MATHEMATICS 391 **FINAL EXAM** **Spring, 2011**

Part I Answer all parts of all five (5) questions in this part.

(1) Compute the general solution of each of the following (7 points each):

(a) $t\,\frac{dy}{dt} + 4y \quad = \quad e^t/t^2 \quad (t > 0).$

(b) $((e^x \cos(y) + (1-x)\sin(y))\,\frac{dy}{dx} + e^x(1+\sin(y)) + \cos(y) \quad = \quad 0.$

(c) $(x-y)\,dy \quad = \quad y\,dx.$

(2) Solve the following initial value problems (8 points each):

(a) $y\,dx + e^{-x}\,dy \quad = \quad 0 \qquad\qquad y(0) = e^{-1}.$

(b) $y'' - 4y' + 5y \quad = \quad 16\cos(t) \qquad\qquad y(0) = 0,\ y'(0) = 0.$

(3) Compute the general solution of each of the following (8 points each):

(a) $x^2y'' - 3xy' + 4y \quad = \quad 0.$

(b) $y'' + y \quad = \quad sec(t).$

(4) For the differential equation:

$$(1+2x^3)y'' - xy' + 2y \quad = \quad 0$$

Compute the recursion formula for the coefficients of the power series solution centered at $x_0 = 0$ and use it to compute the first four nonzero terms of the solution with $y(0) = -6, y'(0) = 0$. (9 points)

(5) A tank with a capacity of 100 gallons initially contains 50 gallons of water with 10 pounds of salt in solution. Fresh water enters at a rate of 2 gallons per minute and a well-stirred mixture is pumped out at the same rate. Compute the amount of salt (in pounds) in the tank 10 minutes after the process begins. (8 points)

PartII Answer all sections of three (3) questions out of the five (5) questions in this part (10 points each).

(6) With y_1 and y_2 two solutions of the equation $y'' + py' + qy = 0$, show that the Wronskian of y_1 and y_2 is either never zero or identically zero (Hint: derive and solve a differential equation which is satisfied by the Wronskian of y_1 and y_2). Include the definition of the Wronskian.

(7) (a) Compute the general solution of the differential equation

$$y^{vi} + 8y''' \quad = \quad 0.$$

(b) Determine the test function $Y(t)$ with the fewest terms to be used to obtain a particular solution of the following equation via the *method of undetermined coefficients.* Do not attempt to determine the coefficients.

$$y^{vi} + 8y''' \quad = \quad t^2 + e^{-2t} - e^{-2t}cos(\sqrt{3}\ t)$$

(8) (a) A 2 pound weight stretches a spring 6 inches to a position $y = 0$. Set up an initial value problem (differential equation and initial conditions) for the position $y(t)$ (t seconds later) if the spring has been stretched down another 6 inches and is released.

(b) Compute the Laplace transform $\mathcal{L}(y)(s)$ where y is the solution of the initial value problem:

$$y\text{"} + 5y' + 3y = 0 \qquad \text{with} \quad y(0) = 1 \quad \text{and} y'(0) = -1.$$

(9) For the differential equation:

$$3xy'' + 2y' + (x+5)y \quad = \quad 0$$

Show that $x_0 = 0$ is a *regular singular point.* Compute the recursion formula for the coefficients of the series solution centered at $x_0 = 0$ which is associated with the larger root of the indicial equation and compute a_1 and a_2 when $a_0 = 1$.

(10) (a) Compute the sine series for the function $f(x) = 1$ on the interval $[0, \pi]$.

(b) Use your answer to part (a) to obtain a series solution $u(t, x)$ for the partial differential equation with x in the interval $[0, \pi]$ and $t > 0$:

$$\begin{aligned} u_t &= 64u_{xx} \quad \text{with} \\ u(t,0) = u(t,\pi) &= 0 \quad \text{for } t > 0 \qquad \text{(boundary conditions)} \\ u(0,x) &= 1 \quad \text{for } 0 < x < \pi \qquad \text{(initial conditions)} \end{aligned}$$