Notes on Interest

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Simple interest with a rate of r per year, means that on a loan of P_0 at time 0, after one year, interest of rP_0 is paid and so the total of $P_1 = P_0 + rP_0$ is returned.

If this is re-invested then next year, the principal is $P_2 = P_1 + rP_1 = P_0 + rP_0 + rP_0 + r^2 P_0$.

However, this is not the way to think of it. Consider the following puzzle:

A store is offering a 10% discount and the state tax rate is 8%. Which way would you do better:

- Take the discount first and add the tax on the discounted amount,
  or
- Add the tax to the original price and then take the discount on the original plus tax?

We usually think of taking the price P and subtracting the discounted amount $.1 \cdot P$, or, for the tax, computing the tax $.08 \cdot P$ and then adding it to P.

Instead, think of the discount as starting with 100% of the price and subtracting 10% to yield 90%. So the new price is $.90 \cdot P$. Similarly, for the tax the new cost is $100\% + 8\% = 108\%$ which is $1.08 \cdot P$.

But $1.08 \cdot .90 \cdot P \ = \ .90 \cdot 1.08 \cdot P$. So the two choices yield the same result.

For one year of simple interest at rate $r\%$ per year, instead of writing $P + rP$, we factor out P and write it as $(1 + r) \cdot P$. Thus, each year we multiply by $(1 + r)$ and so after t years, we have

$$P_t \; = \; (1 + r)^t \cdot P_0.$$

If we compute the interest each month, then the interest is P dollars times $r\%$ per year times $1/12$ of a year. So we multiply by $(1 + \frac{r}{12})$ each month and so in a year we obtain $(1 + \frac{r}{12})^{12} \cdot P$.

In general, compounding n times per year for t years we have

$$P_t \; = \; (1 + \frac{r}{n})^{nt} \cdot P_0.$$

To take the limit as $n \to \infty$, we let $h = \frac{r}{n}$ so that $(1 + \frac{r}{n})^{nt} = [(1 + h)^{1/h}]^{rt}$

$$\ln(1+h)^{1/h} \;=\; \frac{\ln(1+h)}{h} \;=\; \frac{\ln(1+h) - \ln(1)}{h}.$$

So the limit as $h \to 0$ is the derivative of $\ln(x)$ at $x = 1$ which is 1. So the limit of $(1 + h)^{1/h}$ is $e^1 = e$.

$$Lim_{n\to\infty}\,(1+\frac{r}{n})^{nt}\cdot P_0 \;=\; e^{rt}\cdot P_0.$$

Alternatively, we can think of compounding continuously as adding, in the tiny time interval from t to $t + dt$, the interest $P_t \cdot r \cdot dt$. So the change in P, that is dP is $rPdt$. So we have the differential equation

$$dP \;=\; rP\;dt, \quad \text{or} \quad \frac{dP}{dt} \;=\; rP.$$

This is exponential growth at rate r whose solution is given by integrating $\frac{dP}{P} = rdt$ and then exponentiating to get

$$P_t = P_0 e^{rt}$$

as before. This approach becomes especially useful if the interest rate varies with t instead of being constant.

Units

Numbers in science are rarely naked. They are usually attached to units. These come in two varieties.

Counting units measure amounts like gallons, pounds, centimeters, and seconds. These are usually additive.

Put a heap of 5 pounds together with a heap of 3 pounds. The combined heap weighs 8 pounds. A trip of 5 miles following a trip of 3 miles accounts for 8 miles on the odometer.

Ratio units compare amounts. You recognize these by the word *per*, which is literally the Latin word for "through" and in mathematical and scientific use means *divide*. Thus, the units for speed, miles per hour, means miles divided by hours and so is written mi/hr. The unit of price, dollars per pound, written $/lb, means dollars divided by pounds. Finally, percent, written %, means divide by one hundred.

When multiplied, units can be cancelled as in algebraic expressions. Thus, when you buy 2 pounds of hamburger at $4 dollars per pound, you multiply 2 pounds $\times \frac{4\$}{\text{pound}}$ to get a cost of 8 $. Notice that what is added up at the checkout is the total cost. The prices don't add.

Suppose you want to convert 60 miles per hour to the units feet per second. You cancel miles to get feet and cancel hours to get seconds. You do this by multiplying by conversion factors, version of the number 1, but with different units.

$$\frac{60\text{miles}}{\text{hour}} \cdot \frac{\text{feet}}{\text{miles}} \cdot \frac{\text{hours}}{\text{seconds}} = \frac{??\ \text{feet}}{\text{second}}.$$

$$\frac{60\text{miles}}{\text{hour}} \cdot \frac{5280\text{feet}}{1\text{miles}} \cdot \frac{1\text{hours}}{3600\text{seconds}} = \frac{88\ \text{feet}}{\text{second}}.$$

Suppose the prices of goods $g_1, g_2, \ldots$ are $p_1\$/g_1, p_2\$/g_2$, etc. where $p_1\$/g_1$ means p_1 dollars exchanges for 1 unit (pounds, gallons, etc) of good g_1. When you purchase quantities $x_1, x_2, \ldots$ of the goods, the total cost is

$$p_1 \cdot x_1 \; + \; p_2 \cdot x_2 \; + \ldots$$

The interest rate describes the exchange rate between dollars delivered at different dates. So $1\$_0$ is worth $(1+r)^t \; \$_t$ where $\$_0$ is current dollars and $\$_t$ is dollars delivered at time t. With compounding n times per year the price is

$$(1 + \frac{r}{n})^{-nt} \; \$_0/\$_t.$$

and with continuous compounding the price is

$$e^{-rt} \; \$_0/\$_t.$$

The present value of a bundle of dollars delivered at different dates is just a special case for computing the cost.

Suppose that p_t is the current price of $\$_t$. That is, the price is $p_t \$_0 / \$_t$. In the above examples, $p_t = (1 + \frac{r}{n})^{-nt}$ or $= e^{-rt}$. A bundle of assets and liabilities $x_{t_1}, x_{t_2}, \dots$ describes quantities of dollars delivered at different times. Each can be positive or negative with a positive value an asset and a negative value a liability. The present value is just

$$p_{t_1} \cdot x_{t_1} \; + \; p_{t_2} \cdot x_{t_2} \; + \dots$$

EXAMPLE: Suppose you now take out a loan of P_0 dollars at interest rate r and you are paying it off over T years with a payment rate of x dollars per year. However, you are making equally spaced payments n times a year so that each payment is size x/n. Notice that you are paying at different dates. So your first payment consists of $x/n\$_{1/n}$ and the second consists of $x/n\$_{2/n}$, etc. The present value is the stream of payments is equal to the value of the loan. That is,

$$P_0 \ = \ \sum_{k=1}^{nT} (1+\frac{r}{n})^{-k}(x/n).$$

The sum of the geometric series $1+a+a^2+...+a^N = \sum_{k=0}^{N} \ a^k$ is equal to $\frac{1-a^{N+1}}{1-a}$. So $a+a^2+...+a^N = \sum_{k=1}^{N} \ a^k$ is equal to $a \cdot \frac{1-a^N}{1-a}$ (by factoring out the a).

For us $a = (1 + \frac{r}{n})^{-1}$ and $N = nT$.

$$(*)\ P_0 \ = \ (x/n)(1+\frac{r}{n})^{-1}\frac{1-(1+\frac{r}{n})^{-nT}}{1-(1+\frac{r}{n})^{-1}} \ = \ (x/r)[1-(1+\frac{r}{n})^{-nT}].$$

Consider the Vacuum Cleaner Problem.

In this case P_0 is the price of the vacuum cleaner which is \$100. There are $n = 12$ payments over the course of $T = 1$ year. So $x = 108$ with $x/n = 9$.

$$100 = 9 \cdot [1 - (1 + \frac{r}{12})^{-12}]/(r/12)$$

One can solve this graphically for r or check with a calculator that $r = .14$ yields an approximate equality.

There is an indirect but useful way of obtaining the equation (*). When we move from one time unit to the next, the amount still owed changes by the following equation:

$$P_{t+1} = (1 + \frac{r}{n})P_t \ - \ \frac{x}{n}$$

That is, there is an increase due to interest and a decrease due to the payment. We can rewrite this equation as

$$P_{t+1} = P_t - (\frac{x}{n} \ - \ \frac{r}{n}P_t)$$

This says that at time t the payment $\frac{x}{n}$ consists of a piece $\frac{r}{n}P_t$ which is the current interest on the loan and the rest, $\frac{x}{n} \ - \ \frac{r}{n}P_t$ is the amount by which the principal is reduced. In the beginning, a large part of the fixed payment consists of the interest, but as time goes on, P_t gets smaller and so the interest part decreases.

The above equation $P_{t+1} = (1+\frac{r}{n})P_t \ - \ \frac{x}{n}$ is equivalent to

$$(P_{t+1} - \frac{x}{r}) \ = \ (1+\frac{r}{n}) \cdot (P_t - \frac{x}{r})$$

or $X_{t+1} = aX_t$ with $X_t = P_t - \frac{x}{r}$ and $a = (1+\frac{r}{n})$.

$$X_1 = aX_0, \ X_2 = aX_1 = a^2X_0, \ X_3 = aX_2 = a^3X_0, \ldots, X_k = a^kX_0.$$

So

$$P_{nT} - \frac{x}{r} = (1+\frac{r}{n})^{nT}(P_0 - \frac{x}{r}).$$

$$P_{nT} - \frac{x}{r} = (1 + \frac{r}{n})^{nT}(P_0 - \frac{x}{r}).$$

Because $P_{nT} = 0$ we have

$$-\frac{x}{r} = (1 + \frac{r}{n})^{nT}(P_0 - \frac{x}{r}).$$

or

$$-(1 + \frac{r}{n})^{-nT}(\frac{x}{r}) \ + \ (\frac{x}{r}) \ = \ P_0.$$

and this is equation (*).

This is clearer when we use compounding continuously. In a little fraction dt of a year, the change in the amount owed dP is given by

$$dP \;=\; rPdt \;-\; xdt, \quad \text{or} \quad \frac{dP}{dt} = rP - x.$$

Again the increase is due to interest and the decrease is due to the payment.

Notice that $\frac{x}{r}$ has to be larger than P_0, otherwise the amount owed will not decrease.

With $X_t = P_t - \frac{x}{r}$, this equation says $\frac{dX}{dt} = rX$.

This is exponential growth with solution $X_t = X_0 \cdot e^{rt}$ and so

$$P_t = \frac{x}{r} + (P_0 - \frac{x}{r})e^{rt}.$$

Since $P_T = 0$ we have

$$(**) \quad P_0 = \frac{x}{r}(1 - e^{-rT}).$$

or

$$\frac{P_0}{xT} = Z(rT), \quad \text{with} \quad Z(u) = \frac{1 - e^{-u}}{u}.$$

The function Z satisfies

- $Lim_{u \to 0}\ Z(u) = -Lim_{u \to 0} \frac{e^{-u} - e^{0}}{u} = 1.$
- $Lim_{u \to \infty}\ Z(u) = 0.$
- $Z'(u) = \frac{ue^{-u} - (1 - e^{-u})}{u^2} = -\frac{e^{-u}}{u^2} \cdot (e^u - 1 - u)$ and this is negative for $u > 0$.

Thus, P_0 is always smaller than xT and using the function Z we can solve for r.

We can also obtain equation (**) by directly computing the present value of all the payments.

At time t, in the interval from t to $t + dt$ there is a payment of $x\ dt\ \$_t$. The value in $\$_0$ dollars is therefore $x\ e^{-rt}dt$.

Summing these, in the limit as $dt \to 0$, we have the integral

$$P_0 \ = \ \int_0^T \ x \cdot e^{-rt}\ dt \ = \ x \cdot \frac{1 - e^{-rT}}{r}.$$

This is equation (**).

Inflation

Suppose that $p \ \$_0/g_0$ is the current price of a bundle of goods g.

If there is no inflation up to time 1 then the price at time 1 is $p \ \$_1/g_1$.

If the interest rate is r then $(1+r)^{-1} \ \$_0/\$_1$ is the current price of a dollar delivered in a year. It then follows that $(1+r)^{-1} \ g_0/g_1$ is the exchange rate between current goods and goods at time 1. That is, the *real rate* of interest is equal to the *nominal rate* which is the rate on money.

Now suppose that the inflation rate is i. This means that the price of goods has gone up to $(1+i)p \ \$_1/g_1$. Then we have:

$$p^{-1}\ g_0/\$_0\ \cdot (1+r)^{-1}\ \$_0/\$_1 \cdot (1+i)p\ \$_1/g_1 \ = \ \frac{1+i}{1+r} g_0/g_1.$$

The rate $\frac{1+i}{1+r} g_0/g_1$ means that $\frac{1+r}{1+i}$ units of g_1 exchanges for 1 unit of g_0.

That is the real interest rate r_i satisfies:

$$1 + r_i \ = \ \frac{1+r}{1+i} \quad \text{or} \quad r_i = \frac{r-i}{1+i}.$$

This is approximately $r - i$ and so the real rate is obtained by subtracting the inflation rate from the nominal interest rate.

Thus, if you invest \$1000 for a year at a simple interest rate or effective rate of r, then at the end of the year you have $(1+r)\cdot$ \$1000. However, it only buys $\frac{1+r}{1+i}\cdot \$1000 \div p$ units of goods instead of $(1+r)\cdot$ \$1000 $\div\, p$.

I haven't mentioned the *effective rate* before. It is the rate which yields the same amount at simple interest as the current investment.

Thus, if the investment is compounded n times per year, then 1 $\$_0$ yields $(1+\frac{r}{n})^n$ $\$_1$ at the end of the year. So the effective rate r_e is given by

$$1 + r_e \; = \; (1 + \frac{r}{n})^n.$$

If the investment is compounded continuously, then r_e is given by

$$1 + r_e = e^r, \quad \text{or} \quad r_e \ = \ r + \frac{r^2}{2} + \frac{r^3}{3!} + \dots .$$