

### Properties of Rational Expressions

If  $P$ ,  $Q$ , and  $K$  are polynomials with  $Q \neq 0$  and  $K \neq 0$ , then

$$\frac{P}{Q} = \frac{PK}{QK} \quad \text{and} \quad \frac{P}{Q} = \frac{\frac{P}{K}}{\frac{Q}{K}}$$

**Definition:** A **rational function** is any function that can be written in the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where  $P(x)$  and  $Q(x)$  are polynomials and  $Q(x) \neq 0$ .

### Domains:

A polynomial function is function in the form

$$\begin{aligned} P(x) &= a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_3 x^3 + a_2 x^2 + a_1 x^1 + a_0 x^0 \\ &= a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \end{aligned}$$

where  $n$  is a positive integer. The domain of this function is all real numbers,  $(-\infty, \infty)$ .

A standard rational function of the form  $f(x) = \frac{P(x)}{Q(x)}$  where  $P(x)$  and  $Q(x)$  are polynomials and  $Q(x) \neq 0$ ; the domain of this function is all real numbers minus the solution of when we solve  $Q(x) = 0$ .

**Note:** to read function correctly, DO NOT read as an equation, read as rules/operations that must be performed.

For example: Given  $f(x) = x^2 - 5x + 1$

This function should be read as: input to the 2<sup>nd</sup> power minus 5 times the input plus 1.

It should be done this way because if the question is to find  $f(b-3)$ , then the input is  $(b-3)$ . Now applying  $(b-3)$  as input to the rules found above, we get

$$\begin{aligned} f(b-3) &= (b-3)^2 - 5(b-3) + 1 \\ &= (b^2 - 6b + 9) - 5b + 15 + 1 \\ &= b^2 - 11b + 25 \end{aligned}$$

### Additional examples

$$\begin{aligned} 2) \quad g(x) &= \frac{x-2}{x-1} \\ g(0) &= \frac{(0)-2}{(0)-1} = \frac{-2}{-1} = 2 & g(-1) &= \frac{(-1)-2}{(-1)-1} = \frac{-3}{-2} = \frac{3}{2} \\ g(-2) &= \frac{(-2)-2}{(-2)-1} = \frac{-4}{-3} = \frac{4}{3} & g(1) &= \frac{(1)-2}{(1)-1} = \frac{-1}{0} = \text{undefined} \\ g(2) &= \frac{(2)-2}{(2)-1} = \frac{0}{1} = 0 \end{aligned}$$

4) 
$$h(t) = \frac{t-2}{t+1}$$

$$h(0) = \frac{(0)-2}{(0)+1} = \frac{-2}{1} = -2 \quad h(-1) = \frac{(-1)-2}{(-1)+1} = \frac{-3}{0} \text{ undefined}$$

$$h(-2) = \frac{(-2)-2}{(-2)+1} = \frac{-4}{-1} = 4 \quad h(1) = \frac{(1)-2}{(1)+1} = \frac{-1}{2}$$

$$h(2) = \frac{(2)-2}{(2)+1} = \frac{0}{3} = 0$$

6) 
$$f(x) = \frac{x+4}{x-2}$$

Since the numerator and denominator are polynomials, we just have to find all values where the denominator is equal to zero and remove from real numbers.

$$\begin{aligned} x-2 &= 0 & \text{Domain: } (-\infty, 2) \cup (2, \infty) \text{ or } \{x \mid x \neq 2\} \\ x &= 2 \end{aligned}$$

8) 
$$g(x) = \frac{x^2 - 9}{x - 3}$$

Since the numerator and denominator are polynomials, we just have to find all values where the denominator is equal to zero and remove from real numbers.

$$\begin{aligned} x-3 &= 0 & \text{Domain: } (-\infty, 3) \cup (3, \infty) \text{ or } \{x \mid x \neq 3\} \\ x &= 3 \end{aligned}$$

10) 
$$h(t) = \frac{t-5}{t^2 - 25}$$

Since the numerator and denominator are polynomials, we just have to find all values where the denominator is equal to zero and remove from real numbers.

$$\begin{aligned} t^2 - 25 &= 0 \\ (t+5)(t-5) &= 0 & \text{Domain: } (-\infty, -5) \cup (-5, 5) \cup (5, \infty) \text{ or } \{x \mid x \neq -5, x \neq 5\} \\ t+5 &= 0 & t-5 = 0 \\ t &= -5 & t = 5 \end{aligned}$$

12) 
$$\frac{12x-9y}{3x^2 + 3xy}$$

$$\frac{12x-9y}{3x^2 + 3xy} = \frac{3(4x-3y)}{3x(x+y)} = \frac{4x-3y}{x(x+y)}$$

14) 
$$\frac{a^2 - 4a - 12}{a^2 + 8a + 12}$$

$$\frac{a^2 - 4a - 12}{a^2 + 8a + 12} = \frac{(a+2)(a-6)}{(a+2)(a+6)} = \frac{(a-6)}{(a+6)} = \frac{a-6}{a+6}$$

$$16) \quad \frac{20x^2 - 93x + 34}{4x^2 - 9x - 34}$$

$$\frac{20x^2 - 93x + 34}{4x^2 - 9x - 34} = \frac{(5x-2)(4x-17)}{(x+2)(4x-17)} = \frac{(5x-2)}{(x+2)} = \frac{5x-2}{x+2}$$

$$18) \quad \frac{250a + 100ax + 10ax^2}{50a - 2ax^2}$$

$$\frac{250a + 100ax + 10ax^2}{50a - 2ax^2} = \frac{10a(25 + 10x + x^2)}{2a(25 - x^2)} = \frac{10a(5+a)(5+a)}{2a(5+a)(5-a)} = \frac{5(5+a)}{(5-a)} = \frac{5(5+a)}{5-a}$$

$$20) \quad \frac{(x-4)^2(x+3)}{(x+3)^2(x-4)}$$

$$\frac{(x-4)^2(x+3)}{(x+3)^2(x-4)} = \frac{(x-4)(x-4)(x+3)}{(x+3)(x+3)(x-4)} = \frac{(x-4)}{(x+3)} = \frac{x-4}{x+3}$$

$$22) \quad \frac{x^3 - 1}{x^2 - 1}$$

$$\frac{x^3 - 1}{x^2 - 1} = \frac{(x-1)(x^2 + x + 1)}{(x+1)(x-1)} = \frac{(x^2 + x + 1)}{(x-1)} = \frac{x^2 + x + 1}{x-1}$$

$$24) \quad \frac{ad - ad^2}{d - 1}$$

$$\frac{ad - ad^2}{d - 1} = \frac{ad(1-d)}{d - 1} = \frac{ad\{(-d+1)\}}{(d-1)} = \frac{ad\{-(d-1)\}}{(d-1)} = \frac{-ad(d-1)}{(d-1)} = \frac{-ad}{1} = -ad$$

$$26) \quad \frac{6cd - 4c - 9d + 6}{6d^2 - 13d + 6}$$

$$\frac{6cd - 4c - 9d + 6}{6d^2 - 13d + 6} = \frac{2c(3d-2) - 3(3d-2)}{(3d-2)(2d-3)} = \frac{(3d-2)(2c-3)}{(3d-2)(2d-3)} = \frac{(2c-3)}{(2d-3)} = \frac{2c-3}{2d-3}$$

$$28) \quad \frac{36x^2 - 11x - 12}{20x^2 - 39x + 18}$$

$$\frac{36x^2 - 11x - 12}{20x^2 - 39x + 18} = \frac{(9x+4)(4x-3)}{(5x-6)(4x-3)} = \frac{(9x+4)}{(5x-6)} = \frac{9x+4}{5x-6}$$

$$30) \quad \frac{42x^2 + 23x - 10}{14x^2 + 45x - 14}$$

$$\frac{42x^2 + 23x - 10}{14x^2 + 45x - 14} = \frac{(6x+5)(7x-2)}{(2x+7)(7x-2)} = \frac{(6x+5)}{(2x+7)} = \frac{6x+5}{2x+7}$$

32) 
$$\frac{30x^2 - 61x + 30}{60x^2 + 22x - 60}$$

$$\frac{30x^2 - 61x + 30}{60x^2 + 22x - 60} = \frac{30x^2 - 61x + 30}{2(30x^2 + 11x - 30)} = \frac{(5x-6)(6x-5)}{2(5x+6)(6x-5)} = \frac{(5x-6)}{2(5x+6)} = \frac{5x-6}{2(5x+6)}$$

34) 
$$\frac{a^2 - b^2}{a^3 - b^3}$$

$$\frac{a^2 - b^2}{a^3 - b^3} = \frac{(a+b)(a-b)}{(a-b)(a^2 + ab + b^2)} = \frac{(a+b)}{(a^2 + ab + b^2)} = \frac{a+b}{a^2 + ab + b^2}$$

36) 
$$\frac{6x^2 + 7xy - 3y^2}{6x^2 + xy - y^2}$$

$$\frac{6x^2 + 7xy - 3y^2}{6x^2 + xy - y^2} = \frac{(2x+3y)(3x-y)}{(2x+y)(3x-y)} = \frac{(2x+3y)}{(2x+y)} = \frac{2x+3y}{2x+y}$$

38) 
$$\frac{x^2 - 3ax - 2x + 6a}{x^2 - 3ax + 2x - 6a}$$

$$\frac{x^2 - 3ax - 2x + 6a}{x^2 - 3ax + 2x - 6a} = \frac{x(x-3a) - 2(x-3a)}{x(x-3a) + 2(x-3a)} = \frac{(x-3a)(x-2)}{(x-3a)(x+2)} = \frac{(x-2)}{(x+2)} = \frac{x-2}{x+2}$$

40) 
$$\frac{x^3 + 5x^2 - 4x - 20}{x^2 + 7x + 10}$$

$$\frac{x^3 + 5x^2 - 4x - 20}{x^2 + 7x + 10} = \frac{x^2(x+5) - 4(x+5)}{(x+5)(x+2)} = \frac{(x+5)(x^2 - 4)}{(x+5)(x+2)} = \frac{(x+5)(x+2)(x-2)}{(x+5)(x+2)} = \frac{(x-2)}{1} = x-2$$

42) 
$$\frac{2x^4 + 14x^3 + 20x^2}{2x^5 + 4x^4 - 50x^3 - 100x^2}$$

$$\frac{2x^4 + 14x^3 + 20x^2}{2x^5 + 4x^4 - 50x^3 - 100x^2} = \frac{2x^2(x^2 + 7x + 10)}{2x^2\{x^3 + 2x^2 - 25x - 50\}} = \frac{2x^2(x^2 + 7x + 10)}{2x^2\{x^2(x+2) - 25(x+2)\}} = \frac{2x^2(x^2 + 7x + 10)}{2x^2(x+2)\{x^2 - 25\}}$$

$$= \frac{2x^2(x+2)(x+5)}{2x^2(x+2)(x+5)(x-5)} = \frac{1}{(x-5)} = \frac{1}{x-5}$$

44) 
$$\frac{16x^4 - 49}{8x^3 - 12x^2 + 14x - 21}$$

$$\frac{16x^4 - 49}{8x^3 - 12x^2 + 14x - 21} = \frac{16x^4 - 49}{4x^2(2x-3) + 7(2x-3)} = \frac{(4x^2 + 7)(4x^2 - 7)}{(2x-3)(4x^2 + 7)} = \frac{(4x^2 - 7)}{(2x-3)} = \frac{4x^2 - 7}{2x-3}$$

46) 
$$\frac{6-x}{x-6}$$

$$\frac{6-x}{x-6} = \frac{\{6-x\}}{x-6} = \frac{\{-x+6\}}{x-6} = \frac{\{-(x-6)\}}{x-6} = \frac{\{-1\}}{1} = -1$$

48)  $\frac{1-y}{y^2-1}$

$$\frac{1-y}{y^2-1} = \frac{\{1-y\}}{(y+1)(y-1)} = \frac{\{-y+1\}}{(y+1)(y-1)} = \frac{\{-(y-1)\}}{(y+1)(y-1)} = \frac{\{-1\}}{(y+1)} = \frac{-1}{y+1}$$

50)  $\frac{1-a^2}{a^2-2a+1}$

$$\frac{1-a^2}{a^2-2a+1} = \frac{(1+a)(1-a)}{(a-1)(a-1)} = \frac{(1+a)\{(1-a)\}}{(a-1)(a-1)} = \frac{(1+a)\{(-a+1)\}}{(a-1)(a-1)} = \frac{(1+a)\{-(a-1)\}}{(a-1)(a-1)} = \frac{(1+a)\{-1\}}{(a-1)} = \frac{-(1+a)}{a-1}$$

52)  $\frac{5+4r-r^2}{r^2+2r+1}$

$$\frac{5+4r-r^2}{r^2+2r+1} = \frac{\{-r^2+4r+5\}}{r^2+2r+1} = \frac{-\{r^2-4r-5\}}{r^2+2r+1} = \frac{-\{(r+1)(r-5)\}}{(r+1)(r+1)} = \frac{-\{(r-5)\}}{(r+1)} = \frac{-r+5}{r+1} = \frac{5-r}{r+1}$$

54)  $\frac{d^3-c^3}{c^3-d^3}$

method 1 (harder one):

$$\frac{d^3-c^3}{c^3-d^3} = \frac{(d-c)(d^2+cd+c^2)}{(c-d)(c^2+cd+d^2)} = \frac{(-c+d)(c^2+cd+d^2)}{(c-d)(c^2+cd+d^2)} = \frac{-(c-d)(c^2+cd+d^2)}{(c-d)(c^2+cd+d^2)} = \frac{-1}{1} = -1$$

method 2:  $\frac{d^3-c^3}{c^3-d^3} = \frac{(-c^3+d^3)}{(c^3-d^3)} = \frac{-(c^3-d^3)}{(c^3-d^3)} = \frac{-1}{1} = -1$

56)  $\frac{12r^2-25rt+12t^2}{18t^2-32r^2}$

$$\frac{12r^2-25rt+12t^2}{18t^2-32r^2} = \frac{12t^2-25rt+12r^2}{2(9t^2-16r^2)} = \frac{(4t-3r)(3t-4r)}{2(3t+4r)(3t-4r)} = \frac{(4t-3r)}{2(3t+4r)} = \frac{4t-3r}{2(3t+4r)}$$

58)  $\frac{42x^2+47x-55}{7x-5} - \frac{18x^2-15x-88}{3x-8}$

$$\begin{aligned} \frac{42x^2+47x-55}{7x-5} - \frac{18x^2-15x-88}{3x-8} &= \frac{(6x+11)(7x-5)}{(7x-5)} - \frac{(6x+11)(3x-8)}{(3x-8)} = \frac{(6x+11)}{1} - \frac{(6x+11)}{1} \\ &= (6x+11) - (6x+11) = 6x+11 - 6x-11 = 0 \end{aligned}$$

60)  $\frac{x^4-16}{x+2} - \frac{x^4-16}{x-2}$

$$\begin{aligned} \frac{x^4-16}{x+2} - \frac{x^4-16}{x-2} &= \frac{(x^2+4)(x^2-4)}{(x+2)} - \frac{(x^2+4)(x^2-4)}{(x-2)} = \frac{(x^2+4)(x+2)(x-2)}{(x+2)} - \frac{(x^2+4)(x+2)(x-2)}{(x-2)} \\ &= \frac{(x^2+4)(x-2)}{1} - \frac{(x^2+4)(x+2)}{1} = (x^3-2x^2+4x-8) - (x^3+2x^2+4x+8) \\ &= x^3-2x^2+4x-8-x^3-2x^2-4x-8 = -4x^2-16 \end{aligned}$$