Matrix Properties (and some small theorems)

Here are some important matrix properties that we will go through in class. I will from the first test onward, I will assume that you're familiar with all these properties, and can even prove some (or all) of the parts of them. So pay attention, commit to memory and understand!

In all these properties:

- Assume that the sizes of the matrices are such that the operations described are defined.
- If it is not stated that the matrices must be square, do not assume that they must be square.
- Unless otherwise stated: upper case letters represent matrices, lower case letters represent scalars, I_n or I represents the appropriate-sized identity matrix, $\vec{0}$ represents the zero matrix (which may be the matrix with all entries 0, but not necessarily), of the appropriate size. Otherwise, notations are similar to those written in class.
- Definitions will be omitted where I expect you to know what I am talking about. For example, I will talk about inverses, but I expect you to already know the definition for what an inverse is.
- Also, do not assume properties that are not stated! So, for example, it is not stated that matrix multiplication is commutative, that's because it is not! If you assumed it was (because, say, you reason, multiplication of numbers is commutative) you'd be in trouble. Forget the old patterns; everything must be re-proven.
- Several properties can be extended for more terms than you see here, and you should be comfortable doing so, and proving so, usually by applying induction.

<u>Property Set 1: Arithmetic Properties of Matrices</u>

1. $A + B = B + A$ (Commutative law for matrix addition)
2. $A + (B + C) = (A + B) + C$ (Associative law for matrix addition)
3. $A(BC) = (AB)C$ (Associative law for matrix multiplication)
4. $A(B \pm C) = AB \pm AC$ (Left distributive law)
5. $(B \pm C)A = BA \pm CA$ (Right distributive law)
6. $a(B \pm C) = aB \pm aC$
7. $(a \pm b)C = aC \pm bC$
8. $a(bC) = (ab)C$
9. $a(BC) = (aB)C = B(aC)$

<u>Property Set 2: Properties of $\vec{0}$</u>

1. $A + \vec{0} = \vec{0} + A = A$
2. $A - \vec{0} = A$
3. $A - A = A + (-A) = \vec{0}$
4. $0A = \vec{0}$
5. If $cA = \vec{0}$, then $c = 0$ or $A = \vec{0}$

Property Set 3: Properties of Inverse Matrices

Recall, only square, non-singular matrices have inverses, and the inverses are themselves square.

Let A be an invertible matrix and n a non-negative integer.

1. $(AB)^{-1} = B^{-1}A^{-1}$
2. A^{-1} is invertible and $(A^{-1})^{-1} = A$
3. A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$
4. kA is invertible for any non-zero k, and $(kA)^{-1} = k^{-1}A^{-1}$

Small Theorem 1: The inverse of a (square) matrix is unique. That is,

If B and C are inverses of A, then $B = C$.

Property Set 4: Properties of Matrix Transposition

1. $(A^T)^T = A$
2. $(A \pm B)^T = A^T \pm B^T$
3. $(kA)^T = kA^T$
4. $(AB)^T = B^T A^T$
5. $(A^T)^{-1} = (A^{-1})^T$

Small Theorem 2: *If A is invertible, then AA^T and $A^T A$ are invertible.*

Property Set 5: Properties of Symmetric Matrices

1. A^T is symmetric, if A is symmetric
2. $A \pm B$ are symmetric, if A and B are symmetric
3. kA is symmetric, if A is symmetric
4. A^{-1} is symmetric, if A is symmetric

Small Theorem 3: *The product of symmetric matrices is symmetric if and only if the matrices commute (multiplicatively).*