Power Series Part 2: Finding Series Representing a Function

Method 1: Find New Series from Known Series

We began our discussion of power series with the example $\Sigma_{n=0}^{\infty} x^n$. We recognized the series as a geometric series and found, from the formula $S = \dfrac{a}{1-r}$ that the series defined the function $\dfrac{1}{1-x}$. Sometimes we use the word *represents* instead of defines: The series $\Sigma_{n=0}^{\infty} x^n$ represents the function $\dfrac{1}{1-x}$. In this example, we started with a series and found the function it represents. Most of the time, we have a given function and we want to find a power series representing that function.

We will discuss two methods for doing this. The first is to find a power series for a function using another power series for which we already know the function it represents. We illustrate this method with an example.

Example 1: Find a Maclaurin series representing the function $f(x) = \dfrac{x}{1+x^2}$.

Solution: In our discussion on power series thus far, we have described only one function which we have represented as a power series, viz., $\dfrac{1}{1-x} = \Sigma_{n=0}^{\infty} x^n$. We can rewrite $f(x)$ so that it "resembles" $\frac{1}{1-x}$: $\quad f(x) = x\dfrac{1}{1-(-x^2)}$.

In the formula for $\frac{1}{1-x}$, we can **substitute** $-x^2$ wherever we see x to get

$$\begin{aligned} f(x) = x\frac{1}{1-(-x^2)} &= x\Sigma_{n=0}^{\infty}(-x^2)^n = \Sigma_{n=0}^{\infty}(-1)^n x^{2n+1} \\ &= x(1 - x^2 + x^4 - x^6 \pm \ldots) \\ &= x - x^3 + x^5 - x^7 \pm \cdots = \Sigma_{n=0}^{\infty}(-1)^n x^{2n+1} \end{aligned}$$

Here is a slightly more intricate example of finding a series by substitution and multiplication.

Example 2: Find a Maclaurin series representing $f(x) = \frac{x^3}{x+2}$.

Solution: Write

$$\begin{aligned} f(x) &= \frac{x^3}{2} \cdot \frac{1}{1-(-\frac{x}{2})} = \frac{x^3}{2} \Sigma_{n=0}^{\infty} (-x/2)^n \\ &= \frac{x^3}{2} \Sigma_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^n = \Sigma_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3} \\ &= \frac{x^3}{2}(1 - \frac{x}{2} + \frac{x^2}{2^2} - \frac{x^3}{2^3} \pm \dots) \\ &= \frac{x^3}{2} - \frac{x^4}{2^2} + \frac{x^5}{2^3} - \frac{x^6}{2^4} \pm \dots \end{aligned}$$

The following theorem is useful for finding power series.

Theorem: A power series can be differentiated or integrated term by term.

Another way of phrasing this is: The derivative (integral) of the sum is the sum of derivatives (integrals).

Example 3: Find the Maclaurin series for the derivative and integral of $\frac{1}{1-x}$.

Solution:

$$\frac{1}{1-x} = \Sigma_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \Sigma_{n=0}^{\infty} n x^{n-1} = 0 + 1 + 2x + 3x^2 + \ldots$$

$$\int \frac{1}{1-x}\, dx = \Sigma_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$$

Example 4: Find the Maclaurin series for $\ln(1+x)$. Use this to write $\ln 2$ as an infinite series.

Solution:

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \frac{1}{1-(-x)}\,dx$$
$$= \int \Sigma_{n=0}^{\infty}(-1)^n x^n\,dx$$
$$= \Sigma_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \pm \ldots$$

Substituting $x = 1$ into our series, we get

$$\Sigma_{n=0}^{\infty}\frac{(-1)^n 1^{n+1}}{n+1} = \Sigma_{n=0}^{\infty}\frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} \pm \ldots$$

At first you may think the altenating harmonic series is complicated and $\ln 2$ is your friend, something that you are comfortable with. However, if you want to find a numerical value for $\ln 2$, really it is the series that is your friend. By the error estimate for alternating series, the sum of the first 10 terms of the series yields an estimate of $\ln 2$ with an error not more than $1/11$ (the actual error only about $1/20$).

However the alternating series is not a very good friend. To use it to estimate $\ln 2$ with an error of less than $1/10000$ (four decimal places), one must sum 10000 terms.

To obtain a better series to estimate of $\ln 2$, it convenient to write the series with a starting value of $n = 1$ instead of $n = 0$. To do this we repace n by $n - 1$ where ever it occurs in our power series:

$$\Sigma_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \Sigma_{n-1=0}^{\infty} \frac{(-1)^{n-1} x^{n-1+1}}{n-1+1}$$
$$= \Sigma_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

We will first calculate $\ln(\frac{1}{2}) = \ln(1 + (-\frac{1}{2}))$

$$\ln(1 - \frac{1}{2}) = \Sigma_{n=1}^{\infty} \frac{(-1)^{n-1}(-\frac{1}{2})^n}{n}$$
$$= \Sigma_{n=1}^{\infty} \frac{-(\frac{1}{2})^n}{n} = -\Sigma_{n=1}^{\infty} \frac{1}{n2^n}$$
$$\ln 2 = -\ln\frac{1}{2} = \Sigma_{n=1}^{N} \frac{1}{n2^n} + R_N, \quad \text{where}$$
$$R_N = \Sigma_{n=N+1}^{\infty} \frac{1}{n2^n} \leq \Sigma_{n=N+1}^{\infty} \frac{1}{(N+1)2^n} = \frac{1}{N+1}\frac{1}{2^N},$$

since the geometric series $\Sigma_{n=N+1}^{\infty} \frac{1}{2^n} = \frac{1/2^{N+1}}{1-1/2}$.

Summing the first ten terms of this series estimates $\ln 2$ with an error of less than $1/10000$, a great savings over using the harmonic series. For a computer computation that requires very many calculations like $\ln 2$, cutting the amount of arithmetic by a factor of a thousand, can mean the difference between finishing in a tenth of a second instead of almost two minutes. The study of how to make a computation in fewer steps is called numerical analysis.

Example 5: Find the power series centered at $x_0 = 1$ for the function $f(x) = \frac{1}{x}$

Solution:

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \Sigma_{n=0}^{\infty}(1-x)^n = \Sigma_{n=0}^{\infty}(-1)^n(x-1)^n$$

Method 2: Find New Series as Taylor Series

Suppose a function $f(x)$ can be represented by a power series centered at x_0.

$$f(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + c_3(x-x_0)^3 + c_4(x-x_0)^4 + \ldots$$

Then $f(x_0) = c_0 + 0 + 0 + 0 + 0 + \cdots = c_0$.

We will calculate $f'(x)$ and $f'(x_0)$; then $f''(x)$ and $f''(x_0)$; and so forth

$$f'(x) = c_1 + 2c_2(x-x_0) + 3c_3(x-x_0)^2 + 4c_4(x-x_0)^3 + \ldots$$

$$f'(x_0) = c_1 = 1c_1 \quad \text{or} \quad c_1 = f'(x_0)/(1)$$

$$f''(x) = 2c_2 + 3\cdot 2c_3(x-x_0) + 4\cdot 3(x-x_0)^2 + \ldots$$

$$f''(x_0) = 2c_2 \quad \text{or} \quad c_2 = f''(x_0)/(1\cdot 2)$$

$$f'''(x) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-x_0) + \ldots$$

$$f'''(x_0) = 3\cdot 2c_3 \quad \text{or} \quad c_3 = f''(x_0)/(1\cdot 2\cdot 3)$$

We see that the pattern is $c_n = f^{(n)}(x_0)/n!$, where $f^{(n)}$ is the n-th derivative of f for $n \geq 1$ and f^0 means f (taking 0 derivatives). Recall $0! = 1$, so the formula holds for $n = 0$ also. I refer to $f(n)(x_0)/n!$ as the n-th power Taylor coefficient. Substituting these values back into the original expression for $f(x)$ we see that

$$\Sigma_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is the Taylor series for $f(x)$. If $f(x)$ is a closed form function (the only type we will consider), then its Taylor series is the power series representing f.

We showed that if f has a power series centered at x_0, it has to be the Taylor series. Functions that are not closed form function may have a Taylor series that does not represent the function.

Example 6: By calculating its derivatives at 0 as the limit of the difference quotient, it can be seen that for the function $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$, $c_n = 0$ for all n. Thus the Taylor series represents the 0 function, not f.

We give examples showing how to find the power series representation of a function by calculating the Taylor series.

Example 7: Find the Maclaurin series for $f(x) = e^x$.

Solution: Make a table

n	$f^{(n)}(x)$	$c_n = f^{(n)}(x_0)/n!$
0	e^x	$1/0! = 1$
1	e^x	$1/1! = 1$
2	e^x	$1/2!$
3	e^x	$1/3!$

So

$$e^x = \Sigma_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots$$

Example 8: Find the Maclaurin series for $f(x) = \sin x$.

Solution:

n	$f^{(n)}(x)$	$c_n = f^{(n)}(x_0)/n!$
0	$\sin x$	$0/0! = 0$
1	$\cos x$	$1/1! = 1$
2	$-\sin x$	$0/2! = 0$
3	$-\cos x$	$-1/3! = -1/3!$
4	$\sin x$	$0/4! = 0$

So

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \pm \cdots = \Sigma_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

As an exercise, show, as above, that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \pm \cdots = \Sigma_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Notice that $\sin x$ ($\cos x$) is an odd (even) function and its Maclaurin series has only odd (even) powers of x.

You should remember the power series for the four functions $\frac{1}{1-x}$, e^x, $\sin x$ and $\cos x$ and be able to use the substitution method with any of them.

Example 9: Find the Maclaurin series for e^{-x^2}.

Solution:

$$e^{-x^2} = \Sigma_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \Sigma_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$
$$= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} \pm \ldots$$

Example 10: Find the Maclaurin series for $\sin(2x)$.

Solution:

$$\sin(2x) = \Sigma_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \Sigma_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$$
$$= 2x - \frac{2^3 x^3}{3!} - \frac{2^5 x^5}{5!} \pm \ldots$$

Power Series Part 3: Application of Power Series

There are no new ideas in this section. We apply the material from this section to make numerical estimates.

Example 1: (a) Write the sum of the first four nonzero terms of the Maclaurin series for $f(x) = e^{-\frac{x^2}{2}}$.

(b) Use this to estimate $\int_0^1 f(x)\,dx$, with an error of at most .01

(c) Show why the answer your answer to (b) has the required accuracy.

Solution: (a)

$$\begin{aligned} e^{-\frac{x^2}{2}} &= \Sigma_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!} \\ &= 1 - \frac{x^2}{2^1 1!} + \frac{x^4}{2^2 2!} - \frac{x^6}{2^3 3!} \pm \ldots \end{aligned}$$

(b)

$$\begin{aligned} \int_0^1 e^{-\frac{x^2}{2}}\,dx &= \int_0^1 1 - \frac{x^2}{2^1 1!} + \frac{x^4}{2^2 2!} - \frac{x^6}{2^3 3!} \pm \ldots\,dx \\ &= x - \frac{x^3}{2^1 1! 3} + \frac{x^5}{2^2 2! 5} - \frac{x^7}{2^3 3! 7} \pm \ldots \Big|_0^1 \\ &= 1 - \frac{1}{2^1 1! 3} + \frac{1}{2^2 2! 5} - \frac{1}{2^3 3! 7} \pm \ldots \\ &\approx 1 - \frac{1}{6} + \frac{1}{40} \end{aligned}$$

(c) By the alternating series error bound

$$|R_3| \le \left| -\frac{1}{2^3 3! 7} \right| = \frac{1}{336} < 1/100 = .01$$

Example 12: (a) Find the Maclaurin series for $g(x) = \frac{1}{1+x^2}$.

(b) Let $f(x) = \Sigma_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$. Find the Maclaurin series for the derivative $f'(x)$.

(c) Compare (a) and (b) to see that $f'(x) = \frac{1}{1+x^2}$.

(d) Use (c) to show $f(x) = \tan^{-1} x$.

(e) Use (d) to find an estimate of π accuracte to .001

Solution: (a) Change the form for writing g: $g(x) = \frac{1}{1-(-x^2)}$. Substitute $(-x^2)$ for x in the geometric series:

$$g(x) = \Sigma_{n=0}^{\infty}(-x^2)^n = \Sigma_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 \pm \ldots$$

(b–c) $\frac{df}{dx} =$

$$\frac{d}{dx}\left(\Sigma_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}\right) = \Sigma_{n=0}^{\infty}\frac{(-1)^n(2n+1)x^{2n}}{2n+1}$$

$$= \Sigma_{n=0}^{\infty}(-1)^n x^{2n} = \frac{1}{1+x^2}$$

(d) $f(x) = \int f'(x)\,dx = \int \frac{1}{1+x^2}\,dx = \tan^{-1} x$

(e) $\tan^{-1} 1 = \pi/4$, so

$$\pi = 4\tan^{-1} 1$$

$$= 4\Sigma_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}\Big|_{x=1}$$

$$= 4\left(1 - \frac{1}{3} + \frac{1}{5} \pm \cdots \pm \ldots \frac{1}{2n-1}\right) \pm \frac{4}{2n+1}\ldots$$

We can use the alternating series error bound to obtain the required accuracy: $|\pm\frac{4}{2n+1}| = \frac{4}{2n+1} \leqslant .001 = 1/1000$. Clear fractions to obtain $4000 \leqslant 2n+1$ or $2n+1 \geqslant 4000$. So for $n \geqslant 2000 > (4000-1)/2$, summing the alternating series to one less than n will have the desired accuracy:

$$\Sigma_{n=0}^{1999}\frac{4(-1)^n}{2n+1} = 4 - \frac{4}{3} + \cdots - \frac{4}{3999}.$$

Example 13: Let $f(x)$ be the function represented by the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)^2}$.

(a) Find the sum of the first four nonzero term of the Maclaurin series for $f'(x)$.

(b) Approximate $f'(\frac{1}{10})$ with an error at most .001

(c) Show that the approximation has the required accuracy.

Solution: (a) $f(x) = 1 - \frac{x}{2^2} + \frac{x^2}{3^2} - \frac{x^3}{4^2} + \frac{x^4}{5^2} \pm \ldots$

$$f'(x) = 0 - \frac{1}{2^2} + \frac{2x}{3^2} - \frac{3x^2}{4^2} + \frac{4x^3}{5^2} \pm \ldots$$

(b)
$$f'(\frac{1}{10}) = -\frac{1}{4} + \frac{2}{90} - \frac{3}{1600} + \frac{4}{25000} \pm \ldots$$
$$\approx -\frac{1}{4} + \frac{1}{45} - \frac{3}{1600}.$$

(c) $|R_3| \leq \frac{4}{25000} = \frac{1}{6250} < .001$ (but we can only be sure $|R_2| \leq \frac{3}{1600}$, and $\frac{3}{1600} \geq \frac{2}{1600} > \frac{1}{1000}$).

The fact that the Taylor series is the *only* power series representing a function f at a given center has an important consequence. From the fact that $\sum c_n x^n = \sum d_n x^n$ implies $c_n = d_n$ for all n, i.e, one can equate coefficients, and the fact that $0 + 0x + 0x^2 + \ldots$ obviously represents the function which is 0 for all x, we see that $\sum c_n x^n = 0$ for all x implies $c_n = 0$ for all n.

An application of this is that, if we assume a function f has a series representation $\sum c_n x^n$, and we want to solve the differential equation $f + f' = 0$, then by equating coefficients for

$$\sum_{n=0}^{\infty} c_n x^n + \sum_{n=0}^{\infty} n c_n x^{n-1}$$

replace n by n+1

$$= \sum_{n=0}^{\infty} c_n x^n + \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$= \sum_{n=0}^{\infty} c_n x^n + \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

$$= \sum_{n=0}^{\infty} (c_n + (n+1) c_{n+1}) x^n = 0$$

we can obtain a sequence of equations we can solve for all c_n in terms of c_0: for all n, $c_n + (n+1)c_{n+1} = 0$.

Example 14: Let $f(x) = e^{-\frac{1}{x^2}}$ for $x \neq 0$ and let $f(0) = 0$. Using the definition of the derivative as the limit of the difference quotient, one obtains $f^{(n)}(0) = 0$ for all n. Thus the Taylor series converges to 0 for all x.

Suppose $f(x) = \sum c_n x^n$ with radius of convergence R so the series converges at least in $(-R, R)$.

Then $\int \sum c_n x^n = \sum \frac{c_n x^{n+1}}{n+1}$, and this sum has radius of convergence R and the equality holds at least in the open interval $(-R, R)$, but at the endpoints $\pm R$, neither the equality or the convergence of the sum for the integral can be guaranteed. Similar comments hold for $f'(x)$.

In Example 4, we found

$$\ln(1+x) = \int \frac{1}{1+x}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}.$$

The radius of convergence of $1/(1+x)$ is 1. So when we substituted $x = 1$ into this power series, there was no assurance that the series would be convergent, and if it were convergent, that the sum would equal $\ln 2$. In fact, when we subsituted $x = 1$ into the power series for $\ln(1+x)$, we got the alternating harmonic series which we know converges. It happens that the alternating harmonic series does converge to $\ln 2$, but this requires a separate proof.

On the other hand, when we substituted $x = -1/2$, which is in $(-1, 1)$, into the series represenation $\ln(1+x)$, the theorem on integrating series term by term assures us that $\ln 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n}$.

Using this series to estimate the numerical value of $\ln 2$ is not possible, based on our upper bound tests: The series is not alternating, so our alternating series bound is not possible, and, although the conditions for the integral test upper bound are met, the required integral is not a closed form function.

However, there are three standard upper bound theorems (one involving the function, one the integral of the function, and one the derivative of the function) for estimating the error for power series.