MATHEMATICS 391 **FINAL EXAM** **Spring, 2017**

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SECTION The Rebellion

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Do all your work in this answer booklet. If you need extra space, use the facing pages.

Part I Answer all questions in this part.

(1) Compute the general solution of each of the following (7 points each):

(a) $y^{(4)} - y' = 0.$ HOLDE w/ constant coefficients

$r^4 - r = 0$

$r(r^3-1) = 0$

$r(r-1)(r^2+r+1) = 0$

$\Rightarrow r=0, r=1, r = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\Rightarrow \boxed{y = c_1 + c_2 e^t + c_3 e^{-t/2} \cos \frac{\sqrt{3}}{2} t + c_4 e^{-t/2} \sin \frac{\sqrt{3}}{2} t}$

(b) $t \frac{dy}{dt} + 2ty = 1 - y.$ First order linear.

$\Rightarrow ty' + (2t+1)y = 1$

$\Rightarrow y' + (2 + \frac{1}{t})y = \frac{1}{t}$

$\Rightarrow \mu = e^{\int 2 + \frac{1}{t} dt} = e^{2t} \cdot t$

$\Rightarrow (te^{2t} y)' = e^{2t}$

$\Rightarrow te^{2t} y = \frac{1}{2} e^{2t} + C$

$\Rightarrow \boxed{y = \frac{1}{2t} + \frac{C}{te^{2t}}}$

(c) $x^2\,dy - (x^2+xy+y^2)\,dx = 0.$ First order homogeneous.

$$\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$$

$$\Rightarrow \frac{dv}{dx}x + v = 1+v+v^2$$

$$\Rightarrow \frac{dv}{dx}x = 1+v^2$$

$$\Rightarrow \int \frac{dv}{1+v^2} = \int \frac{dx}{x}$$

$$\Rightarrow \tan^{-1} v = \ln|x| + C$$

$$v = \tan(\ln|x|+C)$$

$$\Rightarrow \frac{y}{x} = \tan(\ln|x|+C)$$

$$\Rightarrow \boxed{y = x\tan(\ln|x|+C)}$$

(d) $y'' + 2y' + y = t^{-2}e^{-t}.$ SOLDE w/ constant coefficients and Variation of parameters.

$$r^2+2r+1=0$$

$$\Rightarrow (r+1)^2 = 0$$

$$r_{1,2} = -1$$

$$\Rightarrow y_h = c_1 e^{-t} + c_2 t e^{-t}$$

Now $W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t}-te^{-t} \end{vmatrix} = e^{-2t}$

$$y_p = -y_1\int\frac{y_2\cdot g}{W} + y_2\int\frac{y_1\cdot g}{W}$$

$$= -e^{-t}\int \frac{t\cancel{e^{-t}}\cdot t^{-2}\cancel{e^{-t}}}{\cancel{e^{-2t}}}\,dt + te^{-t}\int\frac{\cancel{e^{-t}}\cdot t^{-2}\cancel{e^{-t}}}{\cancel{e^{-2t}}}\,dt$$

$$= -e^{-t}\int t^{-1}\,dt + te^{-t}\int t^{-2}\,dt$$

$$= -e^{-t}\ln|t| - \cancel{t}e^{-t}\cdot\cancel{t}^{-1} = -e^{-t}\ln|t| - \cancel{e^{-t}}$$

redundant

$$\Rightarrow y = y_h + y_p$$

$$\Rightarrow \boxed{y = c_1e^{-t} + c_2te^{-t} - e^{-t}\ln|t|}$$

(2) Solve the following initial value problems (7 points each):

(a) $(\cos(xy) - xy\sin(xy) + 1)dx + (2y - x^2\sin(xy))dy = 0 \qquad y(0) = 3$ Exact.

$M_y = -x\sin xy - x\sin xy - x^2y\cos xy \qquad N_x = -2x\sin xy - x^2\cos xy$

$= -2x\sin xy - x^2y\cos xy$

$\Rightarrow \Psi = \int \cos xy - xy\sin xy + 1\, dx$

$= \frac{1}{y}\sin xy + x\cos xy - \int \cos xy\, dx + x$

$= \frac{1}{y}\sin xy + x\cos xy - \frac{1}{y}\sin xy + x + C_1$

$= x\cos xy + x + C_1$

also $\Psi = \int 2y - x^2\sin xy\, dy$

$= y^2 + x\cos xy + C_2$

$\Rightarrow x\cos xy + x + y^2 = C$

Applying initial conditions

$\Rightarrow 3^2 = C$

$\Rightarrow \boxed{x\cos xy + x + y^2 = 9}$

(b) $y'' - y' = 6t, \quad y(0) = 0,\ y'(0) = 0.$ $\longrightarrow$ SOLDE w/ constant coefficients (non-homogeneous).

$r^2 - r = 0$

$\Rightarrow r(r-1) = 0$

$r = 0,\ r = 1$

$\Rightarrow y_h = c_1 + c_2e^t$

$y_p = t(At + B) \longrightarrow$ Undetermined coefficients.

$\Rightarrow y_p' = 2At + B$

$y_p'' = 2A$

$\Rightarrow 2A - (2At + B) = 6t$

$\Rightarrow -2At + (2A - B) = 6t + 0$

$\Rightarrow A = -3 \Rightarrow -6 - B = 0 \Rightarrow B = -6$

$\Rightarrow y = c_1 + c_2e^t - 3t^2 - 6t$

$\Rightarrow y' = c_2e^t - 6t - 6$

$\Rightarrow y(0) = 0 = c_1 + c_2$

$\Rightarrow y'(0) = 0 = c_2 - 6 \Rightarrow c_2 = 6,\ c_1 = -6$

$\Rightarrow \boxed{y = -6 + 6e^t - 3t^2 - 6t}$

(3) (9 points)For the differential equation:

$$(2-x^2)y'' \; - \; xy' \; + \; x^2 y \; = \; 0$$

(a) Compute the recursion formula for the coefficients of the power series solution centered at $x_0 = 0$ and use it to compute the first three nonzero terms of the solution with $y(0) = 0, y'(0) = -36$.

$$(2-x^2)\sum_2^\infty n(n-1)a_n x^{n-2} - x\sum_1^\infty n a_n x^{n-1} + x^2\sum_0^\infty a_n x^n = 0$$

$$\Rightarrow \sum_2^\infty 2n(n-1)a_n x^{n-2} - \sum_2^\infty n(n-1)a_n x^n - \sum_1^\infty n a_n x^n + \sum_0^\infty a_n x^{n+2} = 0$$

$$\Rightarrow \sum_0^\infty 2(n+2)(n+1)a_{n+2}x^n - \sum_0^\infty n(n-1)a_n x^n - \sum_0^\infty n a_n x^n + \sum_2^\infty a_{n-2}x^n = 0$$

$$\Rightarrow \sum_0^\infty [2(n+2)(n+1)a_{n+2} - n(n-1)a_n - na_n + a_{n-2}]x^n + \underbrace{\dots}_{\text{sum from 1st 3 series at } n=0,\, n=1} = 0$$

$$\Rightarrow \boxed{a_{n+2} = \frac{n^2 a_n - a_{n-2}}{2(n+1)(n+2)}} \rightarrow \text{Recursion formula.}$$

$$\Rightarrow a_2 = 0 \qquad a_4 = \frac{4a_2 - a_0}{24} = 0$$

$$\Rightarrow a_3 = \frac{a_1 - a_{-1}}{12} = -\frac{36}{12} = -3 \qquad a_5 = \frac{9a_3 - a_1}{2(4)(5)} = \frac{-27+36}{40} = \frac{9}{40}$$

$$\Rightarrow y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$$

$$\Rightarrow \boxed{y = -36x - 3x^3 + \frac{9}{40}x^5 + \dots}$$

(b) Show that the solution given in (a) is an odd function (Hint: what is a_n when n is even ?)

- $a_n = 0$ if n is even, so only odd powers of x remain. Hence we get an odd function (polynomials w/ only odd powers of x are odd).
- Alternatively, note that $y(-x) = -36(-x) - 3(-x)^3 + \frac{9}{40}(-x)^5 + \dots$

$$= -(-36x - 3x^3 + \tfrac{9}{40}x^5 + \dots)$$

$$= -y(x).$$

(4) (a) (5 points) Compute the sine series for the function f such that $f(x) = \pi - x$ on the interval $[0, \pi]$.

$L = \pi$

$$\Rightarrow b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

$$\Rightarrow b_n = \frac{2}{\pi}\int_0^\pi (\pi - x) \sin nx \, dx$$

$\oplus$	-1	$-\frac{1}{n}\cos nx$
$\ominus$	0	$-\frac{1}{n^2}\sin nx$

$$= \frac{2}{\pi}\left[\frac{x-\pi}{n}\cos nx - \frac{1}{n^2}\sin nx\right]\Big|_0^\pi$$

$$= -\frac{2}{\pi}\left[-\frac{\pi}{n}\right] = \frac{2}{n}$$

$$\Rightarrow \boxed{F(f(x)) = \sum_1^\infty \frac{2}{n}\sin nx}$$

(b) (4 points) Compute the solution to the partial differential equation with x in the interval $[0, \pi]$ and $t > 0$:

$$\begin{aligned} u_t &= 9u_{xx} && \text{with} \\ u(0,t) = u(\pi,t) &= 0 \quad \text{for } t > 0 && \text{(boundary conditions)} \\ u(0,x) &= \pi - x \quad \text{for } 0 < x < \pi && \text{(initial conditions)} \end{aligned}$$

$$u(x,t) = \sum_1^\infty b_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\frac{n\pi x}{L}$$

$$\Rightarrow \boxed{u(x,t) = \sum_1^\infty \frac{2}{n} \cdot e^{-9n^2 t} \sin nx}$$

PartII Answer all sections of four (4) questions out of the five (5) questions in this part (10 points each).

(5) For the equation $t^2y'' - 7ty' - 16y = 0$ $(t > 0)$, $y_1(t) = t^4$ is a solution.

(a) Use the method of Reduction of Order to obtain a second, independent solution.

Set $y_2 = v \cdot y_1 = vt^4$

$\Rightarrow y_2' = v't^4 + 4vt^3$

$\Rightarrow y_2'' = v''t^4 + 4v't^3 + 4v't^3 + 12vt^2 = v''t^4 + 8v't^3 + 12vt^2$

$\Rightarrow t^2(v''t^4 + 8v't^3 + 12vt^2) - 7t(v't^4 + 4vt^3) - 16vt^4 = 0$

$\Rightarrow v''t^6 + 8v't^5 + 12vt^4 - 7v't^5 - 28vt^4 - 16vt^4 = 0$

$\vdots$

Something is wrong...

Yup! t^4 is NOT a soln.

This problem has a typo!

(b) Solve the equation directly, using that it is an Euler Equation.

Indicial eq: $r(r-1) - 7r - 16 = 0$

$\Rightarrow r^2 - r - 7r - 16 = 0$

$\Rightarrow r^2 - 8r - 16 = 0$

$\Rightarrow r = \dfrac{8 \pm \sqrt{64 + 4(16)}}{2}$

$= \dfrac{8 \pm \sqrt{64(2)}}{2}$

$= 4 \pm 4\sqrt{2}$

$\Rightarrow y = c_1 t^{4+4\sqrt{2}} + c_2 t^{4-4\sqrt{2}}$

(c) Compute the Wronskian of the pair of solutions.

$$W = \begin{vmatrix} t^{4+4\sqrt{2}} & t^{4-4\sqrt{2}} \\ (4+4\sqrt{2})t^{3+4\sqrt{2}} & (4-4\sqrt{2})t^{3-4\sqrt{2}} \end{vmatrix}$$

$= (4-4\sqrt{2})t^7 - (4+4\sqrt{2})t^7$

$= \boxed{-8\sqrt{2}\, t^7}$

(6) (a) State the definition of the Laplace transform and use it to compute the Laplace transform of the function f with $f(t) = 1$.

Defn: $\boxed{\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt}$

$$\Rightarrow \mathcal{L}\{1\}(s) = \int_0^\infty e^{-st}\,dt$$
$$= -\frac{1}{s}e^{-st}\Big|_0^\infty$$
$$= 0 - \left(-\frac{1}{s}\right)$$
$$= \boxed{\frac{1}{s}}$$

(b) Compute the Laplace transform $\mathcal{L}(y)(s)$ where y is the solution of the initial value problem:

$$2y'' - 5y' + y = 1 \qquad \text{with} \quad y(0) = -3 \quad \text{and } y'(0) = 5.$$

You need not compute the inverse transform.

Let $\mathcal{L}\{y(t)\} = Y$

$$\Rightarrow \mathcal{L}\{y'\} = -y(0) + s\mathcal{L}y = 3 + sY$$
$$\Rightarrow \mathcal{L}\{y''\} = -y'(0) + s\mathcal{L}y' = -5 + s(3 + sY)$$
$$= -5 + 3s + s^2Y$$

$$\Rightarrow -\cancel{10} + \cancel{6}s + 2s^2Y - \cancel{15} - 5sY + Y = \frac{1}{s}$$
$$\Rightarrow (2s^2 - 5s + 1)Y = \frac{1}{s} + 25 - 6s$$
$$\Rightarrow Y = \frac{\frac{1}{s} + 25 - 6s}{2s^2 - 5s + 1}$$
$$\Rightarrow \boxed{Y = \frac{1}{s} \cdot \frac{-6s^2 + 25s + 1}{2s^2 - 5s + 1}}$$

(7) (a) Compute the general solution of the differential equation

$$y^{(4)} + y'' - 6y' + 4y = 0.$$

(Hint: $r^4 + r^2 - 6r + 4 = (r^2 - 2r + 1)(r^2 + 2r + 4)$.)

Char. eq: $r^4 + r^2 - 6r + 4 = 0$

$\Rightarrow (r^2 - 2r + 1)(r^2 + 2r + 4) = 0$

$\Rightarrow (r-1)^2 (r^2 + 2r + 4) = 0$

$\Rightarrow r_{1,2} = 1, \quad r_{3,4} = \dfrac{-2 \pm \sqrt{4 - 4(4)}}{2} = -1 \pm \sqrt{3}\, i$

$$\Rightarrow \boxed{y = c_1 e^{t} + c_2 t e^{t} + c_3 e^{-t} \cos\sqrt{3}t + c_4 e^{-t} \sin\sqrt{3}t}$$

(b) Determine the test function $Y(t)$ with the fewest terms to be used to obtain a particular solution of the following equation via the *method of undetermined coefficients*. Do not attempt to determine the coefficients.

$$y^{(4)} + y'' - 6y' + 4y = 7e^t + te^t \cos(\sqrt{3}\, t) - e^{-t} \sin(\sqrt{3}\, t) + 5.$$

$$\boxed{\begin{aligned} Y = At^2 e^{t} &+ (Bt + C) e^{t} \cos\sqrt{3}t + (Dt + E) e^{t} \sin\sqrt{3}t \\ &+ \left(F e^{-t} \cos\sqrt{3}t + G e^{-t} \sin\sqrt{3}t\right) \cdot t + H \end{aligned}}$$

(8) For the differential equation $3x^2y'' + 2xy' + x^2y = 0$ show that the point $x = 0$ is a regular singular point (either by using the limit definition or by computing the associated Euler equation). Compute the recursion formula for the series solution corresponding to the larger root of the indicial equation. With $a_0 = 1$, compute the first three nonzero terms of the series.

Standard form: $y'' + \frac{2}{3x}y' + \frac{1}{3}y = 0$

$\Rightarrow$ To test $x=0$

$\lim_{x\to 0}(x-0)\cdot\frac{2}{3x} = 2/3$, $\lim_{x\to 0}(x-0)^2\cdot\frac{1}{3} = 0$

$\Rightarrow$ $\boxed{\text{x is a regular singular point.}}$

Indicial eq: $3r(r-1) + 2r = 0$

$3r^2 - r = 0$

$\Rightarrow r(3r-1) = 0$

$\Rightarrow r = 0, r = 1/3$ $\rightarrow$ larger root.

$$\Rightarrow 3x^2\sum_0^\infty (n+r)(n+r-1)a_nx^{n+r-2} + 2x\sum_0^\infty (n+r)a_nx^{n+r-1} + x^2\sum_0^\infty a_nx^{n+r} = 0$$

$$\Rightarrow \sum_0^\infty 3(n+r)(n+r-1)a_nx^{n+r} + \sum_0^\infty 2(n+r)a_nx^{n+r} + \sum_0^\infty a_nx^{n+r+2} = 0$$

$$\Rightarrow 3r(r-1)a_0x^r + 3(1+r)(r)a_1x^{1+r} + 2ra_0x^r + 2(1+r)a_1x^{1+r} + \sum_2^\infty\left[3(n+r)(n+r-1)a_n + 2(n+r)a_n + a_{n-2}\right]x^{n+r} = 0$$

$a_1 = 0$

$$\Rightarrow a_n = \frac{-a_{n-2}}{3(n+r)(n+r-1)+2(n+r)} = \frac{-a_{n-2}}{(n+r)(3n+3r-3+2)} = \boxed{\frac{-a_{n-2}}{(n+r)(3n+3r-1)} = a_n}$$

Recursion formula

$$\text{Set } a_n = \frac{-3a_{n-2}}{(3n+3r)(3n+3r-1)} \xrightarrow{r=1/3} \boxed{a_n = \frac{-3a_{n-2}}{(3n+1)(3n)}}$$

$a_0 = 1$

$a_1 = 0$

$a_2 = \frac{-3a_0}{7\cdot 6} = -\frac{1}{14}$

$a_3 = \frac{-3a_1}{\cdots} = 0$

$a_4 = \frac{-\cancel{3}a_2}{13\cdot \cancel{12}_4} = \frac{1}{14\cdot 13\cdot 4}$

$y = x^r(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots)$

$$\Rightarrow \boxed{y = x^{1/3}\left(1 - \frac{1}{14}x^2 + \frac{1}{14\cdot 13\cdot 4}x^4 + \ldots\right)}$$

(9) A 200 gallon tank initially contains 50 gallons in which are dissolved 5 pounds of salt. The tank is flushed by pumping pure water into the tank at a rate of 3 gallons per minute and a well-mixed solution is pumped out at a rate of 2 gallons per minute. Compute the time when the tank has filled, then write the initial value problems which describes amount of salt in the tank at any time before the tank is full. You need not solve the equation.

$$V = V_0 + (r_{in} - r_{out})t$$
$$= 50 + (3-2)t$$

When $V = 200$, we have

$$200 = 50 + t$$

$$\Rightarrow \boxed{t = 150 \text{ mins}} \longrightarrow \text{time to fill tank.}$$

$$S' = r_{in} c_{in} - \frac{S}{V_0 + (r_{in} - r_{out})t} \cdot r_{out}.$$
$$= 0 - \frac{S}{50+t} \cdot 2$$

$$\Rightarrow \boxed{S' = -\frac{2S}{50+t}, \quad S(0) = 5}$$