Limit Comparison Test

If f and g are continuous and non-negative for $x \geq a$ and $\lim_{x\to\infty} f(x)/g(x) = L$, where $0 < L < \infty$, then

$$\int_a^\infty f(x)\,dx \qquad \text{and} \qquad \int_a^\infty g(x)\,dx$$

both converge or both diverge.

Hereafter, lim means $\lim_{x\to\infty}$.

Proof: (1) Choose F such that $F' = f$, and suppose $\int_a^\infty g\,dx$ is convergent.

(2) From the definition of a limit, $\lim \dfrac{f}{g} = L$ implies there exists a real number x_0 such that, for all $x \geq x_0$,

$$\left|\frac{f(x)}{g(x)} - L\right| < \frac{L}{2}.$$

(3) The inequality in (2) is equivalent to

$$\frac{L}{2}g(x) \leq f(x) \leq \frac{3L}{2}g(x).$$

(4) From the second inequality in (3), for $M \geq x_0$,

$$F(M) - F(x_0) = \int_{x_0}^M f(x)\,dx \leq \frac{3L}{2}\int_{x_0}^M g(x)\,dx \leq \frac{3L}{2}\int_{x_0}^\infty g(x)\,dx.$$

(5) Since $f \geq 0$, $F(M) - f(x_0)$, as a function of M, is increasing, and from (4), it is bounded by $\dfrac{3L}{2}\displaystyle\int_a^\infty g\,dx$. Thus, $F(\infty) = \lim F(M)$ exists and

$$F(\infty) - F(x_0) = \int_{x_0}^\infty f\,dx \leq \frac{3L}{2}\int_{x_0}^\infty g\,dx < \infty.$$

(6) Thus $\int_a^\infty f\,dx$ is also convergent. Similarly, from $g \le \frac{2}{L} f$, we see that the integral for g is convergent if the integral for f is convergent.

Theorem: If $f = U + \sum_{i=1}^m u_i$ and $g = V + \sum_{i=1}^n v_i$, where $f(x)$, $g(x)$,$U(x)$ and $V(x)$ are all continuous and positive for all $x \ge a$, a a real number, and $u_i << U$ and $v_i << V$ for all i, then

$$\int_a^\infty \frac{f(x)}{g(x)}\,dx \qquad \text{and} \qquad \int_a^\infty \frac{U(x)}{V(x)}\,dx$$

both converge or both diverge.

Proof:

$$\lim \frac{f/g}{U/V} = \lim \frac{fV}{gU} = \lim \frac{[UV + \sum (u_i V)]/UV}{[UV + \sum_i (v_i U)]/UV}$$
$$= \frac{1 + 0 + \cdots + 0}{1 + 0 + \cdots + 0} = 1,$$

and the conclusion follows from the limit comparison test.

Example 1: $\int_1^\infty \frac{x^2 + 3\ln x}{x^4 + \ln x}\,dx$ is convergent, since $\ln x << x^2 << x^4$ and $\int_1^\infty \frac{x^2}{x^4}\,dx$ is convergent. If we change x^4 to x^3 in this integral it is then divergent.

Example 2: $\int_0^\infty \frac{xe^x + x^{10} - x^2}{e^{2x} + 1}\,dx$ is convergent, since $1 << x^2 << x^1 0 << \cancel{x}e^x << e^{2x}$ and

$$\int_0^\infty \frac{xe^x}{e^{2x}} = \int_0^\infty \frac{x}{e^x}\,dx$$
$$= x(-e^{-x})|_0^\infty - \int_0^\infty (-e^{-x})\,dx = 0 - 0 - e^{-x}|_0^\infty = 1.$$