Math 392 Test 1B
February 27, 2018

Name: SOLUTIONS

Note that both sides of each page may have printed material.

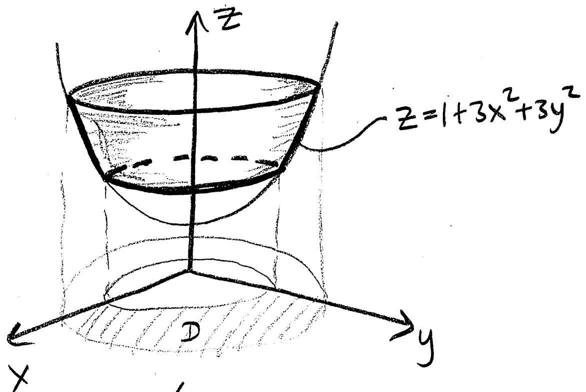
Instructions:

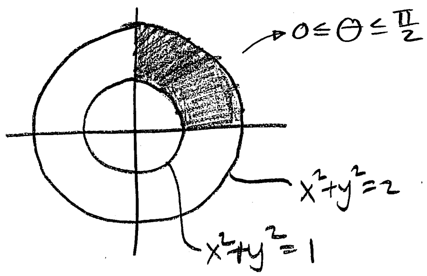
1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, **do NOT panic**! Don't look down, while you're at it.
3. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.
4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.
5. Write neatly so that I am able to follow your sequence of steps and box your answers.
6. Read through the exam and complete the problems that are easy (for you) first!
7. No calculators, notes or other aids allowed! Including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones should be out of sight!**
9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
10. Don't commit any of the blasphemies mentioned in the syllabus!
11. Other than that, have fun and good luck!

    May the force be with you. But you can't ask it to help you with your test.

1. Let S be the part of the surface $z = 1 + 3x^2 + 3y^2$ between the planes $z = 4$ and $z = 7$ in the first octant.

(a) Compute the surface area of S. (10 points)





$$z = f = 1 + 3x^2 + 3y^2$$
$$\Rightarrow f_x = 6x,\ f_y = 6y$$

$\underline{\text{when } z = 4}$

$$4 = 1 + 3x^2 + 3y^2$$
$$\Rightarrow x^2 + y^2 = 1$$
$$\downarrow$$
$$r = 1$$

$\underline{\text{when } z = 7}$

$$7 = 1 + 3x^2 + 3y^2$$
$$\Rightarrow x^2 + y^2 = 2$$
$$\downarrow$$
$$r = \sqrt{2}$$

$$\Rightarrow A = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA$$
$$= \iint_D \sqrt{1 + (6x)^2 + (6y)^2}\, dA$$
$$= \iint_D \sqrt{1 + 36x^2 + 36y^2}\, dA$$

Switch to cylindrical co-ordinates

$$= \int_0^{\frac{\pi}{2}} \int_1^{\sqrt{2}} \sqrt{1 + 36r^2}\, r\, dr\, d\theta$$

$$u = 1 + 36r^2$$
$$du = 72r\,dr \longrightarrow \begin{array}{l} \text{when } r = \sqrt{2},\ u = 73 \\ \text{when } r = 1,\ u = 37 \end{array}$$
$$\frac{1}{72} du = r\,dr$$

$$= \frac{1}{72} \int_0^{\frac{\pi}{2}} \int_{37}^{73} u^{1/2}\, du\, d\theta$$

$$= \frac{1}{72} \cdot \frac{\pi}{\cancel{2}} \cdot \frac{\cancel{2}}{3} \left(u^{3/2} \right) \Big|_{37}^{73}$$

$$= \boxed{\frac{\pi}{216} \left(73^{3/2} - 37^{3/2} \right)}$$

(b) Find the equation of the tangent plane to S at the point $(1,1,7)$. (5 points)

$z = 1+3x^2+3y^2$

$\Rightarrow 1+3x^2+3y^2-z=0$

Set $F = 1+3x^2+3y^2-z$.

$\Rightarrow \vec{n} = \langle a,b,c\rangle = \langle F_x, F_y, F_z\rangle\big|_{(1,1,7)}$

$= \langle 6x, 6y, -1\rangle\big|_{(1,1,7)}$

$= \langle 6, 6, -1\rangle$

also $(x_0, y_0, z_0) = (1,1,7)$

Tangent plane: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$\Rightarrow \boxed{6(x-1)+6(y-1)-(z-7)=0}$

(c) Unrelated to parts (a) and (b): Find the equation of the tangent line (in any form) to the parametrized curve $\vec{r}(t) = \langle \sin t,\ t^2-1,\ 5e^t\rangle$ at the point $(0,-1,5)$. (5 points)

For $(0,-1,5) \to t=0$ (Set $\sin t=0$, $t^2-1=-1$, $5e^t=5$, solve for t).

$\vec{r}(t) = \langle \sin t, t^2-1, 5e^t\rangle \Rightarrow \vec{r}(0) = \langle 0,-1,5\rangle$.

$\vec{r}\,'(t) = \langle \cos t, 2t, 5e^t\rangle \Rightarrow \vec{r}\,'(0) = \langle 1, 0, 5\rangle$.

Tangent line: $\langle x,y,z\rangle = \vec{r}(0) + t\,\vec{r}\,'(0)$

$\Rightarrow \boxed{\langle x,y,z\rangle = \langle 0,-1,5\rangle + t\langle 1,0,5\rangle}$ $\to$ Vector form

OR

$\boxed{x=t,\ y=-1,\ z=5+5t}$ $\to$ Parametric form

OR

$\boxed{x = \dfrac{z-5}{5}\ ;\ y=-1}$ $\to$ Symmetric form.

2. Let $\boldsymbol{F} = < \overbrace{e^{yz} - yz - \cos y}^{f_x},\ \overbrace{xze^{yz} - xz + x\sin y}^{f_y},\ \overbrace{xye^{yz} - xy}^{f_z} >$.

(a) Show that $\boldsymbol{F}$ is conservative by finding a potential function, f, for $\boldsymbol{F}$. (15 points)

$$f = \int e^{yz} - yz - \cos y \, dx = xe^{yz} - xyz - x\cos y + g(y,z)$$

also $f = \int xze^{yz} - xz + x\sin y \, dy = xe^{yz} - xyz - x\cos y + h(x,z)$

also $f = \int xye^{yz} - xy \, dz = xe^{yz} - xyz + k(x,y)$

Comparing we find:

$$\boxed{f = xe^{yz} - xyz - x\cos y + C}$$

Check that f_x, f_y, f_z give the components of $\vec{F}$.
(Very important for this method!)

(b) Find the work done by $\boldsymbol{F}$ in moving an object along the path C, where C is parametrized by two curves: the vector curve $\vec{r}(t) = < \cos t,\ t\sin t,\ e^{t^2} >$ from $(1,0,1)$ to $(-1,0,e^{\pi^2})$ (Note that $t = 0$ for the first point and $t = \pi$ for the second point), followed by the straight line segment from $(-1,0,e^{\pi^2})$ to $(1,1,0)$ (Remember how to parametrize line segments and how to set up the limits!!!). (5 points)

By the fundamental theorem for line integrals,

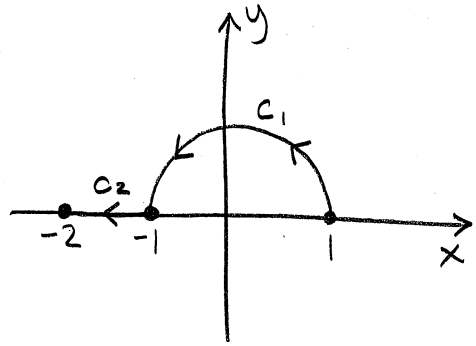
$$W = \int_C \vec{F} \cdot d\vec{r}$$

$$= \overset{\text{final}}{f(1,1,0)} - \overset{\text{initial}}{f(1,0,1)}$$

$$= (1 - \cos 1) - (1 - 1)$$

$$= \boxed{1 - \cos 1}$$

3. Compute $\int_C y^2 - 1\, ds$, where C is the upper half circle $x^2 + y^2 = 1$ from $(1,0)$ to $(-1,0)$ followed by the line segment from $(-1,0)$ to $(-2,0)$. (20 points)



$\underline{C_1}$: $x = \cos t,\ y = \sin t,\quad 0 \le t \le \pi$

$x' = -\sin t,\ y' = \cos t$

$\Rightarrow ds = \sqrt{(x')^2 + (y')^2}\, dt = 1\, dt = dt$

$$\Rightarrow \int_{C_1} = \int_0^\pi (\sin^2 t - 1)\, dt$$

$$= -\int_0^\pi \cos^2 t\, dt$$

$$= -\frac{1}{2}\int_0^\pi 1 + \cos 2t\, dt$$

$$= -\frac{1}{2}\left(t + \frac{1}{2}\sin 2t\right)\Big|_0^\pi$$

$$= -\frac{\pi}{2}$$

$\underline{C_2}$: $(-1,0) \to (-2,0)$

$x = -1 - t,\ y = 0,\quad 0 \le t \le 1$

$x' = -1,\ y' = 0$

$\Rightarrow ds = \sqrt{(x')^2 + (y')^2}\, dt = 1\, dt = dt$

$$\Rightarrow \int_{C_2} = \int_0^1 (0^2 - 1)\, dt$$

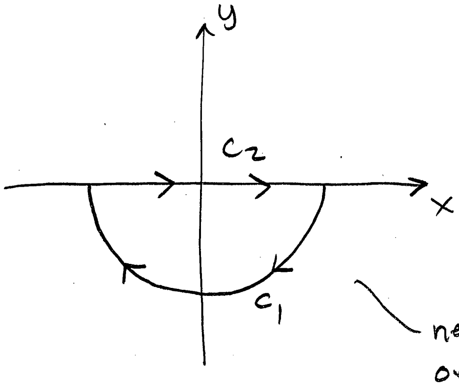
$$= -\int_0^1 1\, dt$$

$$= -1$$

$$\Rightarrow \int_C = \int_{C_1} + \int_{C_2} = \boxed{-\frac{\pi}{2} - 1}$$

4. Let R be the region bounded by the lower semi-circle $x^2 + y^2 = 4$ and the line $y = 0$. Let C be the boundary of R oriented *clockwise*.

Compute $\oint_C -2y\,dx + 3x\,dy$ directly as a line integral. Include a sketch of the region in your answer. (20 points)



negatively oriented

$\underline{C_1}$: $x = 2\cos t,\quad y = 2\sin t,\quad \pi \le t \le 2\pi$

$dx = -2\sin t\,dt,\quad dy = 2\cos t\,dt$

$$\Rightarrow \int_{C_1} = -\int_{\pi}^{2\pi} (-4\sin t)(-2\sin t) + (6\cos t)(2\cos t)\,dt$$

$$= -\int_{\pi}^{2\pi} 8\sin^2 t + 12\cos^2 t\,dt$$

$$= -\int_{\pi}^{2\pi} 4(1-\cos 2t) + 6(1+\cos 2t)\,dt$$

$$= -\int_{\pi}^{2\pi} 10 + 2\cos 2t\,dt$$

$$= -(10t + \sin 2t)\Big|_{\pi}^{2\pi}$$

$$= -10\pi$$

$\underline{C_2}$: $(-2,0) \to (2,0)$

$x = -2+4t,\quad y = 0,\quad 0 \le t \le 1$

$dx = 4\,dt,\quad dy = 0\,dt$

$$\Rightarrow \int_{C_2} = \int_0^1 -2(0)(4) + 3(-2+4t)(0)\,dt$$

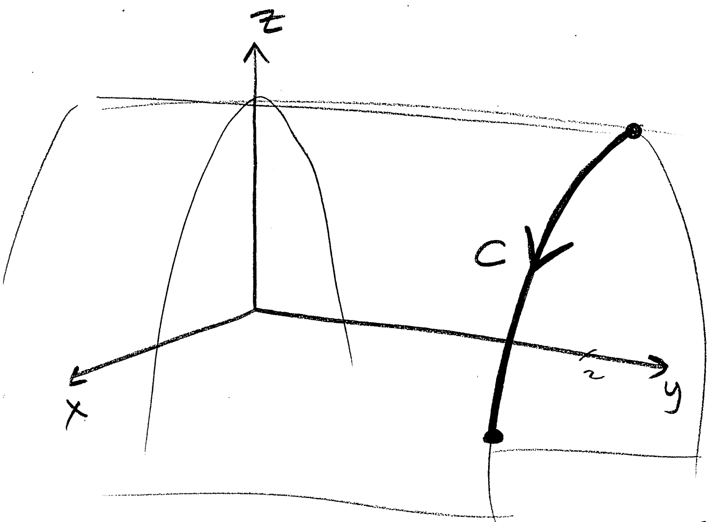
$$= \int_0^1 0\,dt$$

$$= 0$$

$$\Rightarrow \int_C = \int_{C_1} + \int_{C_2} = \boxed{-10\pi}$$

5. Let S be the part of the surface $z = 1 - x^2$ in the first octant with $0 \le y \le 2$. Let C be the curve of intersection of S with the plane $y = 2$. If $\boldsymbol{F} = \langle 1,\ y,\ x^2 \rangle$, find $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

(C is oriented so that the final point is in the xy-plane). (20 points)



$C:\ x = t,\ y = 2,\ z = 1 - t^2,\ 0 \le t \le 1$

$\Rightarrow \vec{r}(t) = \langle t, 2, 1 - t^2 \rangle$

$\Rightarrow d\vec{r} = \langle 1, 0, -2t \rangle dt$

also $\vec{F}(\vec{r}(t)) = \langle 1, 2, t^2 \rangle$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \langle 1, 2, t^2 \rangle \cdot \langle 1, 0, -2t \rangle dt$$

$$= \int_0^1 1 - 2t^3 \, dt$$

$$= t - \frac{t^4}{2} \Big|_0^1$$

$$= \boxed{\frac{1}{2}}$$

Bonus Problems: Note that you must complete all problems in the actual test to be eligible to attempt the bonus problems. Otherwise, anything you write on this page will be disregarded.

1. Verify your answer in problem 4. by using Green's Theorem. (5 points)

From problem 4, $P=-2y \quad Q=3x$

$\Rightarrow P_y=-2 \quad \Rightarrow Q_x=3$

By Green's Thm: $\int_C = -\iint_D Q_x - P_y \, dA$

negative orientation!

$= -\iint_D 5 \, dA$

Area of D

(or long way: $\int_\pi^{2\pi}\int_0^2 r \, dr \, d\theta$)

$= -5\iint_D dA$

$= -5 \frac{\pi(4)}{2}$

$= \boxed{-10\pi}$

2. Compute the arclength along $\vec{r}(t) = < t^2,\ 2t,\ \ln t >$ from the point $(1,2,0)$ to $(e^4, 2e^2, 2)$. (5 points)

Set $t^2=1, 2t=2, \ln t=0 \Rightarrow (1,2,0) \to t=1$

Set $t^2=e^4, 2t=2e^2, \ln t=2 \Rightarrow (e^4,2e^2,2) \to t=e^2$

$\Rightarrow L = \int_1^{e^2} |\vec{r}'(t)| \, dt$

$= \int_1^{e^2} \sqrt{(2t)^2+(2)^2+(\frac{1}{t})^2} \, dt$

$= \int_1^{e^2} \sqrt{4t^2+4+\frac{1}{t^2}} \, dt$

$= \int_1^{e^2} \sqrt{(2t+\frac{1}{t})^2} \, dt$

$= \int_1^{e^2} |2t+\frac{1}{t}| \, dt$

$= \int_1^{e^2} 2t+\frac{1}{t} \, dt$ since $2t+\frac{1}{t}>0$ for $1 \le t \le e^2$

$= t^2 + \ln|t| \Big|_1^{e^2}$

$= \boxed{e^4+1}$

3. Let $F = < y^2z^3,\ 2xyz^3,\ 3xy^2z^2 >$. Compute (5 points each):

(a) $\mathrm{curl}\vec{F}$ $= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$

$= \langle 6xyz^2 - 6xyz^2,\ 3y^2z^2 - 3y^2z^2,\ 2yz^3 - 2yz^3 \rangle$

$= \boxed{\langle 0, 0, 0 \rangle}$

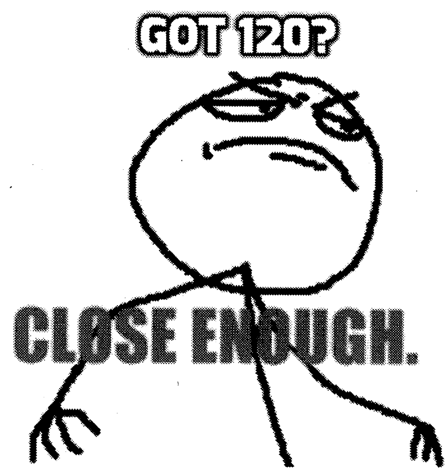
OR

$= \boxed{\vec{0}}$

(b) $\mathrm{div}\vec{F}$ $= \frac{\partial}{\partial x}(y^2z^3) + \frac{\partial}{\partial y}(2xyz^3) + \frac{\partial}{\partial z}(3xy^2z^2)$

$= 0 + 2xz^3 + 6xy^2z$

$= \boxed{2xz^3 + 6xy^2z}$


GOT 120?
CLOSE ENOUGH.