Power Series Part 1

Let $f_0(x), f_1(x), \ldots, f_n(x) \ldots$ be a sequence of real valued functions. For example, $f_n(x) = x^n$ describes the sequence $1, x, x^2, \ldots$. Note that here x^0 mean the constant function 1, even if $x = 0$, not the indeterminate form 0^0.

From this sequence of functions we can define a new function $f(x)$, denoted by

$$f(x) = \Sigma_{n=0}^{\infty} f_n(x),$$

which is defined for any fixed value c of x to be

$$f(c) = \Sigma_{n=0}^{\infty} f_n(c),$$

provided this infinite series is convergent. If the series is divergent for the value c, then c is not in the domain of f. In other words, the the domain of f is exactly all numbers c such that the series $\Sigma f_n(c)$ is convergent.

Example 1: If $f_n(x) = x^n$ and $f(x) = \Sigma_{n=0}^{\infty} x^n$, then
$f(\frac{1}{2}) = \Sigma_{n=0}^{\infty} (\frac{1}{2})^n = \frac{1}{1-\frac{1}{2}} = 2$
(a geometric series with $a = 1$ and $r = 1/2$); and $f(x)$ is the geometric series with initial term $a = 1$ and common ratio $r = x$, so that

$$f(x) = \frac{1}{1-x} \quad \text{for} \quad |x| < 1,$$

with $-1 < x < 1$ as its domain.

We are only going to consider special types of series of functions. Namely, only $f_n(x) = c_n(x - x_0)^n$.

Definition: A *power series* (also called a Taylor series) is a series

$$\Sigma_{n=0}^{\infty} c_n (x - x_0)^n,$$

where x is a variable and $x_0, c_0, c_1, c_2, \ldots$ are constants. The constant x_0 is called the *center* of the series and the constants c_n are called the *coefficients* of the series. If $x_0 = 0$, the power series is referred to as a *Maclaurin* series.

Domain or Interval of Convergence of a Power Series

Using the ratio test, we will see that every power series has a domain (i.e., set of all values of x that when substituted for x yield a convergent infinite series of numbers) which is an interval of the form $(x_0 - R, x_0 + R)$, with possibly one or both endpoints added. The case $R = 0$, corresponding to $[x_0, x_0]$, that is, just converges at x_0 and $R = \infty$, i.e, converges for $(-\infty, \infty)$, which is the set of all real numbers, are both possible. The form of the domain is the reason for calling x_0 the center of the series. Notice that substituting $x = x_0$ into the $\sum_{n=0}^{\infty} c_0(x - x_0)^n$ yields $c_0 + 0 + 0 + \cdots + 0 = c_0$.

We apply the ratio test to find whether, for some fixed value of x, the power series converges. The nth term of the series is $a_n = c_n(x - x_0)^n$, and the ratio test says that if

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1,$$

then the series will converge. Substitute the values of a_n and a_{n+1} to get

$$L = \lim_{n\to\infty} \left| \frac{c_{n+1}(x - x_0)^{n+1}}{c_n(x - x_0)^n} \right| < 1,$$

Cancel n powers of $(x - x_0)$ top and bottom to get:

$$L = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n}(x - x_0) \right| < 1,$$

The absolute value of a product is the product of the absolute values, and $|x - x_0|$ doesn't change as n increases, i.e., it is a constant when n is the variable, so

$$L = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| |x - x_0| < 1,$$

$$L = \left[\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|\right]|x - x_0| < 1, \qquad (*)$$

Define L_0 as

$$L_0 = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|,$$

and let $R = 1/L_0$. Then the inequality $(*)$ becomes

$$L_0|x - x_0| < 1 \qquad \text{or} \qquad |x - x_0| < 1/L_0 = R.$$

A basic statement about inequalities, usually discussed in a first semester calculus course says the inequality $|u(x)| < a$ is equivalent to the pair of inequalities $-a < u < a$, so the inequality $|x - x_0| < R$, is equivalent to

$$-R < x - x_0 < R.$$

Adding x_0 throughout this inequality yields $x_0 - R < x < x_0 + R$ which describes the open interval $(x_0 - R, x_0 + R)$ and which is the set of points with $L < 1$ in the ratio test; the set for which $L > 1$ is the set of points not in the closed interval $[x_0 - R, x_0 + R]$; at the endpoints, $x_0 - R$ and $x_0 + R$, $L = 1$ and the ratio test fails.

R is called the *radius of convergence*. Notice that $(x_0 - R, x_0 + R)$ is the set of all points x on the real line that are a distance less than R from the center x_0. Thus the interval can be considered a one-dimensionl circle, just as one can think of a sphere as a three-dimensional circle.

The part of the ratio test which says $L > 1$ implies that a series is divergent, tells us, in a manner analogous to what we showed for $L < 1$, that a power series diverges if x is not in the interval $[x_0 - R, x_0 + R]$. The ratio test fails when x is one of the endpoints $x_0 - R$ or $x_0 + R$.

Example 2: Find the set of all points x for which the series $\Sigma_{n=0}^{\infty} \dfrac{(-1)^n (x-1)^n}{2^n (n+1)}$ converges. Remember to check the endpoints if necessary.

Solution: In this power series, the center is $x_0 = 1$ and the coefficients are $c_n = \dfrac{(-1)^n}{2^n (n+1)}$. We calculate the limit L_0:

$$L_0 = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \frac{1}{2^{n+1}(n+2)} \cdot \frac{2^n (n+1)}{1}$$
$$= \lim_{n\to\infty} \frac{1}{2(n+2)} \cdot \frac{n+1}{1} = \frac{1}{2}.$$

Thus, $R = 1/L_0 = 2$ and the power series converges at least in the interval $(1-2, 1+2) = (-1, 3)$. Now we have to see if the power series converges at either of the endpoints:

$$x = -1: \quad \Sigma_{n=0}^{\infty} \frac{(-1)^n (-1-1)^n}{2^n (n+1)} = \Sigma_{n=0}^{\infty} \frac{2^n}{2^n (n+1)}$$
$$= \Sigma_{n=0}^{\infty} \frac{1}{n+1},$$

which is divergent because it is the harmonic series.

$$x = 3: \quad \Sigma_{n=0}^{\infty} \frac{(-1)^n (3-1)^n}{2^n (n+1)} = \Sigma_{n=0}^{\infty} \frac{(-1)^n 2^n}{2^n (n+1)}$$
$$= \Sigma_{n=0}^{\infty} \frac{(-1)^n}{n+1},$$

which is convergent by the alternating series test. Thus, $(-1, 3]$ is the set (interval) of convergence.

Example 3: Same question for the series $\Sigma_{n=0}^{\infty} \dfrac{(-1)^n (x-1)^n}{2^n (n+1)^2}$.

Solution: By a calculation similar to that in Example 2, $L_0 = \lim_{n\to\infty} \frac{1}{2(n+2)^2} \cdot \frac{(n+1)^2}{1} = \frac{1}{2}$, so again the series converges in the interval $(-1, 3)$ and possibly one or both endpoints.

Substituting $x = -1$, yields the series $\Sigma_{n=0}^{\infty} \dfrac{1}{(n+1)^2}$, which is convergent ($p$-series, $p = 2$).

Substituting $x = 3$, yields the series $\Sigma_{n=0}^{\infty} \dfrac{(-1)^n}{(n+1)^2}$, which is absolutely convergent.

Therefore, the interval of convergence is $[-1, 3]$.

From Examples 1 (for $\dfrac{1}{1-x}$), 2 and 3, we had, respectively, intervals of convergence $(-1, 1)$, $(-1, 3]$ and $[-1, 3]$, showing that it is possible to have neither, one, or both endpoints be part of the interval of convergence.

Example 4: Same question for the series
$\Sigma_{n=0}^{\infty} \dfrac{(-1)^n(x+1)^n}{2^n(n+1)}$.

Solution: This is the same series as in Example 2, except that $(x-1)$, has been changed to $(x+1)$. We write $x+1$ as $x-(-1)$, to see that the center $x_0 = -1$. With the same calculations as in the solution to Example 2, we see that the series converges at least in the interval $(-1-2, -1+2) = (-3, 1)$, and the same check at the endpoints gives a final answer of $(-3, 1]$.

Example 5: Same question, again, for $\Sigma_{n=0}^{\infty} \dfrac{x^n}{n!}$.

Solution: Here $x_0 = 0$, $c_n = \frac{1}{n!}$, and

$$L_0 = \lim_{n\to\infty} \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

We let $R = 1/L_0 = 1/0^+ = \infty$, so the interval of convergence is $(-\infty, \infty)$, i.e., the set of all real numbers.

Example 6: Same question for $\sum_{n=0}^{\infty} n!x^n$.

Solution: $x_0 = 0$, $c_n = n!$, and

$$L_0 = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty.$$

We let $R = 1/L_0 = 1/\infty = 0$, and the interval of convergence is $[0,0]$, i.e., the series converges only at $x = 0$.

A power series with a radius of convergence of zero is worthless. It only says that $f(x_0) = c_0$.

Example 7: Find the set of all points x for which the series $\sum_{n=0}^{\infty} \dfrac{(-1)^n(2x-2)^n}{n!}$ converges. Remember to check the endpoints if necessary.

Solution: As it is written the series does not have the form of a power series. However, if we write

$$(2x-2)^n = (2(x-1))^n = 2^n(x-1)^n,$$

then the original series can written as $\sum_0^{\infty} \dfrac{(-1)^n 2^n}{n!}(x-1)^n$, a power series with coefficients $c_n = \dfrac{(-1)^n 2^n}{n!}$. Then

$$\begin{aligned} L_0 = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| &= \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \\ &= \lim_{n\to\infty} \frac{2}{n+1} = 0 \end{aligned}$$

Thus, $R = 1/L_0 = \infty$ and the power series converges for all real x.

Example 8: Find the interval of convergence:

$$\Sigma_{n=0}^{\infty} \frac{\ln (n+1)}{\ln (n+2)} x^n.$$

Solution: Use L'Hôpital's rule:

$$L_0 = \lim_{n \to \infty} \frac{\ln(n+2)}{\ln(n+3)} \cdot \frac{\ln (n+2)}{\ln (n+1)} = 1 \cdot 1 = 1.$$

So, $x_0 = 0$, $R = 1/1 = 1$, and the series converges in $(0-1, 0+1) = (-1, 1)$.

Substituting $x = -1$, yields the series $\Sigma_{n=0}^{\infty} \frac{\ln (n+1)}{\ln (n+2)} (-1)^n$. The absolute value of the nth term of this series is $\frac{\ln (n+1)}{\ln (n+2)}$ which we noted above converges to 1, not 0. from which we conclude the series is divergent by the test for divergence.

Substituting $x = 1$, yields the series $\Sigma_{n=0}^{\infty} \frac{\ln (n+1)}{\ln (n+2)}$, which also diverges since the nth term converges to 1. The interval of convergence is $(-1, 1)$