Math 392 Test 1B

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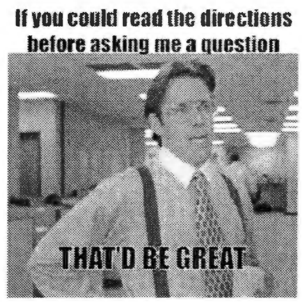
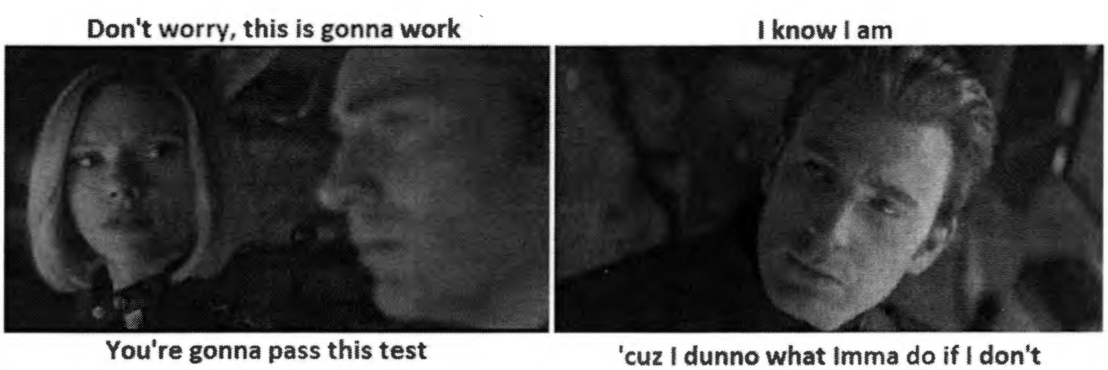
Note that both sides of each page may have printed material.



Instructions:

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, **do NOT panic**! Don't look down, while you're at it.
3. Complete all problems.
4. Show **ALL** your work to receive full credit, unless otherwise stated. You will get 0 credit for simply writing down the answers.
5. Write neatly so that I am able to follow your sequence of steps and box your answers.
6. Read through the exam and complete the problems that are easy (for you) first!
7. You are NOT allowed to use notes, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones and watches should be out of sight! If I catch you with a cellphone or a watch, you will fail the exam. If I catch you with any other paper than this exam, you will fail the exam. No calculators.**
9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
10. Other than that, have fun and good luck!



1. Let $\boldsymbol{F} = < 2x\ln y - yz\,, \frac{x^2}{y} - xz\,, -xy >$. Note: $y>0$ so $|y|=y$.

(a) Show that $\boldsymbol{F}$ is conservative by finding a potential function, f, for $\boldsymbol{F}$. (15 points)

$$f = \int 2x\ln y - yz\,dx = x^2\ln y - xyz + h(y,z)$$

$$f = \int \frac{x^2}{y} - xz\,dy = x^2\ln|y| - xyz + i(x,z) = x^2\ln y - xyz + i(x,z)$$

$$f = \int -xy\,dz = -xyz + j(x,y)$$

$$\Rightarrow \boxed{f(x,y,z) = x^2\ln y - xyz + C}$$

check that f_x, f_y, f_z give the components of $\vec{F}$,
(very important for this method!)

(b) Find the work done by $\boldsymbol{F}$ in moving an object along the path C, where C is parametrized by two curves: the vector curve $\vec{r}(t) = < \cos t\,,\ t\sin t + 1,\ e^{t^2} >$ from $(1,1,1)$ to $(-1,1,e^{\pi^2})$ (Note that $t = 0$ for the first point and $t = \pi$ for the second point), followed by the straight line segment from $(-1,1,e^{\pi^2})$ to $(1,1,0)$ (Remember how to parametrize line segments and how to set up the limits!!!). (5 points)

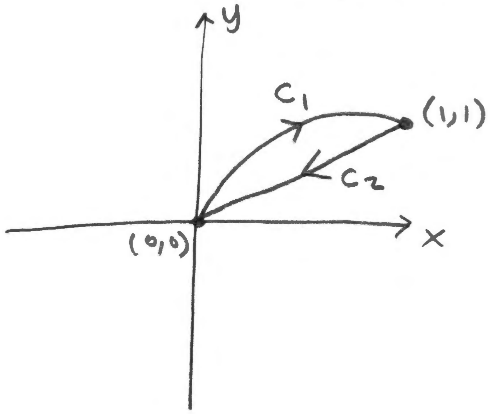
$$\text{Work} = \int_C \vec{F}\cdot d\vec{r}$$

$$= f(1,1,0) - f(1,1,1), \text{ by the FT of line Integrals}$$

$$= 0 - (-1)$$

$$= \boxed{1}$$

2. Compute $\int_C \sqrt{x+y}\, ds$, where C is the closed curve in the plane consisting of a parabolic arc along $x = y^2$ from (0,0) to (1,1), followed by a line segment from (1,1) back to (0,0). Include a sketch of C in your answer. (20 points)



C_1: $x = t^2$, $y = t$, $0 \le t \le 1$

$\Rightarrow x' = 2t$, $y' = 1$

$$\Rightarrow \int_{C_1} = \int_0^1 (t+t)\sqrt{1+4t^2}\, dt$$

$$= \int_0^1 2t\sqrt{1+4t^2}\, dt$$

$$= \frac{1}{6}(1+4t^2)^{3/2}\Big|_0^1$$

$$= \frac{1}{6}(5^{3/2}-1)$$

C_2: $x = 1-t$, $y = 1-t$, $0 \le t \le 1$

$x' = -1$, $y' = -1$

$$\Rightarrow \int_{C_2} = \int_0^1 (\sqrt{1-t} + 1 - t)\sqrt{2}\, dt$$

$$= \sqrt{2}\left[-\frac{2}{3}(1-t)^{3/2} + t - \frac{t^2}{2}\right]\Big|_0^1$$

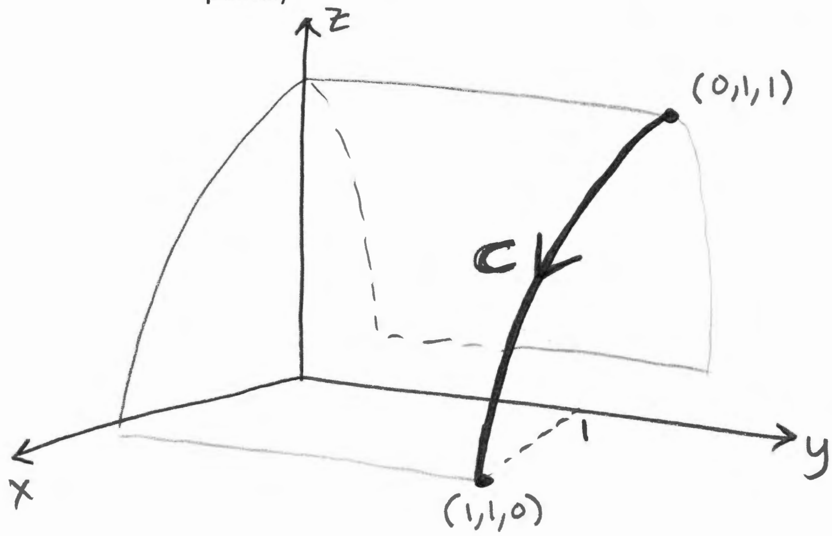
$$= \sqrt{2}\left(1 - \frac{1}{2} + \frac{2}{3}\right)$$

$$= \frac{7\sqrt{2}}{6}$$

$$\Rightarrow \int_C = \int_{C_1} + \int_{C_2}$$

$$= \boxed{\frac{1}{6}(5^{3/2} + 7\sqrt{2} - 1)}$$

3. Let S be the part of the surface $z = 1 - x^2$ in the first octant with $0 \le y \le 1$. Let C be the curve of intersection of S with the plane $y = 1$. If $\boldsymbol{F} = < x,\ y,\ x^2 >$, find the work done by $\boldsymbol{F}$ in moving a particle along the curve C; assuming C is oriented so that the final point is in the xy-plane. (20 points)



$$\underline{C}: \vec{r}(t) = \langle t, 1, 1-t^2 \rangle,\quad 0 \le t \le 1$$

$$d\vec{r} = \langle 1, 0, -2t \rangle$$

$$\vec{F}(\vec{r}(t)) = \langle t, 1, t^2 \rangle$$

$$\Rightarrow \text{Work} = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^1 \langle t, 1, t^2 \rangle \cdot \langle 1, 0, -2t \rangle \, dt$$

$$= \int_0^1 t - 2t^3 \, dt$$

$$= \frac{t^2}{2} - \frac{t^4}{2} \Big|_0^1$$

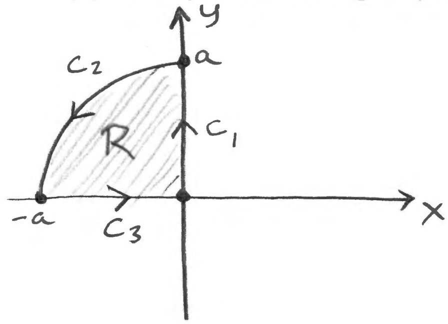
$$= \frac{1}{2} - \frac{1}{2}$$

$$= \boxed{0}$$

4. Let R be the region in the second quadrant bounded by the circle of radius a, $y = 0$, and $x = 0$. Let C be the boundary curve of R, oriented *counter-clockwise*.

Compute $\displaystyle\oint_C xy\,dx - xy\,dy$ in two ways:

(a) Directly as a line integral. (10 points)



C_1: $x = 0$, $y = t$, $0 \le t \le a$

$\Rightarrow dx = 0\,dt$, $dy = dt$

$\Rightarrow \int_{C_1} = \int_0^a 0\,dt = 0$

C_2: $x = a\cos t$, $y = a\sin t$, $\frac{\pi}{2} \le t \le \pi$

$\Rightarrow dx = -a\sin t\,dt$, $dy = a\cos t\,dt$

$$\Rightarrow \int_{C_2} = \int_{\frac{\pi}{2}}^{\pi} (a\cos t)(a\sin t)(-a\sin t) - (a\cos t)(a\sin t)(a\cos t)\,dt$$

$$= -a^3 \int_{\frac{\pi}{2}}^{\pi} \cos t \sin^2 t + \sin t \cos^2 t\,dt$$

$$= -\frac{a^3}{3}\left(\sin^3 t - \cos^3 t\right)\Big|_{\pi/2}^{\pi}$$

$$= -\frac{a^3}{3}(1 - 1)$$

$$= 0$$

C_3: $x = -a + at$, $y = 0$, $0 \le t \le 1$

$\Rightarrow dx = a\,dt$, $dy = 0\,dt$

$\Rightarrow \int_{C_3} = \int_0^1 0\,dt = 0$

$$\Rightarrow \int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

$$= \boxed{0}$$

(This is problem 4 recopied for your convenience. Do part (b) on this page!)
Let R be the region in the second quadrant bounded by the circle of radius a, $y = 0$, and $x = 0$. Let C be the boundary curve of R, oriented *counter-clockwise.*

Compute $\oint_C xy\,dx - xy\,dy$ in two ways:

(b) As a double integral using Green's Theorem. (10 points)

By Green's,

$$\int_C = \iint_R Q_x - P_y \, dA$$

where $P = xy$, and $Q = -xy$

$\Rightarrow P_y = x$, and $Q_x = -y$

$$= -\iint_R x + y \, dA$$

$$= -\int_{\frac{\pi}{2}}^{\pi}\int_0^a (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

$$= -\int_{\pi/2}^{\pi} \frac{r^3}{3}(\cos\theta + \sin\theta)\Big|_0^a \, d\theta$$

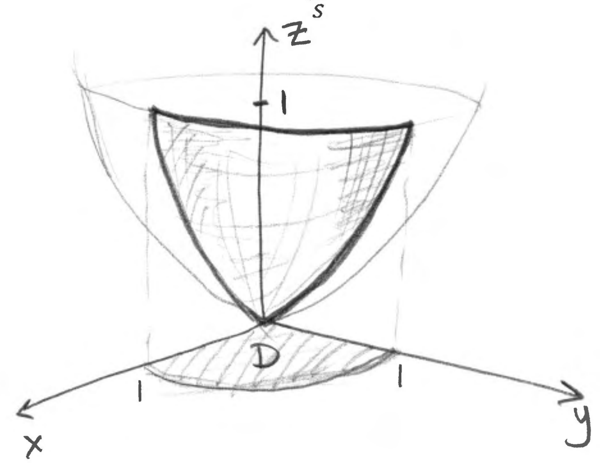
$$= -\frac{a^3}{3}\int_{\pi/2}^{\pi} \cos\theta + \sin\theta \, d\theta$$

$$= -\frac{a^3}{3}(\sin\theta - \cos\theta)\Big|_{\pi/2}^{\pi}$$

$$= -\frac{a^3}{3}\left[(0+1) - (1-0)\right]$$

$$= \boxed{0}$$

5. Compute $\iint_S y\,dS$, where S is the portion of $z = x^2 + y^2$ in the first octant for $z \leq 1$. (10 points)



Set $z = x^2+y^2 = f(x,y)$

Set $\vec{n} = \langle -f_x, -f_y, 1\rangle$

$\Rightarrow \vec{n} = \langle -2x, -2y, 1\rangle$

$\Rightarrow |\vec{n}| = \sqrt{4x^2+4y^2+1}$

$$\Rightarrow \iint_S y\,dS = \iint_D y\sqrt{4x^2+4y^2+1}\,dA$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 r\sin\theta\sqrt{4r^2+1}\;r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\sin\theta\,d\theta\int_0^1 r^2\sqrt{4r^2+1}\,dr$$

$$= -\cos\theta\Big|_0^{\frac{\pi}{2}}\int_0^1 r^2\sqrt{4r^2+1}\,dr$$

$$= \int_0^1 r^2\sqrt{4r^2+1}\,dr$$

$$= \frac{1}{8}\int_0^{\tan^{-1}2}\tan^2 t\sec^3 t\,dt \quad \text{———— (1)}$$

$$= \frac{1}{8}\int_0^{\tan^{-1}2}\sec^5 t - \sec^3 t\,dt \quad \text{———— (2)}$$

$$= \boxed{\frac{1}{64}\left(18\sqrt{5} - \ln|2+\sqrt{5}|\right)} \;*$$

* This integral was more difficult that I wanted it to be because of a typo. I meant to type $\iint_S y^2\,dS$, and that would result in an easy substitution integral (try it!). Here, in line (1), I got there using a trig sub of $2r = \tan t$. I then used the identity $\tan^2 t = \sec^2 t - 1$ to obtain line (2). If you get to line (1) or (2), I'll give you full credit. If you got to the line before line (1), I'll give you 7 points. If you completed the integral, I'll give you extra credit. See the next page for some helpful calculation tips.

Set $I = \int \tan^2 x \sec^3 x \, dx$

$$= \int (\sec^2 x - 1)\sec^3 x \, dx$$

$$= \int \sec^5 x - \sec^3 x \, dx$$

$$= \int \sec^5 x \, dx - \int \sec^3 x \, dx$$

$\int \sec^3 x \, dx$ was memorized in calc III

$$= \int \sec^2 x \sec^3 x \, dx - \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$$

Int by parts, with $u = \sec^3 x$, $dv = \sec^2 x \, dx$.

$$= \sec^3 x \tan x - 3\underbrace{\int \tan^2 x \sec^3 x \, dx}_{I} - \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$$

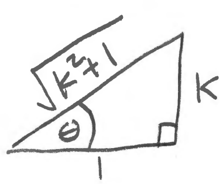
$$\Rightarrow \boxed{I = \frac{1}{4}\left(\sec^3 x \tan x - \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x|\right) + C}$$

Let k be a positive integer. To compute $\sec(\tan^{-1} k)$,

set $\tan^{-1} k = \theta$

$$\Rightarrow \tan\theta = k = \frac{k}{1} = \frac{\text{opp}}{\text{adj}}$$

$\Rightarrow$



$$\Rightarrow \sec(\tan^{-1} k) = \sec\theta$$

$$= \frac{\text{hyp}}{\text{adj}}$$

$$= \sqrt{k^2+1}$$

$$\Rightarrow \boxed{\sec(\tan^{-1} k) = \sqrt{k^2+1}}$$

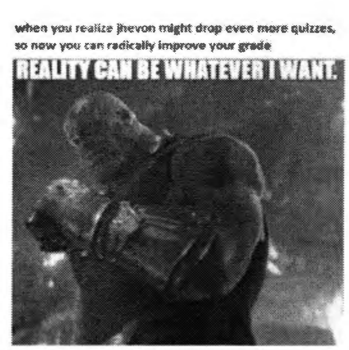
6. Let $\vec{F} = < y\cos x\,,\ z\ln y\ ,\ x^2yz >$. Compute (5 points each):

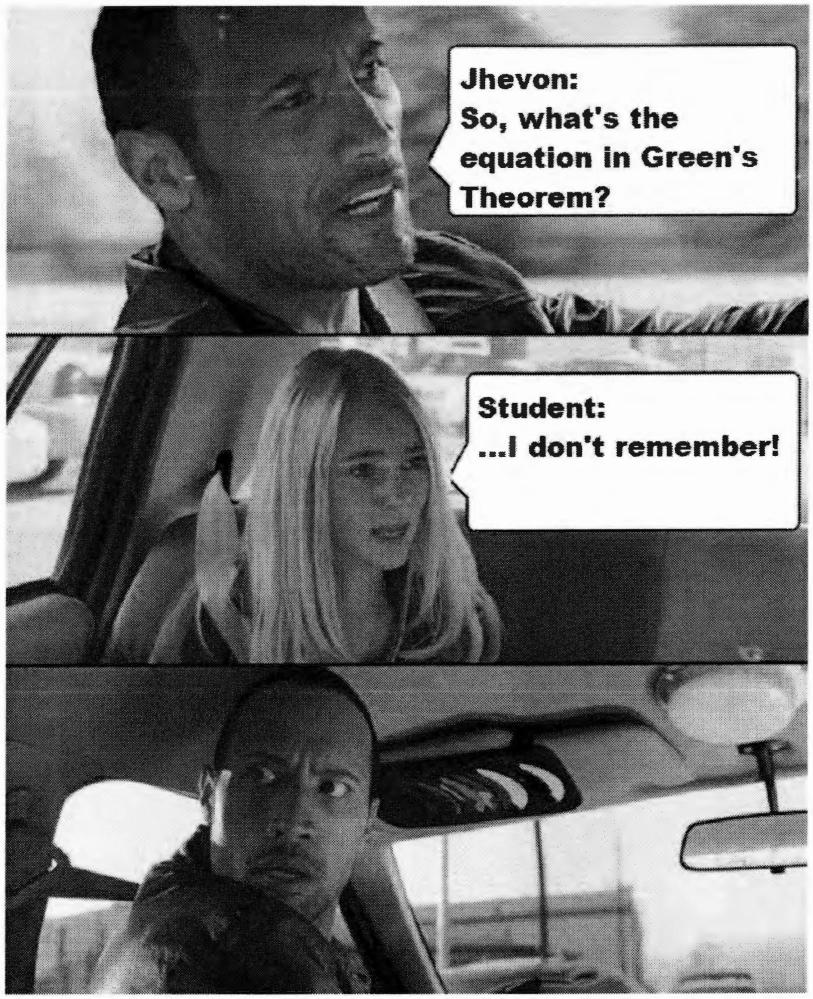
(a) $\mathrm{curl}\vec{F}$ $= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\cos x & z\ln y & x^2yz \end{vmatrix}$

$= \boxed{\langle x^2z - \ln y,\ -2xyz,\ -\cos x \rangle}$

(b) $\mathrm{div}\vec{F} = \frac{\partial}{\partial x}(y\cos x) + \frac{\partial}{\partial y}(z\ln y) + \frac{\partial}{\partial z}(x^2yz)$

$= \boxed{-y\sin x + \frac{z}{y} + x^2y}$




Jhevon:
So, what's the equation in Green's Theorem?
Student:
...I don't remember!