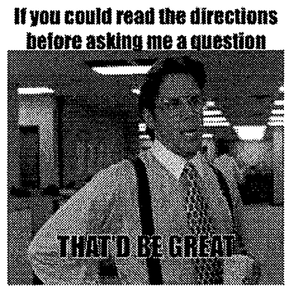
**Math 392 Test 2A**
March 29, 2018

**Name:** SOLUTIONS

**Note that both sides of each page may have printed material.**




**Instructions:**

1. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.

2. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.

3. Write neatly so that I am able to follow your sequence of steps and box your answers.

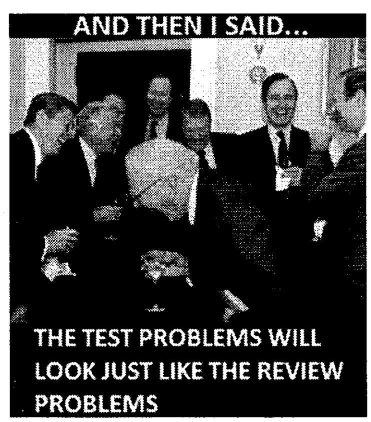
4. Read through the exam and complete the problems that are easy (for you) first!

5. No calculators, notes or other aids allowed! Including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting. In fact, **cell phones should be out of sight!**

6. Use the correct notation and write what you mean! $x^2$ and $x2$ are not the same thing, for example, and I will grade accordingly.

7. Don't commit any of the blasphemies mentioned in the syllabus!

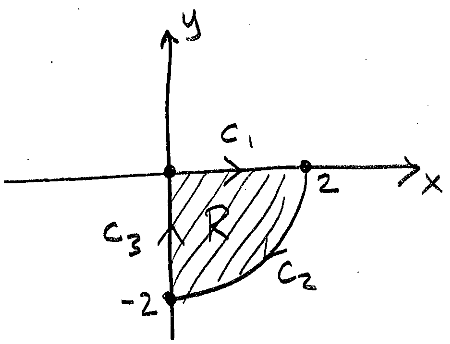
8. Other than that, have fun and good luck!



1. Let $R$ be the region in the fourth quadrant bounded by $x^2+y^2=4$, $y=0$, and $x=0$. Let $C$ be the boundary curve of $R$, oriented *clockwise*.

Compute $\displaystyle\oint_C 4\,dx + \left(\frac{x}{2}-1\right)dy$ in two ways:

(a) Directly as a line integral. (10 points)




$\underline{C_1}$: $x = 2t,\ y = 0,\ 0 \le t \le 1$

$dx = 2dt,\ dy = 0dt$

$$\Rightarrow \int_{C_1} = \int_0^1 8 + (t-1)(0)\,dt$$

$$= 8t\Big|_0^1$$

$$= 8$$

$\underline{C_2}$: $x = 2\cos t,\ y = 2\sin t,\ t: 0 \to -\frac{\pi}{2}$

$dx = -2\sin t\,dt,\ dy = 2\cos t\,dt$

$$\Rightarrow \int_{C_2} = \int_0^{-\frac{\pi}{2}} -8\sin t + (\cos t - 1)(2\cos t)\,dt$$

$$= \int_0^{-\frac{\pi}{2}} -8\sin t + 2\cos^2 t - 2\cos t\,dt$$

$$= \int_0^{-\pi/2} -8\sin t + 1 + \cos 2t - 2\cos t\,dt$$

$$= 8\cos t + t + \tfrac{1}{2}\sin 2t - 2\sin t\Big|_0^{-\pi/2}$$

$$= -\frac{\pi}{2} + 2 - (8)$$

$$= -6 - \frac{\pi}{2}$$

$\underline{C_3}$: $x = 0,\ y = -2 + 2t,\ 0 \le t \le 1$

$dx = 0dt,\ dy = 2dt$

$$\Rightarrow \int_{C_3} = \int_0^1 4(0) + (-1)(2)\,dt$$

$$= \int_0^1 (-2)\,dt$$

$$= -2$$

$$\therefore \int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

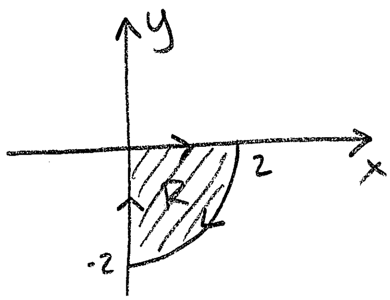
$$= 8 - 6 - \frac{\pi}{2} - 2$$

$$= \boxed{-\frac{\pi}{2}}$$

**(This is problem 1 recopied for your convenience. Do part (b) on this page!)**
Let $R$ be the region in the fourth quadrant bounded by $x^2 + y^2 = 4$, $y = 0$, and $x = 0$. Let $C$ be the boundary curve of $R$, oriented *clockwise*.

Compute $\oint_C 4\,dx + \left(\frac{x}{2} - 1\right) dy$ in two ways:

(b) As a double integral using Green's Theorem. (10 points)




negatively oriented!

$$\int \overset{P}{4}\,dx + \overset{Q}{\left(\frac{x}{2}-1\right)}dy = -\iint_R (Q_x - P_y)\,dA$$

$$= -\iint_R \frac{1}{2}\,dA$$

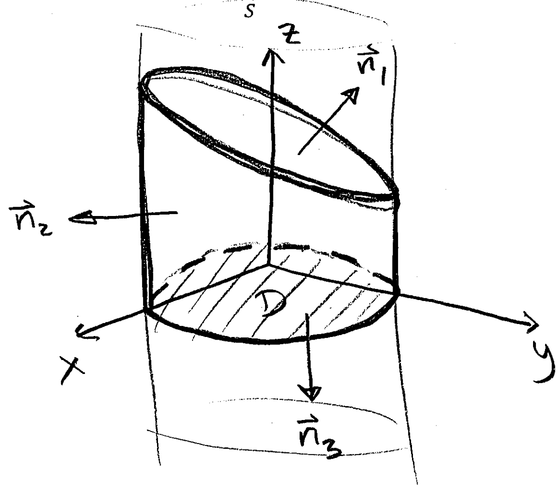
$$= -\frac{1}{2}\iint_R dA$$

$$= -\frac{1}{2}\cdot\frac{1}{4}\pi(2)^2$$

$$= \boxed{-\frac{\pi}{2}}$$

2. Let $E$ be the solid region bounded by $x^2+y^2=1$, $z=0$, and $z+2y=4$. Let $S$ be the positively oriented, three-part boundary of $E$. Let $\vec{F}=<x,y,z>$.

Compute $\iint_S \vec{F}\cdot d\vec{S}$. (20 points)




$\underline{S_1}$: $z=4-2y$

$\Rightarrow \vec{n}_1=\langle -f_x,-f_y,1\rangle=\langle 0,2,1\rangle$

$\Rightarrow \iint_{S_1}=\iint_D \langle x,y,z\rangle\cdot\langle 0,2,1\rangle\,dA$

$=\iint_D 2y+z\,dA$

$=\iint_D \cancel{2y}+(4-\cancel{2y})\,dA$

$=4\iint_D dA$

$=4\pi$

$\underline{S_2}$: $x^2+y^2=1$

$\vec{r}(u,v)=\langle \cos u,\sin u,v\rangle$

$0\le u\le 2\pi,\quad 0\le v\le 4-2\sin u$

$\Rightarrow \vec{r}_u=\langle -\sin u,\cos u,0\rangle$

$\vec{r}_v=\langle 0,\ 0,\ 1\rangle$

$\Rightarrow \vec{n}_2=\langle \cos u,\sin u,0\rangle$

$\Rightarrow \iint_{S_2}=\iint_R \langle \cos u,\sin u,v\rangle\cdot\langle \cos u,\sin u,0\rangle\,dA$

$=\iint_R dA$

$=\int_0^{2\pi}\int_0^{4-2\sin u} dv\,du$

$=\int_0^{2\pi} 4-\overset{0}{\cancel{2\sin u}}\,du$

$=8\pi$

$\underline{S_3}$: $z=0$

$\Rightarrow \vec{n}_3=\langle 0,0,-1\rangle$

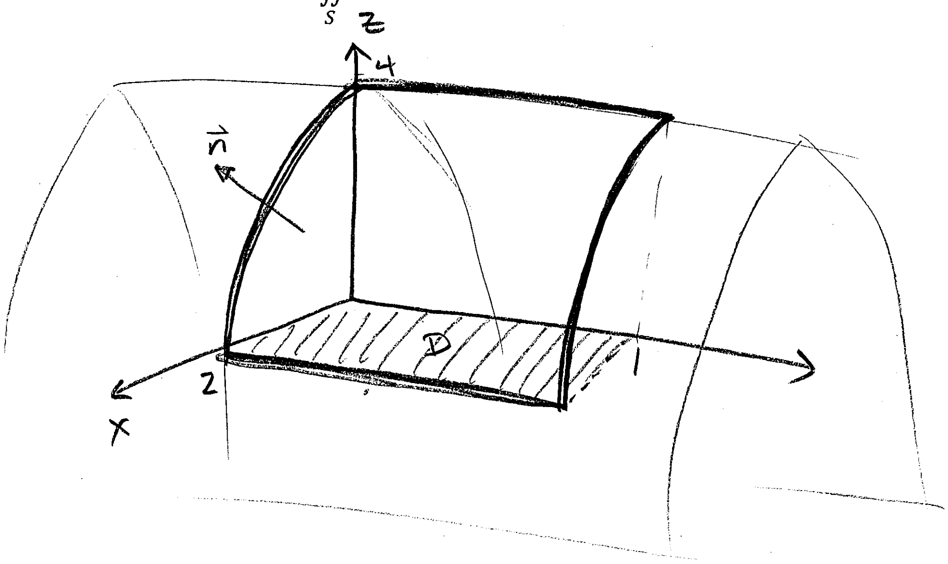
$\Rightarrow \iint_{S_3}=\iint_D \langle x,y,z\rangle\cdot\langle 0,0,-1\rangle\,dA$

$=\iint_D(-z)\,dA$

$=\iint_D 0\,dA=0$

$\Rightarrow \iint_S=\iint_{S_1}+\iint_{S_2}+\iint_{S_3}=\boxed{12\pi}$

3. Compute $\iint_S x\, dS$, where $S$ is the portion of $z = 4 - x^2$ in the first octant for $0 \le y \le 1$. (20 points)




Set $\vec{n} = \langle -f_x, -f_y, 1 \rangle$

$\Rightarrow \vec{n} = \langle 2x, 0, 1 \rangle$

$\Rightarrow |\vec{n}| = \sqrt{4x^2+1}$

$$\Rightarrow \iint_S = \iint_D x\sqrt{4x^2+1}\, dA$$

$$= \int_0^1\int_0^2 x\sqrt{4x^2+1}\, dx\, dy$$

$$= \int_0^2 x\sqrt{4x^2+1}\, dx$$

$u^2 = 4x^2+1$

$\Rightarrow 2u\,du = 8x\,dx$

$\Rightarrow \frac{1}{4}u\,du = x\,dx$

$$= \frac{1}{4}\int_1^{\sqrt{17}} u^2\, du$$

$$= \frac{u^3}{12}\Big|_1^{\sqrt{17}}$$

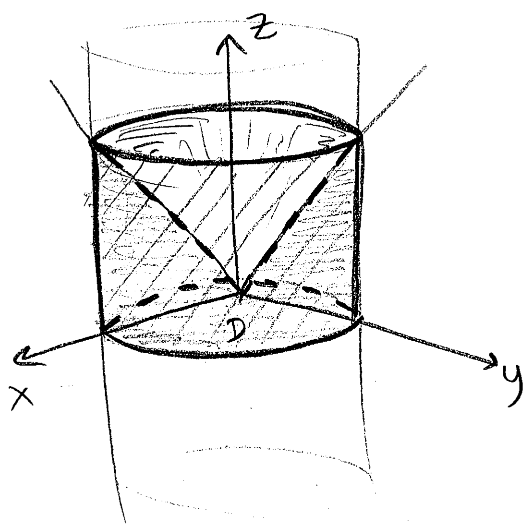
$$= \boxed{\frac{17^{3/2}}{12} - \frac{1}{12}}$$

4. Let $T_1$ be the region below the cone $z = \sqrt{x^2+y^2}$, inside the cylinder $x^2+y^2=1$, above the $xy$-plane. Let $T_2$ be the region inside the sphere $x^2+y^2+z^2=1$, but below the cone $z=\sqrt{x^2+y^2}$.

(a) Set up two integrals: $\iiint_{T_1} (x^2+y^2+z^2)\,dV$ and $\iiint_{T_2} (x^2+y^2+z^2)\,dV$.

Use coordinate systems of your choice. (10 points)

To set-up $T_1$




$0 \le r \le 1$

$0 \le \theta \le 2\pi$

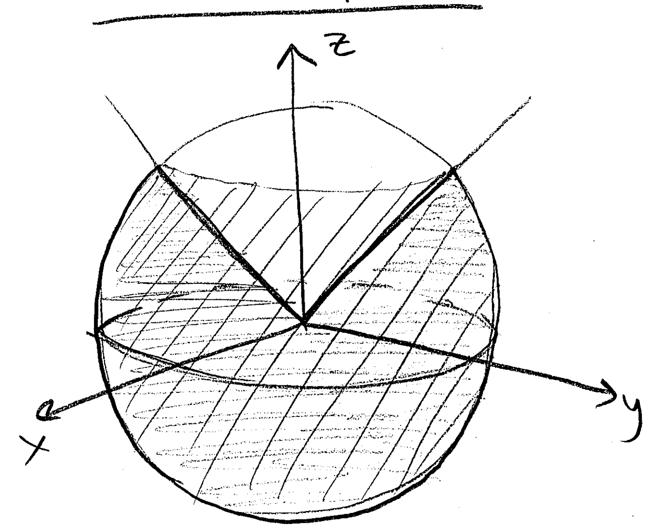
bottom surf $\le z \le$ top surf

$\Rightarrow \quad 0 \le z \le \sqrt{x^2+y^2} = r$

$$\Rightarrow \boxed{\iiint_{T_1} = \int_0^{2\pi}\int_0^1\int_0^r (r^2+z^2)\,r\,dz\,dr\,d\theta}$$

We used cylindrical coordinates

To set-up $T_2$




$\frac{\pi}{4} \le \phi \le \pi$

$0 \le \theta \le 2\pi$

$0 \le \rho \le 1$

$$\Rightarrow \boxed{\iiint_{T_2} = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\pi}\int_0^1 \rho^2 \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}$$

We used spherical coordinates.

(b) Evaluate either of the integrals set up in 4(a). (10 points)

If you chose to evaluate $\iiint_{T_1}$

$$\iiint_{T_1} = \int_0^{2\pi}\int_0^1\int_0^r (r^2+z^2)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3 z + r\frac{z^3}{3}\Big|_0^r dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^4 + \frac{r^4}{3}\, dr\, d\theta$$

$$= 2\pi \cdot \int_0^1 \frac{4}{3} r^4\, dr$$

$$= 2\pi\left(\frac{4}{15} r^5\right)\Big|_0^1$$

$$= \boxed{\frac{8\pi}{15}}$$

If you chose to evaluate $\iiint_{T_2}$

$$\iiint_{T_2} = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\pi}\int_0^1 \rho^4 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\pi} \frac{\rho^5}{5}\sin\phi\Big|_0^1 d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/4}^{\pi} \frac{1}{5}\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{5}\cos\phi\Big|_{\pi/4}^{\pi} d\theta$$

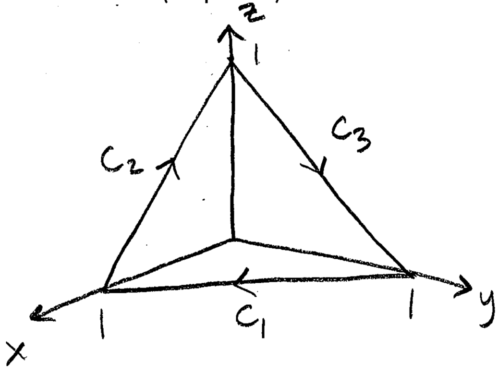
$$= 2\pi\left(\frac{1}{5} + \frac{1}{5}\frac{\sqrt{2}}{2}\right)$$

$$= 2\pi\left(\frac{2+\sqrt{2}}{10}\right)$$

$$= \boxed{\frac{(2+\sqrt{2})\pi}{5}}$$

5. Let $S$ be the part of the plane $x + y + z = 1$ in the first octant. Let $C$ be the boundary of $S$ oriented *clockwise* when viewed from above. If $\mathbf{F} = \langle x,\ y,\ xyz \rangle$, find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

(20 points)




$\underline{C_1}$: line $(0,1,0) \to (1,0,0)$

$\vec{r}(t) = \langle t, 1-t, 0 \rangle,\ 0 \le t \le 1$

$d\vec{r} = \langle 1, -1, 0 \rangle$

$\vec{F}(\vec{r}(t)) = \langle t, 1-t, 0 \rangle$

$\Rightarrow \int_{C_1} = \int_0^1 t + t - 1\, dt$

$= \int_0^1 2t - 1\, dt$

$= t^2 - t \Big|_0^1$

$= 0$

$\underline{C_2}$: line $(1,0,0) \to (0,0,1)$

$\vec{r}(t) = \langle 1-t, 0, t \rangle,\ 0 \le t \le 1$

$d\vec{r} = \langle -1, 0, 1 \rangle$

$\vec{F}(\vec{r}(t)) = \langle 1-t, 0, 0 \rangle$

$\Rightarrow \int_{C_2} = \int_0^1 t - 1\, dt$

$= \frac{t^2}{2} - t \Big|_0^1$

$= -\frac{1}{2}$

$\underline{C_3}$: line $(0,0,1) \to (0,1,0)$

$\vec{r}(t) = \langle 0, t, 1-t \rangle$

$d\vec{r} = \langle 0, 1, -1 \rangle$

$\vec{F}(\vec{r}(t)) = \langle 0, t, 0 \rangle$

$\Rightarrow \int_{C_3} = \int_0^1 t\, dt$

$= \frac{t^2}{2} \Big|_0^1$

$= \frac{1}{2}$

$\Rightarrow \int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$

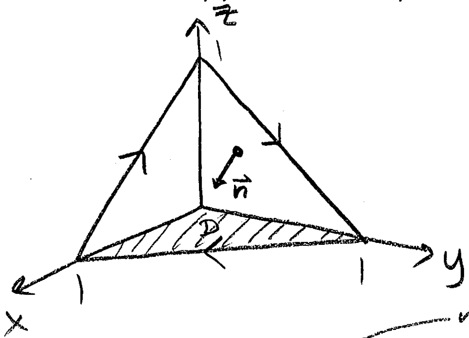
$= 0 - \frac{1}{2} + \frac{1}{2}$

$= \boxed{0}$

All that work for nothing!

**Bonus Problems: Note that you must attempt all problems in the actual test to be eligible to attempt the bonus problems. Otherwise, anything you write on this page will be disregarded.**

1. Verify your answer to problem 5 by using Stokes' Theorem. (10 points)




negatively oriented!

$$\int_C \vec{F}\cdot d\vec{r} = -\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & xyz \end{vmatrix}$$

$$= \langle xz, -yz, 0\rangle$$

$$z = 1 - x - y$$

$$\Rightarrow \vec{n} = \langle f_x, f_y, -1\rangle$$

$$= \langle -1, -1, -1\rangle$$

$$\Rightarrow \int_C = -\iint_D \langle xz, -yz, 0\rangle \cdot \langle -1,-1,-1\rangle\, dA$$

$$= -\int_0^1\int_0^{1-x} yz - xz \; dy\, dx$$

$$= -\int_0^1\int_0^{1-x} y(1-x-y) - x(1-x-y)\; dy\,dx$$

$$= -\int_0^1\int_0^{1-x} y - \cancel{xy} - y^2 - x + x^2 + \cancel{xy}\; dy\,dx$$

$$= -\int_0^1 \frac{y^2}{2} - \frac{y^3}{3} - (x-x^2)y \Big|_0^{1-x} dx$$

$$-\int_0^1 \frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} - (x-x^2)(1-x)\, dx$$

$$= -\int_0^1 \frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} - x(1-x)^2\, dx$$

$$\Rightarrow \quad u = 1-x \Rightarrow x = 1-u$$
$$du = -dx$$

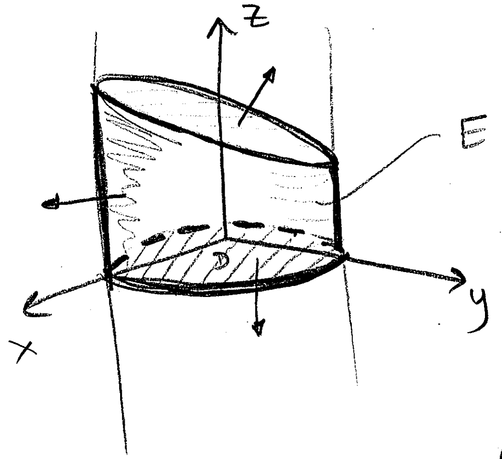
$$= \int_1^0 \frac{u^2}{2} - \frac{u^3}{3} - (1-u)u^2\, du$$

$$= \int_1^0 -\frac{u^2}{2} + \frac{2u^3}{3}\, du$$

$$= -\frac{u^3}{6} + \frac{u^4}{6}\Big|_1^0$$

$$= \boxed{0}$$

2. Verify your answer to problem 2 by using the Divergence Theorem (10 points)




$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV = \iiint_E 3\, dV$$

$$= \int_0^{2\pi}\int_0^1\int_0^{4-2r\sin\theta} 3r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 3r(4-\cancel{2r\sin\theta}^{\,0})\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 12r\, dr\, d\theta$$ 12 times area of D, or

$$= \int_0^{2\pi} 6r^2\Big|_0^1\, d\theta$$

$$= 2\pi \cdot 6$$

$$= \boxed{12\pi}$$

3. What is another name for the Divergence Theorem? (1 point)

Gauss' Theorem <u>or</u> Ostrogradsky's Theorem.

**Me after test 2: Can you curve my grade?**

**Jhevon:**