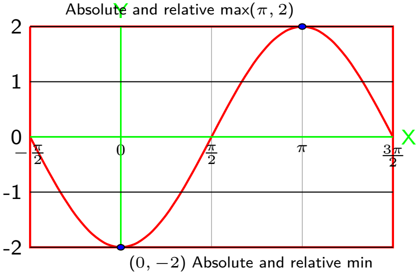
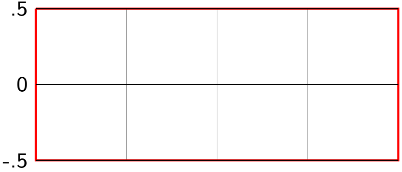
- Angle $Bx + C = x + \frac{\pi}{2}$ goes from 0 to 2π.



Example 11c: Graph the standard wave of $y = \frac{\cos(2x - \pi/2)}{2} = \frac{1}{2}\cos(2x - \pi/2)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = \frac{1}{2}$; $B = 2$; $C = -\pi/2$. Draw the 4 by 2 grid.

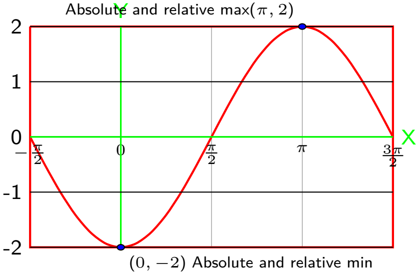


Graphing the standard wave of $y = A\sin(Bx + C)$ by first finding its domain and range.

11b: Graph the standard wave of $y = -2\sin(x + \pi/2)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = -2$; $B = 1$; $C = \pi/2$. Draw the 4 by 2 grid.

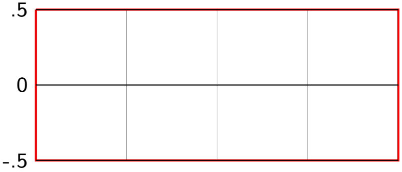
- The amplitude is $|A| = 2$: y goes from -2 to 2.
- The period is $\frac{2\pi}{B} = 2\pi$.
- The phase shift is the x-value that solves $2 + \frac{\pi}{2} = 0$, namely $-\frac{\pi}{2}$.
- The standard wave has domain the interval [phase shift, phase shift + period] = $[-\frac{\pi}{2}, \frac{\pi}{2} + 2\pi] = [-\frac{\pi}{2}, \frac{9\pi}{2}]$.
- Angle $Bx + C = x + \frac{\pi}{2}$ goes from 0 to 2π.



Example 11c: Graph the standard wave of $y = \frac{\cos(2x-\pi/2)}{2} = \frac{1}{2}\cos(2x - \pi/2)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = \frac{1}{2}$; $B = 2$; $C = -\pi/2$. Draw the 4 by 2 grid.

- The amplitude is $|A| = \frac{1}{2}$, so y goes from $\frac{1}{2}$ to $\frac{1}{2}$.

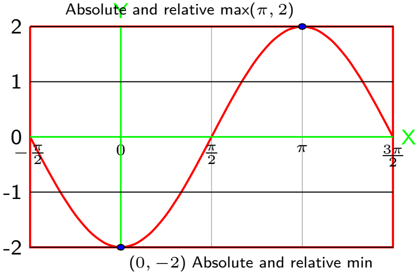


Graphing the standard wave of $y = A\sin(Bx + C)$ by first finding its domain and range.

11b: Graph the standard wave of $y = -2\sin(x + \pi/2)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = -2$; $B = 1$; $C = \pi/2$. Draw the 4 by 2 grid.

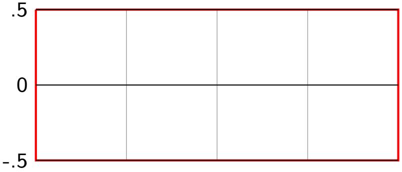
- The amplitude is $|A| = 2$: y goes from -2 to 2.
- The period is $\frac{2\pi}{B} = 2\pi$.
- The phase shift is the x-value that solves $2 + \frac{\pi}{2} = 0$, namely $-\frac{\pi}{2}$.
- The standard wave has domain the interval [phase shift, phase shift + period] = $[-\frac{\pi}{2}, \frac{\pi}{2} + 2\pi] = [-\frac{\pi}{2}, \frac{9\pi}{2}]$.
- Angle $Bx + C = x + \frac{\pi}{2}$ goes from 0 to 2π.



Example 11c: Graph the standard wave of $y = \frac{\cos(2x-\pi/2)}{2} = \frac{1}{2}\cos(2x - \pi/2)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = \frac{1}{2}$; $B = 2$; $C = -\pi/2$. Draw the 4 by 2 grid.

- The amplitude is $|A| = \frac{1}{2}$, so y goes from $\frac{1}{2}$ to $\frac{1}{2}$.
- The period is $\frac{2\pi}{B} = \frac{2\pi}{2} = \pi$.

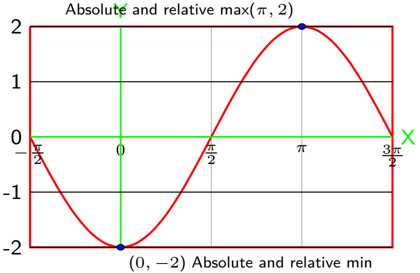


Graphing the standard wave of $y = A\sin(Bx + C)$ by first finding its domain and range.

11b: Graph the standard wave of $y = -2\sin(x + \pi/2)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = -2$; $B = 1$; $C = \pi/2$. Draw the 4 by 2 grid.

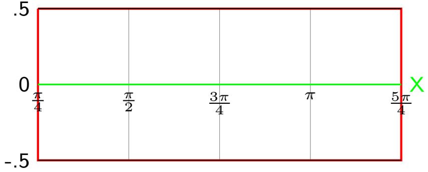
- The amplitude is $|A| = 2$: y goes from -2 to 2.
- The period is $\frac{2\pi}{B} = 2\pi$.
- The phase shift is the x-value that solves $2 + \frac{\pi}{2} = 0$, namely $-\frac{\pi}{2}$.
- The standard wave has domain the interval [phase shift, phase shift + period] = $[-\frac{\pi}{2}, \frac{\pi}{2} + 2\pi] = [-\frac{\pi}{2}, \frac{9\pi}{2}]$.
- Angle $Bx + C = x + \frac{\pi}{2}$ goes from 0 to 2π.



Example 11c: Graph the standard wave of $y = \frac{\cos(2x-\pi/2)}{2} = \frac{1}{2}\cos(2x - \pi/2)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = \frac{1}{2}$; $B = 2$; $C = -\pi/2$. Draw the 4 by 2 grid.

- The amplitude is $|A| = \frac{1}{2}$, so y goes from $\frac{1}{2}$ to $\frac{1}{2}$.
- The period is $\frac{2\pi}{B} = \frac{2\pi}{2} = \pi$.
- The phase shift is the x-value that solves $2x - \frac{\pi}{2} = 0$, namely $\frac{\pi}{4}$.

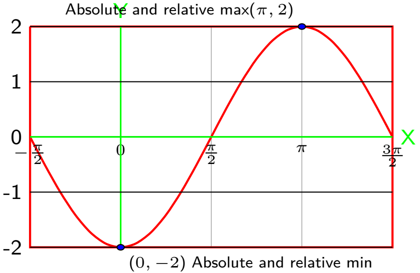


Graphing the standard wave of $y = A\sin(Bx + C)$ by first finding its domain and range.

11b: Graph the standard wave of $y = -2\sin(x + \pi/2)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = -2$; $B = 1$; $C = \pi/2$. Draw the 4 by 2 grid.

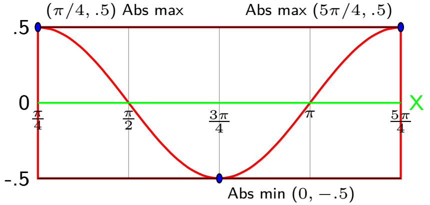
- The amplitude is $|A| = 2$: y goes from -2 to 2.
- The period is $\frac{2\pi}{B} = 2\pi$.
- The phase shift is the x-value that solves $2 + \frac{\pi}{2} = 0$, namely $-\frac{\pi}{2}$.
- The standard wave has domain the interval [phase shift, phase shift + period] = $[-\frac{\pi}{2}, \frac{\pi}{2} + 2\pi] = [-\frac{\pi}{2}, \frac{9\pi}{2}]$.
- Angle $Bx + C = x + \frac{\pi}{2}$ goes from 0 to 2π.



Example 11c: Graph the standard wave of $y = \frac{\cos(2x-\pi/2)}{2} = \frac{1}{2}\cos(2x - \pi/2)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = \frac{1}{2}$; $B = 2$; $C = -\pi/2$. Draw the 4 by 2 grid.

- The amplitude is $|A| = \frac{1}{2}$, so y goes from $\frac{1}{2}$ to $\frac{1}{2}$.
- The period is $\frac{2\pi}{B} = \frac{2\pi}{2} = \pi$.
- The phase shift is the x-value that solves $2x - \frac{\pi}{2} = 0$, namely $\frac{\pi}{4}$.
- The standard wave has domain the interval [phase shift, phase shift + period] = $[\frac{\pi}{4}, \frac{5\pi}{4}]$.

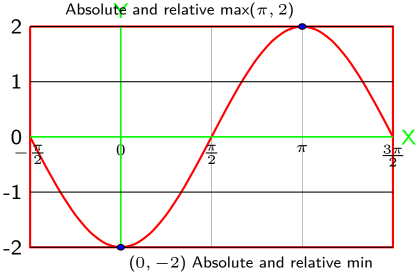


Graphing the standard wave of $y = A\sin(Bx + C)$ by first finding its domain and range.

11b: Graph the standard wave of $y = -2\sin(x + \pi/2)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = -2$; $B = 1$; $C = \pi/2$. Draw the 4 by 2 grid.

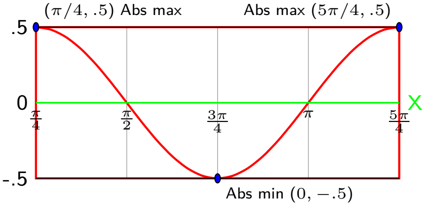
- The amplitude is $|A| = 2$: y goes from -2 to 2.
- The period is $\frac{2\pi}{B} = 2\pi$.
- The phase shift is the x-value that solves $2 + \frac{\pi}{2} = 0$, namely $-\frac{\pi}{2}$.
- The standard wave has domain the interval [phase shift, phase shift + period] = $[-\frac{\pi}{2}, \frac{\pi}{2} + 2\pi] = [-\frac{\pi}{2}, \frac{9\pi}{2}]$.
- Angle $Bx + C = x + \frac{\pi}{2}$ goes from 0 to 2π.



Example 11c: Graph the standard wave of $y = \frac{\cos(2x - \pi/2)}{2} = \frac{1}{2}\cos(2x - \pi/2)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = \frac{1}{2}$; $B = 2$; $C = -\pi/2$. Draw the 4 by 2 grid.

- The amplitude is $|A| = \frac{1}{2}$, so y goes from $\frac{1}{2}$ to $\frac{1}{2}$.
- The period is $\frac{2\pi}{B} = \frac{2\pi}{2} = \pi$.
- The phase shift is the x-value that solves $2x - \frac{\pi}{2} = 0$, namely $\frac{\pi}{4}$.
- The standard wave has domain the interval [phase shift, phase shift + period] = $[\frac{\pi}{4}, \frac{5\pi}{4}]$.
- Angle $Bx + C = 2x - \frac{\pi}{2}$ goes from 0 to 2π.

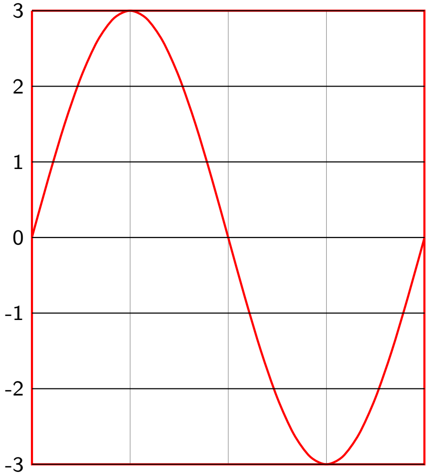


Graphing the standard wave of $y = A\sin(Bx + C)$ by first finding its domain and range.

Example 11d: Graph the standard wave of $y = 3\sin(2x + \pi/3)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = 3$; $B = 2$; $C = \pi/3$. Draw the 4 by 2 grid.

- The amplitude is $|A| = 3$, so y goes from -3 to 3.



Graphing the standard wave of $y = A\sin(Bx + C)$ by first finding its domain and range.

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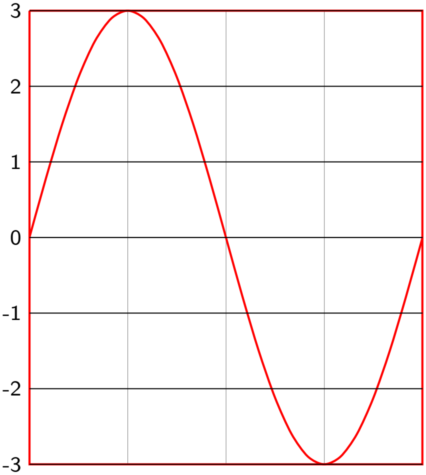
- The amplitude is $|A| = 3$, so y goes from -3 to 3.
- The period is $\frac{2\pi}{B} = \frac{2\pi}{2|} = \pi$.



Graphing the standard wave of $y = A\sin(Bx + C)$ by first finding its domain and range.

Example 11d: Graph the standard wave of $y = 3\sin(2x + \pi/3)$.

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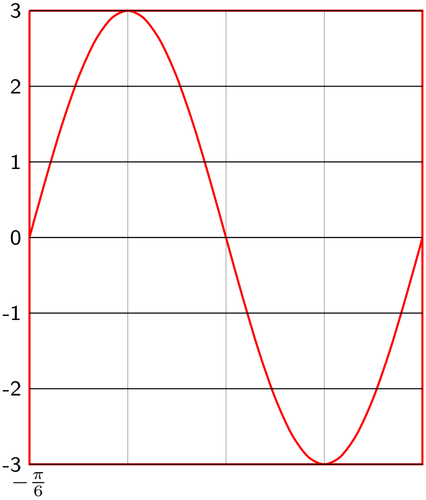
- The amplitude is $|A| = 3$, so y goes from -3 to 3.
- The period is $\frac{2\pi}{B} = \frac{2\pi}{2|} = \pi$.
- The phase shift is the x-value that solves $2x + \frac{\pi}{3} = 0$, namely $-\frac{\pi}{6}$.



Graphing the standard wave of $y = A\sin(Bx + C)$ by first finding its domain and range.

Example 11d: Graph the standard wave of $y = 3\sin(2x + \pi/3)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = 3$; $B = 2$; $C = \pi/3$. Draw the 4 by 2 grid.

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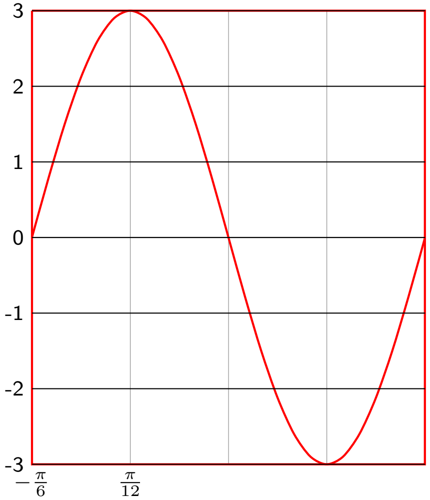
- The standard wave has domain the interval [phase shift, phase shift + period] $= [-\frac{\pi}{6}, \frac{5\pi}{6}]$.



Graphing the standard wave of $y = A\sin(Bx + C)$ by first finding its domain and range.

Example 11d: Graph the standard wave of $y = 3\sin(2x + \pi/3)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = 3$; $B = 2$; $C = \pi/3$. Draw the 4 by 2 grid.

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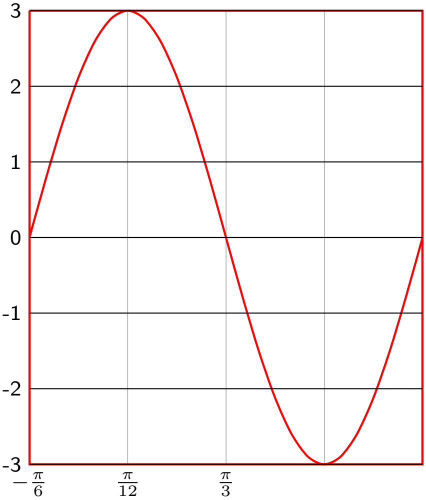
- Angle $Bx + C = 2x + \frac{\pi}{3}$ should go from 0 to 2π.

To solve	$0 \le 2x + \frac{\pi}{3} \le 2\pi$.
Subtract $\frac{\pi}{3}$	$0 - \frac{\pi}{3} \le 2x \le 2\pi - \frac{\pi}{3}$
Simplify	$-\frac{\pi}{3} \le 2x \le \frac{5\pi}{3}$
Divide by 2	$-\frac{\pi}{6} \le x \le \frac{5\pi}{6}$



Graphing the standard wave of $y = A\sin(Bx + C)$ by first finding its domain and range.

Example 11d: Graph the standard wave of $y = 3\sin(2x + \pi/3)$.

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- The period is $\frac{2\pi}{B} = \frac{2\pi}{2|} = \pi$.
- The phase shift is the x-value that solves $2x + \frac{\pi}{3} = 0$, namely $-\frac{\pi}{6}$.
- The standard wave has domain the interval [phase shift, phase shift + period] = $[-\frac{\pi}{6}, \frac{5\pi}{6}]$.
- Angle $Bx + C = 2x + \frac{\pi}{3}$ should go from 0 to 2π.

To solve	$0 \leq 2x + \frac{\pi}{3} \leq 2\pi$.
Subtract $\frac{\pi}{3}$	$0 - \frac{\pi}{3} \leq 2x \leq 2\pi - \frac{\pi}{3}$
Simplify	$-\frac{\pi}{3} \leq 2x \leq \frac{5\pi}{3}$
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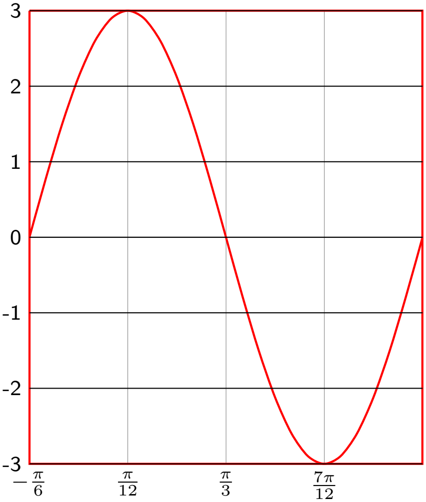

Graphing the standard wave of $y = A\sin(Bx + C)$ by first finding its domain and range.

Example 11d: Graph the standard wave of $y = 3\sin(2x + \pi/3)$.

Solution: The function is $y = A\sin(Bx + C)$ with $A = 3$; $B = 2$; $C = \pi/3$. Draw the 4 by 2 grid.

- The amplitude is $|A| = 3$, so y goes from -3 to 3.
- The period is $\frac{2\pi}{B} = \frac{2\pi}{2|} = \pi$.
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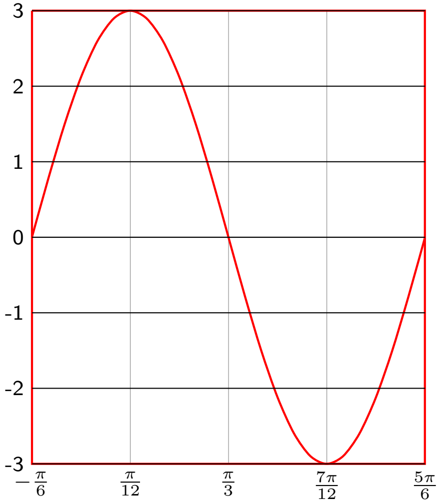

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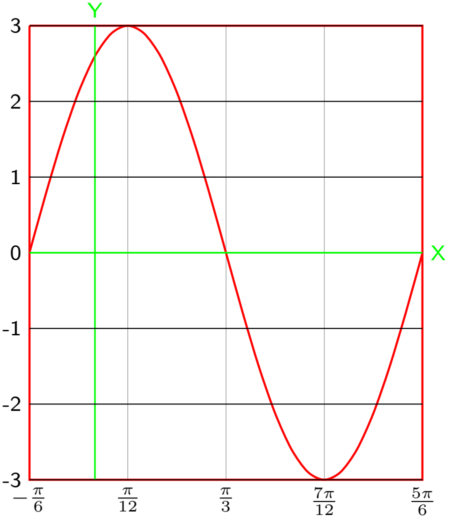

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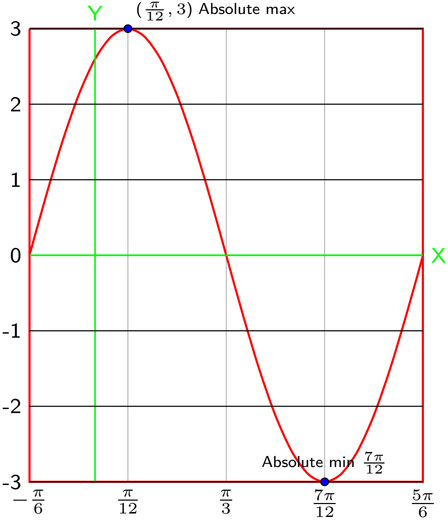

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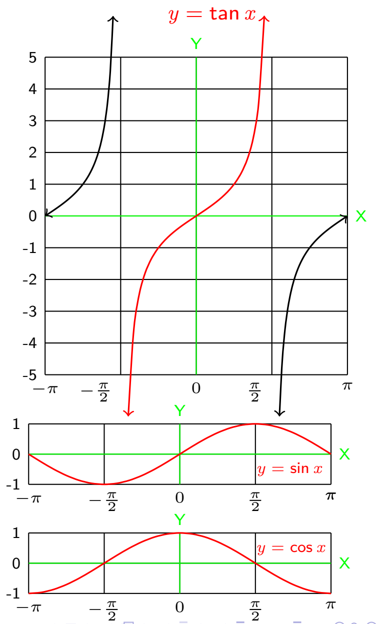

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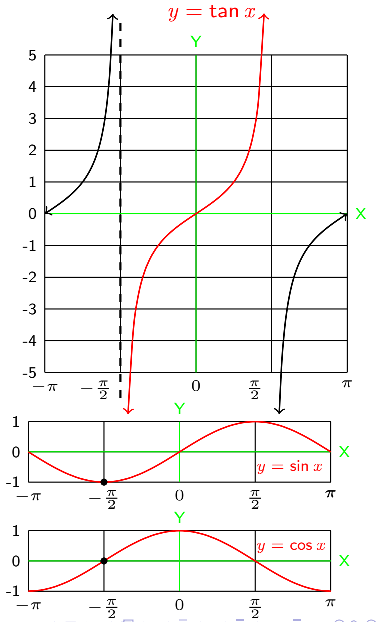

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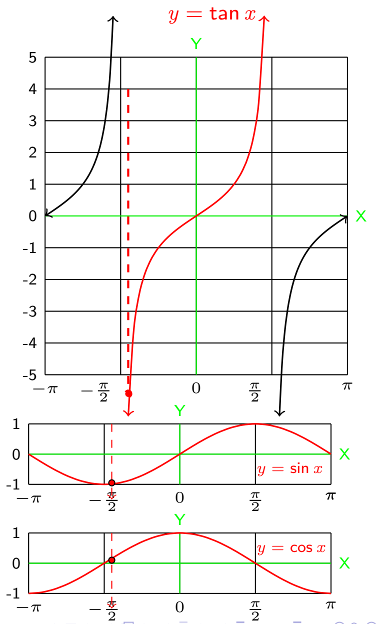

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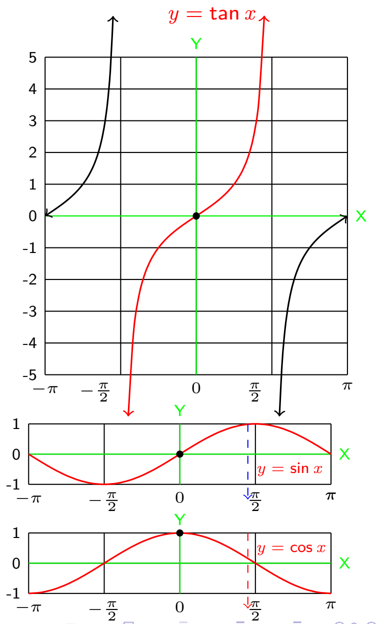

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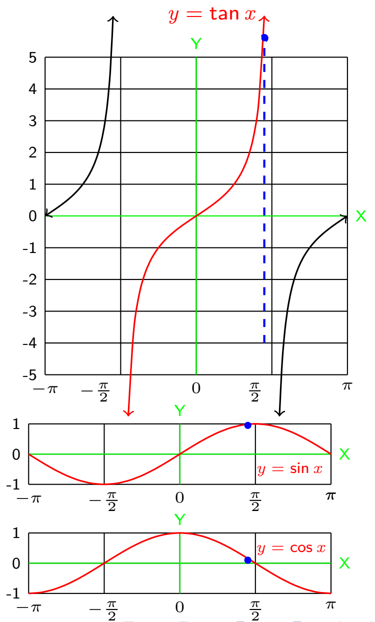

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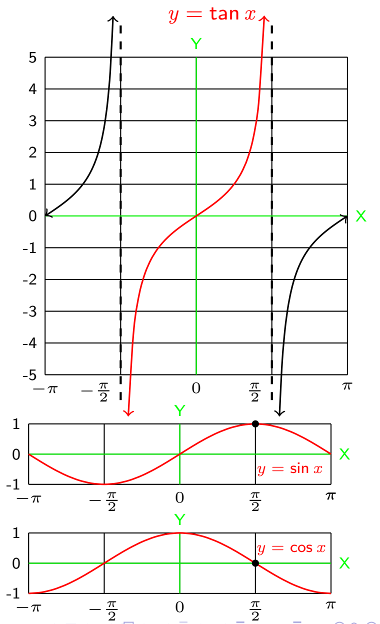

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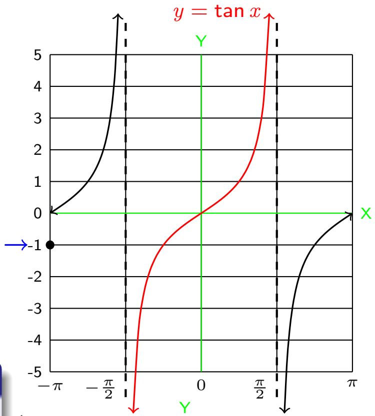

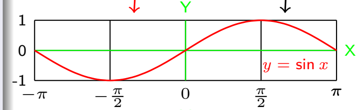

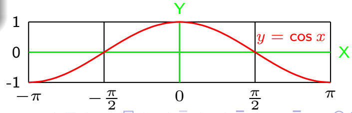

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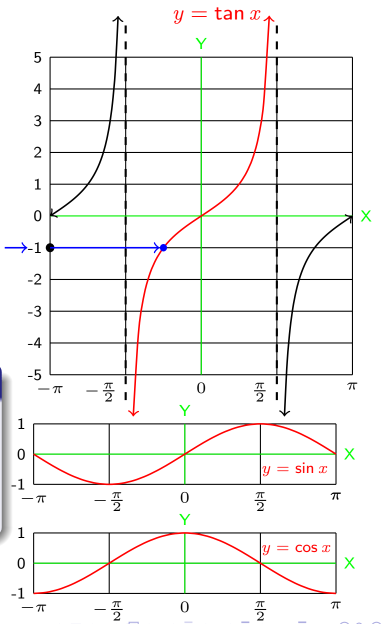

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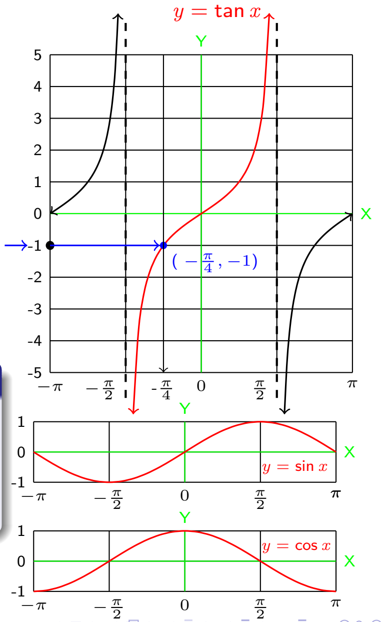

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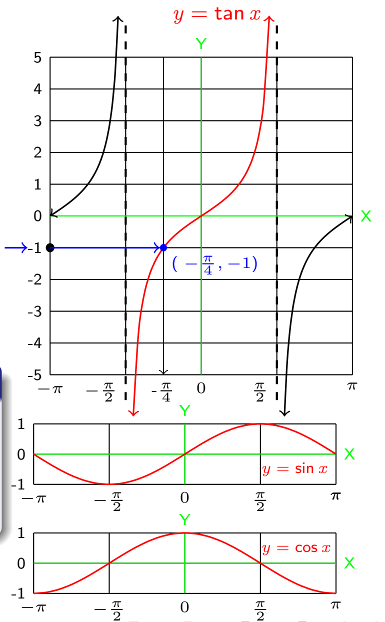


$y = \tan x$ has period π. This means: $\tan(x + \pi) = \tan x$ provided both are defined.
$y = \sin x$ and $y = \cos x$ have period 2π.
One period of each of the 3 graphs has been drawn in red.

- 4.5.1: Solving simple trigonometric equations
- 4.5.2: Inverse trigonometric functions
- 4.5.3: Solving harder trigonometric equations
- 4.5.4: Section 4.5 Quiz

Section 4.5 Preview: Definitions and Procedures

- Definition 4.5.1: Reference angle principle
- Definition 4.5.2: The inverse cosine function $\theta = \text{arccos}(x)$
- Definition 4.5.3: The inverse sine function $\theta = \text{arcsin}(x)$
- Definition 4.5.4: The inverse tangent function $\theta = \text{arctan}(x)$
- Definition 4.5.5: If the endpoint of angle θ is on the circle] $x^2 + y^2 = 1$ is (x, y), then $\tan\theta = \frac{y}{x}$.
- Definition 4.5.6: If $-2\pi < \theta < 0$ and , then Ref $\theta =$
- Procedure 4.5.1: To find values of arcsin, arccos, or arctan :
- Procedure 4.5.2: To *d*raw the terminal line of an angle $\theta < 0$:

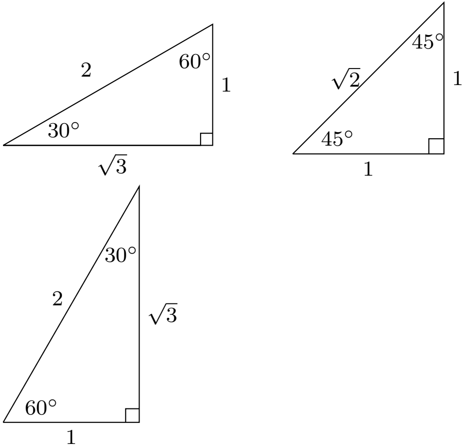
4.5.1 Solving simple trigonometric equations

Section 4.3 showed how to find the trig functions of a given angle. The procedure in reverse: given a trig function value, what angle's trig function has that value? That's easy if the trig function's value is positive: use the table below or the triangles at the right.

Example 1: Find an angle θ with $0 < \theta < 90^\circ$ and $\sin(\theta) = \frac{1}{2}$.
Solution: Consult the table below or the triangle pictures at the right to conclude $\theta = 30^\circ$.

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$\frac{\pi}{6} = 30^\circ$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4} = 45^\circ$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3} = 60^\circ$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Please memorize the following triangles.



Reminder: Angle θ is called acute whenever $0 < \theta < 90°$.

Reference angle principle (RAP)

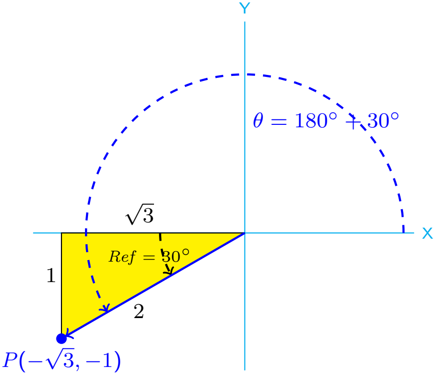
For any angle θ, the reference angle Ref θ is

- the unique acute angle with $\cos(\text{Ref } \theta) = |\cos(\theta)|$;
- the unique acute angle with $\sin(\text{Ref } \theta) = |\sin(\theta)|$;
- the unique acute angle with $\tan(\text{Ref } \theta) = |\tan(\theta)|$.

Example 2: Suppose $\sin(\theta) = -\frac{1}{2}$.

a) Find Ref θ. b) If θ is in Q3, find θ.

Solution:
a) Since $|\sin(\theta)| = |-\frac{1}{2}| = \frac{1}{2} = \sin(30°)$,
RAP says that Ref $\theta = 30°$.
b) From the picture at the right, a quadrant 3 angle with reference angle $30°$ is $180° + 30° = 210°$.



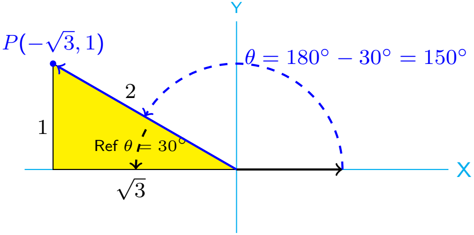
However, any coterminal angle (obtained by adding or subtracting a multiple of a complete circle $= 360°$) will also be in Q3 and have the same sine. Therefore

$$\theta = 210° \pm k \cdot 360°; k = 0, 1, 2, \ldots$$

Example 3: Suppose $\cos(\theta) = -\frac{\sqrt{3}}{2}$, where $0 < \theta < 2\pi$. Find θ.

Solution:

- Since $|\cos(\theta)| = |-\frac{\sqrt{3}}{2}| = \frac{\sqrt{3}}{2} = \cos(30°)$, RAP says that Ref $\theta = 30°$.
- $\cos(\theta) = -\frac{\sqrt{3}}{2}$ is negative. By ASTC θ is in Q2 or Q3.
- The diagram shows that a Q2 angle and its reference angle add to $180°$. Therefore $\theta + 30° = 180°$ and so $\theta = 180° - 30° = 150°$ is the solution in Q2.



Example 3: Suppose $\cos(\theta) = -\frac{\sqrt{3}}{2}$, where $0 < \theta < 2\pi$. Find θ.

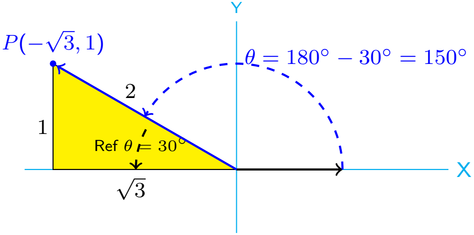
Solution:

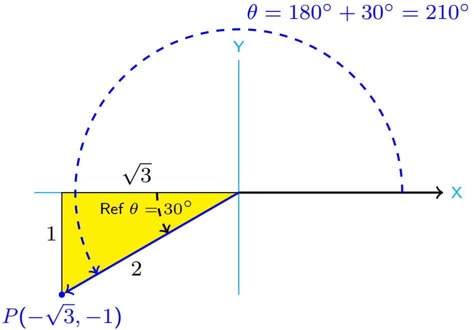
• Since $|\cos(\theta)| = |-\frac{\sqrt{3}}{2}| = \frac{\sqrt{3}}{2} = \cos(30^\circ)$, RAP says that Ref $\theta = 30^\circ$.

• $\cos(\theta) = -\frac{\sqrt{3}}{2}$ is negative. By ASTC θ is in Q2 or Q3.

• The diagram shows that a Q2 angle and its reference angle add to 180°. Therefore $\theta + 30^\circ = 180^\circ$ and so $\theta = 180^\circ - 30^\circ = 150^\circ$ is the solution in Q2.

• A similar diagram shows that a Q3 angle equals its reference angle $+\ 180^\circ$, and so the Q3 solution is $180^\circ + 30^\circ = 210^\circ$.



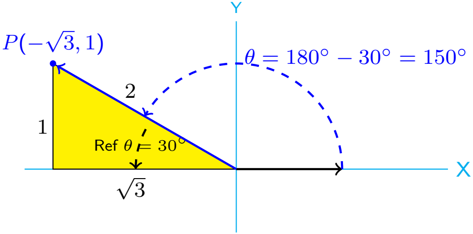


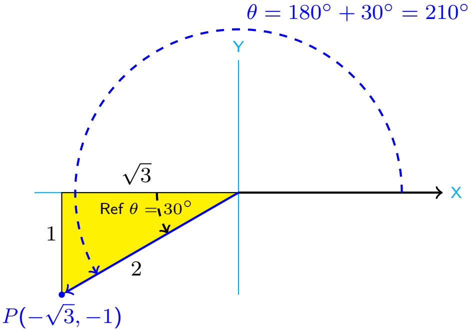
Example 3: Suppose $\cos(\theta) = -\frac{\sqrt{3}}{2}$, where $0 < \theta < 2\pi$. Find θ.

Solution:

- Since $|\cos(\theta)| = |-\frac{\sqrt{3}}{2}| = \frac{\sqrt{3}}{2} = \cos(30°)$, RAP says that Ref $\theta = 30°$.
- $\cos(\theta) = -\frac{\sqrt{3}}{2}$ is negative. By ASTC θ is in Q2 or Q3.
- The diagram shows that a Q2 angle and its reference angle add to $180°$. Therefore $\theta + 30° = 180°$ and so $\theta = 180° - 30° = 150°$ is the solution in Q2.
- A similar diagram shows that a Q3 angle equals its reference angle + $180°$, and so the Q3 solution is $180° + 30° = 210°$.
- Summary: angles $150°$ and $210°$ are solutions. They should be converted to radians because the question asks for angles θ with $0 \le \theta < 2\pi$.

Answer: $\boxed{\theta = \frac{5\pi}{6} \text{ and } \theta = \frac{7\pi}{6}.}$





4.5.2 Inverse trigonometric functions

We showed earlier that $f(x) = x^2$ has an inverse function $f^{-1}(x) = \sqrt{x}$ provided we narrow (restrict) the domain of f from $(-\infty, \infty)$ to $[0, \infty)$.

Now we define inverse functions for $\cos, \sin, \tan$ by carefully restricting the domains of those trigonometric functions.

Many books write $\tan^{-1}, \cos^{-1}$, and $\sin^{-1}$ for the inverse trig functions. But that can be confusing, since $\sin^{-1}(x)$ does NOT equal $\frac{1}{\sin(x)}$.

Instead, we will use the names $\arccos, \arcsin, \arctan$ for the inverse functions of $\cos, \sin$, and $\tan$, respectively.

Let *arctrig* be $\arcsin, \arccos,$ or $\arctan$.
To find arctrig(x), solve trig$(\theta) = x$
for θ in the domain specified at the right.

- θ must be specified in radians, not degrees.
- Since arctrig(x) is a function, the question
  Find $\theta = \text{arctrig}(x)$
  has either no solution or exactly one solution.
- For all x : trig(arctrig x) $= x$ BUT
- arctrig(trig θ)$= \theta$ ONLY if θ is in the domain specified at the right.

The inverse cosine function $\theta = \arccos(x)$

- is defined for x in $[-1, 1]$.
- To find $\theta = \arccos(x)$ solve $x = \cos\theta$ for θ in the domain $0 \leq \theta \leq \pi$.

- $\theta = \arccos(-\frac{1}{2})$ has one solution $\theta = \frac{2\pi}{3}$ since $\cos(\frac{2\pi}{3}) = -\frac{1}{2}$ *and* $0 \leq \frac{2\pi}{3} \leq \pi$.
- $\theta = \arccos(2)$ is undefined, since $-1 \leq \cos(\theta) \leq 1$ for all angles θ. .

The inverse sine function $\theta = \arcsin(x)$

- is defined for x in $[-1, 1]$.
- To find $\theta = \arcsin(x)$ solve $x = \sin\theta$ for θ in the domain $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

The inverse tangent function $\theta = \arctan(x)$

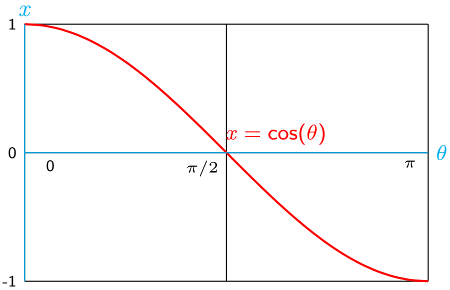
- is defined for all x in $(-\infty, \infty)$.
- To find $\theta = \arctan(x)$ solve $x = \tan\theta$ for θ in the domain $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

To find arccos(x) or arcsin(x) when $x = 0, 1,$ or -1, draw the graph of cos(θ) or sin(θ) in the required domain.

Example 4 : Find arccos(-1), arccos(0), arccos(1).
Solution:

- Graph $x = \cos(\theta)$ for θ in : $0 \leq \theta \leq \pi$.
- Clearly $\cos(0) = 1, \cos(\frac{\pi}{2}) = 0$, and $\cos(\pi) = -1$.
- Now exchange inputs and outputs:

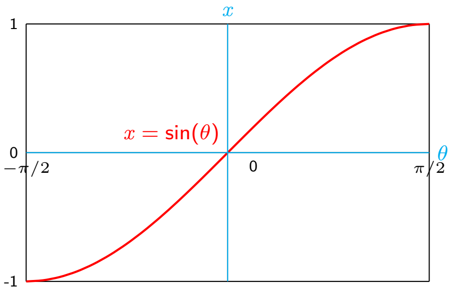
Answer: $\arccos(1) = 0, \arccos(0) = \frac{\pi}{2}$, $\arccos(-1) = \pi$.



Example 5: Find arcsin(-1), arcsin(0), arcsin(1)
Solution:

- Graph $x = \sin(\theta)$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
- Clearly $\sin(\frac{\pi}{2}) = 1, \sin(0) = 0$, and $\sin(-\frac{\pi}{2}) = -1$.
- Now exchange inputs and outputs:

Answer: $\arcsin(1) = \frac{\pi}{2}, \arcsin(0) = 0$, $\arcsin(-1) = -\frac{\pi}{2}$

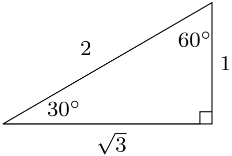


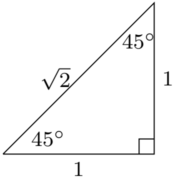
Example 6: Find

a) $\arcsin(\frac{\sqrt{3}}{2})$ b) $\arctan(\sqrt{3})$

c) $\arccos(\frac{1}{\sqrt{2}})$ d) $\arctan(1)$.

Solution: All of these problems are easy because the requested trig function value is positive. Since the values of arctan, arccos, and arcsin can always be an acute angle (between 0 and $\pi/2$), each answer is the unique acute angle with the given sine, cosine, or tangent.





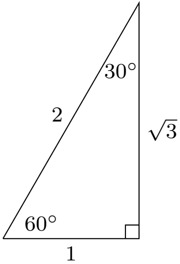
Answers:

a) $\arcsin(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$ since $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$.

b) $\arctan(\sqrt{3}) = \frac{\pi}{3}$ since $\tan(\frac{\pi}{3}) = \sqrt{3}$.

c) $\arccos(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$ since $\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

d) $\arctan(1) = \frac{\pi}{4}$ since $\tan(\frac{\pi}{4}) = 1$.



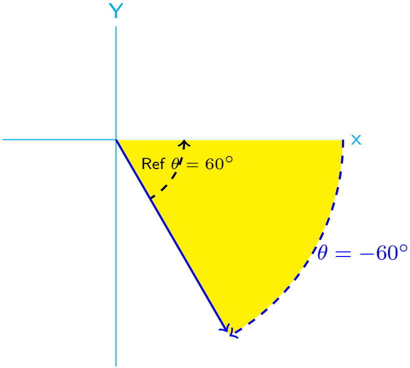
The problems on the following slides will be harder because the requested trig function value is in each case negative.

Example 7: Find $\arcsin(-\frac{\sqrt{3}}{2})$.

Solution: Let $\theta = \arcsin(-\frac{\sqrt{3}}{2})$.

The final answer must be in radians, but you may prefer using degrees until the final step.

- By definition of arcsin, $\sin(\theta) = -\frac{\sqrt{3}}{2}$ *and* $-90° \leq \theta \leq 90°$, so θ is in Q4 or Q1.
- $\sin(\theta)$ is negative. By ASTC, θ is an angle (i.e., has terminal line) in Q3 or Q4.
- Thus θ is in Q4 with $\sin(\theta) = -\frac{\sqrt{3}}{2}$.
- Since $|\sin(\theta)| = |-\frac{\sqrt{3}}{2}| = \frac{\sqrt{3}}{2} = \sin(60°)$, RAP says that Ref $\theta = 60°$.
- From the diagram, the unique Q4 angle with $-90° \leq \theta \leq 90°$ and reference angle $60°$ is $-60° = \boxed{-\frac{\pi}{3}}$.

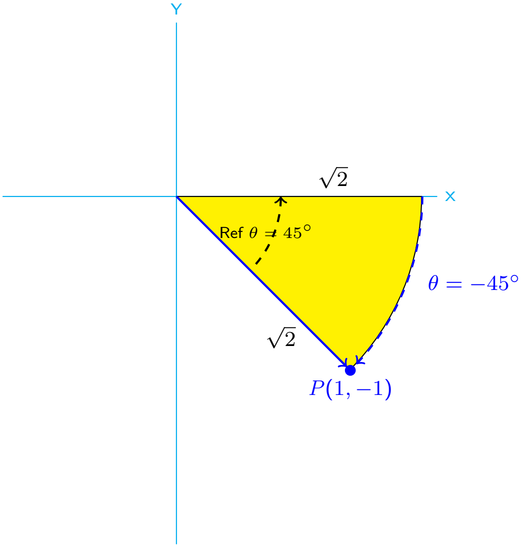


Error warning: The blue line is also the terminal line of the Q4 angle $\theta = 300°$, but that answer is incorrect because it doesn't satisfy $-90° < \theta < 90°$.

Example 8: Find $\arctan(-1)$.

Solution: Let $\theta = \arctan(-1)$. By definition of arctan,

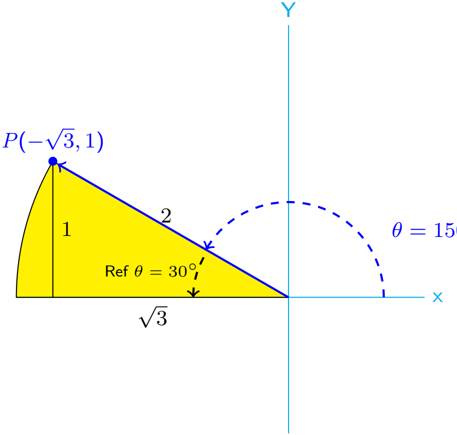
- $\tan(\theta) = -1$ *and* $-90^\circ \leq \theta \leq 90^\circ$ so θ is in Q4 or Q1.
- ASTC: since $\tan(\theta)$ is negative, θ has terminal line in Q2 or Q4.
- Therefore θ is in Q4 and $\tan(\theta) = -1$.
- Since $|\tan(\theta)| = |-1| = 1 = \tan(45^\circ)$, RAP says that Ref $\theta = 45^\circ$.
- From the picture at the right, the only Q4 angle θ with $-90^\circ < \theta < 90^\circ$ and reference angle 45° is $-45^\circ = \boxed{-\frac{\pi}{4}}$.



Example 9: Find $\arccos(-\frac{\sqrt{3}}{2})$.

Solution: Let $\theta = \arccos(-\frac{\sqrt{3}}{2})$.

- By definition of arccos, $\cos(\theta) = -\frac{\sqrt{3}}{2}$ *and* $0 \leq \theta \leq 180°$, so θ is in Q1 or Q2.
- $\cos(\theta)$ is negative: By ASTC, θ is in Q2 or Q3.
- Therefore θ is in Q2 and $\cos(\theta) = -\frac{\sqrt{3}}{2}$.
- Since $|\cos(\theta)| = |-\frac{\sqrt{3}}{2}| = \frac{\sqrt{3}}{2} = \cos(30°)$, RAP says that Ref $\theta = 30°$.
- From the picture at the right, the unique Q2 angle with $0 < \theta < 180°$ and reference angle $30°$ is $180° - 30° = 150° = \boxed{\frac{5\pi}{6}}$.



4.5.3 Solving harder trigonometric equations

Finding the value of arccos, arcsin, arctan in the previous section always produced one solution. General equations with possibly more than one solution are solved using similar techniques.

Example 10 : Solve $\cos^2\theta = \frac{1}{2}$ for $0 \le \theta \le 2\pi$.

Solution: First solve $\cos^2\theta = \frac{1}{2}$ for $\cos\theta = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$.

Since $|\pm\frac{1}{\sqrt{2}}| = \frac{1}{\sqrt{2}} = \cos 45°$, RAP says the reference angle of θ is $45° = \frac{\pi}{4}$ Therefore

- In Q1, $\theta = \frac{\pi}{4}$
- In Q2, $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$
- In Q3 , $\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$
- In Q4, $\theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

$$\boxed{\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.}$$

Example 11: Solve $\sin^2\theta = \frac{3}{4}$.

Solution: $\sin\theta = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$.

Since $|\pm\frac{\sqrt{3}}{2}| = \frac{\sqrt{3}}{2} = \sin 60°$, RAP says the reference angle of θ is $60° = \frac{\pi}{3}$.

- In Q1, $\theta = \frac{\pi}{3}$
- In Q2, $\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$
- In Q3, $\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$
- In Q4, $\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

Since there is no restriction on θ, we must add/subtract any whole number of circles (each measuring 2π radians) to/from these answers.

$$\boxed{\theta = \frac{\pi}{3} \pm 2k\pi, \quad \frac{2\pi}{3} \pm 2k\pi, \quad \frac{4\pi}{3} \pm 2k\pi, \frac{5\pi}{3} \pm 2k\pi \quad \text{for } k = 0, 1, 2, 3, \ldots}$$

Example 12 : Solve $\tan^3\theta = 3\tan\theta$ for $0 \le \theta \le 2\pi$.

Dividing by $\tan\theta$ is an error, since doing so could omit answers with $\tan\theta = 0$.

Solution:

$\tan^3\theta - 3\tan\theta = 0$

$\tan\theta(\tan^2\theta - 3) = 0$

Either $\tan\theta = 0$ or $\tan^2\theta = 3$.

$\tan\theta = 0$ or $\tan^2\theta = \pm\sqrt{3}$.

First solve $\tan\theta = 0$ by recalling:

If the endpoint of angle θ on the circle

$x^2 + y^2 = 1$ is (x, y), then $\tan\theta = \frac{y}{x}$.

In our example, $\tan\theta = \frac{y}{x} = 0$ requires $y = 0$ and $x \ne 0$

Since (x, y) is on the circle $x^2 + y^2 = 1$, $x^2 + 0^2 = 1$ and so $x = -1$ or 1.

The required endpoints on the circle are $(x, y) = (1, 0)$ and $(-1, 0)$. For $0 \le \theta \le 2\pi$, these are the endpoints of angles $\theta = 0, \theta = \pi, \theta = 2\pi$.

Therefore $\tan\theta = 0$ when $\theta = 0$ or $\theta = \pi$ or $\theta = 2\pi$

To solve $\tan\theta = \pm\sqrt{3}$, use

$|\tan\theta| = \sqrt{3} = \tan\frac{\pi}{3}$. By RAP, Ref $\theta = \frac{\pi}{3}$.

Since $\tan\theta = \pm\sqrt{3}$ is positive *or* negative,

ASTC says θ can be in Q1, Q2, Q3, or Q4.

In Q1, $\theta =$ Ref $(\theta) = \frac{\pi}{3}$

In Q2, $\theta = \pi -$Ref$(\theta) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

In Q3, $\theta = \pi +$Ref$(\theta) = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

In Q4, $\theta = 2\pi -$Ref$(\theta) = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

$\tan^2\theta = 3$ when $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

The solutions of $\tan^3\theta = 3\tan\theta$ are the 4 solutions of $\tan^2\theta = 3$ and

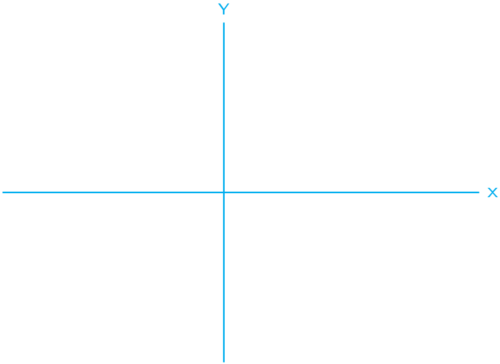
the 3 solutions of $\tan\theta = 0$.

In order, these are $\boxed{\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi.}$

Example 13 : Solve $\sin^2\theta = \frac{1}{2}$ for $-\pi \le \theta \le 0$.

Solution: $\sin\theta = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$. Thus

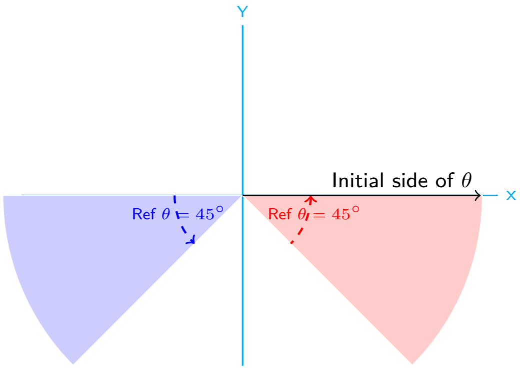
- $|\sin\theta| = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$ so Ref $\theta = \frac{\pi}{4}$ by RAP.
- $-\pi \le \theta \le 0$ says θ is a Q3 or Q4 angle.



Example 13 : Solve $\sin^2\theta = \frac{1}{2}$ for $-\pi \le \theta \le 0$.

Solution: $\sin\theta = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$. Thus

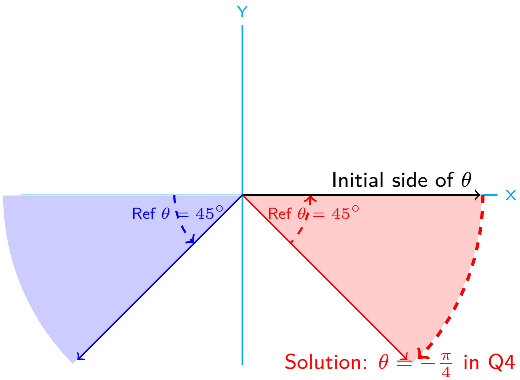
- $|\sin\theta| = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$ so Ref $\theta = \frac{\pi}{4}$ by RAP.
- $-\pi \le \theta \le 0$ says θ is a Q3 or Q4 angle.
- Draw reference angles $45^\circ = \frac{\pi}{4}$ in Q3 and Q4.



Example 13 : Solve $\sin^2\theta = \frac{1}{2}$ for $-\pi \le \theta \le 0$.

Solution: $\sin\theta = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$. Thus

- $|\sin\theta| = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$ so Ref $\theta = \frac{\pi}{4}$ by RAP.
- $-\pi \le \theta \le 0$ says θ is a Q3 or Q4 angle.
- Draw reference angles $45^\circ = \frac{\pi}{4}$ in Q3 and Q4.
- Since θ is a negative angle, draw it clockwise starting at its black initial side.
  Rotating the initial side to the red ray gives $\theta = -45^\circ = -\frac{\pi}{4}$ has Ref $\theta = \frac{\pi}{4}$ as desired.

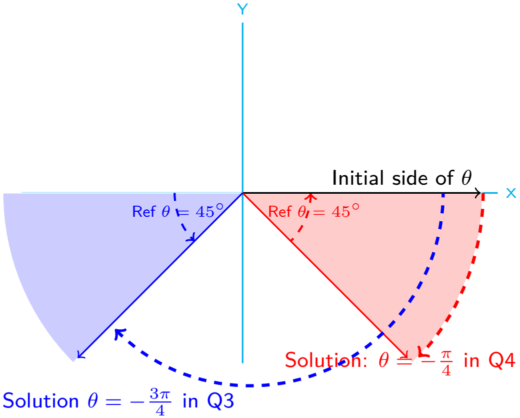


Example 13 : Solve $\sin^2\theta = \frac{1}{2}$ for $-\pi \le \theta \le 0$.

Solution: $\sin\theta = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$. Thus

- $|\sin\theta| = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$ so Ref $\theta = \frac{\pi}{4}$ by RAP.
- $-\pi \le \theta \le 0$ says θ is a Q3 or Q4 angle.
- Draw reference angles $45^\circ = \frac{\pi}{4}$ in Q3 and Q4.
- Since θ is a negative angle, draw it clockwise starting at its black initial side.
  Rotating the initial side to the red ray gives $\theta = -45^\circ = -\frac{\pi}{4}$ has Ref $\theta = \frac{\pi}{4}$ as desired.
- Rotating the initial side to the blue ray gives $\theta = -135^\circ = -\frac{3\pi}{4}$ has Ref $\theta = \frac{\pi}{4}$ as desired.

$$\boxed{\theta = -\pi/4; -3\pi/4}$$

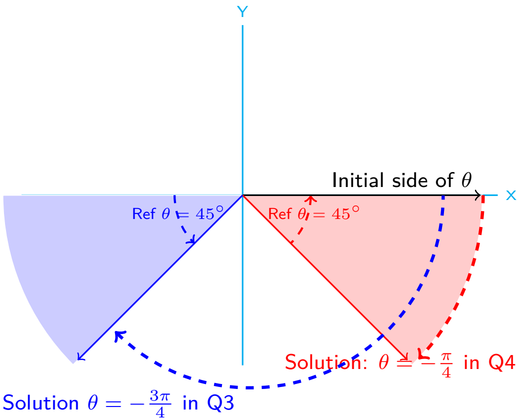


Example 13 : Solve $\sin^2\theta = \frac{1}{2}$ for $-\pi \le \theta \le 0$.

Solution: $\sin\theta = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$. Thus

- $|\sin\theta| = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$ so Ref $\theta = \frac{\pi}{4}$ by RAP.
- $-\pi \le \theta \le 0$ says θ is a Q3 or Q4 angle.
- Draw reference angles $45^\circ = \frac{\pi}{4}$ in Q3 and Q4.
- Since θ is a negative angle, draw it clockwise starting at its black initial side.
  Rotating the initial side to the red ray gives $\theta = -45^\circ = -\frac{\pi}{4}$ has Ref $\theta = \frac{\pi}{4}$ as desired.
- Rotating the initial side to the blue ray gives $\theta = -135^\circ = -\frac{3\pi}{4}$ has Ref $\theta = \frac{\pi}{4}$ as desired.

$$\boxed{\theta = -\pi/4; -3\pi/4}$$



To draw the terminal line of an angle $\theta < 0$

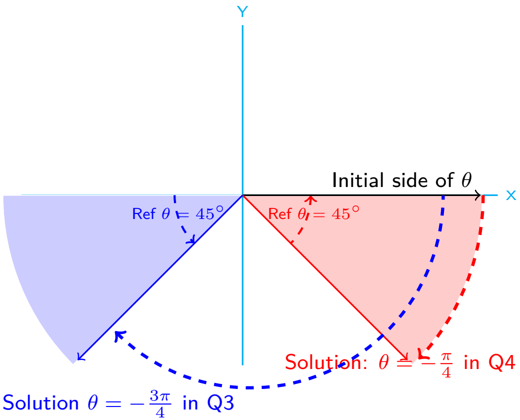
rotate the positive x-axis $|\theta|$ radians clockwise.

Example 13 : Solve $\sin^2\theta = \frac{1}{2}$ for $-\pi \le \theta \le 0$.

Solution: $\sin\theta = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$. Thus

- $|\sin\theta| = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$ so Ref $\theta = \frac{\pi}{4}$ by RAP.
- $-\pi \le \theta \le 0$ says θ is a Q3 or Q4 angle.
- Draw reference angles $45^\circ = \frac{\pi}{4}$ in Q3 and Q4.
- Since θ is a negative angle, draw it clockwise starting at its black initial side.
  Rotating the initial side to the red ray gives $\theta = -45^\circ = -\frac{\pi}{4}$ has Ref $\theta = \frac{\pi}{4}$ as desired.
- Rotating the initial side to the blue ray gives $\theta = -135^\circ = -\frac{3\pi}{4}$ has Ref $\theta = \frac{\pi}{4}$ as desired.

$\theta = -\pi/4; -3\pi/4$



To draw the terminal line of an angle $\theta < 0$

rotate the positive x-axis $|\theta|$ radians clockwise.

To deal with negative angles, verify the following.

If $-2\pi < \theta < 0$ and the terminal line of angle θ is in

- Q4, $\theta = -$ Ref θ
- Q2, $\theta = -$ Ref $\theta - \pi$
- Q3, $\theta =$ Ref $\theta - \pi$
- Q1, $\theta =$ Ref $\theta - 2\pi$

Example 14: Solve $\cos^2\theta = \frac{3}{4}$ for $-2\pi < \theta < 0$.

Solution:

Method 1: Use the method of Example 13.

Method 2: Find all solutions with $0 < \theta < 2\pi$ and subtract 2π from each of those answers.

Example 15: Solve $2\sin^2\theta + \sin\theta = 0$.
Solution: Rewrite as $(\sin\theta)(2\sin\theta + 1) = 0$.
Set each factor to zero: $\sin\theta = 0$ or $2\sin\theta + 1 = 0$.
Therefore $\sin\theta = 0$ or $= -\frac{1}{2}$.

To solve $\sin\theta = 0$, draw the sine graph below. Clearly $\sin\theta = 0$ precisely when $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

To solve $\sin\theta = \frac{1}{2}$: In Example 10, for $0 \le \theta \le 2\pi$ we found $\sin\theta = -\frac{1}{2}$ for $\theta = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$. Since the current problem states no restriction on θ, add arbitrary multiples of 2π to get

$$\theta = k\pi;\ \frac{7\pi}{6} + 2k\pi;\ \frac{11\pi}{6} + 2k\pi, \text{ for } k = 0, \pm 1, \pm 2, \ldots$$

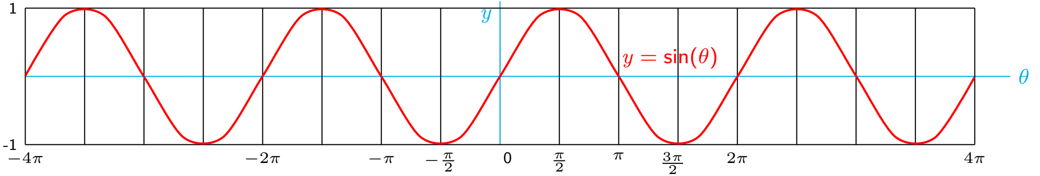
Example 16: Solve $\sin^2\theta - 5\sin\theta = 6$.

Solution: Substitute u for $\sin\theta$. Then $u^2 - 5u = 6 \Rightarrow u^2 - 5u - 6 = 0 \Rightarrow (u-6)(u+1) = 0 \Rightarrow u = 6, u = -1$.
Therefore $\sin\theta = 6$ or -1.

But $\sin\theta = 6$ is impossible, since $-1 \le \sin\theta \le 1$ for all angles θ. Therefore $\sin\theta = -1$.
Look at the graph below. $\sin\theta = -1$ at $\theta = -\frac{\pi}{2}, \frac{3\pi}{2}$.
Since the problem places no restrictions on θ, add arbitrary multiples of 2π to these solutions:

$$\theta = -\frac{\pi}{2} \pm 2k\pi,\ \frac{3\pi}{2} \pm 2k\pi \text{ for } k = 0, 1, 2, \ldots$$

Exercise: In both of these problems,you don't need the graph to solve $\sin\theta = 0, 1,$ or -1. Instead, substitute for $y = \sin\theta$ in $x^2 + y^2 = 1$ to obtain $x = \pm 1, 0, 0$, respectively and then figure out the angle θ.



Section 4.5 Quiz

Example 4.5.1: Find an angle θ with $0 < \theta < 90°$ and $\sin(\theta) = \frac{1}{2}$.

Example 4.5.2: Suppose $\sin(\theta) = -\frac{1}{2}$. a) Find Ref θ. b) If θ is in Q3, find θ.

Example 4.5.3: Suppose $\cos(\theta) = -\frac{\sqrt{3}}{2}$, where $0 < \theta < 2\pi$. Find θ.

Example 4.5.4: Find $\arccos(-1), \arccos(0)$, and $\arccos(1)$.

Example 4.5.5: Find $\arcsin(-1), \arcsin(0)$, and $\arcsin(1)$

Example 4.5.6: Find a) $\arcsin(\frac{\sqrt{3}}{2})$ b) $\arctan(\sqrt{3})$ c) $\arccos(\frac{1}{\sqrt{2}})$ d) $\arctan(1)$.

Example 4.5.7: Find $\arcsin(-\frac{\sqrt{3}}{2})$.

Example 4.5.8: Find $\arctan(-1)$.

Example 4.5.9: Find $\arccos(-\frac{\sqrt{3}}{2})$.

Example 4.5.10: Solve $\cos^2\theta = \frac{1}{2}$ for $0 \le \theta \le 2\pi$.

Example 4.5.11: Solve $\sin^2\theta = \frac{3}{4}$.

Example 4.5.12: Solve $\tan^3\theta = 3\tan\theta$ for $0 \le \theta \le 2\pi$.

Example 4.5.13: Solve $\sin^2\theta = -\frac{1}{2}$. for $-\pi \le \theta \le 0$.

Example 4.5.14: Find all solutions of $\sin\theta = -\frac{1}{2}$ with $0 \le \theta \le 2\pi$.

Example 4.5.15: Solve $2\sin^2\theta + \sin\theta = 0$.

Example 4.5.16: Solve $\sin^2\theta - 5\sin\theta = 6$.

Section 4.5 Review: Inverse trigonometric functions

Example 4.5.1: Find angle θ with

- $0 < \theta < 90^\circ$; $\sin\theta = \frac{1}{2}$. $\Rightarrow$
- $\pi < \theta < \frac{3\pi}{2}$; $\tan\theta = \sqrt{3}$. $\Rightarrow$
- $90^\circ < \theta < 180^\circ$; $\cos\theta = -\frac{1}{\sqrt{2}}$. $\Rightarrow$
- $-\pi < \theta < 0$; $\sec\theta = 2$. $\Rightarrow$

Section 4.5 Review: Inverse trigonometric functions

Example 4.5.1: Find angle θ with

- $0 < \theta < 90^\circ$; $\sin\theta = \frac{1}{2}$. $\Rightarrow \theta = \frac{\pi}{6}$
- $\pi < \theta < \frac{3\pi}{2}$; $\tan\theta = \sqrt{3}$.$\Rightarrow \theta = \frac{4\pi}{3}$
- $90^\circ < \theta < 180^\circ$; $\cos\theta = -\frac{1}{\sqrt{2}}$.$\Rightarrow \theta = \frac{3\pi}{4}$
- $-\pi < \theta < 0$; $\sec\theta = 2$. $\Rightarrow \theta = -\frac{\pi}{3}$

Section 4.5 Review: Inverse trigonometric functions

Example 4.5.1: Find angle θ with

- $0 < \theta < 90^\circ$; $\sin\theta = \frac{1}{2}$. $\Rightarrow \theta = \frac{\pi}{6}$
- $90^\circ < \theta < 180^\circ$; $\cos\theta = -\frac{1}{\sqrt{2}}$. $\Rightarrow \theta = \frac{3\pi}{4}$
- $\pi < \theta < \frac{3\pi}{2}$; $\tan\theta = \sqrt{3}$. $\Rightarrow \theta = \frac{4\pi}{3}$
- $-\pi < \theta < 0$; $\sec\theta = 2$. $\Rightarrow \theta = -\frac{\pi}{3}$

Example 4.5.2: Find Ref θ and θ in $(0, 2\pi)$ if

- $\sin\theta = -\frac{1}{2}$; θ in Q3 $\Rightarrow$
- $\cos\theta = -\frac{\sqrt{2}}{2}$, ; θ in Q2 $\Rightarrow$
- $\sec\theta = -\sqrt{2}$, ; θ in Q3 $\Rightarrow$
- $\sin\theta = -\frac{\sqrt{3}}{2}$; θ in Q4 $\Rightarrow$

Section 4.5 Review: Inverse trigonometric functions

Example 4.5.1: Find angle θ with

- $0 < \theta < 90^\circ$; $\sin\theta = \frac{1}{2}$. $\Rightarrow \theta = \frac{\pi}{6}$
- $90^\circ < \theta < 180^\circ$; $\cos\theta = -\frac{1}{\sqrt{2}}$. $\Rightarrow \theta = \frac{3\pi}{4}$
- $\pi < \theta < \frac{3\pi}{2}$; $\tan\theta = \sqrt{3}$. $\Rightarrow \theta = \frac{4\pi}{3}$
- $-\pi < \theta < 0$; $\sec\theta = 2$. $\Rightarrow \theta = -\frac{\pi}{3}$

Example 4.5.2: Find Ref θ and θ in $(0, 2\pi)$ if

- $\sin\theta = -\frac{1}{2}$; θ in Q3 $\Rightarrow$ Ref $\theta = \frac{\pi}{6}; \theta = \frac{7\pi}{6}$
- $\cos\theta = -\frac{\sqrt{2}}{2}$, ; θ in Q2 $\Rightarrow$ Ref $\theta = \frac{\pi}{4}; \theta = \frac{3\pi}{4}$
- $\sec\theta = -\sqrt{2}$, ; θ in Q3 $\Rightarrow$ Ref $\theta = \frac{\pi}{4}; \theta = \frac{5\pi}{4}$
- $\sin\theta = -\frac{\sqrt{3}}{2}$; θ in Q4 $\Rightarrow$ Ref $\theta = \frac{\pi}{3}; \theta = \frac{5\pi}{3}$

Section 4.5 Review: Inverse trigonometric functions

Example 4.5.1: Find angle θ with

- $0 < \theta < 90^\circ$; $\sin\theta = \frac{1}{2}$. $\Rightarrow \theta = \frac{\pi}{6}$
- $90^\circ < \theta < 180^\circ$; $\cos\theta = -\frac{1}{\sqrt{2}}$. $\Rightarrow \theta = \frac{3\pi}{4}$
- $\pi < \theta < \frac{3\pi}{2}$; $\tan\theta = \sqrt{3}$. $\Rightarrow \theta = \frac{4\pi}{3}$
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- $\sec\theta = -\sqrt{2}$, ; θ in Q3 $\Rightarrow$ Ref $\theta = \frac{\pi}{4}; \theta = \frac{5\pi}{4}$
- $\sin\theta = -\frac{\sqrt{3}}{2}$; θ in Q4 $\Rightarrow$ Ref $\theta = \frac{\pi}{3}; \theta = \frac{5\pi}{3}$

Example 4.5.3: Find all possible values of θ with $0 < \theta < 2\pi$ if

- $\cos\theta = -\frac{\sqrt{3}}{2}$ $\Rightarrow$
- $\sin\theta = \frac{\sqrt{3}}{2}$ $\Rightarrow$
- $\tan\theta = -1$ $\Rightarrow$
- $\sec\theta = 2$ $\Rightarrow$

Section 4.5 Review: Inverse trigonometric functions

Example 4.5.1: Find angle θ with

- $0 < \theta < 90^\circ$; $\sin\theta = \frac{1}{2}$. $\Rightarrow \theta = \frac{\pi}{6}$
- $\pi < \theta < \frac{3\pi}{2}$; $\tan\theta = \sqrt{3}$. $\Rightarrow \theta = \frac{4\pi}{3}$
- $90^\circ < \theta < 180^\circ$; $\cos\theta = -\frac{1}{\sqrt{2}}$. $\Rightarrow \theta = \frac{3\pi}{4}$
- $-\pi < \theta < 0$; $\sec\theta = 2$. $\Rightarrow \theta = -\frac{\pi}{3}$

Example 4.5.2: Find Ref θ and θ in $(0, 2\pi)$ if

- $\sin\theta = -\frac{1}{2}$; θ in Q3 $\Rightarrow$ Ref $\theta = \frac{\pi}{6}$; $\theta = \frac{7\pi}{6}$
- $\sec\theta = -\sqrt{2}$, ; θ in Q3 $\Rightarrow$ Ref $\theta = \frac{\pi}{4}$; $\theta = \frac{5\pi}{4}$
- $\cos\theta = -\frac{\sqrt{2}}{2}$, ; θ in Q2 $\Rightarrow$ Ref $\theta = \frac{\pi}{4}$; $\theta = \frac{3\pi}{4}$
- $\sin\theta = -\frac{\sqrt{3}}{2}$; θ in Q4 $\Rightarrow$ Ref $\theta = \frac{\pi}{3}$; $\theta = \frac{5\pi}{3}$

Example 4.5.3: Find all possible values of θ with $0 < \theta < 2\pi$ if

- $\cos\theta = -\frac{\sqrt{3}}{2}$ $\Rightarrow \theta = \frac{5\pi}{6}; \frac{7\pi}{6}$
- $\tan\theta = -1$ $\Rightarrow \theta = \frac{3\pi}{4}; \frac{7\pi}{4}$
- $\sin\theta = \frac{\sqrt{3}}{2}$ $\Rightarrow \theta = \frac{\pi}{3}; \frac{2\pi}{3}$
- $\sec\theta = 2$ $\Rightarrow \theta = \frac{\pi}{3}; \frac{5\pi}{3}$

Section 4.5 Review: Inverse trigonometric functions

Example 4.5.1: Find angle θ with

- $0 < \theta < 90°$; $\sin\theta = \frac{1}{2}$. $\Rightarrow \theta = \frac{\pi}{6}$
- $90° < \theta < 180°$; $\cos\theta = -\frac{1}{\sqrt{2}}$. $\Rightarrow \theta = \frac{3\pi}{4}$
- $\pi < \theta < \frac{3\pi}{2}$; $\tan\theta = \sqrt{3}$. $\Rightarrow \theta = \frac{4\pi}{3}$
- $-\pi < \theta < 0$; $\sec\theta = 2$. $\Rightarrow \theta = -\frac{\pi}{3}$

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- $\cos\theta = -\frac{\sqrt{3}}{2}$ $\Rightarrow \theta = \frac{5\pi}{6}; \frac{7\pi}{6}$
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- $\tan\theta = -1$ $\Rightarrow \theta = \frac{3\pi}{4}; \frac{7\pi}{4}$
- $\sec\theta = 2$ $\Rightarrow \theta = \frac{\pi}{3}; \frac{5\pi}{3}$

Example 4.5.4: Find

- $\arccos(-1) =$
- $\arccos(0) =$
- $\arccos(1) =$

Section 4.5 Review: Inverse trigonometric functions

Example 4.5.1: Find angle θ with

- $0 < \theta < 90^\circ$; $\sin\theta = \frac{1}{2}$. $\Rightarrow \theta = \frac{\pi}{6}$
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- $\cos\theta = -\frac{\sqrt{2}}{2}$, ; θ in Q2 $\Rightarrow$ Ref $\theta = \frac{\pi}{4}; \theta = \frac{3\pi}{4}$
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- $\sin\theta = \frac{\sqrt{3}}{2}$ $\Rightarrow \theta = \frac{\pi}{3}; \frac{2\pi}{3}$
- $\tan\theta = -1$ $\Rightarrow \theta = \frac{3\pi}{4}; \frac{7\pi}{4}$
- $\sec\theta = 2$ $\Rightarrow \theta = \frac{\pi}{3}; \frac{5\pi}{3}$

Example 4.5.4: Find

- $\arccos(-1) = \pi$
- $\arccos(0) = \frac{\pi}{2}$
- $\arccos(1) = 0$

Section 4.5 Review: Inverse trigonometric functions

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- $0 < \theta < 90°$; $\sin\theta = \frac{1}{2}$. $\Rightarrow \theta = \frac{\pi}{6}$
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- $\sec\theta = -\sqrt{2}$, ; θ in Q3 $\Rightarrow$ Ref $\theta = \frac{\pi}{4}; \theta = \frac{5\pi}{4}$
- $\sin\theta = -\frac{\sqrt{3}}{2}$; θ in Q4 $\Rightarrow$ Ref $\theta = \frac{\pi}{3}; \theta = \frac{5\pi}{3}$

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- $\cos\theta = -\frac{\sqrt{3}}{2}$ $\Rightarrow \theta = \frac{5\pi}{6}; \frac{7\pi}{6}$
- $\sin\theta = \frac{\sqrt{3}}{2}$ $\Rightarrow \theta = \frac{\pi}{3}; \frac{2\pi}{3}$
- $\tan\theta = -1$ $\Rightarrow \theta = \frac{3\pi}{4}; \frac{7\pi}{4}$
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Section 4.5 Review: Inverse trigonometric functions

Example 4.5.1: Find angle θ with

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- $-\pi < \theta < 0$; $\sec\theta = 2$. $\Rightarrow \theta = -\frac{\pi}{3}$

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- $\sin\theta = -\frac{1}{2}$; θ in Q3 $\Rightarrow$ Ref $\theta = \frac{\pi}{6}; \theta = \frac{7\pi}{6}$
- $\sec\theta = -\sqrt{2}$, ; θ in Q3 $\Rightarrow$ Ref $\theta = \frac{\pi}{4}; \theta = \frac{5\pi}{4}$
- $\cos\theta = -\frac{\sqrt{2}}{2}$, ; θ in Q2 $\Rightarrow$ Ref $\theta = \frac{\pi}{4}; \theta = \frac{3\pi}{4}$
- $\sin\theta = -\frac{\sqrt{3}}{2}$; θ in Q4 $\Rightarrow$ Ref $\theta = \frac{\pi}{3}; \theta = \frac{5\pi}{3}$

Example 4.5.3: Find all possible values of θ with $0 < \theta < 2\pi$ if

- $\cos\theta = -\frac{\sqrt{3}}{2}$ $\Rightarrow \theta = \frac{5\pi}{6}; \frac{7\pi}{6}$
- $\tan\theta = -1$ $\Rightarrow \theta = \frac{3\pi}{4}; \frac{7\pi}{4}$
- $\sin\theta = \frac{\sqrt{3}}{2}$ $\Rightarrow \theta = \frac{\pi}{3}; \frac{2\pi}{3}$
- $\sec\theta = 2$ $\Rightarrow \theta = \frac{\pi}{3}; \frac{5\pi}{3}$

Example 4.5.4: Find

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- $\arccos(0) = \frac{\pi}{2}$
- $\arccos(1) = 0$

Example 4.5.5: Find

- $\arcsin(-1) = -\frac{\pi}{2}$
- $\arcsin(0) = 0$
- $\arcsin(1) = \frac{\pi}{2}$

Section 4.5 Review: Inverse trigonometric functions

Example 4.5.1: Find angle θ with

- $0 < \theta < 90^\circ$; $\sin\theta = \frac{1}{2}$. $\Rightarrow \theta = \frac{\pi}{6}$
- $90^\circ < \theta < 180^\circ$; $\cos\theta = -\frac{1}{\sqrt{2}}$. $\Rightarrow \theta = \frac{3\pi}{4}$
- $\pi < \theta < \frac{3\pi}{2}$; $\tan\theta = \sqrt{3}$. $\Rightarrow \theta = \frac{4\pi}{3}$
- $-\pi < \theta < 0$; $\sec\theta = 2$. $\Rightarrow \theta = -\frac{\pi}{3}$

Example 4.5.2: Find Ref θ and θ in $(0, 2\pi)$ if

- $\sin\theta = -\frac{1}{2}$; θ in Q3 $\Rightarrow$ Ref $\theta = \frac{\pi}{6}; \theta = \frac{7\pi}{6}$
- $\cos\theta = -\frac{\sqrt{2}}{2}$, ; θ in Q2 $\Rightarrow$ Ref $\theta = \frac{\pi}{4}; \theta = \frac{3\pi}{4}$
- $\sec\theta = -\sqrt{2}$, ; θ in Q3 $\Rightarrow$ Ref $\theta = \frac{\pi}{4}; \theta = \frac{5\pi}{4}$
- $\sin\theta = -\frac{\sqrt{3}}{2}$; θ in Q4 $\Rightarrow$ Ref $\theta = \frac{\pi}{3}; \theta = \frac{5\pi}{3}$

Example 4.5.3: Find all possible values of θ with $0 < \theta < 2\pi$ if

- $\cos\theta = -\frac{\sqrt{3}}{2}$ $\Rightarrow \theta = \frac{5\pi}{6}; \frac{7\pi}{6}$
- $\sin\theta = \frac{\sqrt{3}}{2}$ $\Rightarrow \theta = \frac{\pi}{3}; \frac{2\pi}{3}$
- $\tan\theta = -1$ $\Rightarrow \theta = \frac{3\pi}{4}; \frac{7\pi}{4}$
- $\sec\theta = 2$ $\Rightarrow \theta = \frac{\pi}{3}; \frac{5\pi}{3}$

Example 4.5.4: Find

- $\arccos(-1) = \pi$
- $\arccos(0) = \frac{\pi}{2}$
- $\arccos(1) = 0$

Example 4.5.5: Find

- $\arcsin(-1) = -\frac{\pi}{2}$
- $\arcsin(0) = 0$
- $\arcsin(1) = \frac{\pi}{2}$

Example 4.5.6: Find

- $\arcsin(\frac{\sqrt{3}}{2}) =$
- $\arctan(\sqrt{3}) =$
- $\arccos(\frac{1}{\sqrt{2}}) =$
- $\arctan(1) =$

Section 4.5 Review: Inverse trigonometric functions

Example 4.5.1: Find angle θ with

- $0 < \theta < 90^\circ$; $\sin\theta = \frac{1}{2}$. $\Rightarrow \theta = \frac{\pi}{6}$
- $\pi < \theta < \frac{3\pi}{2}$; $\tan\theta = \sqrt{3}$.$\Rightarrow \theta = \frac{4\pi}{3}$
- $90^\circ < \theta < 180^\circ$; $\cos\theta = -\frac{1}{\sqrt{2}}$.$\Rightarrow \theta = \frac{3\pi}{4}$
- $-\pi < \theta < 0$; $\sec\theta = 2$. $\Rightarrow \theta = -\frac{\pi}{3}$

Example 4.5.2: Find Ref θ and θ in $(0, 2\pi)$ if

- $\sin\theta = -\frac{1}{2}$; θ in Q3 $\Rightarrow$ Ref $\theta = \frac{\pi}{6}; \theta = \frac{7\pi}{6}$
- $\sec\theta = -\sqrt{2}$, ; θ in Q3 $\Rightarrow$ Ref $\theta = \frac{\pi}{4}; \theta = \frac{5\pi}{4}$
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- $\sin\theta = -\frac{\sqrt{3}}{2}$; θ in Q4 $\Rightarrow$ Ref $\theta = \frac{\pi}{3}; \theta = \frac{5\pi}{3}$

Example 4.5.3: Find all possible values of θ with $0 < \theta < 2\pi$ if

- $\cos\theta = -\frac{\sqrt{3}}{2}$ $\Rightarrow$ $\theta = \frac{5\pi}{6}; \frac{7\pi}{6}$
- $\tan\theta = -1$ $\Rightarrow$ $\theta = \frac{3\pi}{4}; \frac{7\pi}{4}$
- $\sin\theta = \frac{\sqrt{3}}{2}$ $\Rightarrow$ $\theta = \frac{\pi}{3}; \frac{2\pi}{3}$
- $\sec\theta = 2$ $\Rightarrow$ $\theta = \frac{\pi}{3}; \frac{5\pi}{3}$

Example 4.5.4: Find

- $\arccos(-1) = \pi$
- $\arccos(0) = \frac{\pi}{2}$
- $\arccos(1) = 0$

Example 4.5.5: Find

- $\arcsin(-1) = -\frac{\pi}{2}$
- $\arcsin(0) = 0$
- $\arcsin(1) = \frac{\pi}{2}$

Example 4.5.6: Find

- $\arcsin(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$
- $\arctan(\sqrt{3}) = \frac{\pi}{3}$
- $\arccos(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$
- $\arctan(1) = \frac{\pi}{4}$

Section 4.5 Review: Inverse trigonometric functions

Example 4.5.1: Find angle θ with

- $0 < \theta < 90^\circ$; $\sin\theta = \frac{1}{2}$. $\Rightarrow \theta = \frac{\pi}{6}$
- $90^\circ < \theta < 180^\circ$; $\cos\theta = -\frac{1}{\sqrt{2}}$. $\Rightarrow \theta = \frac{3\pi}{4}$
- $\pi < \theta < \frac{3\pi}{2}$; $\tan\theta = \sqrt{3}$. $\Rightarrow \theta = \frac{4\pi}{3}$
- $-\pi < \theta < 0$; $\sec\theta = 2$. $\Rightarrow \theta = -\frac{\pi}{3}$

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- $\cos\theta = -\frac{\sqrt{2}}{2}$, ; θ in Q2 $\Rightarrow$ Ref $\theta = \frac{\pi}{4}; \theta = \frac{3\pi}{4}$
- $\sec\theta = -\sqrt{2}$, ; θ in Q3 $\Rightarrow$ Ref $\theta = \frac{\pi}{4}; \theta = \frac{5\pi}{4}$
- $\sin\theta = -\frac{\sqrt{3}}{2}$; θ in Q4 $\Rightarrow$ Ref $\theta = \frac{\pi}{3}; \theta = \frac{5\pi}{3}$

Example 4.5.3: Find all possible values of θ with $0 < \theta < 2\pi$ if

- $\cos\theta = -\frac{\sqrt{3}}{2}$ $\Rightarrow \theta = \frac{5\pi}{6}; \frac{7\pi}{6}$
- $\sin\theta = \frac{\sqrt{3}}{2}$ $\Rightarrow \theta = \frac{\pi}{3}; \frac{2\pi}{3}$
- $\tan\theta = -1$ $\Rightarrow \theta = \frac{3\pi}{4}; \frac{7\pi}{4}$
- $\sec\theta = 2$ $\Rightarrow \theta = \frac{\pi}{3}; \frac{5\pi}{3}$

Example 4.5.4: Find

- $\arccos(-1) = \pi$
- $\arccos(0) = \frac{\pi}{2}$
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- $\arcsin(-1) = -\frac{\pi}{2}$
- $\arcsin(0) = 0$
- $\arcsin(1) = \frac{\pi}{2}$

Example 4.5.6: Find

- $\arcsin(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$
- $\arctan(\sqrt{3}) = \frac{\pi}{3}$
- $\arccos(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$
- $\arctan(1) = \frac{\pi}{4}$
- $\arcsin(\frac{1}{2}) = \frac{\pi}{6}$
- $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$
- $\arccos(\frac{1}{2}) = \frac{\pi}{3}$
- $\arccos(\frac{\sqrt{3}}{2}) = \frac{\pi}{6}$

Example 4.5.7: Find

- $\arcsin(-\frac{\sqrt{3}}{2}) =$
- $\arccos(-\frac{1}{\sqrt{2}}) =$
- $\arctan(-\frac{1}{\sqrt{3}}) =$
- $\arctan(-1) =$

Example 4.5.7: Find

- arcsin$(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- arccos$(-\frac{1}{\sqrt{2}}) = \frac{3\pi}{4}$
- arctan$(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- arctan$(-1) = -\frac{\pi}{4}$

Example 4.5.7: Find

- $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- $\arccos(-\frac{1}{\sqrt{2}}) = \frac{3\pi}{4}$
- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-1) = -\frac{\pi}{4}$

Example 4.5.8: Find

- $\arctan(-\frac{1}{\sqrt{3}}) =$
- $\arctan(-\sqrt{3}) =$
- $\arccos(-\frac{\sqrt{3}}{2}) =$
- $\arcsin(-\sqrt{2}) =$

Example 4.5.7: Find

- arcsin$(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- arccos$(-\frac{1}{\sqrt{2}}) = \frac{3\pi}{4}$
- arctan$(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- arctan$(-1) = -\frac{\pi}{4}$

Example 4.5.8: Find

- arctan$(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- arctan$(-\sqrt{3}) = -\frac{\pi}{3}$
- arccos$(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$
- arcsin$(-\sqrt{2})=$ ND

Example 4.5.7: Find

- $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- $\arccos(-\frac{1}{\sqrt{2}}) = \frac{3\pi}{4}$
- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-1) = -\frac{\pi}{4}$

Example 4.5.8: Find

- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-\sqrt{3}) = -\frac{\pi}{3}$
- $\arccos(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$
- $\arcsin(-\sqrt{2}) =$ ND
- $\arctan(\sqrt{3}) =$
- $\arctan(0) =$
- $\arctan(1) =$
- $\arccos(\sqrt{2}) =$

Example 4.5.7: Find

- arcsin$(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- arccos$(-\frac{1}{\sqrt{2}}) = \frac{3\pi}{4}$
- arctan$(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- arctan$(-1) = -\frac{\pi}{4}$

Example 4.5.8: Find

- arctan$(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- arctan$(-\sqrt{3}) = -\frac{\pi}{3}$
- arccos$(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$
- arcsin$(-\sqrt{2}) =$ ND
- arctan$(\sqrt{3}) = \frac{\pi}{3}$
- arctan$(0) = 0$
- arctan$(1) = \frac{\pi}{4}$
- arccos$(\sqrt{2}) =$ ND

Example 4.5.7: Find

- $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- $\arccos(-\frac{1}{\sqrt{2}}) = \frac{3\pi}{4}$
- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-1) = -\frac{\pi}{4}$

Example 4.5.8: Find

- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-\sqrt{3}) = -\frac{\pi}{3}$
- $\arccos(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$
- $\arcsin(-\sqrt{2}) =$ ND
- $\arctan(\sqrt{3}) = \frac{\pi}{3}$
- $\arctan(0) = 0$
- $\arctan(1) = \frac{\pi}{4}$
- $\arccos(\sqrt{2}) =$ ND

Example 4.5.9: Find

- $\arccos(0) =$
- $\arcsin(-\frac{\sqrt{3}}{2}) =$
- $\arcsin(2) =$
- $\arccos(-\frac{1}{2}) =$

Example 4.5.7: Find

- $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- $\arccos(-\frac{1}{\sqrt{2}}) = \frac{3\pi}{4}$
- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-1) = -\frac{\pi}{4}$

Example 4.5.8: Find

- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-\sqrt{3}) = -\frac{\pi}{3}$
- $\arccos(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$
- $\arcsin(-\sqrt{2}) =$ ND
- $\arctan(\sqrt{3}) = \frac{\pi}{3}$
- $\arctan(0) = 0$
- $\arctan(1) = \frac{\pi}{4}$
- $\arccos(\sqrt{2}) =$ ND

Example 4.5.9: Find

- $\arccos(0) = \frac{\pi}{2}$
- $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- $\arcsin(2) =$ ND
- $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$

Example 4.5.7: Find

- $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- $\arccos(-\frac{1}{\sqrt{2}}) = \frac{3\pi}{4}$
- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-1) = -\frac{\pi}{4}$

Example 4.5.8: Find

- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-\sqrt{3}) = -\frac{\pi}{3}$
- $\arccos(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$
- $\arcsin(-\sqrt{2}) =$ ND
- $\arctan(\sqrt{3}) = \frac{\pi}{3}$
- $\arctan(0) = 0$
- $\arctan(1) = \frac{\pi}{4}$
- $\arccos(\sqrt{2}) =$ ND

Example 4.5.9: Find

- $\arccos(0) = \frac{\pi}{2}$
- $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- $\arcsin(2) =$ ND
- $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$

Example 4.5.10: Solve for $0 \leq \theta \leq 2\pi$.

- $\cos^2\theta = \frac{1}{2} \Rightarrow \theta =$
- $\sin^2\theta = \frac{1}{2} \Rightarrow \theta =$
- $\cos^2\theta = \frac{3}{4} \Rightarrow \theta =$
- $\tan^2\theta = 1 \Rightarrow \theta =$

Example 4.5.7: Find

- $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- $\arccos(-\frac{1}{\sqrt{2}}) = \frac{3\pi}{4}$
- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-1) = -\frac{\pi}{4}$

Example 4.5.8: Find

- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-\sqrt{3}) = -\frac{\pi}{3}$
- $\arccos(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$
- $\arcsin(-\sqrt{2}) =$ ND
- $\arctan(\sqrt{3}) = \frac{\pi}{3}$
- $\arctan(0) = 0$
- $\arctan(1) = \frac{\pi}{4}$
- $\arccos(\sqrt{2}) =$ ND

Example 4.5.9: Find

- $\arccos(0) = \frac{\pi}{2}$
- $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- $\arcsin(2) =$ ND
- $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$

Example 4.5.10: Solve for $0 \le \theta \le 2\pi$.

- $\cos^2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.
- $\cos^2\theta = \frac{3}{4} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.
- $\sin^2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.
- $\tan^2\theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Example 4.5.7: Find

- $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- $\arccos(-\frac{1}{\sqrt{2}}) = \frac{3\pi}{4}$
- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-1) = -\frac{\pi}{4}$

Example 4.5.8: Find

- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-\sqrt{3}) = -\frac{\pi}{3}$
- $\arccos(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$
- $\arcsin(-\sqrt{2}) =$ ND
- $\arctan(\sqrt{3}) = \frac{\pi}{3}$
- $\arctan(0) = 0$
- $\arctan(1) = \frac{\pi}{4}$
- $\arccos(\sqrt{2}) =$ ND

Example 4.5.9: Find

- $\arccos(0) = \frac{\pi}{2}$
- $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- $\arcsin(2) =$ ND
- $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$

Example 4.5.10: Solve for $0 \le \theta \le 2\pi$.

- $\cos^2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.
- $\sin^2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.
- $\cos^2\theta = \frac{3}{4} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.
- $\tan^2\theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Example 4.5.11: Solve for all real θ by first finding solutions with $0 \le \theta \le 2\pi$

- $\sin^2\theta = \frac{3}{4} \Rightarrow \theta =$
- $2\sin^2\theta + \sin\theta = 0 \Rightarrow \theta =$
- $2\cos\theta\sin\theta + \sin\theta = 2 \Rightarrow \theta =$
- $\sin\theta\cos\theta = 0 \Rightarrow \theta =$

Example 4.5.7: Find

- $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- $\arccos(-\frac{1}{\sqrt{2}}) = \frac{3\pi}{4}$
- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-1) = -\frac{\pi}{4}$

Example 4.5.8: Find

- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-\sqrt{3}) = -\frac{\pi}{3}$
- $\arccos(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$
- $\arcsin(-\sqrt{2}) =$ ND
- $\arctan(\sqrt{3}) = \frac{\pi}{3}$
- $\arctan(0) = 0$
- $\arctan(1) = \frac{\pi}{4}$
- $\arccos(\sqrt{2}) =$ ND

Example 4.5.9: Find

- $\arccos(0) = \frac{\pi}{2}$
- $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- $\arcsin(2) =$ ND
- $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$

Example 4.5.10: Solve for $0 \leq \theta \leq 2\pi$.

- $\cos^2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.
- $\cos^2\theta = \frac{3}{4} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.
- $\sin^2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.
- $\tan^2\theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Example 4.5.11: Solve for all real θ by first finding solutions with $0 \leq \theta \leq 2\pi$

- $\sin^2\theta = \frac{3}{4} \Rightarrow \theta = \frac{\pi}{3} \pm 2k\pi, \frac{2\pi}{3} \pm 2k\pi, \frac{4\pi}{3} \pm 2k\pi, \frac{5\pi}{3} \pm 2k\pi,$ for $k = 0, 1, 2, \ldots$
- $2\sin^2\theta + \sin\theta = 0 \Rightarrow \theta = \pm k\pi, \frac{7\pi}{6} \pm 2k\pi, \frac{11\pi}{6} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$
- $2\cos\theta\sin\theta + \sin\theta = 2 \Rightarrow \theta = \pm k\pi, \frac{2\pi}{3} \pm 2k\pi, \frac{4\pi}{3} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$
- $\sin\theta\cos\theta = 0 \Rightarrow \theta = \pm\frac{k\pi}{2}$ for $k = 0, 1, 2, \ldots$

Example 4.5.7: Find

- $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- $\arccos(-\frac{1}{\sqrt{2}}) = \frac{3\pi}{4}$
- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-1) = -\frac{\pi}{4}$

Example 4.5.8: Find

- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-\sqrt{3}) = -\frac{\pi}{3}$
- $\arccos(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$
- $\arcsin(-\sqrt{2})=$ ND
- $\arctan(\sqrt{3}) = \frac{\pi}{3}$
- $\arctan(0) = 0$
- $\arctan(1) = \frac{\pi}{4}$
- $\arccos(\sqrt{2}) =$ ND

Example 4.5.9: Find

- $\arccos(0) = \frac{\pi}{2}$
- $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- $\arcsin(2) =$ ND
- $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$

Example 4.5.10: Solve for $0 \le \theta \le 2\pi$.

- $\cos^2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.
- $\cos^2\theta = \frac{3}{4} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.
- $\sin^2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.
- $\tan^2\theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Example 4.5.11: Solve for all real θ by first finding solutions with $0 \le \theta \le 2\pi$

- $\sin^2\theta = \frac{3}{4} \Rightarrow \theta = \frac{\pi}{3} \pm 2k\pi, \frac{2\pi}{3} \pm 2k\pi, \frac{4\pi}{3} \pm 2k\pi, \frac{5\pi}{3} \pm 2k\pi,$ for $k = 0, 1, 2, \ldots$
- $2\sin^2\theta + \sin\theta = 0 \Rightarrow \theta = \pm k\pi, \frac{7\pi}{6} \pm 2k\pi, \frac{11\pi}{6} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$
- $2\cos\theta\sin\theta + \sin\theta = 2 \Rightarrow \theta = \pm k\pi, \frac{2\pi}{3} \pm 2k\pi, \frac{4\pi}{3} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$
- $\sin\theta\cos\theta = 0 \Rightarrow \theta = \pm\frac{k\pi}{2}$ for $k = 0, 1, 2, \ldots$

Example 4.5.12: Solve for $0 \le \theta \le 2\pi$.

- $\tan^3\theta = 3\tan\theta \Rightarrow \theta =$
- $\sin^2\theta - 5\sin\theta = 6 \Rightarrow \theta =$

Example 4.5.7: Find

- $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- $\arccos(-\frac{1}{\sqrt{2}}) = \frac{3\pi}{4}$
- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-1) = -\frac{\pi}{4}$

Example 4.5.8: Find

- $\arctan(-\frac{1}{\sqrt{3}}) = -\frac{\pi}{6}$
- $\arctan(-\sqrt{3}) = -\frac{\pi}{3}$
- $\arccos(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$
- $\arcsin(-\sqrt{2}) =$ ND
- $\arctan(\sqrt{3}) = \frac{\pi}{3}$
- $\arctan(0) = 0$
- $\arctan(1) = \frac{\pi}{4}$
- $\arccos(\sqrt{2}) =$ ND

Example 4.5.9: Find

- $\arccos(0) = \frac{\pi}{2}$
- $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$
- $\arcsin(2) =$ ND
- $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$

Example 4.5.10: Solve for $0 \le \theta \le 2\pi$.

- $\cos^2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$
- $\cos^2\theta = \frac{3}{4} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}.$
- $\sin^2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$
- $\tan^2\theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$

Example 4.5.11: Solve for all real θ by first finding solutions with $0 \le \theta \le 2\pi$

- $\sin^2\theta = \frac{3}{4} \Rightarrow \theta = \frac{\pi}{3} \pm 2k\pi, \frac{2\pi}{3} \pm 2k\pi, \frac{4\pi}{3} \pm 2k\pi, \frac{5\pi}{3} \pm 2k\pi,$ for $k = 0, 1, 2,...$
- $2\sin^2\theta + \sin\theta = 0 \Rightarrow \theta = \pm k\pi, \frac{7\pi}{6} \pm 2k\pi, \frac{11\pi}{6} \pm 2k\pi$ for $k = 0, 1, 2, ...$
- $2\cos\theta\sin\theta + \sin\theta = 2 \Rightarrow \theta = \pm k\pi, \frac{2\pi}{3} \pm 2k\pi, \frac{4\pi}{3} \pm 2k\pi$ for $k = 0, 1, 2, ...$
- $\sin\theta\cos\theta = 0 \Rightarrow \theta = \pm\frac{k\pi}{2}$ for $k = 0, 1, 2, ...$

Example 4.5.12: Solve for $0 \le \theta \le 2\pi$.

- $\tan^3\theta = 3\tan\theta \Rightarrow \theta = \ \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi.$
- $\cos^2\theta - 5\sin\theta = 5 \Rightarrow \theta = \frac{3\pi}{2}$
- $\sin^2\theta - 5\sin\theta = 6 \Rightarrow \theta = \frac{3\pi}{2}$
- $2\sin^2\theta - \sin\theta = 0 \Rightarrow \theta = 0, \pi, 2\pi, \frac{\pi}{6}, \frac{5\pi}{6}$

Example 4.5.13: Solve for all real θ by first finding solutions with $0 \leq \theta \leq 2\pi$

- $\cos^2\theta = \frac{1}{2}$ for $-\pi < \theta < \pi$ $\Rightarrow \theta =$
- $\sin^2\theta = \frac{1}{2}$ for $-\pi < \theta < 0.$ $\Rightarrow \theta =$
- $2\cos^2\theta - 2\sin^2\theta = 1$ for $0 \leq \theta \leq 2\pi$ $\Rightarrow \theta =$
- $\cos^2\theta + \sin^2\theta = 1$

Example 4.5.13: Solve for all real θ by first finding solutions with $0 \leq \theta \leq 2\pi$

- $\cos^2\theta = \frac{1}{2}$ for $-\pi < \theta < \pi$ $\Rightarrow \theta = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$
- $\sin^2\theta = \frac{1}{2}$ for $-\pi < \theta < 0.$ $\Rightarrow \theta = -\frac{3\pi}{4}, -\frac{\pi}{4}$
- $2\cos^2\theta - 2\sin^2\theta = 1$ for $0 \leq \theta \leq 2\pi$ $\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
- $\cos^2\theta + \sin^2\theta = 1$ $\Rightarrow$ all θ in $(-\infty, \infty)$

Example 4.5.13: Solve for all real θ by first finding solutions with $0 \leq \theta \leq 2\pi$

- $\cos^2\theta = \frac{1}{2}$ for $-\pi < \theta < \pi$ $\Rightarrow \theta = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$
- $\sin^2\theta = \frac{1}{2}$ for $-\pi < \theta < 0.$ $\Rightarrow \theta = -\frac{3\pi}{4}, -\frac{\pi}{4}$
- $2\cos^2\theta - 2\sin^2\theta = 1$ for $0 \leq \theta \leq 2\pi$ $\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
- $\cos^2\theta + \sin^2\theta = 1$ $\Rightarrow$ all θ in $(-\infty, \infty)$

Example 4.5.14: Solve for all real θ by first finding solutions with $0 \leq \theta \leq 2\pi$

- $\cos^2\theta = \frac{3}{4}$ for $-2\pi \leq \theta \leq 0$ $\Rightarrow \theta =$
- $2\sin\theta + 1 = 0$ with $0 \leq \theta \leq 4\pi.$ $\Rightarrow \theta =$
- $\sin^2\theta - 3\cos^2\theta + 1 = 0$ with $0 \leq \theta \leq 2\pi.$ $\Rightarrow \theta =$
- $\sin^2\theta - 5\sin\theta + 6 = 0$ $\Rightarrow$

Example 4.5.13: Solve for all real θ by first finding solutions with $0 \le \theta \le 2\pi$

- $\cos^2\theta = \frac{1}{2}$ for $-\pi < \theta < \pi$ $\Rightarrow \theta = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$
- $\sin^2\theta = \frac{1}{2}$ for $-\pi < \theta < 0.$ $\Rightarrow \theta = -\frac{3\pi}{4}, -\frac{\pi}{4}$
- $2\cos^2\theta - 2\sin^2\theta = 1$ for $0 \le \theta \le 2\pi$ $\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
- $\cos^2\theta + \sin^2\theta = 1$ $\Rightarrow$ all θ in $(-\infty, \infty)$

Example 4.5.14: Solve for all real θ by first finding solutions with $0 \le \theta \le 2\pi$

- $\cos^2\theta = \frac{3}{4}$ for $-2\pi \le \theta \le 0$ $\Rightarrow \theta = -\frac{\pi}{6}, -\frac{5\pi}{6}, -\frac{7\pi}{6}, -\frac{11\pi}{6}$
- $2\sin\theta + 1 = 0$ with $0 \le \theta \le 4\pi.$ $\Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$
- $\sin^2\theta - 3\cos^2\theta + 1 = 0$ with $0 \le \theta \le 2\pi.$ $\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
- $\sin^2\theta - 5\sin\theta + 6 = 0$ $\Rightarrow$ No solution

Example 4.5.13: Solve for all real θ by first finding solutions with $0 \leq \theta \leq 2\pi$

- $\cos^2\theta = \frac{1}{2}$ for $-\pi < \theta < \pi$ $\Rightarrow \theta = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$
- $\sin^2\theta = \frac{1}{2}$ for $-\pi < \theta < 0.$ $\Rightarrow \theta = -\frac{3\pi}{4}, -\frac{\pi}{4}$
- $2\cos^2\theta - 2\sin^2\theta = 1$ for $0 \leq \theta \leq 2\pi$ $\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
- $\cos^2\theta + \sin^2\theta = 1$ $\Rightarrow$ all θ in $(-\infty, \infty)$

Example 4.5.14: Solve for all real θ by first finding solutions with $0 \leq \theta \leq 2\pi$

- $\cos^2\theta = \frac{3}{4}$ for $-2\pi \leq \theta \leq 0$ $\Rightarrow \theta = -\frac{\pi}{6}, -\frac{5\pi}{6}, -\frac{7\pi}{6}, -\frac{11\pi}{6}$
- $2\sin\theta + 1 = 0$ with $0 \leq \theta \leq 4\pi.$ $\Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$
- $\sin^2\theta - 3\cos^2\theta + 1 = 0$ with $0 \leq \theta \leq 2\pi.$ $\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
- $\sin^2\theta - 5\sin\theta + 6 = 0$ $\Rightarrow$ No solution

Example 4.5.15: Solve for all real θ by first finding solutions with $0 \leq \theta \leq 2\pi$

- $2\sin^2\theta + \sin\theta = 0$ $\Rightarrow \theta =$
- $2\sin^2\theta - \sin\theta = 0$ $\Rightarrow \theta =$
- $\tan^2\theta - \tan\theta = 0$ $\Rightarrow \theta =$
- $2\sin^3\theta - \sin\theta = 0$ $\Rightarrow \theta =$

Example 4.5.13: Solve for all real θ by first finding solutions with $0 \le \theta \le 2\pi$

- $\cos^2\theta = \frac{1}{2}$ for $-\pi < \theta < \pi$ $\Rightarrow \theta = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$
- $\sin^2\theta = \frac{1}{2}$ for $-\pi < \theta < 0.$ $\Rightarrow \theta = -\frac{3\pi}{4}, -\frac{\pi}{4}$
- $2\cos^2\theta - 2\sin^2\theta = 1$ for $0 \le \theta \le 2\pi$ $\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
- $\cos^2\theta + \sin^2\theta = 1$ $\Rightarrow$ all θ in $(-\infty, \infty)$

Example 4.5.14: Solve for all real θ by first finding solutions with $0 \le \theta \le 2\pi$

- $\cos^2\theta = \frac{3}{4}$ for $-2\pi \le \theta \le 0$ $\Rightarrow \theta = -\frac{\pi}{6}, -\frac{5\pi}{6}, -\frac{7\pi}{6}, -\frac{11\pi}{6}$
- $2\sin\theta + 1 = 0$ with $0 \le \theta \le 4\pi.$ $\Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$
- $\sin^2\theta - 3\cos^2\theta + 1 = 0$ with $0 \le \theta \le 2\pi.$ $\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
- $\sin^2\theta - 5\sin\theta + 6 = 0$ $\Rightarrow$ No solution

Example 4.5.15: Solve for all real θ by first finding solutions with $0 \le \theta \le 2\pi$

- $2\sin^2\theta + \sin\theta = 0$ $\Rightarrow \theta = \pm k\pi, \frac{7\pi}{6} \pm 2k\pi, \frac{11\pi}{6} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$
- $2\sin^2\theta - \sin\theta = 0$ $\Rightarrow \theta = \pm k\pi, \frac{\pi}{6} \pm 2k\pi, \frac{5\pi}{6} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$
- $\tan^2\theta - \tan\theta = 0$ $\Rightarrow \theta = \pm k\pi, \frac{\pi}{4} \pm 2k\pi, \frac{5\pi}{4} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$
- $2\sin^3\theta - \sin\theta = 0$ $\Rightarrow \theta = \pm k\pi, \frac{\pi}{4} \pm 2k\pi, \frac{3\pi}{4} \pm 2k\pi, \frac{5\pi}{4} \pm 2k\pi, \frac{7\pi}{4} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$

Example 4.5.13: Solve for all real θ by first finding solutions with $0 \le \theta \le 2\pi$

- $\cos^2\theta = \frac{1}{2}$ for $-\pi < \theta < \pi$ $\Rightarrow \theta = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$
- $\sin^2\theta = \frac{1}{2}$ for $-\pi < \theta < 0.$ $\Rightarrow \theta = -\frac{3\pi}{4}, -\frac{\pi}{4}$
- $2\cos^2\theta - 2\sin^2\theta = 1$ for $0 \le \theta \le 2\pi$ $\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
- $\cos^2\theta + \sin^2\theta = 1$ $\Rightarrow$ all θ in $(-\infty, \infty)$

Example 4.5.14: Solve for all real θ by first finding solutions with $0 \le \theta \le 2\pi$

- $\cos^2\theta = \frac{3}{4}$ for $-2\pi \le \theta \le 0$ $\Rightarrow \theta = -\frac{\pi}{6}, -\frac{5\pi}{6}, -\frac{7\pi}{6}, -\frac{11\pi}{6}$
- $2\sin\theta + 1 = 0$ with $0 \le \theta \le 4\pi.$ $\Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$
- $\sin^2\theta - 3\cos^2\theta + 1 = 0$ with $0 \le \theta \le 2\pi.$ $\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
- $\sin^2\theta - 5\sin\theta + 6 = 0$ $\Rightarrow$ No solution

Example 4.5.15: Solve for all real θ by first finding solutions with $0 \le \theta \le 2\pi$

- $2\sin^2\theta + \sin\theta = 0$ $\Rightarrow \theta = \pm k\pi, \frac{7\pi}{6} \pm 2k\pi, \frac{11\pi}{6} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$
- $2\sin^2\theta - \sin\theta = 0$ $\Rightarrow \theta = \pm k\pi, \frac{\pi}{6} \pm 2k\pi, \frac{5\pi}{6} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$
- $\tan^2\theta - \tan\theta = 0$ $\Rightarrow \theta = \pm k\pi, \frac{\pi}{4} \pm 2k\pi, \frac{5\pi}{4} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$
- $2\sin^3\theta - \sin\theta = 0$ $\Rightarrow \theta = \pm k\pi, \frac{\pi}{4} \pm 2k\pi, \frac{3\pi}{4} \pm 2k\pi, \frac{5\pi}{4} \pm 2k\pi, \frac{7\pi}{4} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$

Example 4.5.16: Solve for all real θ by first finding solutions with $0 \le \theta \le 2\pi$

- $\sin^2\theta - 5\sin\theta = 6$ $\Rightarrow \theta =$
- $\cos^2\theta - 2\sin^2\theta = 1$ $\Rightarrow \theta =$
- $3\tan^2\theta = 1$ $\Rightarrow \theta =$
- $4\sin^2\theta = 3$ $\Rightarrow \theta =$

Example 4.5.13: Solve for all real θ by first finding solutions with $0 \leq \theta \leq 2\pi$

- $\cos^2\theta = \frac{1}{2}$ for $-\pi < \theta < \pi$ $\Rightarrow \theta = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$
- $\sin^2\theta = \frac{1}{2}$ for $-\pi < \theta < 0.$ $\Rightarrow \theta = -\frac{3\pi}{4}, -\frac{\pi}{4}$
- $2\cos^2\theta - 2\sin^2\theta = 1$ for $0 \leq \theta \leq 2\pi$ $\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
- $\cos^2\theta + \sin^2\theta = 1$ $\Rightarrow$ all θ in $(-\infty, \infty)$

Example 4.5.14: Solve for all real θ by first finding solutions with $0 \leq \theta \leq 2\pi$

- $\cos^2\theta = \frac{3}{4}$ for $-2\pi \leq \theta \leq 0$ $\Rightarrow \theta = -\frac{\pi}{6}, -\frac{5\pi}{6}, -\frac{7\pi}{6}, -\frac{11\pi}{6}$
- $2\sin\theta + 1 = 0$ with $0 \leq \theta \leq 4\pi.$ $\Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$
- $\sin^2\theta - 3\cos^2\theta + 1 = 0$ with $0 \leq \theta \leq 2\pi.$ $\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
- $\sin^2\theta - 5\sin\theta + 6 = 0$ $\Rightarrow$ No solution

Example 4.5.15: Solve for all real θ by first finding solutions with $0 \leq \theta \leq 2\pi$

- $2\sin^2\theta + \sin\theta = 0$ $\Rightarrow \theta = \pm k\pi, \frac{7\pi}{6} \pm 2k\pi, \frac{11\pi}{6} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$
- $2\sin^2\theta - \sin\theta = 0$ $\Rightarrow \theta = \pm k\pi, \frac{\pi}{6} \pm 2k\pi, \frac{5\pi}{6} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$
- $\tan^2\theta - \tan\theta = 0$ $\Rightarrow \theta = \pm k\pi, \frac{\pi}{4} \pm 2k\pi, \frac{5\pi}{4} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$
- $2\sin^3\theta - \sin\theta = 0$ $\Rightarrow \theta = \pm k\pi, \frac{\pi}{4} \pm 2k\pi, \frac{3\pi}{4} \pm 2k\pi, \frac{5\pi}{4} \pm 2k\pi, \frac{7\pi}{4} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$

Example 4.5.16: Solve for all real θ by first finding solutions with $0 \leq \theta \leq 2\pi$

- $\sin^2\theta - 5\sin\theta = 6$ $\Rightarrow \theta = \frac{3\pi}{2} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$
- $\cos^2\theta - 2\sin^2\theta = 1$ $\Rightarrow \theta = k\pi$ for $k = 0, 1, 2, \ldots$
- $3\tan^2\theta = 1$ $\Rightarrow \theta = \frac{\pi}{6} \pm 2k\pi, \frac{5\pi}{6} \pm 2k\pi, \frac{7\pi}{6} \pm 2k\pi, \frac{11\pi}{6} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$
- $4\sin^2\theta = 3$ $\Rightarrow \theta = \frac{\pi}{3} \pm 2k\pi, \frac{2\pi}{3} \pm 2k\pi, \frac{4\pi}{3} \pm 2k\pi, \frac{5\pi}{3} \pm 2k\pi$ for $k = 0, 1, 2, \ldots$

Section 4.6: Trigonometric identities

- 4.6.1: Trig functions of $30°, 45°$, and $60°$.
- 4.6.2: Basic trigonometric identities
- 4.6.3: Even-odd identities
- 4.6.4: Even and odd functions
- 4.6.5 Simplifying trigonometric expressions
- 4.6.6 Section 4.6 Quiz Review

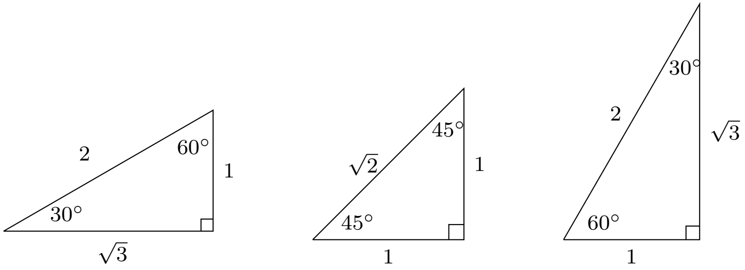
Section 4.6 Preview: Definitions

- Definition 4.6.1: Even and odd identities:
- Definition 4.6.2: Even and odd functions:

4.6.1 Trig functions of $30°, 45°$, and $60°$.

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$\frac{\pi}{6} = 30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4} = 45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3} = 60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Reminder: the special triangles below allow you to figure out the information in the table at the right.



4.6.2 Basic trigonometric identities

In the diagram, the terminal side of angle θ goes from the origin to $P(x, y)$. The distance from P to the origin is r. By the Pythagorean theorem, $x^2 + y^2 = r^2$. The six trig functions are:

- $\cos\theta = \frac{x}{r}$
- $\sin\theta = \frac{y}{r}$
- $\tan\theta = \frac{y}{x}$
- $\sec\theta = \frac{r}{x}$
- $\csc\theta = \frac{r}{y}$
- $\cot\theta = \frac{x}{y}$

All trig functions can be expressed in terms of $\sin\theta$ and $\cos\theta$:

- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$
- $\sec\theta = \dfrac{1}{\cos\theta}$
- $\csc\theta = \dfrac{1}{\sin\theta}$

Divide $x^2 + y^2 = r^2$ by r^2: $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$. Therefore

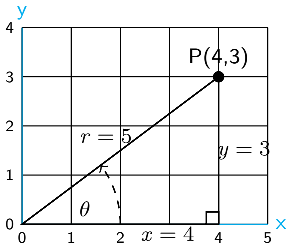
$\cos^2(\theta) + \sin^2(\theta) = 1$. Rearrange: $\cos^2(\theta) = 1 - \sin^2\theta$ and $\sin^2\theta = 1 - \cos^2\theta$

Divide $x^2 + y^2 = r^2$ by x^2: $1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$. Therefore

$1 + \tan^2(\theta) = \sec^2(\theta)$. Rearrange: $\tan^2\theta = \sec^2\theta - 1$ and $\sec^2\theta - \tan^2\theta = 1$

Divide $x^2 + y^2 = r^2$ by y^2: $\left(\frac{r}{y}\right)^2 + 1 = \left(\frac{r}{x}\right)^2$. Therefore

$1 + \cot^2(\theta) = \csc^2(\theta)$. Rearrange: $\cot^2\theta = \csc^2\theta - 1$ and $\csc^2\theta - \cot^2\theta = 1$



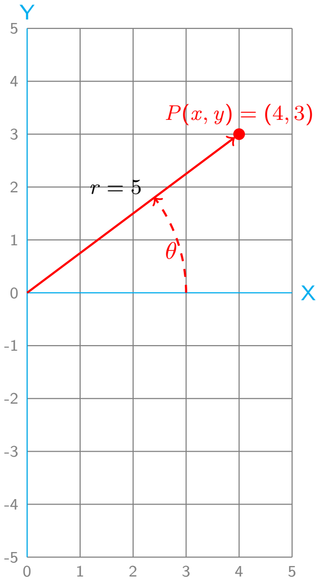
- $\cos\theta = \frac{x}{r}$
- $\sin\theta = \frac{y}{r}$
- $x^2 + y^2 = r^2$

4.6.3 Even-odd identities

Even-odd identities. For any angle θ

- $\cos(-\theta) = \cos\theta$
- $\sin(-\theta) = -\sin\theta$
- $\tan(-\theta) = -\tan\theta$

In the diagram at the right:



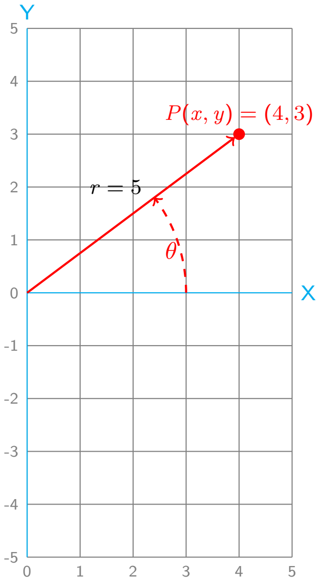
4.6.3 Even-odd identities

Even-odd identities. For any angle θ

- $\cos(-\theta) = \cos\theta$
- $\sin(-\theta) = -\sin\theta$
- $\tan(-\theta) = -\tan\theta$

In the diagram at the right:

- The endline of θ goes from the origin to point $(x, y) = (4, 3)$ in Quadrant 1.
  $\cos\theta = \frac{x}{r} = \frac{4}{5}$ and $\sin\theta = \frac{y}{r} = \frac{3}{5}$



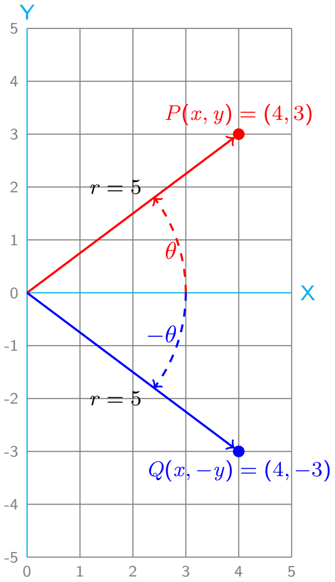
4.6.3 Even-odd identities

Even-odd identities. For any angle θ

- $\cos(-\theta) = \cos\theta$
- $\sin(-\theta) = -\sin\theta$
- $\tan(-\theta) = -\tan\theta$

In the diagram at the right:

- The endline of θ goes from the origin to point $(x, y) = (4, 3)$ in Quadrant 1.
  $\cos\theta = \frac{x}{r} = \frac{4}{5}$ and $\sin\theta = \frac{y}{r} = \frac{3}{5}$
- The endline of angle $-\theta$ goes from the origin to point $(x, -y) = (4, -3)$ in Quadrant 4. Then
  - $\cos(-\theta) = \frac{x}{r} = \frac{4}{5} = \cos\theta$
  - $\sin(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -\frac{3}{5} = -\sin\theta$
  - $\tan(-\theta) = \frac{-y}{x} = -\frac{y}{x} = -\frac{3}{4} = -\tan\theta$



4.6.3 Even-odd identities

Even-odd identities. For any angle θ

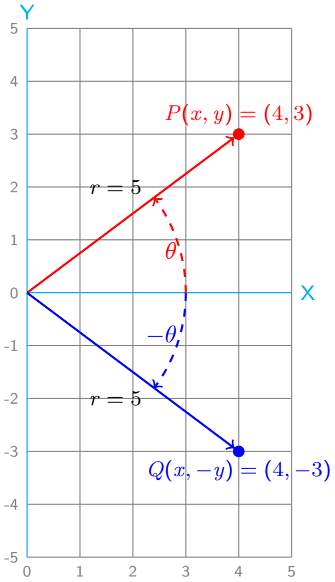
- $\cos(-\theta) = \cos\theta$
- $\sin(-\theta) = -\sin\theta$
- $\tan(-\theta) = -\tan\theta$

In the diagram at the right:

- The endline of θ goes from the origin to point $(x, y) = (4, 3)$ in Quadrant 1.
  $\cos\theta = \frac{x}{r} = \frac{4}{5}$ and $\sin\theta = \frac{y}{r} = \frac{3}{5}$
- The endline of angle $-\theta$ goes from the origin to point $(x, -y) = (4, -3)$ in Quadrant 4. Then
  - $\cos(-\theta) = \frac{x}{r} = \frac{4}{5} = \cos\theta$
  - $\sin(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -\frac{3}{5} = -\sin\theta$
  - $\tan(-\theta) = \frac{-y}{x} = -\frac{y}{x} = -\frac{3}{4} = -\tan\theta$

  Or, use the identity $\tan = \frac{\sin}{\cos}$ to find

  $$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\frac{\sin(\theta)}{\cos\theta} = -\tan\theta$$



4.6.4 Even and odd functions

Even and odd functions

A function $y = f(x)$ is

- even if for all x in its domain, $f(-x) = f(x)$.

  The graph remains unchanged when reflected through the y-axis: The graph is y-axis symmetric.

  If point (x, y) is on the graph, then point $(-x, y)$ is on the graph.

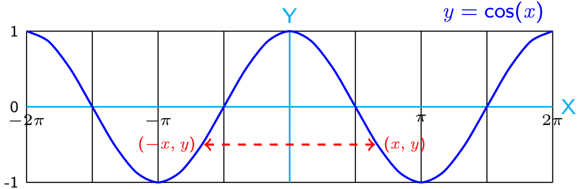
- odd if for all x in its domain, $f(-x) = -f(x)$.

  The graph remains unchanged when reflected through the origin: The graph is origin symmetric.

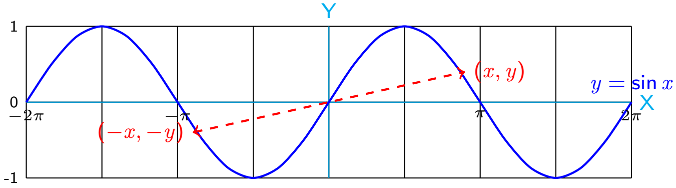
  If point (x, y) is on the graph, then point $(-x, -y)$ is on the graph

The last slide showed that cosine is an even function, while sine and tangent are odd functions. To better understand what this means, look at the graphs below.

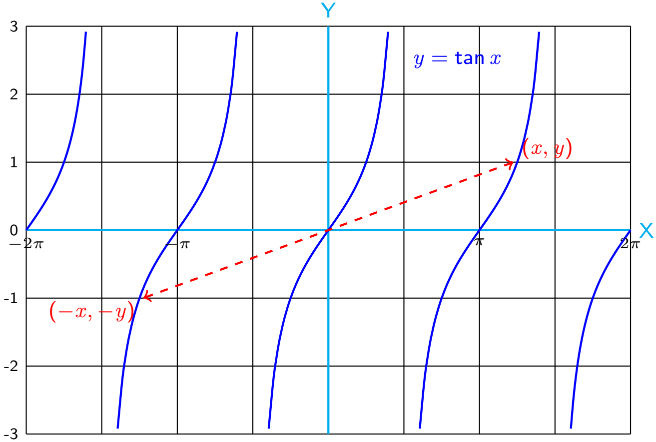
The function $y = f(x) = \cos(x)$ is even: for all x, $\cos(-x) = \cos(x)$. Its graph is y-axis symmetric. If point (x, y) is on the graph, so is point $(-x, y)$



The function $y = f(x) = \sin(x)$ is odd: for all x, $\sin(-x) = -\sin(x)$. Its graph is origin symmetric. If point (x, y) is on the graph, so is point $(-x, -y)$



The function $y = f(x) = \tan(x)$ is odd: for all x, $\tan(-x) = -\tan(x)$. Its graph is origin symmetric. If point (x, y) is on the graph, so is point $(-x, -y)$



4.6.5 Simplifying trigonometric expressions

Sometimes it is important to transform one trig expression into another. To do so, you need to build new identities.

Method 1:
Rewrite expressions in terms of sine and cosine.

Example 1: Simplify and rewrite $\cos^2\theta(1+\tan^2\theta)$ in terms of sine and cosine.

Solution:

To simplify	$\cos^2\theta(1+\tan^2\theta)$
Use $1+\tan^2\theta = \sec^2\theta$	$= \cos^2\theta\sec^2\theta$
Use $\cos\theta = \dfrac{1}{\sec\theta}$	$= \cos^2\theta\cdot(\dfrac{1}{\cos\theta})^2$
Simplify	$= \boxed{1}$

Example 2: Simplify and rewrite $\dfrac{\sec^2\theta - 1}{\sec^2\theta}$ in terms of sine and cosine.

Solution:

To simplify	$\dfrac{\sec^2\theta - 1}{\sec^2\theta}$
Separate into two fractions	$= \dfrac{\sec^2\theta}{\sec^2\theta} - \dfrac{1}{\sec^2\theta}$
Use $\cos\theta = \dfrac{1}{\sec\theta}$	$= 1-\cos^2\theta$
Use $\sin^2\theta = 1-\cos^2\theta$	$= \boxed{\sin^2\theta}$

Example 3:

Combine fractions and simplify $\dfrac{1+\sin u}{\cos u} + \dfrac{\cos u}{1+\sin u}$

Solution:

Start with $\dfrac{1+\sin u}{\cos u} + \dfrac{\cos u}{1+\sin u}$

Add fractions $= \dfrac{(1+\sin u)(1+\sin u) + \cos u \cos u}{\cos u(1+\sin u)}$

Expand $= \dfrac{1+2\sin u + \sin^2 u + \cos^2 u}{\cos u(1+\sin u)}$

$= \dfrac{1+2\sin u + 1}{\cos u(1+\sin u)}$ since $\sin^2 + \cos^2 = 1$

Rewrite $= \dfrac{2(1+\sin u)}{\cos u(1+\sin u)}$

Cancel $= \dfrac{2}{\cos u} = \boxed{2\sec u}$

The last step was a required simplification, since it removed a fraction from the answer.

Reminder: a fraction equals its numerator times the reciprocal of its denominator: $\frac{3}{4} = 3\cdot\frac{1}{4}$ and $\frac{u}{v} = u\cdot\frac{1}{v}$

Two examples:

- $\sec u = \dfrac{1}{\cos u} \Rightarrow \cos u = \dfrac{1}{\sec u}$
- $\csc u = \dfrac{1}{\sin u} \Rightarrow \sin u = \dfrac{1}{\csc u}$

Example 4:

Combine fractions and simplify $\dfrac{\cos u}{\sec u} + \dfrac{\sin u}{\csc u}$

Solution:

To simplify $\dfrac{\cos u}{\sec u} + \dfrac{\sin u}{\csc u}$

Rewrite fractions $= \cos u \cdot \dfrac{1}{\sec u} + \sin u \cdot \dfrac{1}{\csc u}$

Use reminder above $= \cos u \cos u + \sin u \sin u$

Pythagorean identity $= \cos^2 u + \sin^2 u = \boxed{1}$

The solution of the next example creates a complex fraction. To simplify it, multiply its numerator and denominator by the LCD of any nested (little) fractions that it contains.

Example 5: Simplify $\dfrac{\sin u + 1}{\csc u + 1}$

Solution:

To simplify	$\dfrac{\sin u + 1}{\csc u + 1}$
Rewrite $\csc u$	$= \dfrac{\sin u + 1}{\frac{1}{\sin u} + 1}$
Multiiply numerator and denominator by $\sin u$	$= \dfrac{\sin u(\sin u + 1)}{\sin u(\frac{1}{\sin u} + 1)}$
Multiply out on bottom	$= \dfrac{\sin u(\sin u + 1)}{1 + \sin u}$
Cancel:	$= \boxed{\sin u}$

Example 6: Simplify $\dfrac{2 + \cot^2 u}{\csc^2 u}$

Solution:

To simplify	$\dfrac{2 + \cot^2 u}{\csc^2 u}$
Rewrite $\cot u$ and $\csc u$	$= \dfrac{2 + \left(\frac{\cos u}{\sin u}\right)^2 u}{\left(\frac{1}{\sin u}\right)^2}$
Square the fractions	$= \dfrac{2 + \frac{\cos^2 u}{\sin^2 u}}{\frac{1}{\sin^2 u}}$
Multiiply numerator and denominator by $\sin^2 u$	$= \dfrac{2\sin^2 u + \cos^2 u}{1}$
Use $\cos^2 u = 1 - \sin^2 u$	$= 2\sin^2 u + 1 - \sin^2 u$
	$= \boxed{\sin^2 u + 1}$
Or use $sin^2 u = 1 - \cos^2 u$	$= \boxed{2 - cos^2 u}$

Example 7: Another approach: simplify $\dfrac{2+\cot^2 u}{\csc^2 u}$ by using $\csc^2 u = 1+\cot^2 u \Rightarrow \cot^2 u = \csc^2 u - 1$

Solution:

$$
\begin{aligned}
&\text{To simplify} && \frac{2+\cot^2 u}{\csc^2 u} \\
&\text{Rewrite the numerator} && = \frac{2+\csc^2 u - 1}{\csc^2 u} \\
& && = \frac{1+\csc^2 u}{\csc^2 u} \\
&\text{Split the fraction in two} && = \frac{1}{\csc^2 u} + \frac{\csc^2 u}{\csc^2 u} \\
&\text{Use } \frac{1}{\csc u} = \sin u && = \boxed{\sin^2 u + 1}
\end{aligned}
$$

Section 4.6 Quiz

Example 4.6.1: Simplify and rewrite $\cos^2\theta(1+\tan^2\theta)$ in terms of sine and cosine.

Example 4.6.2: Simplify and rewrite $\dfrac{\sec^2\theta - 1}{\sec^2\theta}$ in terms of sine and cosine.

Example 4.6.3: Combine fractions and simplify $\dfrac{1+\sin u}{\cos u} + \dfrac{\cos u}{1+\sin u}$

Example 4.6.4: Combine fractions and simplify $\dfrac{\cos u}{\sec u} + \dfrac{\sin u}{\csc u}$

Example 4.6.5: Simplify $\dfrac{\sin u + 1}{\csc u + 1}$

Example 4.6.6 : Simplify $\dfrac{2+\cot^2 u}{\csc^2 u}$ (use complex fraction simplification)

Example 4.6.7 : Simplify $\dfrac{2+\cot^2 u}{\csc^2 u}$ (use a trig identity involving $\cot u$ and $\csc u$.)

Section 4.6 Review: Trigonometric identities

Example 4.6.1: Simplify and rewrite in terms of sine and cosine.

- $\cos^2\theta(1+\tan^2\theta) =$
- $\sin^2\theta(1+\tan^2\theta)\cot^2\theta =$
- $\sin\theta\cos^2\theta\tan^2\theta =$
- $\dfrac{\cos^2\theta}{1-\sin\theta} =$

Section 4.6 Review: Trigonometric identities

Example 4.6.1: Simplify and rewrite in terms of sine and cosine.

- $\cos^2\theta(1+\tan^2\theta) = 1$
- $\sin^2\theta(1+\tan^2\theta)\cot^2\theta = 1$
- $\sin\theta\cos^2\theta\tan^2\theta = \sin^3\theta$
- $\dfrac{\cos^2\theta}{1-\sin\theta} = 1+\sin\theta$

Section 4.6 Review: Trigonometric identities

Example 4.6.1: Simplify and rewrite in terms of sine and cosine.

- $\cos^2\theta(1+\tan^2\theta) = 1$
- $\sin^2\theta(1+\tan^2\theta)\cot^2\theta = 1$
- $\sin\theta\cos^2\theta\tan^2\theta = \sin^3\theta$
- $\dfrac{\cos^2\theta}{1-\sin\theta} = 1+\sin\theta$

Example 4.6.2: Simplify and rewrite in terms of sine and cosine.

- $\dfrac{\sec^2\theta - 1}{\sec^2\theta} =$
- $\dfrac{-1}{\sec\theta(1+\tan^2\theta)} =$
- $\dfrac{1-\cos^2\theta}{\tan^2\theta} =$
- $\csc^2\theta(1-\cos^2\theta) =$

Section 4.6 Review: Trigonometric identities

Example 4.6.1: Simplify and rewrite in terms of sine and cosine.

- $\cos^2\theta(1+\tan^2\theta) = 1$
- $\sin^2\theta(1+\tan^2\theta)\cot^2\theta = 1$
- $\sin\theta\cos^2\theta\tan^2\theta = \sin^3\theta$
- $\dfrac{\cos^2\theta}{1-\sin\theta} = 1+\sin\theta$

Example 4.6.2: Simplify and rewrite in terms of sine and cosine.

- $\dfrac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$
- $\dfrac{-1}{\sec\theta(1+\tan^2\theta)} = -\cos^3\theta$
- $\dfrac{1-\cos^2\theta}{\tan^2\theta} = \cos^2\theta$
- $\csc^2\theta(1-\cos^2\theta) = 1$

Section 4.6 Review: Trigonometric identities

Example 4.6.1: Simplify and rewrite in terms of sine and cosine.

- $\cos^2\theta(1+\tan^2\theta) = 1$
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Example 4.6.2: Simplify and rewrite in terms of sine and cosine.

- $\dfrac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$
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- $\dfrac{1-\cos^2\theta}{\tan^2\theta} = \cos^2\theta$
- $\csc^2\theta(1-\cos^2\theta) = 1$

Example 4.6.3: Combine and simplify to a single trig function or a constant.

- $\dfrac{1+\sin u}{\cos u} + \dfrac{\cos u}{1+\sin u} =$
- $\dfrac{\cos u}{1-\sin u} + \dfrac{\cos u}{1+\sin u} =$
- $\dfrac{-\cos u}{1-\sin u} + \dfrac{1+\sin u}{\cos u} =$
- $\dfrac{\cos^2 u}{1-\sin^2 u} + \dfrac{\sin^2 u}{1-\cos^2 u} =$

Section 4.6 Review: Trigonometric identities

Example 4.6.1: Simplify and rewrite in terms of sine and cosine.

- $\cos^2\theta(1+\tan^2\theta) = 1$
- $\sin\theta\cos^2\theta\tan^2\theta = \sin^3\theta$
- $\sin^2\theta(1+\tan^2\theta)\cot^2\theta = 1$
- $\dfrac{\cos^2\theta}{1-\sin\theta} = 1+\sin\theta$

Example 4.6.2: Simplify and rewrite in terms of sine and cosine.

- $\dfrac{\sec^2\theta-1}{\sec^2\theta} = \sin^2\theta$
- $\dfrac{1-\cos^2\theta}{\tan^2\theta} = \cos^2\theta$
- $\dfrac{-1}{\sec\theta(1+\tan^2\theta)} = -\cos^3\theta$
- $\csc^2\theta(1-\cos^2\theta) = 1$

Example 4.6.3: Combine and simplify to a single trig function or a constant.

- $\dfrac{1+\sin u}{\cos u} + \dfrac{\cos u}{1+\sin u} = 2\sec u$
- $\dfrac{-\cos u}{1-\sin u} + \dfrac{1+\sin u}{\cos u} = 0$
- $\dfrac{\cos u}{1-\sin u} + \dfrac{\cos u}{1+\sin u} = 2\sec u$
- $\dfrac{\cos^2 u}{1-\sin^2 u} + \dfrac{\sin^2 u}{1-\cos^2 u} = 2$

Section 4.6 Review: Trigonometric identities

Example 4.6.1: Simplify and rewrite in terms of sine and cosine.

- $\cos^2\theta(1+\tan^2\theta) = 1$
- $\sin\theta\cos^2\theta\tan^2\theta = \sin^3\theta$
- $\sin^2\theta(1+\tan^2\theta)\cot^2\theta = 1$
- $\dfrac{\cos^2\theta}{1-\sin\theta} = 1+\sin\theta$

Example 4.6.2: Simplify and rewrite in terms of sine and cosine.

- $\dfrac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$
- $\dfrac{1-\cos^2\theta}{\tan^2\theta} = \cos^2\theta$
- $\dfrac{-1}{\sec\theta(1+\tan^2\theta)} = -\cos^3\theta$
- $\csc^2\theta(1-\cos^2\theta) = 1$

Example 4.6.3: Combine and simplify to a single trig function or a constant.

- $\dfrac{1+\sin u}{\cos u} + \dfrac{\cos u}{1+\sin u} = 2\sec u$
- $\dfrac{-\cos u}{1-\sin u} + \dfrac{1+\sin u}{\cos u} = 0$
- $\dfrac{\cos u}{1-\sin u} + \dfrac{\cos u}{1+\sin u} = 2\sec u$
- $\dfrac{\cos^2 u}{1-\sin^2 u} + \dfrac{\sin^2 u}{1-\cos^2 u} = 2$

Example 4.6.4: Combine and simplify to a single trig function or a constant.

- $\dfrac{\cos u}{\sec u} + \dfrac{\sin u}{\csc u} =$
- $\sin u\left(\dfrac{\cos u}{\sin u} + \dfrac{\sin u}{\cos u}\right) =$
- $\tan u\sin u + \cos u =$
- $\sin u\cos u\sec u\cot u =$

Section 4.6 Review: Trigonometric identities

Example 4.6.1: Simplify and rewrite in terms of sine and cosine.

- $\cos^2\theta(1+\tan^2\theta) = 1$
- $\sin\theta\cos^2\theta\tan^2\theta = \sin^3\theta$
- $\sin^2\theta(1+\tan^2\theta)\cot^2\theta = 1$
- $\dfrac{\cos^2\theta}{1-\sin\theta} = 1+\sin\theta$

Example 4.6.2: Simplify and rewrite in terms of sine and cosine.

- $\dfrac{\sec^2\theta-1}{\sec^2\theta} = \sin^2\theta$
- $\dfrac{1-\cos^2\theta}{\tan^2\theta} = \cos^2\theta$
- $\dfrac{-1}{\sec\theta(1+\tan^2\theta)} = -\cos^3\theta$
- $\csc^2\theta(1-\cos^2\theta) = 1$

Example 4.6.3: Combine and simplify to a single trig function or a constant.

- $\dfrac{1+\sin u}{\cos u} + \dfrac{\cos u}{1+\sin u} = 2\sec u$
- $\dfrac{-\cos u}{1-\sin u} + \dfrac{1+\sin u}{\cos u} = 0$
- $\dfrac{\cos u}{1-\sin u} + \dfrac{\cos u}{1+\sin u} = 2\sec u$
- $\dfrac{\cos^2 u}{1-\sin^2 u} + \dfrac{\sin^2 u}{1-\cos^2 u} = 2$

Example 4.6.4: Combine and simplify to a single trig function or a constant.

- $\dfrac{\cos u}{\sec u} + \dfrac{\sin u}{\csc u} = 1$
- $\sin u\left(\dfrac{\cos u}{\sin u} + \dfrac{\sin u}{\cos u}\right) = \sec u$
- $\tan u\sin u + \cos u = \sec u$
- $\sin u\cos u\sec u\cot u = \cos u$

Section 4.6 Review: Trigonometric identities

Example 4.6.5: Simplify

- $\dfrac{\sin u + 1}{\csc u + 1} =$
- $\dfrac{1 - \sin^2 u}{\cos^3 u} =$
- $\dfrac{\sin^2 u}{\cos^2 u - 1} =$
- $\dfrac{\sec^2 u - 1}{\sin^2 u} =$

Section 4.6 Review: Trigonometric identities

Example 4.6.5: Simplify

- $\dfrac{\sin u + 1}{\csc u + 1} = \sin u$
- $\dfrac{1 - \sin^2 u}{\cos^3 u} = \sec u$
- $\dfrac{\sin^2 u}{\cos^2 u - 1} = -1$
- $\dfrac{\sec^2 u - 1}{\sin^2 u} = \sec^2 u$

Section 4.6 Review: Trigonometric identities

Example 4.6.5: Simplify

- $\dfrac{\sin u + 1}{\csc u + 1} = \sin u$
- $\dfrac{\sin^2 u}{\cos^2 u - 1} = -1$
- $\dfrac{1 - \sin^2 u}{\cos^3 u} = \sec u$
- $\dfrac{\sec^2 u - 1}{\sin^2 u} = \sec^2 u$

Example 4.6.6 : Use complex fraction simplification to simplify

- $\dfrac{2 + \cot^2 u}{\csc^2 u} =$
- $\dfrac{1 + \cot^2 u}{\csc^3 u} =$
- $\dfrac{2 + \tan^2 u}{\sec^2 u} =$
- $\dfrac{1 - \tan^2 u}{\sec^2 u} =$

Section 4.6 Review: Trigonometric identities

Example 4.6.5: Simplify

- $\dfrac{\sin u+1}{\csc u+1}=\sin u$
- $\dfrac{\sin^2 u}{\cos^2 u-1}=-1$
- $\dfrac{1-\sin^2 u}{\cos^3 u}=\sec u$
- $\dfrac{\sec^2 u-1}{\sin^2 u}=\sec^2 u$

Example 4.6.6 : Use complex fraction simplification to simplify

- $\dfrac{2+\cot^2 u}{\csc^2 u}= \sin^2 u+1$
- $\dfrac{1+\cot^2 u}{\csc^3 u}= \sin u$
- $\dfrac{2+\tan^2 u}{\sec^2 u}= \cos^2 u+1$
- $\dfrac{1-\tan^2 u}{\sec^2 u}= 1-2\sin^2 u$

Section 4.6 Review: Trigonometric identities

Example 4.6.5: Simplify

- $\dfrac{\sin u + 1}{\csc u + 1} = \sin u$
- $\dfrac{1 - \sin^2 u}{\cos^3 u} = \sec u$
- $\dfrac{\sin^2 u}{\cos^2 u - 1} = -1$
- $\dfrac{\sec^2 u - 1}{\sin^2 u} = \sec^2 u$

Example 4.6.6 : Use complex fraction simplification to simplify

- $\dfrac{2 + \cot^2 u}{\csc^2 u} = \sin^2 u + 1$
- $\dfrac{2 + \tan^2 u}{\sec^2 u} = \cos^2 u + 1$
- $\dfrac{1 + \cot^2 u}{\csc^3 u} = \sin u$
- $\dfrac{1 - \tan^2 u}{\sec^2 u} = 1 - 2\sin^2 u$

Example 4.6.7 : Redo Exercise 4.6.6 by using trig identities involving $\cot, \csc, \sec$.

Section 4.7: Sum and difference formulas

- 4.7.1: Trig functions of general angles
- 4.7.2: Sum and difference formulas
- 4.7.3: Multiples of 15°
- 4.7.4: Section 4.7 Quiz Review

Section 4.7 Preview: Procedures

- Procedure 4.7.1: To find $\sin(\theta), \cos(\theta), \tan(\theta)$ of any angle θ:
- Procedure 4.7.2: To find trig functions of the sum of angles:
- Procedure 4.7.3: To find trig functions of the difference of angles:

4.7.1 Finding trig functions of general angles

How to find $\sin(\theta), \cos(\theta), \tan(\theta)$ of any angle θ

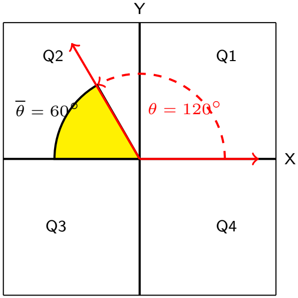
1. Draw the endline of angle θ.
2. The reference angle Ref θ is the acute angle between the endline of angle θ and the x-axis.

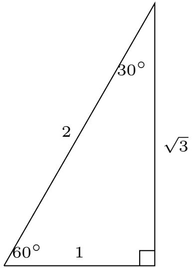
- $\cos(\theta) = \quad \cos(\text{Ref } \theta)$ if θ's endline is in Q1 or Q4,where cosine is $+$, or $\cos(\theta) = -\cos(\text{Ref } \theta)$ if θ's endline is in Q2 or Q3, where cosine is $-$.
- $\sin(\theta) = \quad \sin(\text{Ref } \theta)$ if θ's endline is in Q1 or Q2, where sine is $+$, or $\sin(\theta) = -\sin(\text{Ref } \theta)$ if θ's endline is in Q3 or Q4, where sine is $-$.
- $\tan(\theta) = \quad \tan(\text{Ref } \theta)$ if θ's endline is in Q1 or Q3, where tangent is $+$, or $\tan(\theta) = -\tan(\text{Ref } \theta)$ if θ's endline is in Q2 or Q4, where tangent is $-$.

Note: these statements also hold when $\cos, \sin, \tan$ are replaced by $\sec, \csc, \cot$ respectively.

Example 1: Find $\cos 120°$.
Solution: $\theta = 120°$ has endline in Q2.
Its (yellow) reference angle is
Ref $(\theta) = 180° - 120° = 60°$.
The triangle below at the right shows that $\cos 60° = \frac{1}{2}$.
Since cosine is negative in Q2,

$$\cos 120° = \cos\theta = -\cos \text{Ref}(\theta) = -\cos 60° = \boxed{-\tfrac{1}{2}.}$$





4.7.2 Sum and difference trig formulas

To find trig functions of the sum of angles:

Let u and v be any angles, both expressed in radians, or both in degrees. Then:

- $\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v)$
- $\sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)$
- $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$

Exercise: Use the first two formulas above to derive the third. Hint: start with $\tan(u+v) = \frac{\sin(u+v)}{\cos(u+v)}$ and use the first two bullets. Then divide top and bottom of the fraction by $\cos(u)\cos(v)$ and simplify.

It is best to memorize the above three identities. Other identities can be easily derived from them. For example, recall that $\cos(-u) = \cos(u)$ and $\sin(-v) = -\sin(v)$. Then we can figure out $\cos(u-v)$ by writing $u-v$ as the sum $u + -v$. The result is:

$\cos(u-v) = \cos(u + -v)$
$= \cos(u)\cos(-v) - \sin(u)\sin(-v)$
$= \cos(u)\cos(v) + \sin(u)\sin(v)$.

To find trig functions of the difference of angles:

Let u and v be any angles, both expressed in radians, or both in degrees. Then:

- $\cos(u-v) = \cos(u)\cos(v) + \sin(u)\sin(v)$
- $\sin(u-v) = \sin(u)\cos(v) - \cos(u)\sin(v)$
- $\tan(u-v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

Next, we find trig functions of any angle with reference angle 15° and 75° by applying sum and difference formulas to angles $30^\circ, 45^\circ$, or 60°.

Example 2: Rewrite each of the four angles $15^\circ, 75^\circ, 105^\circ, 165^\circ$ as a sum or difference of angles with reference angle $30^\circ, 45^\circ$, and 60°.

Solution:
$15^\circ = 45^\circ - 30^\circ$; $75^\circ = 45^\circ + 30^\circ$;
$105^\circ = 45^\circ + 60^\circ$; $165^\circ = 120^\circ + 45^\circ$.

Example 3: Use these answers to find $\cos(15^\circ), \sin(75^\circ), \cos(105^\circ)$, and $\sin(165^\circ)$.
Solution:
Apply the appropriate sum or difference formula:

Worked examples

- $\cos 15^\circ = \cos(45^\circ - 30^\circ)$
  $= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$
  $\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \boxed{\frac{1+\sqrt{3}}{2\sqrt{2}}}$
- $\sin 75^\circ = \sin(45^\circ + 30^\circ)$
  $= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin(30^\circ)$
  $= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \boxed{\frac{\sqrt{3}+1}{2\sqrt{2}}}$
- $\cos 105^\circ = \cos(45^\circ + 60^\circ)$
  $= \cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ$
  $= \frac{1}{\sqrt{2}} \cdot \frac{1}{2} - \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} = \boxed{\frac{1-\sqrt{3}}{2\sqrt{2}}}$

  The following example uses the fact that 120° is a Quadrant 2 angle with reference angle $180^\circ - 120^\circ = 60^\circ$. See the diagram on the first slide in this section.
- $\sin 165^\circ = \sin(45^\circ + 120^\circ)$
  $= \sin 45^\circ \cos 120^\circ + \cos 45^\circ \sin 120^\circ$
  $= \frac{1}{\sqrt{2}} \cdot (-\cos 60^\circ) + \frac{1}{\sqrt{2}} \cdot \sin 60^\circ =$
  $= \frac{1}{\sqrt{2}} \cdot (-\frac{1}{2}) + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} = \boxed{\frac{-1+\sqrt{3}}{2\sqrt{2}}}$

To rationalize the answers above, multiply numerator and denominator of each answer by $\sqrt{2}$. For example

$\sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{(\sqrt{3}+1)\sqrt{2}}{2\sqrt{2}(\sqrt{2})} = \frac{\sqrt{3}\sqrt{2}+\sqrt{2}}{2(2)} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}.$

These examples used degree measures for angles. Of course, you should also be able to find trig functions of angles expressed in radians.

Example 4. Find $\cos\frac{\pi}{12}$, $\sin\frac{5\pi}{12}$, $\cos\frac{7\pi}{12}$, and $\sin\frac{11\pi}{12}$

Solution: Each angle is an integer multiple of $\frac{\pi}{12} = 15^\circ$. Answers are at the left.

- $\cos\frac{\pi}{12} = \cos(15^\circ) = \cos(45^\circ - 30^\circ)$
- $\sin\frac{5\pi}{12} = \sin(5 \cdot 15^\circ) = \sin 75^\circ = \sin(45^\circ + 30^\circ)$
- $\cos\frac{7\pi}{12} = \cos(7 \cdot 15^\circ) = \cos 105^\circ = \cos(45^\circ + 60^\circ)$
- $\sin\frac{11\pi}{12} \sin(11 \cdot 15^\circ) = \sin 165^\circ = \sin(45^\circ + 120^\circ)$

4.7.3 Trig functions of multiples of 15°

We can now find trig functions of any angle that is a multiple of $15^\circ = \frac{\pi}{12}$.

Example 5: Find sin$(\frac{67\pi}{12})$
Solution: First find a coterminal angle between 0 and 2π by subtracting (twice) $2\pi = \frac{24\pi}{12}$. The result is

$\frac{67\pi}{12} - \frac{24\pi}{12} - \frac{24\pi}{12} = \frac{(67-24-24)\pi}{12} = \frac{19\pi}{12}$.

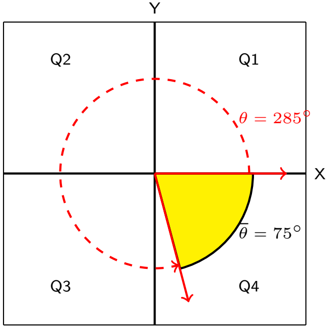
This is a bit better: $\frac{67\pi}{12} = \frac{19\pi}{12} = 19 \cdot 15^\circ = 285^\circ$.

To rewrite this as a sum or difference, refer to the diagram at the right.

285° is between 270° and 360°.
285° is in Quadrant 4.
Its reference angle is $360^\circ - 285^\circ = 75^\circ$.
Since 285° is in Quadrant 4, its sine is negative.
Thus $\sin(285^\circ) = -\sin 75^\circ$.

Use a sum formula to find
$\sin 75^\circ = \sin(45^\circ + 30^\circ) = \frac{1+\sqrt{3}}{2\sqrt{2}}$; see Example 2.

$$\sin \frac{67\pi}{12} = -\frac{1+\sqrt{3}}{2\sqrt{2}}$$



Sometimes you need to work backwards, from a numerical expression to a formula:

Example 6: Find the exact value of
$\sin 16^\circ \cos 44^\circ + \cos 16^\circ \sin 44^\circ$.
Solution: This is a situation where variables are easier to understand than numbers.
Let $u = 16^\circ$ and $v = 44^\circ$. You are being asked to find the exact value of
$\sin 16^\circ \cos 44^\circ + \cos 16^\circ \sin 44^\circ = \sin u \cos v + \cos u \sin v$,
which is exactly $\sin(u + v)$.
Therefore you are being asked to find

$$\sin(u + v) = \sin(16^\circ + 44^\circ) = \boxed{\sin(60^\circ) = \frac{\sqrt{3}}{2}}.$$

Example 7: Prove the identity $\sin(x - \frac{\pi}{2}) = -\cos x$.
Solution: Since this is a formula involving the sine of a difference of angles, write
$\sin(u - v) = \sin u \cos v - \cos u \sin v$.

Substitute $u = x$ and $v = \frac{\pi}{2}$ to obtain

$$\boxed{\sin(x - \tfrac{\pi}{2}) = \sin x \cos \tfrac{\pi}{2} - \cos x \sin \tfrac{\pi}{2} = -\cos(x)}$$

because $\cos \frac{\pi}{2} = 0$ and $\sin \frac{\pi}{2} = 1$.

Section 4.7 Quiz

Example 1: Find $\cos 120°$.

Example 2: Rewrite each of the four angles $15°, 75°, 105°, 165°$ as a sum or difference of angles with reference angle $30°, 45°$, or $60°$.

Example 3: Use the answers to Example 1 to find $\cos(15°), \sin(75°), \cos(105°)$, and $\sin(165°)$.

Example 4. Find $\cos\frac{\pi}{12}, \sin\frac{5\pi}{12}, \cos\frac{7\pi}{12}$, and $\sin\frac{11\pi}{12}$

Example 5: Find $\sin(\frac{67\pi}{12})$.

Example 6: Find the exact value of $\sin 16° \cos 44° + \cos 16° \sin 44°$.

Example 7: Prove the identity $\sin(x - \frac{\pi}{2}) = -\cos x$.

Section 4.7 Review: Sum and difference formulas

Example 4.7.1: Find

- $\cos 120° =$
- $\sin 120° =$
- $\sin 225° =$
- $\cos 300° =$

Section 4.7 Review: Sum and difference formulas

Example 4.7.1: Find

- $\cos 120° = -\frac{1}{2}$
- $\sin 120° = \frac{\sqrt{3}}{2}$
- $\sin 225° = -\frac{1}{\sqrt{2}}$
- $\cos 300° = \frac{1}{2}$

Section 4.7 Review: Sum and difference formulas

Example 4.7.1: Find

- $\cos 120^\circ = -\frac{1}{2}$ • $\sin 120^\circ = \frac{\sqrt{3}}{2}$ • $\sin 225^\circ = -\frac{1}{\sqrt{2}}$ • $\cos 300^\circ = \frac{1}{2}$

Example 4.7.2: Rewrite each angle as a sum or difference of angles with reference angle $30^\circ, 45^\circ$, or 60°.

- $15^\circ =$ • $75^\circ =$ • $105^\circ =$ • $165^\circ =$

Section 4.7 Review: Sum and difference formulas

Example 4.7.1: Find

• $\cos 120^\circ = -\frac{1}{2}$ • $\sin 120^\circ = \frac{\sqrt{3}}{2}$ • $\sin 225^\circ = -\frac{1}{\sqrt{2}}$ • $\cos 300^\circ = \frac{1}{2}$

Example 4.7.2: Rewrite each angle as a sum or difference of angles with reference angle $30^\circ, 45^\circ$, or 60°.

• $15^\circ = 45^\circ - 30^\circ$ • $75^\circ = 300^\circ + 45^\circ$ • $105^\circ = 60^\circ + 45^\circ$ • $165^\circ = 120^\circ + 45^\circ$

Section 4.7 Review: Sum and difference formulas

Example 4.7.1: Find

• $\cos 120^\circ = -\frac{1}{2}$ • $\sin 120^\circ = \frac{\sqrt{3}}{2}$ • $\sin 225^\circ = -\frac{1}{\sqrt{2}}$ • $\cos 300^\circ = \frac{1}{2}$

Example 4.7.2: Rewrite each angle as a sum or difference of angles with reference angle $30^\circ, 45^\circ$, or 60°.

• $15^\circ = 45^\circ - 30^\circ$ • $75^\circ = 300^\circ + 45^\circ$ • $105^\circ = 60^\circ + 45^\circ$ • $165^\circ = 120^\circ + 45^\circ$

Example 4.7.3: Use the answers to Example 4.7.2 to find

• $\cos(15^\circ) =$ • $\sin(75^\circ) =$ • $\cos(105^\circ) =$ • $\sin(165^\circ) =$

Section 4.7 Review: Sum and difference formulas

Example 4.7.1: Find

• $\cos 120^\circ = -\frac{1}{2}$ • $\sin 120^\circ = \frac{\sqrt{3}}{2}$ • $\sin 225^\circ = -\frac{1}{\sqrt{2}}$ • $\cos 300^\circ = \frac{1}{2}$

Example 4.7.2: Rewrite each angle as a sum or difference of angles with reference angle $30^\circ, 45^\circ$, or 60°.

• $15^\circ = 45^\circ - 30^\circ$ • $75^\circ = 300^\circ + 45^\circ$ • $105^\circ = 60^\circ + 45^\circ$ • $165^\circ = 120^\circ + 45^\circ$

Example 4.7.3: Use the answers to Example 4.7.2 to find

• $\cos(15^\circ) = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\sin(75^\circ) = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\cos(105^\circ) = \frac{1-\sqrt{3}}{2\sqrt{2}}$ • $\sin(165^\circ) = \frac{-1+\sqrt{3}}{2\sqrt{2}}$

Section 4.7 Review: Sum and difference formulas

Example 4.7.1: Find

- $\cos 120^\circ = -\frac{1}{2}$ • $\sin 120^\circ = \frac{\sqrt{3}}{2}$ • $\sin 225^\circ = -\frac{1}{\sqrt{2}}$ • $\cos 300^\circ = \frac{1}{2}$

Example 4.7.2: Rewrite each angle as a sum or difference of angles with reference angle $30^\circ, 45^\circ$, or 60°.

- $15^\circ = 45^\circ - 30^\circ$ • $75^\circ = 300^\circ + 45^\circ$ • $105^\circ = 60^\circ + 45^\circ$ • $165^\circ = 120^\circ + 45^\circ$

Example 4.7.3: Use the answers to Example 4.7.2 to find

- $\cos(15^\circ) = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\sin(75^\circ) = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\cos(105^\circ) = \frac{1-\sqrt{3}}{2\sqrt{2}}$ • $\sin(165^\circ) = \frac{-1+\sqrt{3}}{2\sqrt{2}}$

Example 4.7.4. Find

- $\cos \frac{\pi}{12} =$ • $\sin \frac{5\pi}{12} =$ • $\cos \frac{7\pi}{12} =$ • $\sin \frac{11\pi}{12} =$

Section 4.7 Review: Sum and difference formulas

Example 4.7.1: Find

- $\cos 120^\circ = -\frac{1}{2}$ • $\sin 120^\circ = \frac{\sqrt{3}}{2}$ • $\sin 225^\circ = -\frac{1}{\sqrt{2}}$ • $\cos 300^\circ = \frac{1}{2}$

Example 4.7.2: Rewrite each angle as a sum or difference of angles with reference angle $30^\circ, 45^\circ$, or 60°.

- $15^\circ = 45^\circ - 30^\circ$ • $75^\circ = 300^\circ + 45^\circ$ • $105^\circ = 60^\circ + 45^\circ$ • $165^\circ = 120^\circ + 45^\circ$

Example 4.7.3: Use the answers to Example 4.7.2 to find

- $\cos(15^\circ) = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\sin(75^\circ) = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\cos(105^\circ) = \frac{1-\sqrt{3}}{2\sqrt{2}}$ • $\sin(165^\circ) = \frac{-1+\sqrt{3}}{2\sqrt{2}}$

Example 4.7.4. Find

- $\cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\sin\frac{5\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{2\sqrt{2}}$ • $\sin\frac{11\pi}{12} = \frac{-1+\sqrt{3}}{2\sqrt{2}}$

Section 4.7 Review: Sum and difference formulas

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Example 4.7.4. Find

- $\cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\sin\frac{5\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{2\sqrt{2}}$ • $\sin\frac{11\pi}{12} = \frac{-1+\sqrt{3}}{2\sqrt{2}}$

Example 4.7.5: Find

- $\sin(\frac{67\pi}{12}) =$ • $\cos(\frac{31\pi}{12}) =$ • $\sin(\frac{-33\pi}{12}) =$ • $\sin(\frac{29\pi}{12}) =$

Section 4.7 Review: Sum and difference formulas

Example 4.7.1: Find

- $\cos 120° = -\frac{1}{2}$ • $\sin 120° = \frac{\sqrt{3}}{2}$ • $\sin 225° = -\frac{1}{\sqrt{2}}$ • $\cos 300° = \frac{1}{2}$

Example 4.7.2: Rewrite each angle as a sum or difference of angles with reference angle $30°, 45°$, or $60°$.

- $15° = 45° - 30°$ • $75° = 300° + 45°$ • $105° = 60° + 45°$ • $165° = 120° + 45°$

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- $\cos(15°) = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\sin(75°) = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\cos(105°) = \frac{1-\sqrt{3}}{2\sqrt{2}}$ • $\sin(165°) = \frac{-1+\sqrt{3}}{2\sqrt{2}}$

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- $\cos \frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\sin \frac{5\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\cos \frac{7\pi}{12} = \frac{1-\sqrt{3}}{2\sqrt{2}}$ • $\sin \frac{11\pi}{12} = \frac{-1+\sqrt{3}}{2\sqrt{2}}$

Example 4.7.5: Find

- $\sin(\frac{67\pi}{12}) = -\frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\cos(\frac{31\pi}{12}) = \frac{1-\sqrt{3}}{2\sqrt{2}}$ • $\sin(\frac{-33\pi}{12}) = -\frac{1}{\sqrt{2}}$ • $\sin(\frac{29\pi}{12}) = \frac{1+\sqrt{3}}{2\sqrt{2}}$

Section 4.7 Review: Sum and difference formulas

Example 4.7.1: Find

- $\cos 120^\circ = -\frac{1}{2}$ • $\sin 120^\circ = \frac{\sqrt{3}}{2}$ • $\sin 225^\circ = -\frac{1}{\sqrt{2}}$ • $\cos 300^\circ = \frac{1}{2}$

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- $\sin(\frac{67\pi}{12}) = -\frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\cos(\frac{31\pi}{12}) = \frac{1-\sqrt{3}}{2\sqrt{2}}$ • $\sin(\frac{-33\pi}{12}) = -\frac{1}{\sqrt{2}}$ • $\sin(\frac{29\pi}{12}) = \frac{1+\sqrt{3}}{2\sqrt{2}}$

Example 4.7.6: Find the exact value of

- $\sin 16^\circ \cos 44^\circ + \cos 16^\circ \sin 44^\circ =$
- $\cos 25^\circ \cos 35^\circ - \sin 35^\circ \sin 25^\circ =$
- $\sin 40^\circ \cos 10^\circ - \sin 10^\circ \cos 40^\circ =$
- $\sin 120^\circ \cos 165^\circ - \cos 120^\circ \sin 165^\circ =$

Section 4.7 Review: Sum and difference formulas

Example 4.7.1: Find

- $\cos 120^\circ = -\frac{1}{2}$ • $\sin 120^\circ = \frac{\sqrt{3}}{2}$ • $\sin 225^\circ = -\frac{1}{\sqrt{2}}$ • $\cos 300^\circ = \frac{1}{2}$

Example 4.7.2: Rewrite each angle as a sum or difference of angles with reference angle $30^\circ, 45^\circ$, or 60°.

- $15^\circ = 45^\circ - 30^\circ$ • $75^\circ = 300^\circ + 45^\circ$ • $105^\circ = 60^\circ + 45^\circ$ • $165^\circ = 120^\circ + 45^\circ$

Example 4.7.3: Use the answers to Example 4.7.2 to find

- $\cos(15^\circ) = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\sin(75^\circ) = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\cos(105^\circ) = \frac{1-\sqrt{3}}{2\sqrt{2}}$ • $\sin(165^\circ) = \frac{-1+\sqrt{3}}{2\sqrt{2}}$

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Example 4.7.5: Find

- $\sin(\frac{67\pi}{12}) = -\frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\cos(\frac{31\pi}{12}) = \frac{1-\sqrt{3}}{2\sqrt{2}}$ • $\sin(\frac{-33\pi}{12}) = -\frac{1}{\sqrt{2}}$ • $\sin(\frac{29\pi}{12}) = \frac{1+\sqrt{3}}{2\sqrt{2}}$

Example 4.7.6: Find the exact value of

- $\sin 16^\circ \cos 44^\circ + \cos 16^\circ \sin 44^\circ = \frac{\sqrt{3}}{2}$ • $\cos 25^\circ \cos 35^\circ - \sin 35^\circ \sin 25^\circ = \frac{1}{2}$
- $\sin 40^\circ \cos 10^\circ - \sin 10^\circ \cos 40^\circ = \frac{1}{2}$ • $\sin 120^\circ \cos 165^\circ - \cos 120^\circ \sin 165^\circ = -\frac{1}{\sqrt{2}}$

Section 4.7 Review: Sum and difference formulas

Example 4.7.1: Find

- $\cos 120° = -\frac{1}{2}$ • $\sin 120° = \frac{\sqrt{3}}{2}$ • $\sin 225° = -\frac{1}{\sqrt{2}}$ • $\cos 300° = \frac{1}{2}$

Example 4.7.2: Rewrite each angle as a sum or difference of angles with reference angle $30°, 45°$, or $60°$.

- $15° = 45° - 30°$ • $75° = 300° + 45°$ • $105° = 60° + 45°$ • $165° = 120° + 45°$

Example 4.7.3: Use the answers to Example 4.7.2 to find

- $\cos(15°) = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\sin(75°) = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\cos(105°) = \frac{1-\sqrt{3}}{2\sqrt{2}}$ • $\sin(165°) = \frac{-1+\sqrt{3}}{2\sqrt{2}}$

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- $\cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\sin\frac{5\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{2\sqrt{2}}$ • $\sin\frac{11\pi}{12} = \frac{-1+\sqrt{3}}{2\sqrt{2}}$

Example 4.7.5: Find

- $\sin(\frac{67\pi}{12}) = -\frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\cos(\frac{31\pi}{12}) = \frac{1-\sqrt{3}}{2\sqrt{2}}$ • $\sin(\frac{-33\pi}{12}) = -\frac{1}{\sqrt{2}}$ • $\sin(\frac{29\pi}{12}) = \frac{1+\sqrt{3}}{2\sqrt{2}}$

Example 4.7.6: Find the exact value of

- $\sin 16° \cos 44° + \cos 16° \sin 44° = \frac{\sqrt{3}}{2}$ • $\cos 25° \cos 35° - \sin 35° \sin 25° = \frac{1}{2}$
- $\sin 40° \cos 10° - \sin 10° \cos 40° = \frac{1}{2}$ • $\sin 120° \cos 165° - \cos 120° \sin 165° = -\frac{1}{\sqrt{2}}$

Example 4.7.7: Prove each of the following identities

- $\sin(x - \frac{\pi}{2}) =$ • $\cos(x + \frac{\pi}{2}) =$ • $\sin(x + \frac{3\pi}{2}) =$ • $\sin(x + \pi) =$

Section 4.7 Review: Sum and difference formulas

Example 4.7.1: Find

- $\cos 120^\circ = -\frac{1}{2}$ • $\sin 120^\circ = \frac{\sqrt{3}}{2}$ • $\sin 225^\circ = -\frac{1}{\sqrt{2}}$ • $\cos 300^\circ = \frac{1}{2}$

Example 4.7.2: Rewrite each angle as a sum or difference of angles with reference angle $30^\circ, 45^\circ$, or 60°.

- $15^\circ = 45^\circ - 30^\circ$ • $75^\circ = 300^\circ + 45^\circ$ • $105^\circ = 60^\circ + 45^\circ$ • $165^\circ = 120^\circ + 45^\circ$

Example 4.7.3: Use the answers to Example 4.7.2 to find

- $\cos(15^\circ) = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\sin(75^\circ) = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\cos(105^\circ) = \frac{1-\sqrt{3}}{2\sqrt{2}}$ • $\sin(165^\circ) = \frac{-1+\sqrt{3}}{2\sqrt{2}}$

Example 4.7.4. Find

- $\cos \frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\sin \frac{5\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\cos \frac{7\pi}{12} = \frac{1-\sqrt{3}}{2\sqrt{2}}$ • $\sin \frac{11\pi}{12} = \frac{-1+\sqrt{3}}{2\sqrt{2}}$

Example 4.7.5: Find

- $\sin(\frac{67\pi}{12}) = -\frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\cos(\frac{31\pi}{12}) = \frac{1-\sqrt{3}}{2\sqrt{2}}$ • $\sin(\frac{-33\pi}{12}) = -\frac{1}{\sqrt{2}}$ • $\sin(\frac{29\pi}{12}) = \frac{1+\sqrt{3}}{2\sqrt{2}}$

Example 4.7.6: Find the exact value of

- $\sin 16^\circ \cos 44^\circ + \cos 16^\circ \sin 44^\circ = \frac{\sqrt{3}}{2}$ • $\cos 25^\circ \cos 35^\circ - \sin 35^\circ \sin 25^\circ = \frac{1}{2}$
- $\sin 40^\circ \cos 10^\circ - \sin 10^\circ \cos 40^\circ = \frac{1}{2}$ • $\sin 120^\circ \cos 165^\circ - \cos 120^\circ \sin 165^\circ = -\frac{1}{\sqrt{2}}$

Example 4.7.7: Prove each of the following identities

- $\sin(x - \frac{\pi}{2}) = -\cos x$ • $\cos(x + \frac{\pi}{2}) = -\sin x.$ • $\sin(x + \frac{3\pi}{2}) = -\cos x.$ • $\sin(x + \pi) = -\sin x.$

Section 4.8: Double and half-angle formulas

- 4.8.1: Expressing one trig function in terms of another
- 4.8.2: Double-angle formulas
- 4.8.3: Half-angle formulas
- 4.8.4: Composing trig functions and their inverses
- 4.8.5: Section 4.8 Quiz Review

Section 4.6 Preview: Definitions

- Definition 4.8.1: Double-angle formulas for any angle u:
- Definition 4.8.2: More double-angle formulas for cos$(2u)$:
- Definition 4.8.3: Formulas for squares of trig functions:
- Definition 4.8.4: Half-angle formulas for any angle:

4.8.1 Expressing one trig function in terms of another

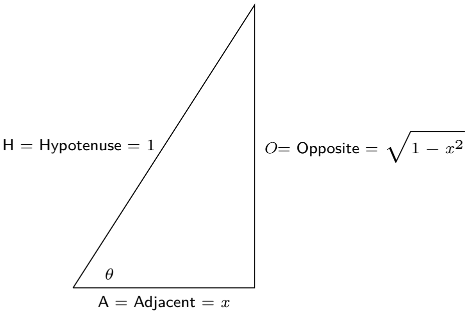
Example 1: If θ is an acute angle with $\cos\theta = x$, find the other five trig functions of θ.

Solution: In the acute triangle at the right, $\cos\theta = \frac{A}{H} = \frac{x}{1} = x$ as required.

Solve $A^2 + O^2 = H^2$ for $O = \sqrt{H^2 - A^2} = \sqrt{1 - x^2}$.

The other trig functions of θ are

- $\sin\theta = \frac{O}{H} = \dfrac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$.
- $\tan\theta = \frac{O}{A} = \dfrac{\sqrt{1-x^2}}{x}$.
- $\cot\theta = \frac{A}{O} = \dfrac{1}{\tan\theta} = \frac{x}{\sqrt{1-x^2}}$.
- $\sec\theta = \frac{H}{A} = \dfrac{1}{\cos\theta} = \frac{1}{x}$.
- $\csc\theta = \frac{H}{O} = \dfrac{1}{\sqrt{1-x^2}}$.



4.8.2 Double-angle formulas

Recall the sum formulas:

- $\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v)$
- $\sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)$
- $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$

In each formula, substitute u for v.

Double-angle formulas for any angle u:

- $\cos(2u) = \cos^2 u - \sin^2 u$
- $\sin(2u) = 2\sin u \cos u$
- $\tan(2u) = \dfrac{2\tan u}{1 - \tan^2 u}$

Rewriting $\cos^2 u + \sin^2 u = 1$ gives $\cos^2 u = 1 - \sin^2 u$ and $\sin^2 u = 1 - \cos^2 u$.

Plugging each of these identities into $\cos(2u) = \cos^2 u - \sin^2 u$ yields

More double-angle formulas for $\cos(2u)$

- $\cos(2u) = 2\cos^2 u - 1$
- $\cos(2u) = 1 - 2\sin^2 u$

Example 2: Suppose $\cos\theta = -\frac{2}{3}$ and θ is in Quadrant 3. Find $\cos(2\theta)$ and $\sin(2\theta)$.

Solution:

- To find $\cos(2\theta)$, use the double-angle formula $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ $= \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1$ $= 2(-\frac{2}{3})^2 - 1 = 2(\frac{4}{9}) - 1 = \frac{8}{9} - \frac{9}{9} = -\frac{1}{9}$.
- To find $\sin(2\theta)$, use the double-angle formula $\sin(2\theta) = 2\sin\theta\cos\theta$. We are given $\cos\theta = -\frac{2}{3}$ but need to figure out $\sin\theta$.
  Since $\sin^2\theta = 1 - \cos^2\theta$, $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$ $= \pm\sqrt{1 - (-\frac{2}{3})^2} = \pm\sqrt{1 - \frac{4}{9}} = \pm\sqrt{\frac{5}{9}} = \pm\frac{\sqrt{5}}{3}$.
  Since θ is a Quadrant 3 angle, its sine is negative.
  Therefore $\sin\theta = -\frac{\sqrt{5}}{3}$ and so
  $\sin(2\theta) = 2\sin\theta\cos\theta = 2(-\frac{\sqrt{5}}{3})(-\frac{2}{3}) = \frac{4\sqrt{5}}{9}$.
  A partial check: $\cos^2 2\theta + \sin^2 2\theta = \frac{1}{81} + \frac{16\cdot 5}{81} = \frac{81}{81} = 1$.

$$\sin(2\theta) = \frac{4\sqrt{5}}{9} \qquad \cos(2\theta) = -\frac{1}{9}$$

4.8.3 Half-angle formulas

Add the identities	$\cos^2\theta - \sin^2\theta = \cos 2\theta$
and	$\cos^2\theta + \sin^2\theta = 1$
to get	$2\cos^2\theta = 1 + \cos(2\theta)$.
Start over with	$\cos^2\theta + \sin^2\theta = 1$
Subtract	$\cos^2\theta - \sin^2\theta = \cos 2\theta$
to get	$2\sin^2\theta = 1 - \cos(2\theta)$

Dividing each result by 2 gives

Formulas for squares of trig functions:

- $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$
- $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$
- $\tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$

The last identity follows from

$\tan^2\theta = \left(\frac{\sin\theta}{\cos\theta}\right)^2 = \frac{\sin^2\theta}{\cos^2\theta} = \frac{\frac{1-\cos 2\theta}{2}}{\frac{1+\cos 2\theta}{2}}$

Substitute $\frac{u}{2}$ for θ in each formula at the left to obtain

Half-angle formulas for any angle u:

- $\cos\dfrac{u}{2} = \pm\sqrt{\dfrac{1+\cos u}{2}}$
- $\sin\dfrac{u}{2} = \pm\sqrt{\dfrac{1-\cos u}{2}}$
- $\tan\dfrac{u}{2} = \pm\sqrt{\dfrac{1-\cos u}{1+\cos u}}$

To figure out the sign, apply ASTC to the quadrant containing the endline of angle $\dfrac{u}{2}$.

Example 3: Find $\sin\frac{\pi}{8}$.

Solution: Solve $\frac{u}{2} = \frac{\pi}{8}$ to obtain $u = \frac{\pi}{4} = 45°$. Then

$$\sin\tfrac{\pi}{8} = \sin\tfrac{u}{2} = \pm\sqrt{\tfrac{1-\cos u}{2}} = \pm\sqrt{\tfrac{1-\cos 45°}{2}} = \pm\sqrt{\tfrac{1-\frac{1}{\sqrt{2}}}{2}}.$$

Since $\frac{\pi}{8} = 22.5°$ is in Quadrant 1, its sine is positive. Therefore

$$\sin\tfrac{\pi}{8} = \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{2\left(1-\frac{1}{\sqrt{2}}\right)}{2\cdot 2}} = \sqrt{\frac{2-\sqrt{2}}{4}}$$

$$\boxed{\sin\tfrac{\pi}{8} = \tfrac{1}{2}\sqrt{2-\sqrt{2}}}.$$

Using the half-angle formulas

Example 4: Find $\cos 112.5°$.

Solution: Set $\frac{u}{2} = 112.5°$ to obtain $u = 225°$. Then

$\cos 112.5° = \cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}} = \pm\sqrt{\frac{1+\cos 225°}{2}}$

The Quadrant 3 angle $225°$ has reference angle $225° - 180° = 45°$. By RAP: $\cos 225° = \pm\cos 45°$.

Since cosine is negative in Quadrant 3,

$\cos 225° = -\cos(45°) = -\frac{1}{\sqrt{2}}$. thus

$\cos 112.5° = \pm\sqrt{\frac{1+\cos 225°}{2}} = \pm\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}}$.

The sign is negative since $112.5°$ is a Quadrant 2 angle.

$$\boxed{\cos 112.5° = -\frac{1}{2}\sqrt{2-\sqrt{2}}}.$$

Example 5: Find $\cos\frac{u}{2}$ if $\sin u = -\frac{3}{5}$ and u is in Quadrant 3.

Solution:

First figure out $\cos u$. Solve $\cos^2 u + \sin^2 u = 1$ to find

$\cos u = \pm\sqrt{1-\sin^2 u} = \pm\sqrt{1-\left(-\frac{3}{5}\right)^2} =$

$\pm\sqrt{1-\frac{9}{25}} = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$.

Since u is in Quadrant 3, its cosine is negative:

$\cos u = -\frac{4}{5}$. By the half-angle formula,

$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}} = \pm\sqrt{\frac{1-\frac{4}{5}}{2}} = \pm\sqrt{\frac{1}{10}}$.

Since u is in Quadrant 3, $180° < u < 270°$.

Divide by 2 to get $90° < \frac{u}{2} < 135°$.

Therefore $\frac{u}{2}$ is in Quadrant 2, where cosine is negative, and so

Answer: $\boxed{\cos\frac{u}{2} = -\sqrt{\frac{1}{10}}}$.

4.8.4 Combining trig functions and their inverses

Reminder: Let trig be cos, sin, or tan.
arctrig(trig θ) need not equal θ.
However, trig (arctrig x) $= x$ for all x.

Example 6: Find $\cos(2\arccos(x))$.
Solution:
Let $\theta = \arccos(x)$. Then $\cos\theta = \cos(\arccos x) = x$ and
$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 =$

$$\boxed{\cos(2\arccos(x)) = 2x^2 - 1}.$$

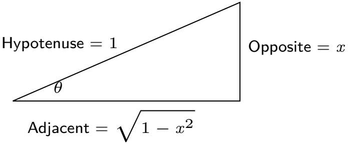
Example 7. Find $\tan(2\arcsin x)$.
Solution: Let $\theta = \arcsin x$. Then
$\sin\theta = \sin(\arcsin x) = x$.

To find $\tan\theta$, use the acute triangle below.

$\sin\theta = \dfrac{O}{H} = \dfrac{x}{1} = x$ as required.

Solve $A^2 + O^2 = H^2$ for $A = \sqrt{H^2 - O^2} = \sqrt{1 - x^2}$.

Then $\tan\theta = \frac{O}{A} = \frac{x}{\sqrt{1-x^2}}$.



Now find $\tan(2\arcsin x) = \tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$.

The numerator is $2\tan\theta = \dfrac{2x}{\sqrt{1-x^2}}$.

The denominator is $1 - \tan^2\theta = 1 - \left(\dfrac{x}{\sqrt{1-x^2}}\right)^2$

$$= 1 - \frac{x^2}{1-x^2} = \frac{1-x^2}{1-x^2} - \frac{x^2}{1-x^2} = \frac{1-2x^2}{1-x^2}.$$

Then $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{\frac{2x}{\sqrt{1-x^2}}}{\frac{1-2x^2}{1-x^2}}$

$$= \frac{2x}{\sqrt{1-x^2}} \cdot \frac{1-x^2}{1-2x^2}$$

$$= \frac{2x \cdot (\sqrt{1-x^2})^2}{(\sqrt{1-x^2}) \cdot (1-2x^2)}$$

$$= \frac{2x \cdot \sqrt{1-x^2}\,\cancel{\sqrt{1-x^2}}}{\cancel{\sqrt{1-x^2}} \cdot (1-2x^2)}$$

$$\boxed{\tan(2\arcsin x\ ) = \frac{2x\sqrt{1-x^2}}{1-2x^2}}.$$

Section 4.8 Quiz

Example 4.8.1: Assume θ is an acute angle. If $\cos\theta = x$, find the other five trig functions of θ.

Example 4.8.2: Suppose $\cos\theta = -\frac{2}{3}$ and θ is in Quadrant 3. Find $\cos(2\theta)$ and $\sin(2\theta)$.

Example 4.8.3: Find $\sin\frac{\pi}{8}$.

Example 4.8.4: Find $\cos 112.5°$.

Example 4.8.5: Find $\cos\frac{u}{2}$ if $\sin u = -\frac{3}{5}$ and u is in Quadrant 3.

Example 4.8.6: Find $\cos(2\arccos(x))$.

Example 4.8.7:Find $\tan(2\arcsin x)$.

Section 4.8 Review: Trigonometric functions of general angles

Example 4.8.1: Assume θ is an acute angle. Find the other five trig functions of θ if

- $\cos\theta = x \Rightarrow$
- $\sin\theta = x \Rightarrow$
- $\tan\theta = x \Rightarrow$
- $\sec\theta = x \Rightarrow$

Section 4.8 Review: Trigonometric functions of general angles

Example 4.8.1: Assume θ is an acute angle. Find the other five trig functions of θ if

- $\cos\theta = x \Rightarrow$ $\sin\theta = \sqrt{1-x^2}$; $\tan\theta = \frac{\sqrt{1-x^2}}{x}$; $\cot\theta = \frac{x}{\sqrt{1-x^2}}$; $\sec\theta = \frac{1}{x}$; $\csc\theta = \frac{1}{\sqrt{1-x^2}}$
- $\sin\theta = x \Rightarrow$ $\cos\theta = \sqrt{1-x^2}$; $\tan\theta = \frac{x}{\sqrt{1-x^2}}$; $\cot\theta = \frac{\sqrt{1-x^2}}{x}$; $\sec\theta = \frac{1}{\sqrt{1-x^2}}$; $\csc\theta = \frac{1}{x}$
- $\tan\theta = x \Rightarrow$ $\cos\theta = \frac{1}{\sqrt{1+x^2}}$; $\sin\theta = \frac{x}{\sqrt{1+x^2}}$; $\cot\theta = \frac{1}{x}$; $\sec\theta = \sqrt{1+x^2}$; $\csc\theta = \frac{\sqrt{1+x^2}}{x}$
- $\sec\theta = x \Rightarrow$ $\cos\theta = \frac{1}{x}$; $\sin\theta = \frac{\sqrt{x^2-1}}{x}$; $\tan\theta = \sqrt{x^2-1}$; $\cot\theta = \frac{1}{\sqrt{x^2-1}}$; $\csc\theta = \frac{x}{\sqrt{x^2-1}}$

Section 4.8 Review: Trigonometric functions of general angles

Example 4.8.1: Assume θ is an acute angle. Find the other five trig functions of θ if

- $\cos\theta = x \Rightarrow$ $\sin\theta = \sqrt{1-x^2}$; $\tan\theta = \frac{\sqrt{1-x^2}}{x}$; $\cot\theta = \frac{x}{\sqrt{1-x^2}}$; $\sec\theta = \frac{1}{x}$; $\csc\theta = \frac{1}{\sqrt{1-x^2}}$
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- $\tan\theta = x \Rightarrow$ $\cos\theta = \frac{1}{\sqrt{1+x^2}}$; $\sin\theta = \frac{x}{\sqrt{1+x^2}}$; $\cot\theta = \frac{1}{x}$; $\sec\theta = \sqrt{1+x^2}$; $\csc\theta = \frac{\sqrt{1+x^2}}{x}$
- $\sec\theta = x \Rightarrow$ $\cos\theta = \frac{1}{x}$; $\sin\theta = \frac{\sqrt{x^2-1}}{x}$; $\tan\theta = \sqrt{x^2-1}$; $\cot\theta = \frac{1}{\sqrt{x^2-1}}$; $\csc\theta = \frac{x}{\sqrt{x^2-1}}$

Example 4.8.2: If

- $\cos\theta = -\frac{2}{3}$ and θ is in Quadrant 3, find $\cos(2\theta) =$ and $\sin(2\theta) =$
- $\sin\theta = \frac{2}{3}$ and θ is in Quadrant 2, find $\cos(2\theta)=$ and $\sin(2\theta)=$
- $\cos\theta = \frac{2}{3}$ and θ is in Quadrant 4, find $\cos(2\theta)=$ and $\sin(2\theta)=$
- $\sin\theta = -\frac{2}{3}$ and θ is in Quadrant 4, find $\cos(2\theta)=$ and $\sin(2\theta)=$

Section 4.8 Review: Trigonometric functions of general angles

Example 4.8.1: Assume θ is an acute angle. Find the other five trig functions of θ if

- $\cos\theta = x \Rightarrow$ $\sin\theta = \sqrt{1-x^2}$; $\tan\theta = \frac{\sqrt{1-x^2}}{x}$; $\cot\theta = \frac{x}{\sqrt{1-x^2}}$; $\sec\theta = \frac{1}{x}$; $\csc\theta = \frac{1}{\sqrt{1-x^2}}$
- $\sin\theta = x \Rightarrow$ $\cos\theta = \sqrt{1-x^2}$; $\tan\theta = \frac{x}{\sqrt{1-x^2}}$; $\cot\theta = \frac{\sqrt{1-x^2}}{x}$; $\sec\theta = \frac{1}{\sqrt{1-x^2}}$; $\csc\theta = \frac{1}{x}$
- $\tan\theta = x \Rightarrow$ $\cos\theta = \frac{1}{\sqrt{1+x^2}}$; $\sin\theta = \frac{x}{\sqrt{1+x^2}}$; $\cot\theta = \frac{1}{x}$; $\sec\theta = \sqrt{1+x^2}$; $\csc\theta = \frac{\sqrt{1+x^2}}{x}$
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Example 4.8.2: If

- $\cos\theta = -\frac{2}{3}$ and θ is in Quadrant 3, find $\cos(2\theta) = -\frac{1}{9}$ and $\sin(2\theta) = \frac{4\sqrt{5}}{9}$
- $\sin\theta = \frac{2}{3}$ and θ is in Quadrant 2, find $\cos(2\theta)= \frac{1}{9}$ and $\sin(2\theta)= -\frac{4\sqrt{5}}{9}$
- $\cos\theta = \frac{2}{3}$ and θ is in Quadrant 4, find $\cos(2\theta)= -\frac{1}{9}$ and $\sin(2\theta)= -\frac{4\sqrt{5}}{9}$
- $\sin\theta = -\frac{2}{3}$ and θ is in Quadrant 4, find $\cos(2\theta)= \frac{1}{9}$ and $\sin(2\theta)= -\frac{4\sqrt{5}}{9}$

Section 4.8 Review: Trigonometric functions of general angles

Example 4.8.1: Assume θ is an acute angle. Find the other five trig functions of θ if

- $\cos\theta = x \Rightarrow$ $\sin\theta = \sqrt{1-x^2}$; $\tan\theta = \frac{\sqrt{1-x^2}}{x}$; $\cot\theta = \frac{x}{\sqrt{1-x^2}}$; $\sec\theta = \frac{1}{x}$; $\csc\theta = \frac{1}{\sqrt{1-x^2}}$
- $\sin\theta = x \Rightarrow$ $\cos\theta = \sqrt{1-x^2}$; $\tan\theta = \frac{x}{\sqrt{1-x^2}}$; $\cot\theta = \frac{\sqrt{1-x^2}}{x}$; $\sec\theta = \frac{1}{\sqrt{1-x^2}}$; $\csc\theta = \frac{1}{x}$
- $\tan\theta = x \Rightarrow$ $\cos\theta = \frac{1}{\sqrt{1+x^2}}$; $\sin\theta = \frac{x}{\sqrt{1+x^2}}$; $\cot\theta = \frac{1}{x}$; $\sec\theta = \sqrt{1+x^2}$; $\csc\theta = \frac{\sqrt{1+x^2}}{x}$
- $\sec\theta = x \Rightarrow$ $\cos\theta = \frac{1}{x}$; $\sin\theta = \frac{\sqrt{x^2-1}}{x}$; $\tan\theta = \sqrt{x^2-1}$; $\cot\theta = \frac{1}{\sqrt{x^2-1}}$; $\csc\theta = \frac{x}{\sqrt{x^2-1}}$

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- $\sin\theta = \frac{2}{3}$ and θ is in Quadrant 2, find $\cos(2\theta)= \frac{1}{9}$ and $\sin(2\theta)= -\frac{4\sqrt{5}}{9}$
- $\cos\theta = \frac{2}{3}$ and θ is in Quadrant 4, find $\cos(2\theta)= -\frac{1}{9}$ and $\sin(2\theta)= -\frac{4\sqrt{5}}{9}$
- $\sin\theta = -\frac{2}{3}$ and θ is in Quadrant 4, find $\cos(2\theta)= \frac{1}{9}$ and $\sin(2\theta)= -\frac{4\sqrt{5}}{9}$

Example 4.8.3: Find

- $\sin\frac{\pi}{8} =$
- $\cos\frac{\pi}{8} =$
- $\sin\frac{\pi}{12} =$
- $\cos\frac{3\pi}{8} =$

Section 4.8 Review: Trigonometric functions of general angles

Example 4.8.1: Assume θ is an acute angle. Find the other five trig functions of θ if

- $\cos\theta = x \Rightarrow$ $\sin\theta = \sqrt{1-x^2}$; $\tan\theta = \frac{\sqrt{1-x^2}}{x}$; $\cot\theta = \frac{x}{\sqrt{1-x^2}}$; $\sec\theta = \frac{1}{x}$; $\csc\theta = \frac{1}{\sqrt{1-x^2}}$
- $\sin\theta = x \Rightarrow$ $\cos\theta = \sqrt{1-x^2}$; $\tan\theta = \frac{x}{\sqrt{1-x^2}}$; $\cot\theta = \frac{\sqrt{1-x^2}}{x}$; $\sec\theta = \frac{1}{\sqrt{1-x^2}}$; $\csc\theta = \frac{1}{x}$
- $\tan\theta = x \Rightarrow$ $\cos\theta = \frac{1}{\sqrt{1+x^2}}$; $\sin\theta = \frac{x}{\sqrt{1+x^2}}$; $\cot\theta = \frac{1}{x}$; $\sec\theta = \sqrt{1+x^2}$; $\csc\theta = \frac{\sqrt{1+x^2}}{x}$
- $\sec\theta = x \Rightarrow$ $\cos\theta = \frac{1}{x}$; $\sin\theta = \frac{\sqrt{x^2-1}}{x}$; $\tan\theta = \sqrt{x^2-1}$; $\cot\theta = \frac{1}{\sqrt{x^2-1}}$; $\csc\theta = \frac{x}{\sqrt{x^2-1}}$

Example 4.8.2: If

- $\cos\theta = -\frac{2}{3}$ and θ is in Quadrant 3, find $\cos(2\theta) = -\frac{1}{9}$ and $\sin(2\theta) = \frac{4\sqrt{5}}{9}$
- $\sin\theta = \frac{2}{3}$ and θ is in Quadrant 2, find $\cos(2\theta) = \frac{1}{9}$ and $\sin(2\theta) = -\frac{4\sqrt{5}}{9}$
- $\cos\theta = \frac{2}{3}$ and θ is in Quadrant 4, find $\cos(2\theta) = -\frac{1}{9}$ and $\sin(2\theta) = -\frac{4\sqrt{5}}{9}$
- $\sin\theta = -\frac{2}{3}$ and θ is in Quadrant 4, find $\cos(2\theta) = \frac{1}{9}$ and $\sin(2\theta) = -\frac{4\sqrt{5}}{9}$

Example 4.8.3: Find

- $\sin\frac{\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}}$ • $\cos\frac{\pi}{8} = \frac{1}{2}\sqrt{2+\sqrt{2}}$ • $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$ • $\cos\frac{3\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}}$

Section 4.8 Review: Trigonometric functions of general angles

Example 4.8.1: Assume θ is an acute angle. Find the other five trig functions of θ if

- $\cos\theta = x \Rightarrow \sin\theta = \sqrt{1-x^2}$; $\tan\theta = \frac{\sqrt{1-x^2}}{x}$; $\cot\theta = \frac{x}{\sqrt{1-x^2}}$; $\sec\theta = \frac{1}{x}$; $\csc\theta = \frac{1}{\sqrt{1-x^2}}$
- $\sin\theta = x \Rightarrow \cos\theta = \sqrt{1-x^2}$; $\tan\theta = \frac{x}{\sqrt{1-x^2}}$; $\cot\theta = \frac{\sqrt{1-x^2}}{x}$; $\sec\theta = \frac{1}{\sqrt{1-x^2}}$; $\csc\theta = \frac{1}{x}$
- $\tan\theta = x \Rightarrow \cos\theta = \frac{1}{\sqrt{1+x^2}}$; $\sin\theta = \frac{x}{\sqrt{1+x^2}}$; $\cot\theta = \frac{1}{x}$; $\sec\theta = \sqrt{1+x^2}$; $\csc\theta = \frac{\sqrt{1+x^2}}{x}$
- $\sec\theta = x \Rightarrow \cos\theta = \frac{1}{x}$; $\sin\theta = \frac{\sqrt{x^2-1}}{x}$; $\tan\theta = \sqrt{x^2-1}$; $\cot\theta = \frac{1}{\sqrt{x^2-1}}$; $\csc\theta = \frac{x}{\sqrt{x^2-1}}$

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- $\cos\theta = -\frac{2}{3}$ and θ is in Quadrant 3, find $\cos(2\theta) = -\frac{1}{9}$ and $\sin(2\theta) = \frac{4\sqrt{5}}{9}$
- $\sin\theta = \frac{2}{3}$ and θ is in Quadrant 2, find $\cos(2\theta) = \frac{1}{9}$ and $\sin(2\theta) = -\frac{4\sqrt{5}}{9}$
- $\cos\theta = \frac{2}{3}$ and θ is in Quadrant 4, find $\cos(2\theta) = -\frac{1}{9}$ and $\sin(2\theta) = -\frac{4\sqrt{5}}{9}$
- $\sin\theta = -\frac{2}{3}$ and θ is in Quadrant 4, find $\cos(2\theta) = \frac{1}{9}$ and $\sin(2\theta) = -\frac{4\sqrt{5}}{9}$

Example 4.8.3: Find

- $\sin\frac{\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}}$ • $\cos\frac{\pi}{8} = \frac{1}{2}\sqrt{2+\sqrt{2}}$ • $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$ • $\cos\frac{3\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}}$

Example 4.8.4: Find

- $\cos 112.5^\circ =$
- $\cos 67.5^\circ =$
- $\cos 165^\circ =$
- $\sin 165^\circ =$

Section 4.8 Review: Trigonometric functions of general angles

Example 4.8.1: Assume θ is an acute angle. Find the other five trig functions of θ if

- $\cos\theta = x \Rightarrow$ $\sin\theta = \sqrt{1-x^2}$; $\tan\theta = \frac{\sqrt{1-x^2}}{x}$; $\cot\theta = \frac{x}{\sqrt{1-x^2}}$; $\sec\theta = \frac{1}{x}$; $\csc\theta = \frac{1}{\sqrt{1-x^2}}$
- $\sin\theta = x \Rightarrow$ $\cos\theta = \sqrt{1-x^2}$; $\tan\theta = \frac{x}{\sqrt{1-x^2}}$; $\cot\theta = \frac{\sqrt{1-x^2}}{x}$; $\sec\theta = \frac{1}{\sqrt{1-x^2}}$; $\csc\theta = \frac{1}{x}$
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- $\cos\theta = \frac{2}{3}$ and θ is in Quadrant 4, find $\cos(2\theta) = -\frac{1}{9}$ and $\sin(2\theta) = -\frac{4\sqrt{5}}{9}$
- $\sin\theta = -\frac{2}{3}$ and θ is in Quadrant 4, find $\cos(2\theta) = \frac{1}{9}$ and $\sin(2\theta) = -\frac{4\sqrt{5}}{9}$

Example 4.8.3: Find

- $\sin\frac{\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}}$ • $\cos\frac{\pi}{8} = \frac{1}{2}\sqrt{2+\sqrt{2}}$ • $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$ • $\cos\frac{3\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}}$

Example 4.8.4: Find

- $\cos 112.5° = -\frac{1}{2}\sqrt{2-\sqrt{2}}$ • $\cos 67.5° = \frac{1}{2}\sqrt{2-\sqrt{2}}$ • $\cos 165° = -\frac{1+\sqrt{3}}{2\sqrt{2}}$ • $\sin 165° = \frac{-1+\sqrt{3}}{2\sqrt{2}}$

Example 4.8.5: if

- $\sin u = -\frac{3}{5}$ and u is in Q3, Find $\cos \frac{u}{2}$ $=$
- $\cos u = \frac{3}{5}$ and u is in Q4, Find $\cos \frac{u}{2}$ $=$
- $\sin u = -\frac{3}{5}$ and u is in Q3, Find $\sin \frac{u}{2}$ $=$
- $\cos u = \frac{3}{5}$ and u is in Q4, Find $\sin \frac{u}{2}$ $=$

Example 4.8.5: if

- $\sin u = -\frac{3}{5}$ and u is in Q3, Find $\cos\frac{u}{2} = -\sqrt{\frac{1}{10}}$
- $\cos u = \frac{3}{5}$ and u is in Q4, Find $\cos\frac{u}{2} = \sqrt{\frac{1}{5}}$
- $\sin u = -\frac{3}{5}$ and u is in Q3, Find $\sin\frac{u}{2} = \sqrt{\frac{9}{10}}$
- $\cos u = \frac{3}{5}$ and u is in Q4, Find $\sin\frac{u}{2} = \sqrt{\frac{4}{5}}$

Example 4.8.5: if

- $\sin u = -\frac{3}{5}$ and u is in Q3, Find $\cos\frac{u}{2} = -\sqrt{\frac{1}{10}}$
- $\cos u = \frac{3}{5}$ and u is in Q4, Find $\cos\frac{u}{2} = \sqrt{\frac{1}{5}}$
- $\sin u = -\frac{3}{5}$ and u is in Q3, Find $\sin\frac{u}{2} = \sqrt{\frac{9}{10}}$
- $\cos u = \frac{3}{5}$ and u is in Q4, Find $\sin\frac{u}{2} = \sqrt{\frac{4}{5}}$

Example 4.8.6: Find

- $\cos(2\arccos(x)) =$
- $\sin(2\arcsin(x)) =$
- $\sin(2\arccos(x)) =$
- $\cos(2\arcsin(x)) =$

Example 4.8.5: if

- $\sin u = -\frac{3}{5}$ and u is in Q3, Find $\cos\frac{u}{2} = -\sqrt{\frac{1}{10}}$ • $\cos u = \frac{3}{5}$ and u is in Q4, Find $\cos\frac{u}{2} = \sqrt{\frac{1}{5}}$
- $\sin u = -\frac{3}{5}$ and u is in Q3, Find $\sin\frac{u}{2} = \sqrt{\frac{9}{10}}$ • $\cos u = \frac{3}{5}$ and u is in Q4, Find $\sin\frac{u}{2} = \sqrt{\frac{4}{5}}$

Example 4.8.6: Find

- $\cos(2\arccos(x)) = 2x^2 - 1$
- $\sin(2\arcsin(x)) = 2x\sqrt{1-x^2}$
- $\sin(2\arccos(x)) = 2x\sqrt{1-x^2}$
- $\cos(2\arcsin(x)) = 1 - 2x^2$

Example 4.8.5: if

- $\sin u = -\frac{3}{5}$ and u is in Q3, Find $\cos \frac{u}{2} = -\sqrt{\frac{1}{10}}$
- $\cos u = \frac{3}{5}$ and u is in Q4, Find $\cos \frac{u}{2} = \sqrt{\frac{1}{5}}$
- $\sin u = -\frac{3}{5}$ and u is in Q3, Find $\sin \frac{u}{2} = \sqrt{\frac{9}{10}}$
- $\cos u = \frac{3}{5}$ and u is in Q4, Find $\sin \frac{u}{2} = \sqrt{\frac{4}{5}}$

Example 4.8.6: Find

- $\cos(2\arccos(x)) = 2x^2 - 1$
- $\sin(2\arcsin(x)) = 2x\sqrt{1-x^2}$
- $\sin(2\arccos(x)) = 2x\sqrt{1-x^2}$
- $\cos(2\arcsin(x)) = 1 - 2x^2$

Example 4.8.7: Find

- $\tan(2\arcsin x) =$
- $\tan(2\arctan x) =$
- $\cos(2\arctan(x)) =$
- $\tan(2\arccos(x)) =$

Example 4.8.5: if

- $\sin u = -\frac{3}{5}$ and u is in Q3, Find $\cos\frac{u}{2} = -\sqrt{\frac{1}{10}}$
- $\cos u = \frac{3}{5}$ and u is in Q4, Find $\cos\frac{u}{2} = \sqrt{\frac{1}{5}}$
- $\sin u = -\frac{3}{5}$ and u is in Q3, Find $\sin\frac{u}{2} = \sqrt{\frac{9}{10}}$
- $\cos u = \frac{3}{5}$ and u is in Q4, Find $\sin\frac{u}{2} = \sqrt{\frac{4}{5}}$

Example 4.8.6: Find

- $\cos(2\arccos(x)) = 2x^2 - 1$
- $\sin(2\arcsin(x)) = 2x\sqrt{1-x^2}$
- $\sin(2\arccos(x)) = 2x\sqrt{1-x^2}$
- $\cos(2\arcsin(x)) = 1 - 2x^2$

Example 4.8.7: Find

- $\tan(2\arcsin x) = \dfrac{2x\sqrt{1-x^2}}{1-2x^2}$
- $\tan(2\arctan x) = -\frac{2x}{x^2-1}$
- $\cos(2\arctan(x)) = \frac{1-x^2}{x^2+1}$
- $\tan(2\arccos(x)) = \frac{2x\sqrt{1-x^2}}{2x^2-1}$

Chapter 4 Review

- Precalculus Section 4.1 Review: Right triangle trigonometry
- Precalculus Section 4.2 Review: Angles and circles
- Precalculus Section 4.3 Review: General angles
- Precalculus Section 4.4 Review: Trig functions and graphs
- Precalculus Section 4.5 Review: Inverse trig functions
- Precalculus Section 4.6 Review: Trigonometric identities
- Precalculus Section 4.7 Review: Sum and difference formulas
- Precalculus Section 4.8 Review: Trig functions of general angles

To review a section listed above:

Click on its button to view the first Example in that section as well as three similar questions. Work out the answers, then click again to see if you are correct. If so, keep on clicking.

If you have trouble answering a question, click on the button to its left to access its solution in the text. Then click on the faint ⥀ Adobe control at the bottom right of the text screen to continue your review.