T2. ***Sylvester's inequality*** states that if A and B are $n \times n$ matrices with rank r_A and r_B, respectively, then the rank r_{AB} of AB satisfies the inequality

$$r_A + r_B - n \le r_{AB} \le \min(r_A, r_B)$$

where $\min(r_A, r_B)$ denotes the smaller of r_A and r_B or their common value if the two ranks are the same. Use your technology utility to confirm this result for some matrices of your choice.

4.9 Basic Matrix Transformations in R^2 and R^3

In this section we will continue our study of linear transformations by considering some basic types of matrix transformations in R^2 and R^3 that have simple geometric interpretations. The transformations we will study here are important in such fields as computer graphics, engineering, and physics.

There are many ways to transform the vector spaces R^2 and R^3, some of the most important of which can be accomplished by matrix transformations using the methods introduced in Section 1.8. For example, rotations about the origin, reflections about lines and planes through the origin, and projections onto lines and planes through the origin can all be accomplished using a linear operator T_A in which A is an appropriate 2×2 or 3×3 matrix.

Reflection Operators

Some of the most basic matrix operators on R^2 and R^3 are those that map each point into its symmetric image about a fixed line or a fixed plane that contains the origin; these are called ***reflection operators***. Table 1 shows the standard matrices for the reflections about the coordinate axes in R^2, and Table 2 shows the standard matrices for the reflections about the coordinate planes in R^3. In each case the standard matrix was obtained using the following procedure introduced in Section 1.8: Find the images of the standard basis vectors, convert those images to column vectors, and then use those column vectors as successive columns of the standard matrix.

Table 1

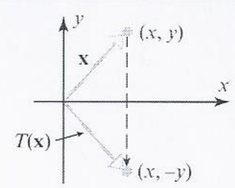
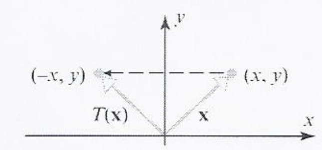
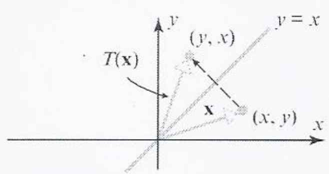
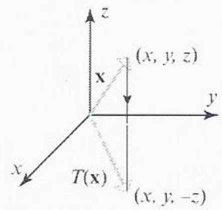
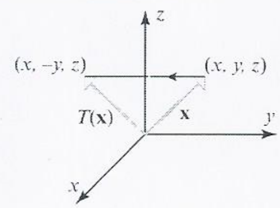
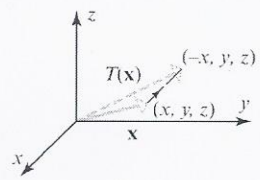
Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Reflection about the x-axis $T(x, y) = (x, -y)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, -1)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the y-axis $T(x, y) = (-x, y)$		$T(\mathbf{e}_1) = T(1, 0) = (-1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the line $y = x$ $T(x, y) = (y, x)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 1)$ $T(\mathbf{e}_2) = T(0, 1) = (1, 0)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Table 2

Operator	Illustration	Images of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$	Standard Matrix
Reflection about the xy-plane $T(x, y, z) = (x, y, -z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, -1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about the xz-plane $T(x, y, z) = (x, -y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, -1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about the yz-plane $T(x, y, z) = (-x, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (-1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Projection Operators Matrix operators on R^2 and R^3 that map each point into its orthogonal projection onto a fixed line or plane through the origin are called ***projection operators*** (or more precisely, ***orthogonal projection*** operators). Table 3 shows the standard matrices for the orthogonal projections onto the coordinate axes in R^2, and Table 4 shows the standard matrices for the orthogonal projections onto the coordinate planes in R^3.

Table 3

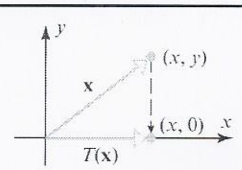
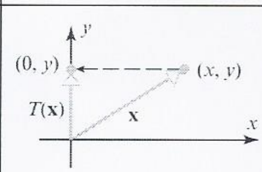
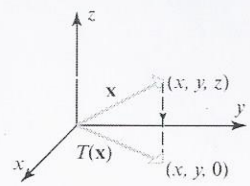
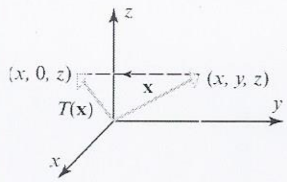
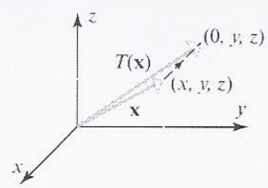
Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Orthogonal projection onto the x-axis $T(x, y) = (x, 0)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection onto the y-axis $T(x, y) = (0, y)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

*Table 4

Operator	Illustration	Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	Standard Matrix
Orthogonal projection onto the xy-plane $T(x, y, z) = (x, y, 0)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal projection onto the xz-plane $T(x, y, z) = (x, 0, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 0, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal projection onto the yz-plane $T(x, y, z) = (0, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (0, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rotation Operators Matrix operators on R^2 and R^3 that move points along arcs of circles centered at the origin are called ***rotation operators***. Let us consider how to find the standard matrix for the rotation operator $T: R^2 \to R^2$ that moves points *counterclockwise* about the origin through a positive angle θ. As illustrated in Figure 4.9.1, the images of the standard basis vectors are

$$T(\mathbf{e}_1) = T(1, 0) = (\cos\theta, \sin\theta) \quad \text{and} \quad T(\mathbf{e}_2) = T(0, 1) = (-\sin\theta, \cos\theta)$$

so it follows from Formula (14) of Section 1.8 that the standard matrix for T is

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

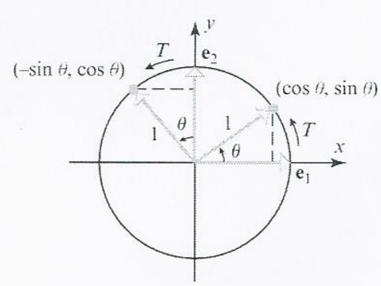


Figure 4.9.1

In keeping with common usage we will denote this operator by R_θ and call

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{1}$$