SUPPLEMENTARY NOTES FOR MATH 212

Integration

From first semester calculus, students are expected to know three techniques (theorems) for evaluating integrals.

1 Linearity. Integration is linear, viz., for a constant c and continuous functions $f(x)$ and $g(x)$,

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

2 Substitution Theorem. For continuous function $f(x)$, differentiable $u(x)$, $\int f(u(x))u'(x)\,dx = \int f(u)\,du$. The meaning of the right side of this equality is that if F is an antiderivative of f, then $\int f(u)\,du = F(u(x))$.

3 The Fundamental Theorem of Calculus. If $f(x)$ is a differentiable function and a is a constant, then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \quad \text{and} \quad \int_a^x \frac{df}{dt}\,dt = f(x) - f(a).$$

(The Fundamental Theorem could be stated more generally, but this is is sufficient to produce entries in a table of integrals in the text.)

Example: Since $\frac{d}{dx}\sin x = \cos x$, we have $\int \cos x\,dx = \sin x + C$.

Example: Evaluate $\int x(x^2-4)^9\,dx$.

Solution 1: We rewrite the integral in a form for which the substitution theorem applies, with $f(x) = x^9$, and $u(x) = x^2 - 4$.

$$\int x(x^2-4)^9\,dx = \int \frac{1}{2}2x(x^2-4)^9\,dx = \frac{1}{2}\int 2x(x^2-4)^9\,dx$$
$$=\frac{1}{2}\int u^9\,du = \frac{1}{2}\frac{u^{10}}{10} = \frac{(x^2-4)^{10}}{20} + C$$

Solution 2 (which shows why the theorem is named the Substitution Theorem): The symbol $\frac{df}{dx}$ means the derivative of f, but we treat it as if it were a fraction.

$$\begin{aligned}
&\int x(x^2-4)^9\,dx \\
&= \int u^9\left(\frac{1}{2}\,du\right) \\
&= \frac{1}{2}\frac{u^{10}}{10} \\
&= \frac{(x^2-4)^{10}}{20} + C
\end{aligned}
\qquad
\begin{aligned}
u &= x^2 - 4 \\
\frac{du}{dx} &= 2x \\
du &= 2x\,dx \\
\frac{1}{2}du &= x\,dx
\end{aligned}$$

For a definite integral, one must distinguish between limits for x and limits for u. We illustrate with the function above integrated from 2 to 3. As x ranges from 2 to 3, $u(x)$ ranges from $u(2) = 0$ to $u(3) = 5$, so

$$\int_{x=2}^{3} x(x^2-4)^9\,dx = \int_{u=0}^{5} u^9\,du = \frac{u^{10}}{20}\Big|_0^5 = \frac{5^{10}}{20} \quad \text{or}$$

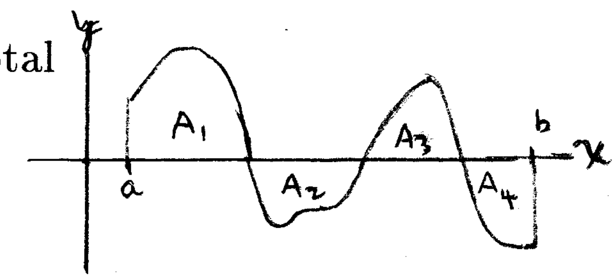
$$\int_{x=2}^{3} x(x^2-4)^9\,dx = \frac{(x^2-4)^{10}}{20}\Big|_2^3 = \frac{5^{10}}{20}.$$

but **NOT** $\frac{u^{10}}{20}\Big|_2^3 = \frac{3^{10}-2^{10}}{20}$.

Some elementary facts about integration are listed below.

1. For constants a, b and c, $\int_a^b c\,dx = c(b-a)$.

2. $\int_a^b f(x)\,dx$ is equal to the total area below the curve when it is above the x-axis minus the total area above the curve where it is below the x-axis. For example, for the function whose graph is on the right, the value of the integral is $A_1 + A_3 - (A_2 + A_4)$.

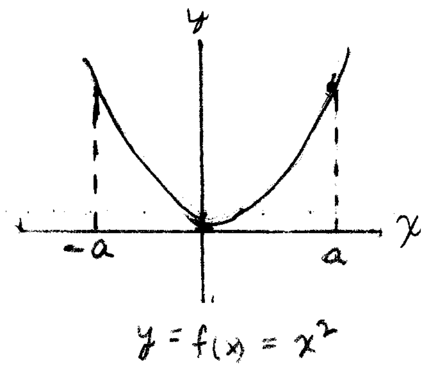


3. For $a \leq b$ and $\int_a^b f(x)\,dx = F(x)|_a^b$, where $F' = f$,
$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx = F(x)|_b^a$.

Example: Instead of
$\int_a^b \sin x\,dx = -\cos x|_a^b = -\cos b - (-\cos a) = \cos a - \cos b$
one can write
$\int_a^b \sin x\,dx = -\cos x|_a^b = \cos x|_b^a = \cos a - \cos b$.

4. A function f is called *even* if $f(-x) = f(x)$ for all x in it's domain; $f(x) = x^2$ is even. A function is even if and only if it has symmetry about the y-axis.

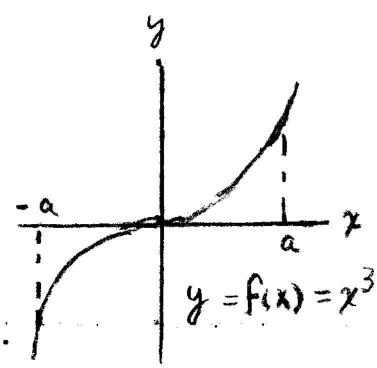
If f is even, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.



A function f is called *odd* if $f(-x) = -f(x)$ for all x in it's domain; $f(x) = x^3$ is odd. A function is odd if and only if it has symmetry about the origin.

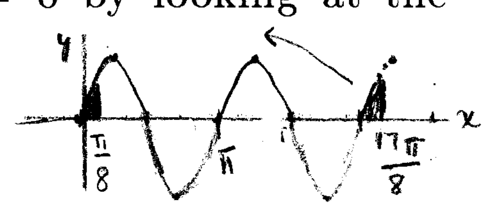
If f is odd, $\int_{-a}^{a} f(x)\,dx = 0$.

A product of two even or two odd functions is even; a product of an even and an odd function is odd. A polynomial is even (odd) if and only if only even (odd) power terms have nonzero coefficients.



5. $\int_a^b \sin(cx)\,dx = 0$ and $\int_a^b \cos(cx)\,dx = 0$ if the length $b - a$ of the interval of integration is an integral multiple of $2\pi/c$. We explain why. First, the period $\sin(cx)$ and $\cos(cx)$ is found by setting $cx = 2\pi$, so the period is $2\pi/c$. Now we illustrate with an example.

Example: Show $\int_{\pi/8}^{17\pi/8} \sin 2x\,dx = 0$ by looking at the graph. The period of $\sin 2x$ is $2\pi/2 = \pi$. By looking at the graph, we see we could move the portion of the graph from 2π to $17\pi/8$ back to the origin to obain two loops of the graph above the x-axis and two loops below the x-axis, and this cut and paste from the end to the beginning works because the entire graph is a whole number (namely 2) periods of the curve.



Change of Base

For integration (and differentiation) of exponential and logarithmic functions, the following change of base formulas can be useful:

$$\log_b c = \frac{\log_a c}{\log_a b} \qquad b^c = a^{c \log_a b}$$

Use of the special case $a = e$ occurs frequently:

$$\log_b c = \frac{\ln c}{\ln b} \qquad b^c = e^{c \ln b}$$

Exponential Growth and Decay

Quantities y whose size can be described by the equation $y(t) = Ae^{kt}$, where A and k are constants are said grow (if $k > 0$) or decay (if $k < 0$) *exponentially*. Letting $t = 0$ in this equation shows $y(0) = A$. Often y_0 is used to denote $y(0)$.

Using the change of base formula $e^{kt} = b^{kt \log_b e}$, we see that the equation above can be described using any base. Frequently, using base e is convenient, but there are times when another base is convenient.

For two values, t_1 and t_2, of t, if we know the values $y_i = y(t_i)$, and we let $\Delta t = t_2 - t_1$, then

$$\frac{y_2}{y_1} = \frac{y_0 e^{kt_2}}{y_0 e^{kt_1}} = e^{k\Delta t}$$

$$\ln\left(\frac{y_2}{y_1}\right) = k\Delta t$$

$$k = \frac{1}{\Delta t} \ln\left(\frac{y_2}{y_1}\right)$$

$$y(t) = y_0 e^{[\frac{1}{\Delta t} \ln(y_2/y_1)]t} = y_0 \left(e^{\ln(y_2/y_1)}\right)^{t/\Delta t}$$

$$y(t) = y_0 \left(\frac{y_2}{y_1}\right)^{t/\Delta t},$$

and this form of expression of $y(t)$ is also sometimes convenient.

Example1: Suppose a group of rabbits initially has 60 rabbits and the number of rabbits in the group doubles every 8 days.

(a) Find a formula for the approximate number, $N(t)$, of rabbits after t days.

(b)Find the approximate number of rabbits after 44 days.

(c) Find the number (the answer is an integer) of rabbits after 48 days.

Solution: (a) $y_0 = 60$, $y_8 = 2(60)$, so

$$y(t) = y_0 \left(\frac{y_8}{y_0}\right)^{t/8} = (60)2^{t/8}.$$

$$\begin{aligned} \text{(b)} \quad y(44) =&(60)2^{44/8} = (60)2^{5+1/2} = (60)(32)\sqrt{2} \\ =&1920\sqrt{2} \approx 2715. \end{aligned}$$

(c) $y(48) = (60)2^6 = (60)(64) = 3840$.

Note: Applying the change of base formula in (a) we get $y(t) = 60e^{[(\ln 2)/8]t}$, and in part (c), we would get $y(48) = 60e^{6\ln 2}$, which does simplify to (60)(64), but perhaps this is less obvious.

Example 2: A yeast culture originally has 29 g., and 30 minutes later has 37 g. How long does it take to double.

Solution: $y(t) = 29\left(\frac{37}{29}\right)^{t/30}$. Let t_2 be the time to double from 29 g to 58 g:

$$y(t_2) = 29\left(\frac{37}{29}\right)^{t_2/30} = 58$$

$$\left(\frac{37}{29}\right)^{t_2/30} = 2$$

$$(t_2/30)\ln\left(\frac{37}{29}\right) = \ln 2$$

$$t = \frac{30\ln 2}{\ln(37/29)}.$$

Example 3: All temperatures are in degrees Fahrenheit. An object is submerged in a liquid maintained at 60°. One hour after being submerged the temperature of the object is 180°, and after three hours, the temperature is 90°. Let $T(t)$ be the temperature of the object at time t hours, and let $(\Delta T)(t) = T(t) - 60$ be the difference between the temperature of the object and the temperature of the liquid. Assume that $(\Delta T)(t)$ satisfies an exponential decay law.

(a) Find the function $T(t)$.

(b) Find, to the nearest half degree without using a calculator or other electronic device, the temperature of the object 6 hours after being submerged.

(c) Find an exact expression for the time t at which the temperature of the object is 61°.

Solution: In tabular form the information given is

t	$T(t)$	$(\Delta T)(t)$
0	?	
1	180	120
3	90	30

So

(a) $$(\Delta T)(t) = (\Delta T)_0 \left(\frac{30}{120}\right)^{t/(3-1)}$$

$$= (\Delta T)_0 \left(\frac{1}{4}\right)^{t/(3-1)} = (\Delta T)_0 \left(\frac{1}{2}\right)^t.$$

$$120 = (\Delta T)(1) = (\Delta T)_0 \left(\frac{1}{2}\right)^1$$

$$(\Delta T)_0 = 240$$

$$T(t) = 60 + (\Delta T)(t) = 60 + 240 \left(\frac{1}{2}\right)^t$$

(b) $$T(6) = 60 + 240 \left(\frac{1}{2}\right)^6 = 60 + \frac{240}{64} = 60 + \frac{15}{4} = 63.75$$

(c) $$61 = T(t) = 60 + (240) \left(\frac{1}{2}\right)^t$$

$$1 = 240 \left(\frac{1}{2}\right)^t$$

$$\left(\frac{1}{2}\right)^t = \frac{1}{240}$$

$$t \ln \frac{1}{2} = \ln \frac{1}{240}$$

$$t = \frac{\ln \frac{1}{240}}{\ln \frac{1}{2}} = \frac{\ln 240}{\ln 2}. \qquad (\approx 7.9 \text{ hours})$$

Rate of Growth of Functions

The material presented here is useful when applying the Limit Comparison for improper integrals, and a discrete analogue of this material will be essential for much of what is done with infinite series.

By the phrase "$f = o(g)$ at a" we mean $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$. Usually, we will be interested in $a = \infty$ and functions f and g for which $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$. Our intuitive interpretation of $f = o(g)$ is that the function g is getting large faster than f as $x \to \infty$.

For now lim will mean $\lim_{x\to\infty}$.

Example 1: If $0 < m < n$, then $x^m = o(x^n)$, i.e.,

$$\lim \frac{x^m}{x^n} = \lim \frac{1}{x^{n-m}} = 0.$$

For a more specific example x^2 and x^3 both get large as x becomes large; 100^2 is ten thousand, quite large, but 100^3 is a million, much larger.

The notation used here, instead of $f = o(g)$, is $f << g$. This is done to call attention to the fact that the relation is transitive: if $f << g$ and $g << h$, then $f << h$. The proof is simply that

$$\lim \frac{f(x)}{h(x)} = \lim \frac{f(x)}{g(x)} \frac{g(x)}{h(x)} = 0.$$

If f and g are non-negative functions, then $\lim(f/g) = 0$ if and only if $\lim(g/f) = \infty$.

Example 2: By L'Hôpital's rule, $\lim \frac{x}{e^x} = \lim \frac{1}{e^x} = 0$. Now $\lim \frac{x^2}{e^x} = \lim \frac{2x}{e^x} = 0$. In the same fashion if $p(x)$ is any polynomial, then $p(x) << e^x$. If $0 < a \leq 1$ and $x \geq 1$, then $x^a \leq x$, so $x^a << e^x$. For $a > 1$, continued iteration of

$$\lim \frac{x^a}{e^x} = \lim \frac{ax^{a-1}}{e^x} = \dots$$

leads to the conclusion that $x^a << e^x$, for all $a > 0$.

Example 3: For $a > 0$,

$$\lim \frac{\ln x}{x^a} = \lim \frac{1/x}{ax^{a-1}} = \lim \frac{1}{ax^a} = 0,$$

i.e. $\ln x << x^a$.

Reasoning as in Examples 1–3, we see that if:

log is $\log_b x$, $b > 1$;
$p(x)$ is a polynomial, or a positive power of x; and
exp is an exponential a^{cx}, $a > 1$, $c > 0$,

then $\log << p(x) << \exp$.

If U, V, u_i, v_i, for all i, are non-negative functions and

$f = U + u_1 + \cdots + u_m$, where $u_i << U$ for all i;
$g = V + v_1 + \cdots + v_n$, where $v_i << V$ for all i; and $U << V$,

then, by dividing both numerator and denominator by V, we see

$$\lim \frac{f}{g} = \lim \frac{\frac{U}{V} + \frac{u_1}{V} + \dots \frac{u_m}{V}}{1 + \frac{v_1}{V} + \dots \frac{v_n}{V}} = \lim \frac{U}{V}.$$

Similarly, the same conclusion is valid of $V << U$ or $0 < \lim \frac{U}{V} < \infty$.

Example 4:

$$\lim \frac{x^4 + 3x^3 + 4}{x^6 + 6x^2 + 1} = \lim \frac{x^4}{x^6} = 0.$$

Example 5:

$$\lim \frac{e^x + x^2 + 1}{x^8 + \ln x} = \lim \frac{e^x}{x^8} = \infty.$$