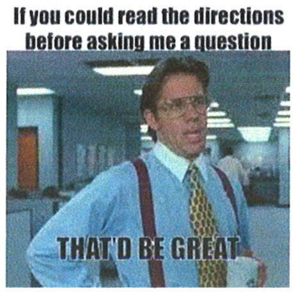
Math 392 Test 2

July 18, 2019

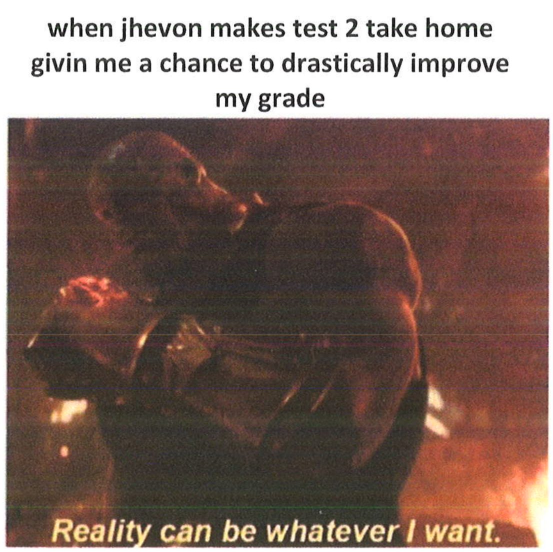
Name: JHEVON SMITH

Note that both sides of each page may have printed material.



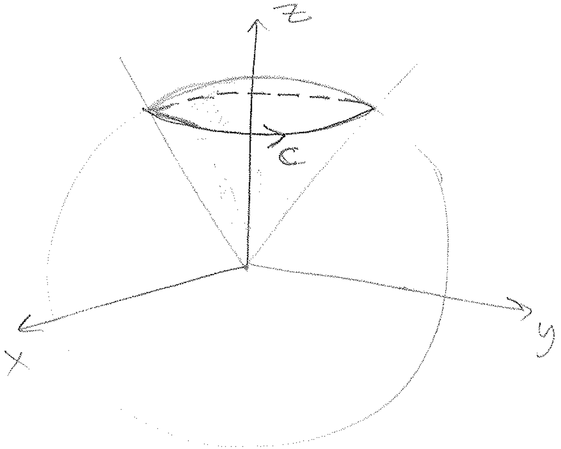
Instructions:

1. Read the instructions.
2. Do this test on your own after studying. It should take you **1 hour and 15 minutes.** Cheating will make your life harder, as if you do it, the final will destroy you and a good grade here won't matter.
3. IMPORTANT: All closed surfaces in this exam are positively oriented.
4. Print out this exam, staple it, and write your full solutions on it in the appropriate spaces.
5. Bring your completed exam to the class on the date above. No extensions or excuses allowed!
6. Complete all problems in the actual test. Bonus problems are optional and will only be counted if all other problems are attempted. All non-bonus problems are equally weighted.
7. Show **ALL** your work to receive full credit, unless otherwise stated. You will get 0 credit for simply writing down the answers. Solutions in which the answer is not indicated or several contradictory answers are present will be considered incorrect.
8. Write neatly so that I am able to follow your sequence of steps and box your answers.
9. Read through the exam and complete the problems that are easy (for you) first!
10. You are NOT allowed to use notes, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
11. Write exact answers only. NOT decimal approximations.
12. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
13. Other than that, have fun and good luck!



1. Let C be the curve of intersection of $\overset{1}{x^2+y^2+z^2=8}$ and $\overset{2}{z=\sqrt{x^2+y^2}}$. Let $\vec{F} = \langle y^3,\ 2x\ ,\ \sin^{-1} z\,\rangle$. Compute the work done by $\vec{F}$ in moving a particle once around C, counter-clockwise when viewed from above, in two ways:

(a) Directly as a line integral. Include a sketch of the surfaces and indicate C. (10 points)



For intersection, plug 2 into 1

$\Rightarrow x^2+y^2+(x^2+y^2)=8$

$\Rightarrow x^2+y^2=4$

When $x^2+y^2=4 \Rightarrow z=\sqrt{x^2+y^2}=2$

$\Rightarrow C:\ \vec{r}(t)=\langle 2\cos t, 2\sin t, 2\rangle,\ 0\le t\le 2\pi$

$\Rightarrow \vec{r}\,'(t)=\langle -2\sin t, 2\cos t, 0\rangle$

$\Rightarrow \vec{F}(\vec{r}(t))=\langle 8\sin^3 t, 4\cos t, \sin^{-1}(2)\rangle$

Now, Work $=\int_C \vec{F}\cdot d\vec{r}$

$=\int_0^{2\pi} -16\sin^4 t + 8\cos^2 t\,dt$

$=\int_0^{2\pi} -16\left[\tfrac{1}{2}(1-\cos 2t)\right]^2 + 4(1+\underbrace{\cancel{\cos 2t}}_{0})\,dt$

$=\int_0^{2\pi} -4(1-2\underbrace{\cancel{\cos 2t}}_{0}+\cos^2 2t)+4\,dt$

$=\int_0^{2\pi} -4-2(1+\underbrace{\cancel{\cos 4t}}_{0})+4\,dt$

$=\int_0^{2\pi} -6+4\,dt$

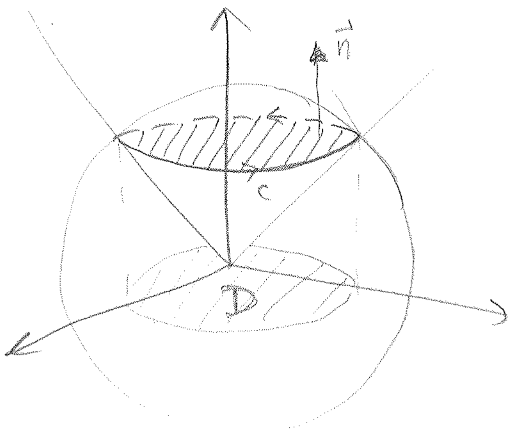
$=2\pi(-2)$

$=\boxed{-4\pi}$

(This is problem 1 recopied for your convenience. Do part (b) on this page!)

Let C be the curve of intersection of $x^2 + y^2 + z^2 = 8$ and $z = \sqrt{x^2 + y^2}$. Let $\vec{F} = \langle y^3,\ 2x\ ,\ \sin^{-1} z \rangle$. Compute the work done by $\vec{F}$ in moving a particle once around C, counter-clockwise when viewed from above, in two ways:

(b) As a double integral using Stokes' Theorem. (10 points)



We will use the surface $z=2$.

$\Rightarrow \vec{n} = \langle 0,0,1 \rangle$

Now $\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 & 2x & \sin^{-1} z \end{vmatrix}$

$= \langle 0, 0, 2-3y^2 \rangle$

Now, Work $= \int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F} \cdot d\vec{S}$

$= \iint_D 2 - 3y^2 \, dA$

$= \int_0^{2\pi} \int_0^2 (2 - 3r^2 \sin^2\theta) r \, dr \, d\theta$

$= \int_0^{2\pi} \int_0^2 (2 - \frac{3}{2} r^2 (1 - \cancel{\cos 2\theta}^{\,0})) r \, dr \, d\theta$

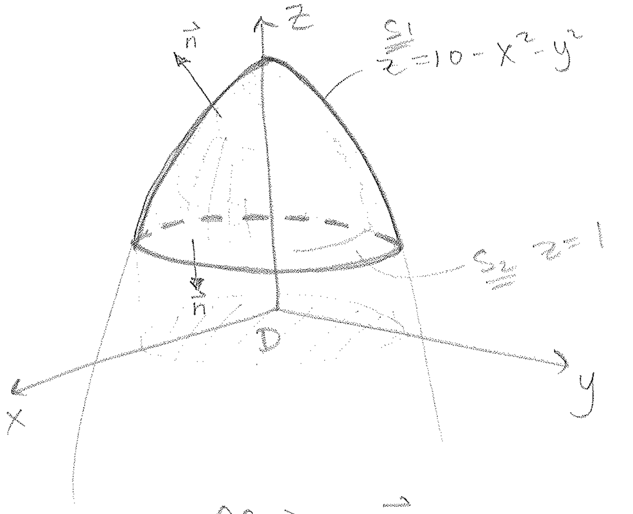
$= 2\pi \int_0^2 2r - \frac{3}{2} r^3 \, dr$

$= 2\pi \left(r^2 - \frac{3}{8} r^4\right)\Big|_0^2$

$= 2\pi (4-6) = \boxed{-4\pi}$

2. Let E be the solid region bounded by the surfaces $z = 10 - x^2 - y^2$ and $z = 1$. Let S be the boundary of E. If $\vec{F} = < x, y, z >$, compute the total flux over E in two ways:

(a) Directly as a surface integral. Include a sketch of E in your answer. (10 points)



$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{S}$$

$\underline{\underline{S_1}}$: $z = 10 - x^2 - y^2$

$$\Rightarrow \vec{n} = \langle -f_x, -f_y, 1 \rangle$$
$$= \langle 2x, 2y, 1 \rangle$$

$$\Rightarrow \iint_{S_1} = \iint_D 2x^2 + 2y^2 + (10 - x^2 - y^2)\,dA$$
$$= \iint_D x^2 + y^2 + 10\,dA$$
$$= \int_0^{2\pi}\int_0^3 (r^2 + 10)\,r\,dr\,d\theta$$
$$= 2\pi\left[\frac{r^4}{4} + 5r^2\right]\Big|_0^3$$
$$= \left(\frac{3^4}{2} + 90\right)\pi$$

$\underline{\underline{S_2}}$: $z = 1$

$$\Rightarrow \vec{n} = \langle 0, 0, -1 \rangle$$

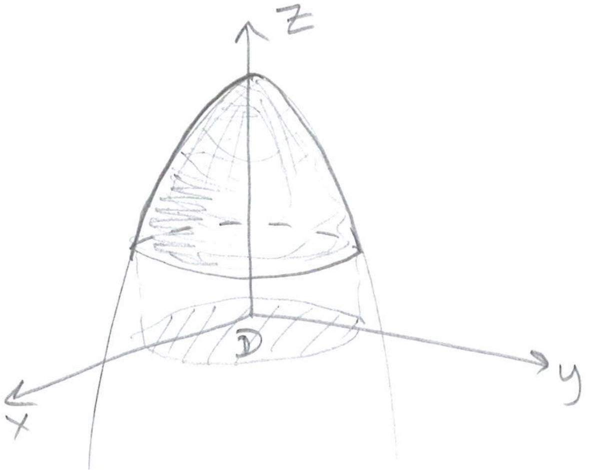
$$\Rightarrow \iint_{S_2} = \iint_D (-z)\,dA$$
$$= \iint_D (-1)\,dA$$
$$= -9\pi$$

$$\Rightarrow \text{Flux} = \iint_{S_1} + \iint_{S_2}$$
$$= \left(\frac{81}{2} + 90 - 9\right)\pi$$
$$= \left(\frac{81}{2} + 81\right)\pi$$
$$= \frac{3(81)\pi}{2} = \boxed{\frac{243\pi}{2}}$$

(This is problem 2 recopied for your convenience. Do part (b) on this page!)

Let E be the solid region bounded by the surfaces $z = 10 - x^2 - y^2$ and $z = 1$. Let S be the boundary of E. If $\vec{F} = < x, y, z >$, compute the total flux over E in two ways:

(b) As a triple integral using the Divergence Theorem. (10 points)



By the Divergence Theorem,

$$\text{Flux} = \iiint_E \operatorname{div} \vec{F}\, dV$$

$$= \iiint_E 3\, dV$$

$$= \int_0^{2\pi}\int_0^3\int_1^{10-r^2} 3r\, dz\, dr\, d\theta$$

$$= 2\pi \int_0^3 3r(9-r^2)\, dr$$

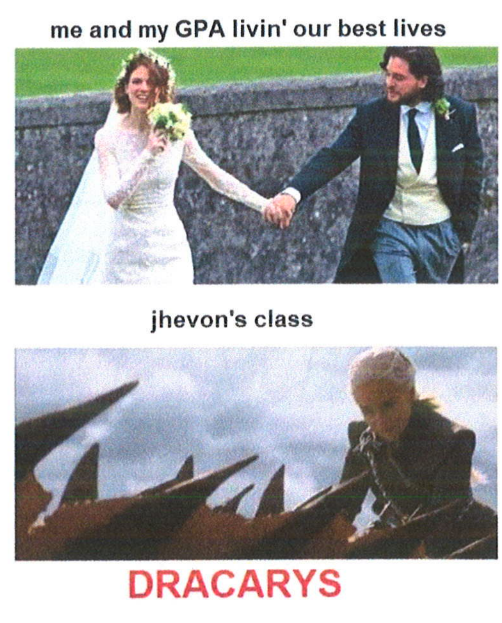
$$= 2\pi \int_0^3 27r - 3r^3\, dr$$

$$= 2\pi \left(\frac{27}{2}r^2 - \frac{3}{4}r^4\right)\Big|_0^3$$

$$= 2\pi\left(\frac{3^5}{2} - \frac{3^5}{4}\right)$$

$$= 2\pi\left(\frac{3^5}{4}\right)$$

$$= \boxed{\frac{3^5\pi}{2}} = \boxed{\frac{243\pi}{2}}$$



4. Let $A = \begin{pmatrix} 2 & -2 & -1 \\ 0 & -2 & -1 \\ 1 & -1 & -1 \end{pmatrix}$.

(a) (15 points) Find the inverse of A.

(b) (5 points) Use your answer to part (a) to solve the system

$$\begin{aligned} 2x - 2y - z &= 2 \\ -2y - z &= 2 \\ x - y - z &= 0 \end{aligned}$$

(a) Row Reduction

$$\left(\begin{array}{ccc|ccc} 2 & -2 & -1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -2 \end{array}\right) \begin{array}{l} R3 \\ R2 \times -1 \\ R1 - 2R3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 2 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 & -2 \end{array}\right) \begin{array}{l} R1 + R3 \\ R2 - R3 \\ R3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1/2 & -1/2 & 1 \\ 0 & 0 & 1 & 1 & 0 & -2 \end{array}\right) \begin{array}{l} 2R1 + R2 \\ R2/2 \\ R3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & -1/2 & -1/2 & 1 \\ 0 & 0 & 1 & 1 & 0 & -2 \end{array}\right)$$

OR Adjoint Method

$$\det A = \begin{vmatrix} 0 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & -1 \end{vmatrix} \; R_1 - 2R_2 \; = \begin{vmatrix} 0 & -2 \\ 1 & -1 \end{vmatrix} = 2.$$

$$\mathrm{Adj}(A) = \begin{pmatrix} \oplus \begin{vmatrix} -2 & -1 \\ -1 & -1 \end{vmatrix} & \ominus \begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} & \oplus \begin{vmatrix} 0 & -2 \\ 1 & -1 \end{vmatrix} \\ \ominus \begin{vmatrix} -2 & -1 \\ -1 & -1 \end{vmatrix} & \oplus \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} & \ominus \begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} \\ \oplus \begin{vmatrix} -2 & -1 \\ -2 & -1 \end{vmatrix} & \ominus \begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} & \oplus \begin{vmatrix} 2 & -2 \\ 0 & -2 \end{vmatrix} \end{pmatrix}^T$$

$$= \begin{pmatrix} 1 & -1 & 2 \\ -1 & -1 & 0 \\ 0 & 2 & -4 \end{pmatrix}^T$$

$$= \begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 2 \\ 2 & 0 & -4 \end{pmatrix}$$

Either way: $\boxed{A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 2 \\ 2 & 0 & -4 \end{pmatrix}}$

(b)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & -1/2 & 1 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}}$$

5. Let $A = \begin{pmatrix} 2 & -2 & -1 \\ 0 & -2 & -1 \\ 1 & -1 & -1 \end{pmatrix}$.

(a) (15 points) Compute: (i) $\det A$, (ii) $\det A^{-1}$, (iii) $\det(3A^2A^TA^{-3})$.

(b) (5 points) Use Cramer's Rule to solve for x only (do not solve for y or z!!!) in the following system. No credit for any other method.

$$\begin{array}{rcrcrcr} 2x & - & 2y & - & z & = & 2 \\ & & -2y & - & z & = & 2 \\ x & - & y & - & z & = & 0 \end{array}$$

(a) (i) $\det A = \boxed{2}$ (done on previous page)

(ii) $\det A^{-1} = \dfrac{1}{\det A} = \boxed{\dfrac{1}{2}}$

(iii) $\det(3A^2A^TA^{-3}) = 3^3 \cancel{(\det A)^2} \cancel{(\det A)} \cancel{(\det A)^{-3}}$

$= \boxed{27}$

(b) By above, $D = 2$.

$D_x = \begin{vmatrix} 2 & -2 & -1 \\ 2 & -2 & -1 \\ 0 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 2 & -2 & -1 \\ 0 & -1 & -1 \end{vmatrix} = 0$, by expanding along top row.

$\Rightarrow$ by Cramer's Rule: $x = \dfrac{D_x}{D}$

$= \dfrac{0}{2}$

$\Rightarrow \boxed{x = 0}$

Bonus Problems: Note that you must attempt all problems in the actual test to be eligible to attempt the bonus problems. Otherwise, anything you write on this page will be disregarded.

1. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$.

(a) (6 points) Find the eigenvalues and corresponding eigenvectors of A. Indicate which vectors correspond with which values using appropriate notation.

$$\det\begin{pmatrix} \lambda-1 & 1 \\ -2 & \lambda-4 \end{pmatrix} = 0$$
$$\Rightarrow (\lambda-1)(\lambda-4)+2 = 0$$
$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$
$$\Rightarrow (\lambda-2)(\lambda-3) = 0$$
$$\Rightarrow \boxed{\lambda_1 = 2, \lambda_2 = 3} \to \text{Eigenvalues}$$

$\underline{\lambda_1 = 2}$

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right)$$
$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$
$$x_2 = t$$
$$\Rightarrow x_1 = -t$$
$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t$$
$$\Rightarrow \boxed{\vec{\lambda}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

↳ eigenvector for $\lambda_1 = 2$.

$\underline{\lambda_2 = 3}$

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & -1 & 0 \end{array}\right)$$
$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$
$$x_2 = t$$
$$\Rightarrow x_1 = -t/2$$
$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} t = \begin{pmatrix} -1 \\ 2 \end{pmatrix} k$$
$$\Rightarrow \boxed{\vec{\lambda}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}}$$

↳ eigenvector for $\lambda_2 = 3$

(b) (4 points) Use your answer in part (a) to find the general solution of the system

$$\begin{aligned} y_1' &= y_1 - y_2 \\ y_2' &= 2y_1 + 4y_2 \end{aligned}$$

$$\boxed{\vec{y} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{3t}}$$

(c) Diagonalize A by finding an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$ (5 points).

Set $\boxed{P = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}}$

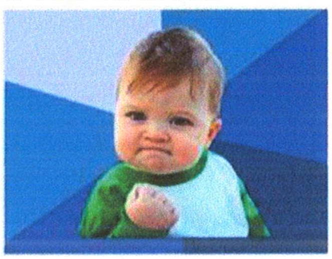
Then $P^{-1} = \frac{1}{-1}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$

(d) Compute A^5 as a 2x2 matrix (5 points).

$$A^5 = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 32 & 0 \\ 0 & 243 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -32 & -243 \\ 32 & 486 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 64-243 & 32-243 \\ -64+486 & -32+486 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -179 & -211 \\ 422 & 454 \end{pmatrix}}$$

mrw jhevon says we have
a take home test



me after taking the
take home test