SUPPLEMENTARY NOTE 5b FOR MATH 212

Conic Sections and Quadric Surfaces

CONIC SECTIONS

A *conic section* is the intersection of a right circular cone and a plane.

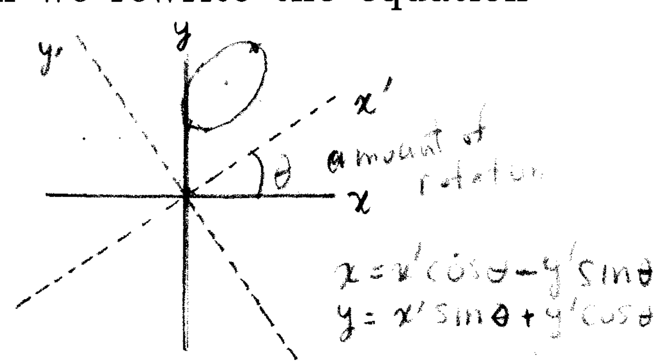
THE GENERAL SECOND DEGREE EQUATION

The degree of a term of a polynomial in more than one variable is the sum of the exponents of the variables in the term. For example, the term $2x^3yz^2$ has degree $3 + 1 + 2 = 6$. By the general second degree polynomial in two variables, we mean all polynomials in two variables whose terms have degree at most two and at least one term is of degree two. The general second degree equation is the equation that a general second degree polynomial equals zero, i.e., it is of the form

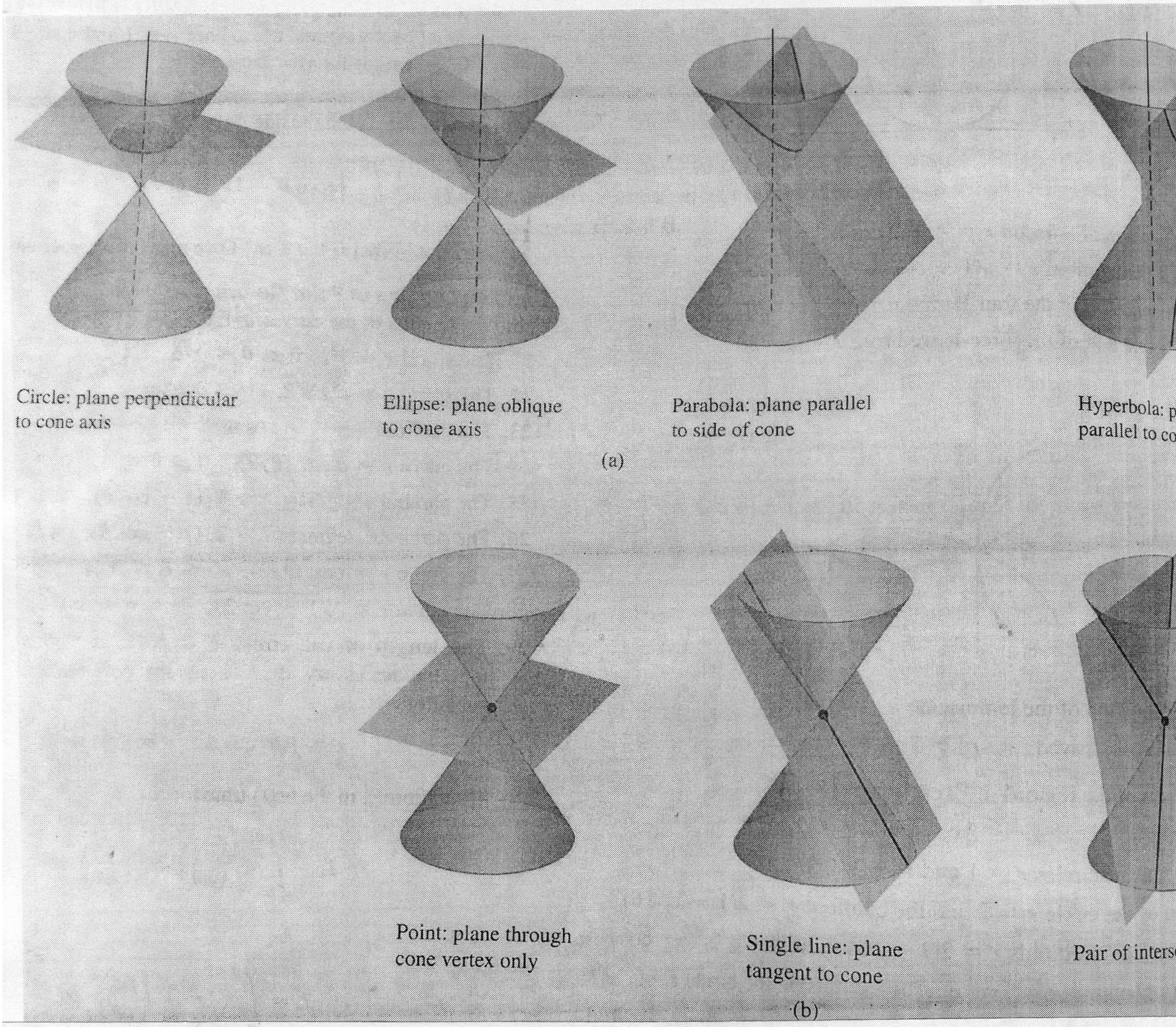
$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0.$$

We ask the question, seemingly unrelated to conic sections, of what are all possible graphs of second degree polynomial equations. To answer this question we rewrite the equation several different ways.

It is possible to express in a rotated coordinate system, any graph of a second degree polynomial so that the coefficient C of the xy term is zero. Thus, we will only consider second degree equations with no xy term.



Conic Sections



If both $A \neq 0$ and $D \neq 0$, then we can complete the square and write

$$Ax^2 + Dx + F = A(x-\alpha)^2 + F'.$$

Similarly we can complete the square if $B \neq 0$ and $E \neq 0$ to get $B(y-\beta)^2 + F'$. Since α and β are shifts, we can consider only the case for which there is square term or a linear term but not both, for both variables, and all possible graphs of the general second degree equation are rotations and/or shifts of equations of the form

$$\begin{cases} Ax^2 \\ Ax \end{cases} + \begin{cases} By^2 \\ By \end{cases} = F,$$

where we pick just one term in each bracketed pair. If $F \neq 0$, we divide both sides of the equation by F to obtain the form

$$\begin{cases} Ax^2 \\ Ax \end{cases} + \begin{cases} By^2 \\ By \end{cases} = \begin{cases} 1 \\ 0 \end{cases}$$

For example,

$$\begin{aligned} 2x^2 + 0y^2 + 4x + y + 6 &= 2[x^2 - 2x] + y + 6 \\ &= 2[(x-1)^2 - 1] + y + 6 \\ &= 2(x-1)^2 + y + (-2+6) = 0 \end{aligned}$$

If we can graph $-\frac{1}{2}x^2 - \frac{1}{4}y = 1$, then we just shift one unit horizontally to obtain the graph of the equation displayed above.

We make one last modification: If $A > 0$, then we may write A in the form $A = \frac{1}{a^2}$ (let $a = 1/\sqrt{A}$). If $A < 0$, write $A = -\frac{1}{a^2}$, where $a = 1/\sqrt{|A|}$ Similarly write $B = \pm\frac{1}{b^2}$, and make this change of notation on the top lines above:

$$\begin{cases} \pm\dfrac{x^2}{a^2} \\ Ax \end{cases} + \begin{cases} \pm\dfrac{y^2}{b^2} \\ By \end{cases} = \begin{cases} 1 \\ 0 \end{cases}$$

We now examine each possibility with last forms written above:

Case 1: The constant is 1.

(a) Both terms on the left are squares

(i) 2 pluses

(ii) 1 plus

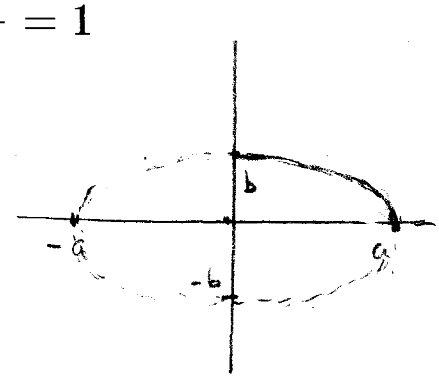
(iii) both minuses

(b) There is only one square term.

(c) No square terms: then equation is linear, not second degree.

Case 1(a)(i)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



We construct the graph. First we find the intercepts. Set $y = 0$ to find the x intercepts: $x^2 = a^2$ and $x = \pm a$. Similarly $y = \pm b$. We graph the portion of the graph in the first quadrant, by solving for y. The first and second derivatives are both negative, so the curve is decreasing and concave down. Finally we see, by replacing first x and then y by their negatives, that the graph has symmetry about both the $x-$ and $y-$axes. This graph is called an *ellipse*. When $a = b$ we get the circle of radius a as a special case.

For use in our discussion of quadric surfaces, we need to look at a slightly more general equation that that given for the ellipse.

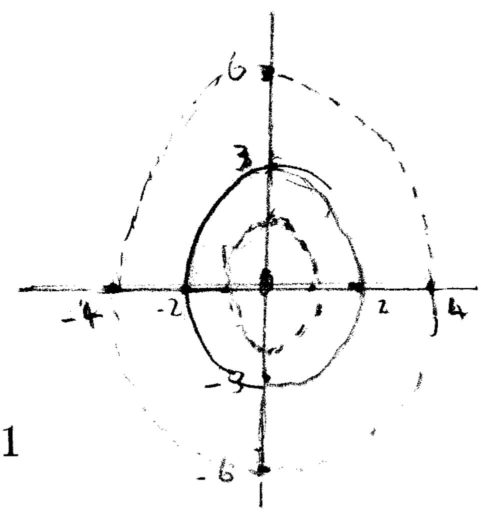
Example: For $d = \frac{1}{2}, 1, 4$ graph the equation

$$\frac{x^2}{4} + \frac{y^2}{9} = d$$

Solution: For $d = 1$ this is the ellipse with vertices $(\pm 2, 0)$ and $(0, \pm 3)$.

For $d = 4$, we write

$$\frac{x^2}{4} + \frac{y^2}{9} = 4$$
$$\frac{x^2}{4 \cdot 4} + \frac{y^2}{4 \cdot 9} = 1$$
$$\frac{x^2}{(\sqrt{4} \cdot 2)^2} + \frac{y^2}{(\sqrt{4} \cdot 3)^2} = 1$$



This an ellipse with parameters a and b that are twice those of the equation with $d = 1$.

For $d = \frac{1}{2}$

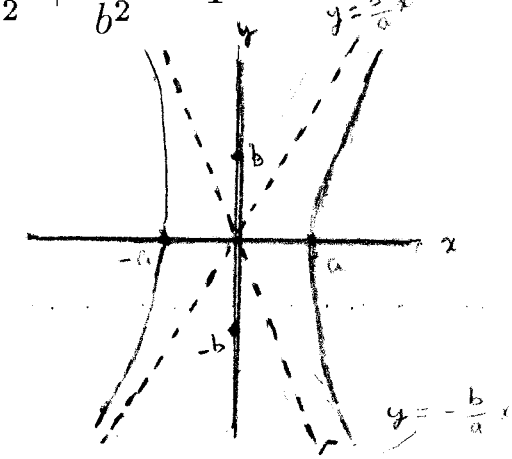
$$\frac{x^2}{4} + \frac{y^2}{9} = \frac{1}{2}$$
$$\frac{x^2}{\frac{1}{2}4} + \frac{y^2}{\frac{1}{2} \cdot 9} = 1,$$

and the parameters here are $\sqrt{\frac{1}{2}}$ times the parameters with $d = 1$.

The conclusion is that the parameters for d are $\sqrt{d}$ times the parameters when $d = 1$.

Case 1(a)(ii)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



We graph the first equation, beginning by finding the intercepts. For $x-$intercepts, $x^2 = a^2$, so $x = \pm a$, For $y-$intercepts $-y^2 = a^2$. Since $-y^2 \leq 0$ and $a^2 > 0$, there are no $y-$intercepts. In the first quadrant $y = b\sqrt{(x^2/a^2) - 1}$. For large values of x, the (-1) term in the radical is very small compared to x^2/a^2, so $y = \pm\frac{b}{a}x$ are asymptotes. Since $y' > 0$ and $y'' < 0$ in the first quadrant, the curve is increasing and concave down. Again, symmetry about both axes, allows us to finish the graph. This graph is called a *hyperbola*.

For the second equation, there will be intercepts $y = \pm b$ and no $x-$intercepts, and the graph will open up and down, instead of left and right.

Case1(a) (iii)

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Since the left side is less than or equal to zero for all x and y, the graph is empty.

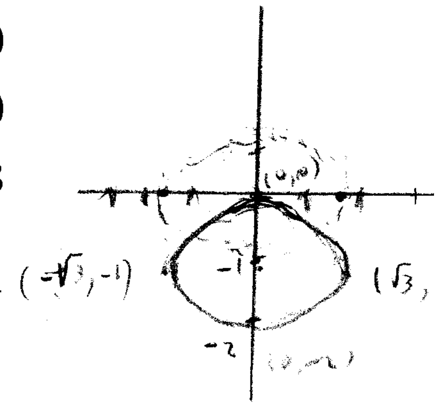
Case 1(b) $x^2/a^2 + By = 1$ or $Ax + y^2/b^2 = 1$ is clearly a parabola opening up or down for the first equation and left or right for the second case.

Case 2, the constant is 0, yields only two degenerate graphs and a duplication of Case 1(b). Thus, the non-degenerate graphs of all second degree equations coincide with the non-degenerate conic sections.

Example 1: Graph $x^2 + 3y^2 + 6y = 0$

Solution: Complete the square:

$$x^2 + 3[y^2 + 2y] = 0$$
$$x^2 + 3[(y+1)^2 - 1] = 0$$
$$x^2 + 3(y+1)^2 - 3 = 0$$
$$x^2 + 3(y+1)^2 = 3$$
$$\frac{x^2}{3} + (y+1)^2 = 1$$
$$\frac{x^2}{(\sqrt{3})^2} + \frac{(y+1)^2}{1} = 1$$



QUADRIC SURFACES

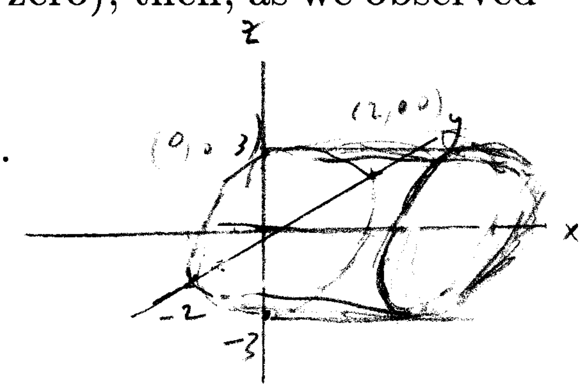
A *quadric surface* is a graph of the general second degree polynomial equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

We first observe that if only two of the variables appear in the second degree equation (i.e., both the square and linear term of one variable have coefficient zero), then, as we observed earlier, the graph is a cylinder.

Example 2: $\frac{y^2}{4} + \frac{z^2}{9} = 1$ in $\mathbf{R}^3$.

Solution:



Example 3: Describe the graph of $(x-1)(x-3) = 0$ in $\mathbf{R}^3$

Solution: It is the two planes $x = 1$ and $x = 3$ which are perpendicular to the $x-$axis. In general the graph of a second degree equation in just one variable is the empty set or one or two planes perpendicular to the axis of the variable that appears in the equation.

We wish to catalogue quadric surfaces in a manner analogous to what we did for conic sections. Since our discussion shows what is possible if only one or two of the variables appear with nonzero coefficients, we assume all three variables appear in the equation.

Two rotations (not requiring advanced mathematical techniques, but intricate enough that it never would be included in an elementary calculus book) will eliminate the cross product terms by describing the graph in the rotated coordinate axes.

Also, as for conics, we can complete the squares and shift to assume our equations are of the form

$$\begin{cases} \pm\dfrac{x^2}{a^2} \\ Ax \end{cases} + \begin{cases} \pm\dfrac{y^2}{b^2} \\ By \end{cases} + \begin{cases} \pm\dfrac{z^2}{c^2} \\ Cz \end{cases} = \begin{cases} 1 \\ 0 \end{cases}$$

Case 1: The constant is 1.

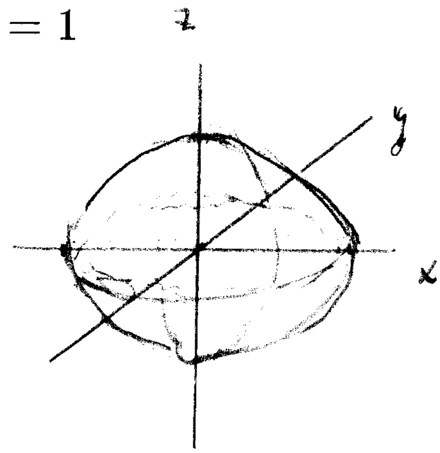
- (a) 3 terms on the left are squares
  - (i) 3 pluses
  - (ii) 2 pluses
  - (iii) 1 plus
  - (iv) no pluses: left side less than 0, so empty graph
- (b) 2 terms on the left are squares
  - (i) no pluses or 2 pluses
  - (ii) 1 plus
- (c) 1 term on the left is a square
- (d) No square terms: then equation is linear, not second degree.

Case 2: The constant is 0.

- (a) 3 terms on the left are squares
  - (i) 3 pluses or no pluses: graph is just the origin
  - (ii) 1 or 2 pluses
- (b) 2 terms on the left are squares: same as Case 1(b)
- (c) 1 term on the left is a square: same as Case 1(c)

Case 1(a)(i)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



We think of how a clay pot may be built in layers. Put a frame up and build the layers around that. For our "frame" we look at the portion of the graph that is in the the plane of the screen, the xz–plane from our perspective, that is, the portion of the graph with $y=0$:

$$\frac{x^2}{a^2}+\frac{z^2}{c^2}=1,$$

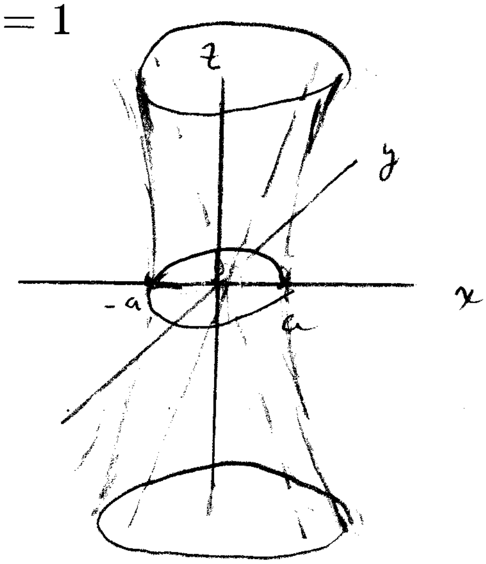
which is an ellipse, as shown at the right. Then we build horizontal strips at height $z=z_0$,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{z_0^2}{c^2}=d$$

This is an ellipse with dimension shrunk in proportion by a factor $\sqrt{d}$. By symmetry the bottom half is symmetric with the top half. This figure is called an *ellipsoid*.

Case 1(a)(ii)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



We chose the z term to be the one "different" in the sense that the x and y terms both have a plus sign, the z term does not. In all our examples we will let z be the one that is different. Our "frame" in the $xz-$ plane is

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1,$$

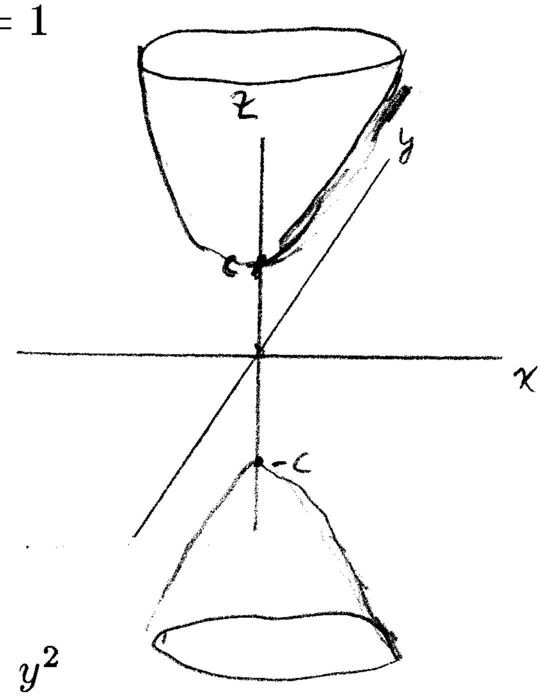
which is a hyperbola with $x-$intercepts at $x = \pm a$, shown on the accompanying diagram. Again, we look at horizontal "strips," more formally referred to as *traces* (or cross-sections) at various heights $z = z_0$;

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{z_0^2}{c^2} = d$$

This is referred to as a *hyperboloid of one sheet*, since the side walls are hyperbola shaped, and the whole surface is one connected piece.

Case 1(a)(iii)

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



Again, for $y = 0$, we have

$$-\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1,$$

a hyperbola with $z-$intercepts $z = \pm c$, as shown in the diagram. The horizontal cross-sections at various heights $z = z_0$, $z_0 \geq c$,

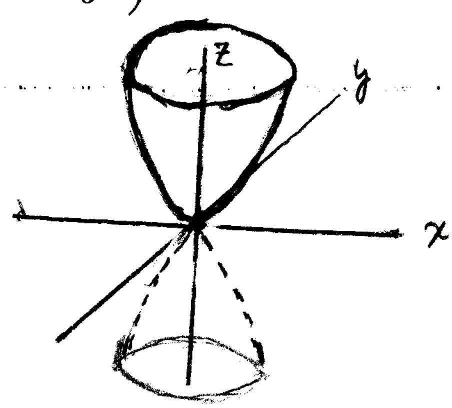
$$d = \frac{z_0^2}{c^2} - 1 = \frac{x^2}{a^2} + \frac{y^2}{b^2},$$

are ellipses of increasing size. This is called a *hyperboloid of two sheets* because the sides walls are hyperbola shaped and the surface consists of two disconnected pieces.

Case 1(b)(i) When both square terms are positive (the first equation below), we can solve for z (the second equation). We can make the shift $z \longrightarrow z + \frac{1}{C}$ and absorb $\frac{1}{|C|}$ into the coefficents of x^2 and y^2 to obtain the form in the third equation below.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + Cz = 1$$
$$z = \frac{1}{C} - \frac{1}{C}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)$$
$$z = \pm\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right).$$

When $y = 0$, we obtain $z = (1/a^2)x^2$, a parabola, shown in the diagram. For a fixed value $z = z_0$,



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \pm z_0 = d > 0,$$

is an ellipse.

This surface is called a *elliptical paraboloid* because the side wall is parabola shaped and the horizontal traces are ellipses.

If both square terms are negative, then

$$z = -\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right),$$

and the bowl shaped figure opens downwards instead of upwards.

Case 1(b)(ii) When one square term is positive, say the x term is positive, then, as in Case 1(b)(i), it suffices to consider the second equation below.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + Cz = 1$$
$$z = \pm\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right).$$

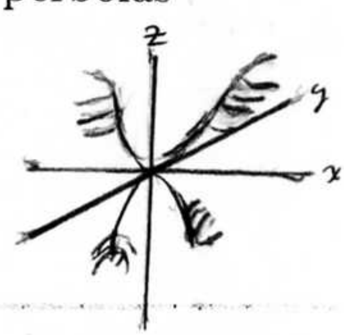
We assume that the "$\pm$" is plus. The situation when "$\pm$" is minus is analogous. The trace in the xz-plane, the plane of the screen from our perspective, is a parabola unfolding upwards from the origin. For a fixed $z = z_0 \geq 0$,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z_0 = d.$$

This is a hyperbola with vertices $(\pm\sqrt{d}a, 0, z_0)$ as shown below.

To see the whole surface, we must also look at the trace in the yz-plane, the vertical plane through the $z-$axis which comes in and out of the screen. There we have a parabola unfolding downwards, and for $z = z_0 < 0$, hyperbolas

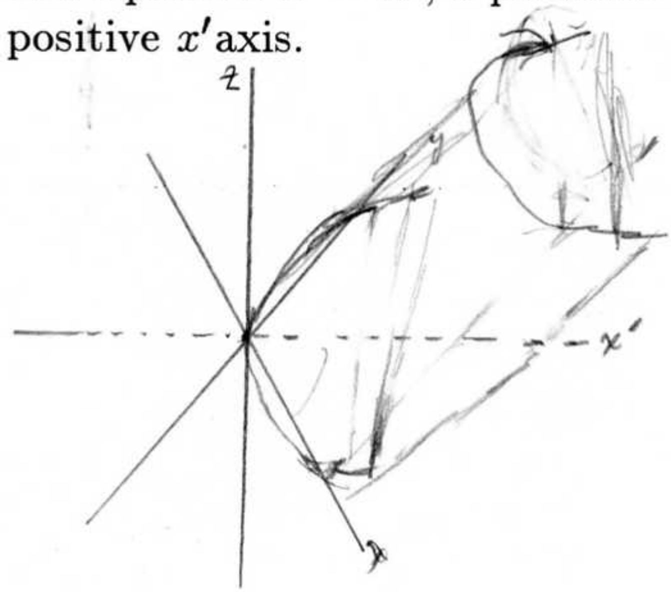
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z_0$$
$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = -z_0 = d,$$



These are hyperbolas unfolding in and out of the screen. The surface looks somewhat like a saddle and is called a *hyperbolic paraboloid*, because the "frame" consists of two parabolas and the cross-sections are hyperbolas.

Case 1(c) One square term: $\frac{z^2}{a^2} + By + Cz = 1$. Graphing this surface requires a rotating the x and y axes about the $z-$axis. Graphs requiring rotations are not part of the syllabus. For the sake of completeness we show an example.

Example 4: The graph of $z^2 = x + y$ is shown. The $x-$axis drawn rotated 45° behind the screen, and x'-axis is in the plane of the board. In terms of a rotated system using (x', y', z) coordinates the graph has equation $z^2 = x'$, a parabolic cylinder unfolding along the positive x'axis.



Case 2(a)(ii) If we multiply the second equation below (with 1 positive term) we obtain the first equation (with two positive terms).

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \quad \text{or}$$
$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$$

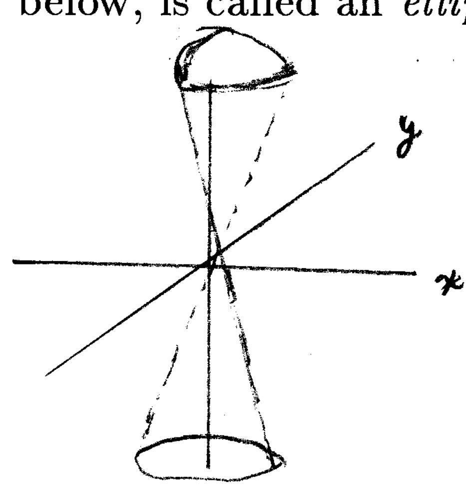
The trace on the xz-plane of the first equation above has equation

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 0$$
$$\frac{x^2}{a^2} = \frac{z^2}{c^2}$$
$$\pm\frac{x}{a} = \frac{z}{c}$$
$$z = \pm\frac{c}{a}x,$$

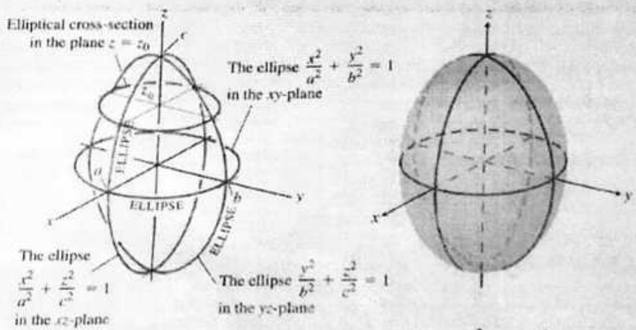
a pair of straight lines. The cross-sections are ellipses:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z_0^2}{c^2} = d$$

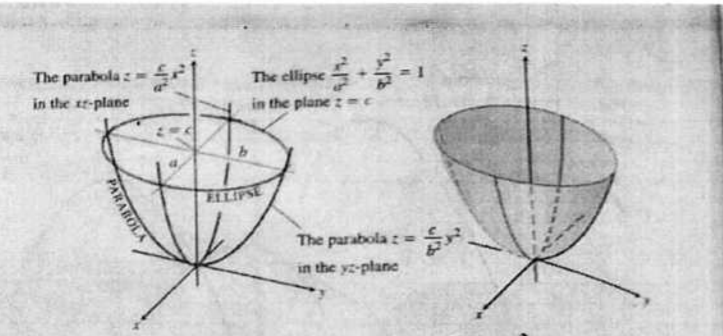
The surface, shown below, is called an *elliptical cone*.



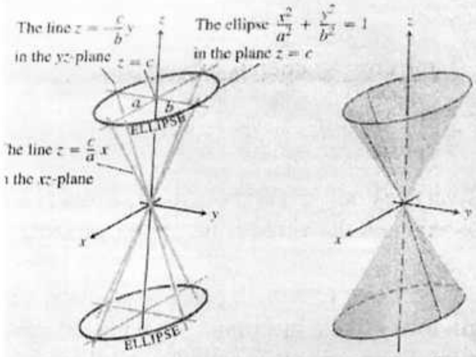
Quadric Surfaces



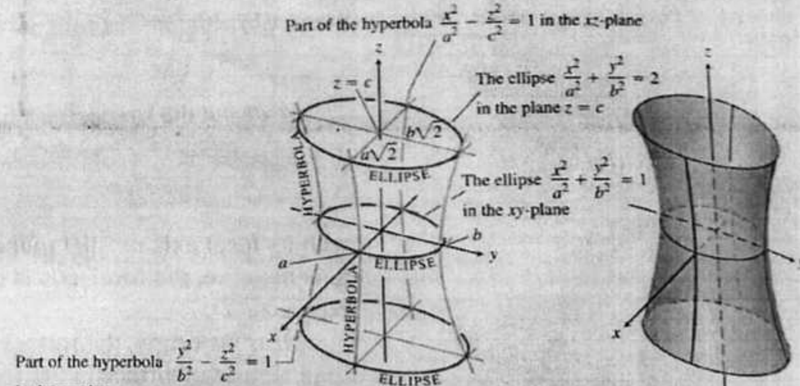
ELLIPSOID $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



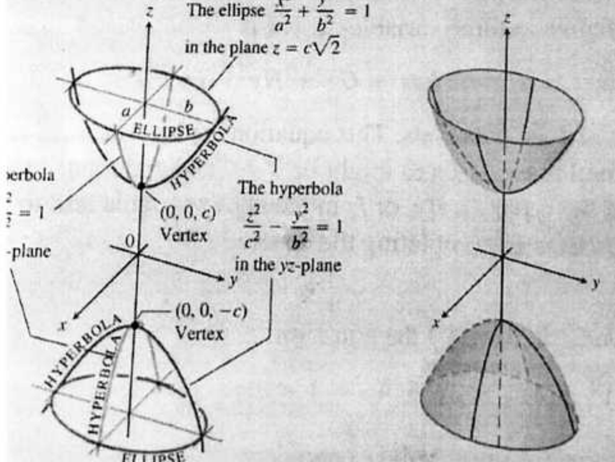
ELLIPTICAL PARABOLOID $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$



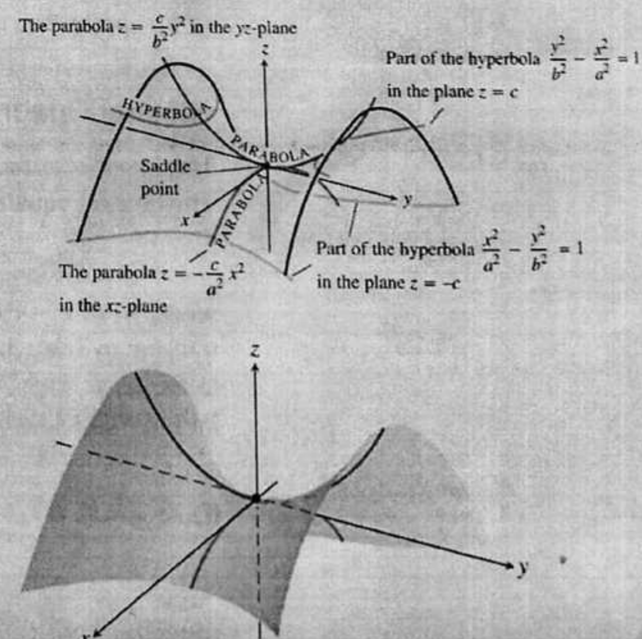
LLIPTICAL CONE $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$



HYPERBOLOID OF ONE SHEET $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



ERBOLOID OF TWO SHEETS $\quad \frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$

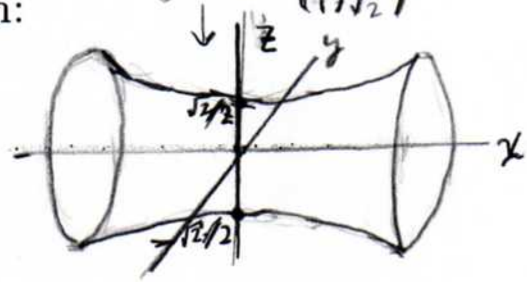
Example 5: (a) Graph and label the center and z-intercepts:

$$y^2 - x^2 + 2z^2 - 8z + 7 = 0$$

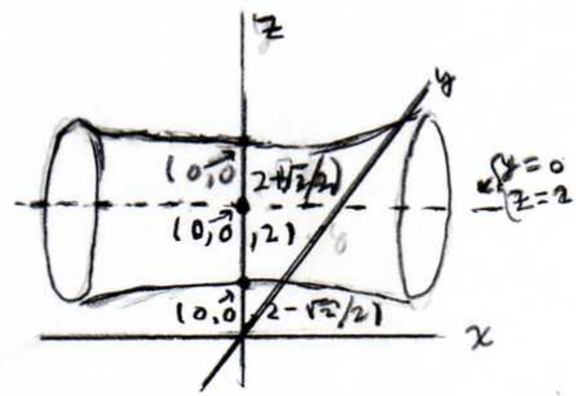
(b) Sketch the graph of the trace of the surface in the $xy-$ and $yz-$planes.

Solution: (a) Write in standard form:

$y^2 - x^2 + \frac{z^2}{(1/\sqrt{2})^2} = 1$



$$y^2 - x^2 + 2z^2 - 8z + 7 = 0$$
$$y^2 - x^2 + 2[z^2 - 4z] + 7 = 0$$
$$y^2 - x^2 + 2[(z-2)^2 - 4] + 7 = 0$$
$$y^2 - x^2 + 2(z-2)^2 - 8 + 7 = 0$$
$$y^2 - x^2 + \frac{(z-2)^2}{(1/\sqrt{2})^2} = 1$$

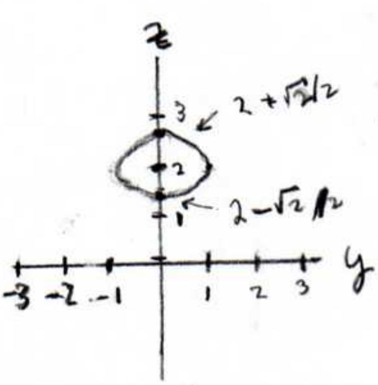


$1/\sqrt{2} = \frac{\sqrt{2}}{2}$

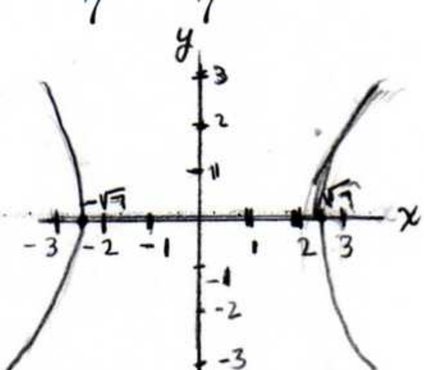
(b) Set $x = 0$, then $z = 0$:

$$y^2 + \frac{(z-2)^2}{(1/\sqrt{2})^2} = 1$$

trace in yz-plane, xz-plane

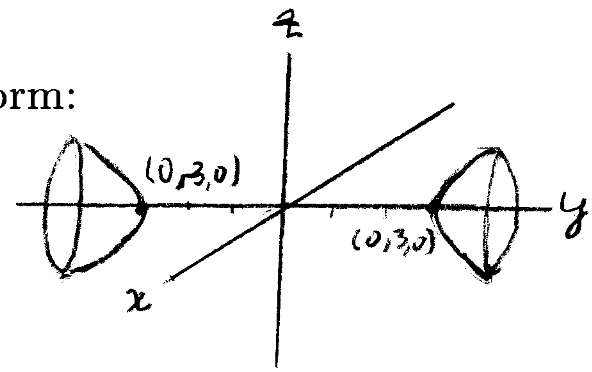


$$y^2 + \frac{(-2)^2}{1/2} - x^2 = 1$$
$$y^2 + 8 - x^2 = 1$$
$$y^2 - x^2 = -7$$
$$-\frac{y^2}{7} + \frac{x^2}{7} = 1$$



Example 6: (a) Graph and label the vertices:

$$\frac{x^2}{4} - y^2 + z^2 + 9 = 0$$

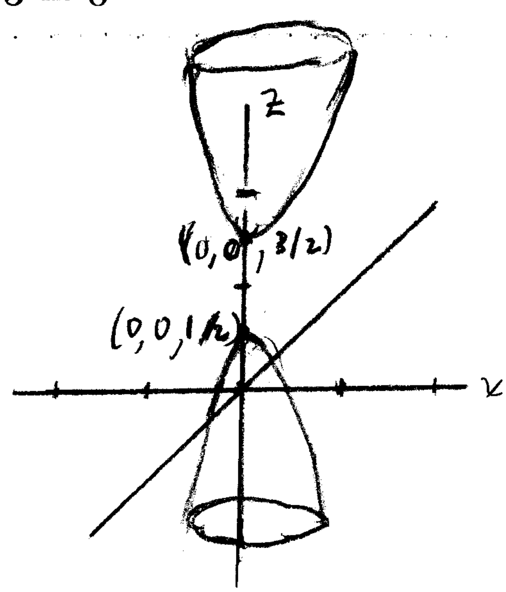


Solution: Write in standard form:

$$-\frac{x^2}{36} + \frac{y^2}{9} - \frac{z^2}{9} = 1$$

Example 7: Graph and label the vertices:

$$x^2 + y^2 - 4z^2 + 8z - 3 = 0$$



Solution: Write in standard form:

$$x^2 + y^2 - 4z^2 + 8z - 3 = 0$$
$$x^2 + y^2 - 4[z^2 - 2z] - 3 = 0$$
$$x^2 + y^2 - 4[(z-1)^2 - 1] - 3 = 0$$
$$x^2 + y^2 - 4(z-1)^2 + 4 - 3 = 0$$
$$x^2 + y^2 - 4(z-1)^2 = -1$$
$$-x^2 - y^2 + 4(z-1)^2 = 1$$
$$-x^2 - y^2 + \frac{(z-1)^2}{(1/2)^2} = 1$$

Example 8: Graph and label all vertices, if there are any:

$$x^2 + 4z^2 + 16y + 8z - 12 = 0$$

Solution: Write in standard form:

$$x^2 + 4z^2 + 16y + 8z - 12 = 0$$

$$x^2 + 4[z^2 + 2z] + 16y - 12 = 0$$

$$x^2 + 4[(z+1)^2 - 1] + 16y - 12 = 0$$

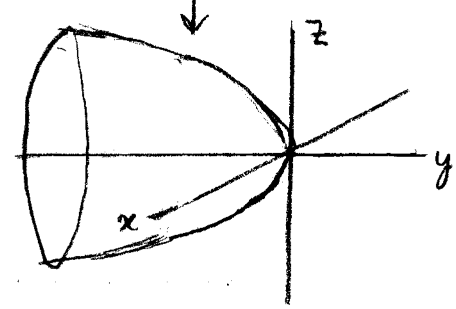
$$x^2 + 4(z+1)^2 - 4 + 16y - 12 = 0$$

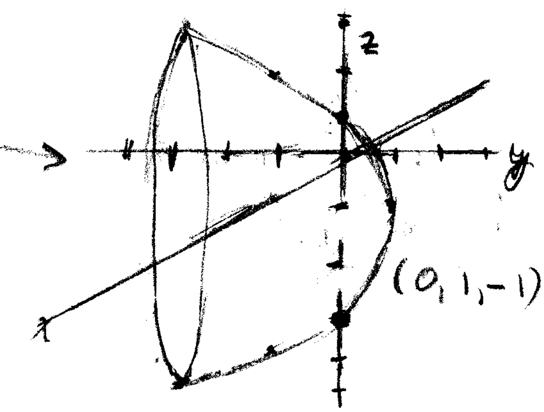
$$x^2 + 4(z+1)^2 - 16 = -16y$$

$$-\frac{x^2}{16} - \frac{(z+1)^2}{4} + 1 = y$$

$$-\left[\frac{x^2}{16} + \frac{(z+1)^2}{4}\right] + 1 = y$$

$$-\left[\frac{x^2}{16} + \frac{z^2}{4}\right] = y$$





Example 9: Graph and label all vertices, if there are any:

$$2x^2 + y^2 - z^2 - 6z - 9 = 0$$

Solution: Write in standard form:

$$2x^2 + y^2 - z^2 - 6z - 9 = 0$$

$$2x^2 + y^2 - [z^2 + 6z] - 9 = 0$$

$$2x^2 + y^2 - [(z+3)^2 - 9] - 9 = 0$$

$$2x^2 + y^2 - (z+3)^2 + 9 - 9 = 0$$

$$2x^2 + y^2 - (z+3)^2 = 0$$

xz – trace:

$$2x^2 - (z+3)^2 = 0$$

$$z = -3 \pm \sqrt{2}x$$